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JANUARY 1969

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EDWARD G. BEGLE

AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR EDWARD G. BEGLE

The Award for Distinguished Service this year has been given to a man who has served the mathematical interests of the nation in the broadest fashion and with the most direct and widespread effects, a man whose services, by their quality, diversity and extent, more than qualify him for the honor.

Born in Saginaw, Michigan, November 27, 1914, Edward Griffith Begle received his pre-doctoral training at the University of Michigan where, under the influence of Ray Wilder, he became interested in topology. Finishing his degree at Princeton with Solomon Lefschetz, and holding a National Research Fellowship for 1941–1942 at Ann Arbor, he then went to Yale where he remained until taking his present position at Stanford in 1961. His research at Princeton, Ann Arbor, and Yale yielded important results in locally connected spaces and generalized manifolds.

In 1951 his work entered wider channels when he was elected Secretary of the American Mathematical Society. In the six years he remained in this sensitive post he rendered noteworthy service in a period that contained chronic (and customary) financial crises, occasional internal dissension, and cloudy days on the international scene, a time when the A.M.S. was suffering pains of growth and adjustment. Throughout, his was a steadying and productive influence, one that helped significantly in the career of the Society.

In 1958, when the School Mathematics Study Group began, he consented to be its director. The leadership he has provided has been consistently devoted to the ideal of making first-rate mathematics appropriate, attractive and available to pre-college students. Initially it was obvious that reform in mathematics curricula was needed and long past due, but it was not at all clear how it might be accomplished. Through the ensuing clouds of discussion and disagreement, and the years of development and devotion (and duodenalism), he remained a balanced perceptive and judicious guide, and it is in the largest measure due to him that S.M.S.G. has been able to generate such solid accomplishments. These benefits have not been confined to curricula. Through S.M.S.G. various components of the mathematical community that had been largely ignorant of each other began working together; each acquired more knowledge and understanding of the others and of the problems facing mathematics education at various levels. The community has become more knowledgeable, more democratic, and more capable of making further progress. Moreover, the information acquired by S.M.S.G., particularly through its testing programs, should provide a base for comprehension of what the precollege mathematics education apparatus of the country really does. In all these endeavors Professor Begle has had an influence frequently catalytic and always most fundamental and discerning.

This influence has been felt beyond this country's precollege mathematics. The direct effects in Latin America, and indirect ones in Africa, the Antipodes, and Europe, attest to this. And to those involved in undergraduate mathematics

here, the effect on our work and its possibilities has become clear; many freshmen now arriving are better trained than we would have thought possible ten years ago.

With the assistance of many individuals and of the components of the mathematical community and with the substantial support of the National Science Foundation he has conducted a national experiment, unprecedented, as far as I know, in its combination of depth, scope and size. He has done so with character and courage, with good judgment and balance, with understanding and endurance, and in a continual searching for the first rate. He is a mixture of Welshman, New Englander, American, mathematician, teacher, and sachem, and we are all in his debt. The mathematical and scientific part of American life in the middle third of this century may well be judged to have been outstanding; if the history of it is ever properly written, E. G. Begle's role therein will clearly have an exceptional place.

B. J. PETTIS

EDITORIAL

Entering the last quarter of its first century, this MONTHLY has a new cover. Few remember that it was ever anything but blue on blue, as introduced by Editor Lester R. Ford, Sr. in 1942. We hope the MONTHLY will be better than ever. If so, this will be due to the excellence of our authors' contributions and the hard work of the Associate and Collaborating Editors and the referees.

The Editor invites criticisms, complaints, and suggestions.

Backlog: Main articles, 11 months; Mathematical Notes, 10 months; Research Problems, 4 months; Classroom Notes, 4 months; Mathematical Education, 4 months.

HARLEY FLANDERS, *Editor*

STATEMENT OF POLICY

From its inception, the MONTHLY has been the journal of college mathematics, designed to compete neither with journals of school mathematics nor with journals of mathematical research. Its founders pledged themselves to promote and advance college mathematics. As have editors in the past so do we again endorse this pledge.

Most of the MONTHLY subscribers teach college mathematics. In our view we can best serve them by using the following criteria for selecting material.

MAIN ARTICLES. We seek expository and survey articles the subject matter of which is relevant to the current mathematical scene.

MATHEMATICAL NOTES. We seek brief papers which give new insights, new proofs of old theorems, and mathematical pearls. Notes should be one or two pages in length.

RESEARCH PROBLEMS. We seek easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics.

It is not the task of the MONTHLY to publish research which is of interest only to a small group of specialists. However, original research papers which are of wide interest, which can be read with profit by nonspecialists, and which meet our standards of exposition will be published.

CLASSROOM NOTES. We seek short papers with mathematical content suitable for classroom presentation at the undergraduate and early graduate levels.

MATHEMATICAL EDUCATION. This is an area of experiment, rapid change, and considerable controversy. We shall encourage open discussion of all professional aspects of education pertinent to our work as teachers of mathematics.

PROBLEMS. From Vol. 1, No. 1, the problem section has been the backbone of the MONTHLY. We seek problems of merit, both elementary and advanced, particularly problems in modern mathematics.

REVIEWS. Textbooks, monographs, and films of interest to our readers will be reviewed. We shall encourage reviewers to be critical in addition to being informative.

HARLEY FLANDERS, *Editor*

PRIZE CONTEST FOR UNDERGRADUATES

A prize will be awarded for the best expository article on the topic

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The winning article will be published in the MONTHLY. It should run 5 to at most 10 pages in print.

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WHAT IS GLOBAL ANALYSIS?

S. SMALE, University of California, Berkeley

There has recently been a lot of activity in that branch of mathematics now referred to as "global analysis." For example, the subject of the 1968 Summer Institute of the American Mathematical Society was global analysis.

My definition of global analysis is simply the study of differential equations, both ordinary and partial, on manifolds and vector space bundles. Thus one might consider global analysis as differential equations from a global, or topological point of view.

Even the earliest studies of differential equations contained an element of global analysis; this element had become quite important for example in the work of Poincaré on ordinary differential equations. G. D. Birkhoff's development of dynamical systems and especially M. Morse's theory of geodesics are both excellent examples of global analysis. After the rapid recent progress in topology, the subject of our exposition has been moving especially fast. After mentioning a couple of references in partial differential equations, I shall devote the rest of my article to an account of a theorem in dynamical systems to illustrate the global analysis point of view.

Recently there have been nice results in the topology of linear elliptic differential operators, especially in the work of Atiyah, Singer, and Bott (see for example [2] and [4]).

One cannot expect to have a satisfactory framework for nonlinear partial differential equations with linear function spaces. Thus it is important that nonlinear partial differential equations are beginning to be attacked by a systematic use of infinite dimensional manifolds of maps. A good survey of this is Eells [3].

The work of Andronov, Pontryagin [1] and Peixoto [5] in dynamical systems (or ordinary differential equations), on one hand can be explained in relatively simple terms and on the other hand gives a real insight into this modern way of looking at differential equations. I shall try to give a brief account of their theory now.

Consider an ordinary differential equation (1st order, autonomous) defined on a domain D in the x, y-plane:

$$\frac{dx}{dt} = P(x, y) \qquad \frac{dy}{dt} = Q(x, y).$$

Stephen Smale received his Ph.D. at the University of Michigan in 1956. He has occupied various positions at the University of Chicago, the Institute for Advanced Study, Columbia University, and his present location, the University of California at Berkeley. For his outstanding research in differential topology and in global analysis, Professor Smale was awarded the Fields Medal of the International Mathematical Union in 1966 and the Veblen Prize of the American Mathematical Society in 1964. *Editor*

We shall assume that these functions P, Q defined on D are continuously differentiable (or of class C^1). Now the fundamental existence theorem of ordinary differential equations yields for each (x_0, y_0) in D and real t sufficiently small in absolute value, $|t| < \epsilon$, functions $f(x_0, y_0, t)$, $g(x_0, y_0, t)$ which satisfy the initial conditions $f(x_0, y_0, 0) = x_0$, $g(x_0, y_0, 0) = y_0$ and the differential equation

$$\left(\frac{df}{dt}\right)(x_0, y_0, t) = P(f(x_0, y_0, t), g(x_0, y_0, t))$$

$$\left(\frac{dg}{dt}\right)(x_0, y_0, t) = Q(f(x_0, y_0, t), g(x_0, y_0, t)).$$

Let us look at this phenomenon from a more geometric point of view and in fact get away from the particular choice of x, y-coordinates.

To each (x, y) in D associate the vector $(P(x, y), Q(x, y))$ of the x, y-plane with the initial point at (x, y). This gives us what is called a C^1 vector field on D. For each point p of D, we will call the associated vector for short $X(p)$. Then the existence theorem we just stated may be interpreted to yield a system of plane curves $\phi_t(p)$ with $\phi_0(p) = p$, and with the property that the tangent of the curve at a point q of D will be the vector $X(q)$ (see Figure 1).

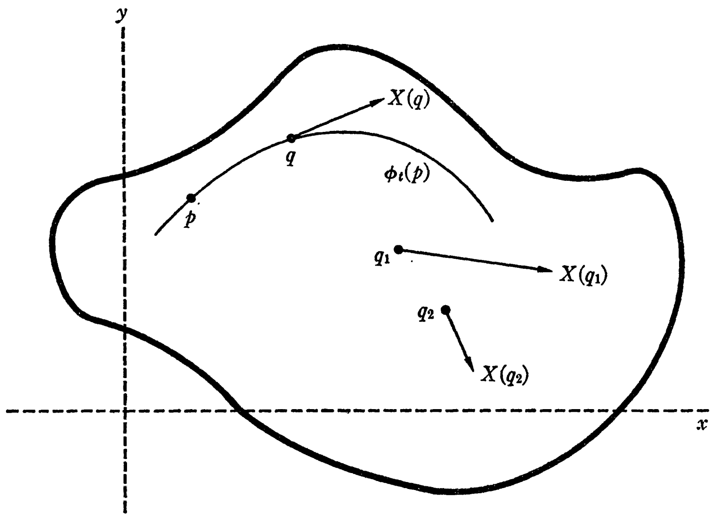

FIG. 1.

The right context for the study of this differential equation becomes clearer now. More generally than a domain of the Euclidean plane, consider a 2-dimensional smooth manifold M. Roughly speaking, one can think of this as a surface in 3-dimensional Euclidean space E^3 or better abstractly as a space on which differentiation makes sense and a neighborhood of each point is a domain in the plane. To each point p of M there is associated a 2-dimensional vector space

$T_p(M)$, the tangent space of M at p. If M is a surface in E^3 then $T_p(M)$ is the plane tangent to M at p.

A vector field X on M is an assignment, continuously differentiable, $p \to X(p)$ for p in M to $X(p)$, a "vector" in $T_p(M)$. The vector field on D defined previously from the differential equation given by the functions P, Q on D is now a vector field on the 2-manifold D in this sense.

To define the basic idea of this article, structural stability of a differential equation, we need to develop two things: one, the space of differential equations on M, $\chi(M)$, and two, an equivalence relation on $\chi(M)$, the phase portrait.

We have seen that the kind of differential equations on M we are studying (which are really pretty general except for the low dimension) correspond to vector fields on M. We call the set of all vector fields (C^1 as usual) on M, $\chi(M)$.

Now $\chi(M)$ has the structure of a vector space, using the fact that for each $p \in M$, the values of all vector fields lie in the same linear space $T_p(M)$. That is if X, Y belong to $\chi(M)$, $(X+Y)(p) = X(p) + Y(p)$. This space $\chi(M)$ will be basic in what follows.

The solution curves $\phi_t(p)$ of a vector field X on M, defined earlier, may be "pieced together " so that for each p, $\phi_t(p)$ will be defined for all $a < t < b$ where the interval (a, b) is maximal. If M is compact, for each p, this interval will be $(-\infty, \infty)$ so that we have a 1-parameter group ϕ_t of transformations on M. Thus for each real t, ϕ_t is a C^1 transformation of M, ϕ_t: $M \to M$, with a C^1 inverse, ϕ_0 is the identity and $\phi_t(\phi_s) = \phi_{t+s}$. In short ϕ_t is a dynamical system.

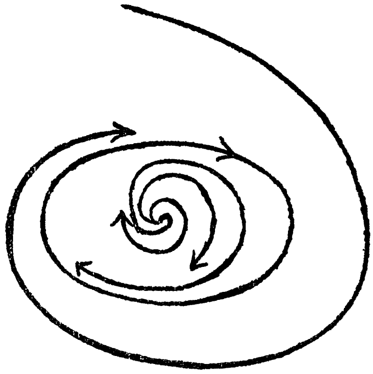

FIG. 2a.

To abstract the qualitative features of a differential equation on M, the concept of a phase portrait becomes important. Usually the phase portrait means the picture of the solution curves of the differential equation. For example, Figure 2a is the phase portrait of a differential equation in the plane.

To give a precise mathematical content to "phase portrait," we proceed as

follows. Say X, Y in $\chi(M)$ are *topologically equivalent* when there is a homeomorphism h: $M \rightarrow M$ taking solution curves of X into those of Y. Thus the differential equation in Figure 2a is topologically equivalent to that described in Figure 2b.

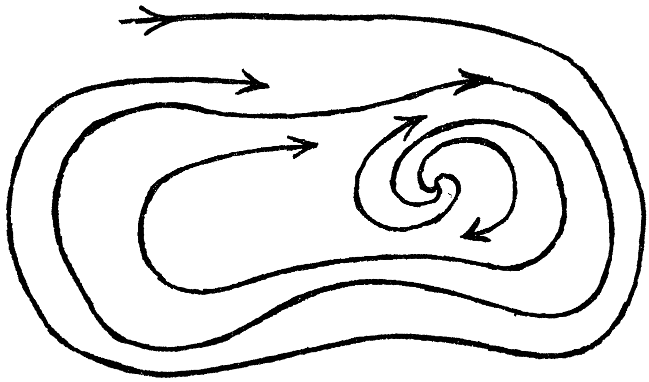

FIG. 2b.

Then two differential equations on M have the same phase portrait if they are topologically equivalent. A definition of *phase portrait* is thus a topological equivalence class of differential equations on M. A main goal of the qualitative study of ordinary differential equations is to obtain information on the phase portrait of differential equations.

To make progress in this direction, one soon sees the need to avoid "degenerate" cases. For example a differential equation that is zero on all of M, or even on some nonempty open set of M should be considered degenerate and excluded from most considerations. I think that engineers and physicists will agree with this statement.

To aid in discussing the question of degeneracy, a topology or metric on $\chi(M)$ is useful. To simplify matters in defining this metric, in the rest of our article, we will assume M compact. This excludes many or even most interesting examples, but on the other hand the main features are not lost.

Assuming M compact define a norm $\|\ \|$ on $\chi(M)$ as follows. Let $U_1, \cdots, U_k$ be a covering of M, $\overline{U}_i \subset V_i$, with each V_i a plane domain. Then on each V_i, X in $\chi(M)$ is represented by $P_i(x, y)$, $Q_i(x, y)$ as at the beginning. Then $\|X\|$ is defined as the maximum of the following finite set of numbers:

$$\sup_{(x,y)\in U_i} |P(x, y)| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i} |Q(x, y)| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i} \left| \frac{\partial P}{\partial x}(x, y) \right| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i} \left| \frac{\partial P}{\partial y}(x, y) \right| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i} \left| \frac{\partial Q}{\partial x}(x, y) \right| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i} \left| \frac{\partial Q}{\partial y}(x, y) \right| \qquad i = 1, \cdots, k.$$

This gives $\chi(M)$ the structure of a complete normed space or a Banach space. A metric on $\chi(M)$ is then defined by $d(X, Y) = \|X - Y\|$.

With this metric on $\chi(M)$ it is possible to say when differential equations are "close." In terms of local coordinate representations, two differential equations are close when the P and Q are uniformly close, with their first derivatives uniformly close as well.

With this background, we say that X in $\chi(M)$ is *structurally stable* when there is a neighborhood $N(X)$ in $\chi(M)$ with the property that every Y in $N(X)$ is topologically equivalent to X. Thus X is structurally stable when nearby differential equations have the same phase portrait. A little thought will indicate that this excludes degeneracy; a structurally stable X cannot be degenerate (in some senses at least). It is an important concept for the engineer who studies qualitative differential equations, since in engineering the differential equations one works with are only approximations of the real equations. The engineer wants the qualitative conclusion he makes to be valid for the actual differential equation which describes his world. In fact the original idea of structural stability was the joint work of an engineer, A. Andronov, and a mathematician, L. Pontryagin.

Thus it becomes important to know if most differential equations are structurally stable.

THEOREM. (M. Peixoto) *If M is a compact 2-dimensional manifold, then the structurally stable differential equations in $\chi(M)$ form an open and dense set.*

This theorem is an excellent theorem in global analysis. One sees in two ways how it is global. First the differential equation is defined over a whole manifold, and structural stability depends on its behavior everywhere. Second, the theorem makes a conclusion about the space of all differential equations on M.

The proof gives much information on the structure of differential equations on 2-manifolds.

We state the main lemma which indicates how this is so.

The nonwandering set $\Omega(X)$ of X is defined as the set of x in M such that for every neighborhood U of x and t_0, there is a $t > t_0$ with $\phi_t(U) \cap U \neq \emptyset$.

MAIN LEMMA. *If M is a compact 2-manifold and X is in $\chi(M)$, then X is structurally stable if and only if the following conditions are met:*

(a) *Each closed orbit and each singular point of X is "nondegenerate." This nondegeneracy is defined in terms of derivatives associated to the closed orbits and singular points.*

(b) *The separatrices of saddle points don't meet.*

(c) *$\Omega(X)$ consists of the finite union of closed orbits and singular points.*

[*Separatrices* are the trajectories which come to and leave from the saddle points.]

If α is a singular point or closed orbit, let $W^s(\alpha)$ be the set of x in M with $\phi_t(x)\to\alpha$ as $t\to\infty$. Then if X is structurally stable, it provides for a decomposition of M as the finite union of $W^s(\alpha)$ as α ranges over the closed orbits and singular points. This decomposition gives a good practical understanding of the differential equation X.

A survey of this subject with many references is [6].

This article is based on an address before the Mathematical Association of America, San Francisco, 26 January, 1968.

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AUTOBIOGRAPHICAL NOTES

GRIFFITH C. EVANS, University of California, Berkeley

In the second semester of his junior year at Harvard, Evans was allowed to teach an independent class. Previously he had graded papers, and his change in station may have been due to his habit of supplementing grades with brief suggestions for improvement. In fact, the graded papers were left on a shelf where the teachers could inspect them. Also this class was itself inspected once by Professor Julian Coolidge.[1] That it went well was surely a tribute to the students who were so courteous to a teacher younger than themselves.

This teaching was paid for by the Department of Mathematics. But the next and later years, off and on, Evans was listed as a regular instructor. In the year 1909–10, while writing his Ph.D. thesis for Maxime Bôcher, away at the time, one of Evans's classes was Bôcher's class on vector analysis, including quaternions, with some additions toward the end. The class of twelve was quite wonderful, containing at ends of the spectrum Norbert Wiener, as informal listener, and William Sidis, freshman (aetatis 11).

How could Evans refrain from becoming a mathematics teacher? The world seemed young, and there was plenty of time to study philosophy and physics, to read for the sake of reading, to be a dilettante in French and German literature. His first paper was a summary in 1909 of the first part of his thesis, the second in 1910, a ten page contribution in physics on Kirchhoff's Law, both published in the Proceedings of the American Academy of Arts and Sciences. The world was bright. The year 1914 was indeed far around the bend.

From 1910 to 1912 Evans studied with Vito Volterra in Rome, and in 1912 was appointed to teach at the Rice Institute by President Edgar Odell Lovett who was in Europe seeking a faculty. Thus Evans became Assistant Professor of Pure Mathematics, and at about the same time P. J. Daniell, Senior Wrangler at Cambridge, became Assistant Professor of Applied Mathematics—sufficient reason of course for Evans to work in applied mathematics and Daniell to invent the Daniell Integral.

The department owed much to Julian Coolidge. He let us have William C. Graustein, who had been in the vector analysis class, and later Lester R. Ford, as assistant professors, the former after post-Ph.D. work in Germany, the latter

1. I still see him standing for a minute or two in the back of the room.

Professor Evans is respected for his continuing research in integral equations, potential theory, complex variables, and theoretical economics. His two Amer. Math. Soc. Colloquium Publications *Functionals and their Applications* (1918) and *The Logarithmic Potential* (1927) are widely quoted. Yet he writes: "Nevertheless I have enjoyed my long life as a teacher." Concerning the present article, Professor Evans says: "I use the third person in what follows in the hope of making it to some extent objective."

This is the first of several articles by senior mathematicians who have made noteworthy contributions to the development of American mathematics. *Editor*

similarly in Edinburgh. Also we acquired Hubert E. Bray and Arthur H. Copeland as teaching assistants. Later, of course, young people came to us from elsewhere: thus, Aristotle Michal, from Roberts College on the Adriatic, via Clark University, where A. G. Webster was professor.

Evans had been reading something on statistics by P. Lévy, and this inspired him to ask Copeland to teach a course on the subject at Rice. While teaching, Copeland developed, entirely on his own, a thesis on statistics to submit to Harvard for the Ph.D. degree. As Ph.D. he returned to Rice to teach again.

Rice had two NRC Fellows in 1932–33. One was Charles B. Morrey, Jr., who came after a year at Harvard. The other was Robert Martin, a pupil of Aristotle Michal: both master and pupil died too young.

In the fall of 1934 Evans transferred to the University of California at Berkeley, although he had made the decision in the spring of 1933. Thus he was able to send Dr. Morrey ahead of him.

California was growing rapidly and some of the older professors were retiring. There was a Reorganization Committee which assisted in the growth of various departments. In mathematics, the Committee had already appointed Alfred L. Foster, likewise an NRC Fellow. It was Evans's fortune, good or bad, to be chairman for fifteen years. Thus, as at Rice, he was pressed for time in writing for himself.

In 1931, during a summer term at the University of Minnesota, Evans had an opportunity to know R. A. Fisher very well. All the more then, remembering Copeland, he was anxious to see mathematical statistics developed at the University of California, and so wrote to President Sproul as early as 1935 or '36. Among four possibilities, the department finally chose Jerzy Neyman, who set up immediately a statistical laboratory, surely a valuable addition to the University of California. Later it became a separate department.

Depression or not, war or not, to the rapidly growing University came fine mathematicians. Hans Lewy had been a Privat-dozent in Germany and was already known for his mathematics. He saw what was coming and became an exile to America and a member of our Faculty. (After the War, he was offered his old position as Privat-dozent. By that time he was Full Professor.) Derrick Lehmer, teaching number theory at Lehigh University, also an NRC Fellow, succeeded his father at the University of California in mathematics and in making calculating machines. Frantisek Wolf was another exile in California: he taught operator theory to many budding Ph.D.'s.

During World War II, a list of refugees was sent by a national committee to various universities. President Sproul set up a local committee, including a mathematician and an engineer, to see what ones might be useful to us. No committee ever worked more assiduously in setting up such a list, although we guessed pretty well what might happen: namely, that those chosen by us would be chosen also and taken by various Eastern universities whose names were more famous abroad. But from knowledge of our own, our committee added two

names not on the list, one in mathematical logic, Alfred Tarski, the other an architect engineer in prestressed concrete. They stayed with us.

Others who came during this chairmanship were Stephen P. Diliberto from Princeton, to teach mathematical astronomy, Edmund Pinney from the California Institute of Technology, in the field of structure and design, Anthony P. Morse from Cornell and the Institute for Advanced Study, the author of *A Theory of Sets* containing a formal treatment of logic, and Raphael M. Robinson, from California and Brown, in logic and complex variables.

But this was only the beginning of the rapid growth and increasing prestige of the Department of Mathematics in the University of California at Berkeley.

ON A THEOREM OF FROBENIUS

RICHARD BRAUER, Harvard University

1. A well-known theorem of Frobenius states that if G is a finite group of order g and if n is a divisor of g, the number of solutions β of the equation $\beta^n=1$ in G is a multiple of n. This can be generalized in various ways. A number of proofs have been given [1–7]. I shall give here still another proof. I found this proof a very long time ago when I was still a student. I did not publish it, but I have used it for many years as material for problems in courses, giving a number of hints. I present the proof here in the hope that other teachers may find it interesting and useful for the same purpose.

2. We start with a lemma.

LEMMA. *Let H be a group. Let A be a normal subgroup of finite order a. If $\sigma\in H$ and if $\alpha\in A$, then σ^a and $(\sigma\alpha)^a$ are conjugate in H.*

Proof. Set $\tau=\sigma\alpha$. For each $\beta\in A$, consider the set S_β of distinct elements $\sigma^{-j}\beta\tau^j$ with $j=0, \pm 1, \pm 2, \cdots$. Since

$$\sigma^{-j}\beta\tau^j \in \sigma^{-j}\beta(\sigma\alpha)^jA = \sigma^{-j}\sigma^jA = A,$$

S_β is a subset of A. In particular, S_β is a finite set. For two integers i and j, we have

$$\sigma^{-i}\beta\tau^i = \sigma^{-j}\beta\tau^j \tag{1}$$

Professor Brauer did his doctoral work under I. Schur at the University of Berlin. He has been on the faculties of the Univ. of Koenigsberg, Univ. of Kentucky, Institute for Advanced Study, Univ. of Toronto, and Univ. of Michigan. Presently he holds the Perkins Professorship at Harvard. On leaves of absence he held positions at the Tata Institute, Nagoya Univ., the Akademie der Wissenschaften Göttingen (Gauss Professor), and the E. T. H. Zürich. He is best known for his contributions to the representation theory of groups and algebras and was awarded the AMS Cole Prize in Algebra in 1949. *Editor*

if and only if $\beta^{-1}\sigma^{j-i}\beta=\tau^{j-i}$. It follows that the number N_β of distinct elements in S_β is the least positive exponent N for which

$$\beta^{-1}\sigma^N\beta = \tau^N \tag{2}$$

and that (1) holds if and only if N divides $i-j$. It is also clear that if β and γ are elements of A, the sets S_β and S_γ are equal or disjoint. Hence A is a disjoint union of sets S_β, say

$$A = S_{\beta_1} \cup S_{\beta_2} \cup \cdots \cup S_{\beta_r} \tag{3}$$

with $\beta_i \in A$. Let m denote the minimal value of N_{β_i} with $1 \leqq i \leqq r$. Then by (2), m is the least positive exponent for which conjugation by an element β of A carries σ^m into τ^m. Moreover, if exactly k of the r numbers N_{β_i} are equal to m, the km elements of the corresponding sets S_{β_i} are exactly the elements β of A for which (2) holds with $N=m$. On the other hand, the number of these elements is the order of the centralizer of σ^m in A and hence divides a. Thus km divides a. Since σ^m and τ^m are conjugate and since m divides a, we see that σ^a and τ^a are conjugate as stated.

Remark. The proof shows that there exist elements $\beta \in A$ which carry σ^a into τ^a.

3. Let G be a group and let H be a subgroup. We shall say that two elements β, γ of G are *equivalent with regard to* H, if $\beta^{-r}\gamma^r \in H$ for all integers r. Clearly this is an equivalence relation. If β is an element of G, denote by F_β the set of all elements $\sigma \in G$ for which $\beta^{-r}\sigma\beta^r \in H$ for all integers r.

Proposition 1. *Let G be a group, H a subgroup, and β an element of G. Then F_β is a subgroup of H and $\beta^{-1}F_\beta\beta = F_\beta$. An element γ of G is equivalent to β with regard to H if and only if $\gamma \in \beta F_\beta$.*

Proof. The first statement is obvious. If $\gamma=\beta\phi$ with $\phi \in F_\beta$, then since F_β is normal in the group generated by β and F_β, for any integer r

$$\beta^{-r}\gamma^r = \beta^{-r}(\beta\phi)^r \in \beta^{-r}\beta^r F_\beta = F_\beta \subseteq H$$

and β and γ are equivalent.

Conversely, if β and γ are equivalent, set

$$\delta_r = \beta^{-r}\gamma^r \in H.$$

Then, for integral s

$$\beta^{-s}\delta_1\beta^s = \beta^{-s}\beta^{-1}\gamma\beta^s = \beta^{-s-1}\gamma^{s+1}\gamma^{-s}\beta^s = \delta_{s+1}\delta_s^{-1} \in H.$$

This shows that $\delta_1 \in F_\beta$. Hence $\gamma=\beta\delta_1 \in \beta F_\beta$ as stated.

Proposition 2. *If H in Proposition 1 has finite order h and if β and γ are equivalent with regard to H, then β^h and γ^h are conjugate.*

Proof. By Proposition 1, $\gamma=\beta\phi$ with $\phi \in F_\beta$. Apply the lemma to the group $H=\langle\beta, F_\beta\rangle$ and its normal subgroup $A=F_\beta$, noting that the order of F_β divides h.

4. As before, let G be a group and H a subgroup. We shall say that two elements β, γ of G are *weakly equivalent with regard to* H, if there exists an element $\tau \in H$ such that $\beta^{-r}\tau\gamma^r \in H$ for all integers r. Again this is an equivalence relation. Since our condition can be written in the form

$$\beta^{-r}(\tau\gamma\tau^{-1})^r \in H,$$

we have

PROPOSITION 3. *Two elements β, γ of G are weakly equivalent with regard to H if and only if β is equivalent with regard to H to an H-conjugate $\tau\gamma\tau^{-1}$ of γ, $(\tau \in H)$.*

It follows from Propositions 3 and 1 that the class of β with regard to weak equivalence consists of the elements of the form

$$\gamma = \tau^{-1}\beta\delta\tau, \qquad \delta \in F_\beta, \qquad \tau \in H. \tag{4}$$

It is clear that it suffices to let τ range over a set T of representatives for right cosets in H modulo F_β. We claim that the corresponding representation (4) of γ is unique. This will be shown, if we can prove the following statement:

PROPOSITION 4. *Let G, H, β, F_β be as before. If $\sigma \in H$ and if for some δ and δ' in F_β*

$$\sigma^{-1}\beta\delta\sigma = \beta\delta' \tag{5}$$

then $\sigma \in F_\beta$.

Proof. It follows from (5) that

$$\beta^{-1}\sigma\beta \in F_\beta\sigma F_\beta.$$

Since β normalizes F_β, then for integral r,

$$\beta^{-r-1}\sigma\beta^{r+1} \in F_\beta\beta^{-r}\sigma\beta^r F_\beta.$$

Since $\sigma \in H$, it follows by induction on r, (both for $r > 0$ and for $r < 0$) that $\beta^{-r}\sigma\beta^r \in H$. Hence $\sigma \in F_\beta$ as claimed.

The next proposition is a consequence of the remark following (4), of Proposition 4, and of Proposition 2.

PROPOSITION 5. *Let G be a group and H a subgroup of finite order h. If $\beta \in G$, the class of β with regard to weak equivalence consists of exactly h elements γ and for all these elements γ, the elements β^h and γ^h are conjugate in G.*

5. One form of the Frobenius' Theorem reads

THEOREM. *Let G be a group of finite order g and let K be a conjugate class of G. If n is a positive integer and $(g, n) = d$, the number N of elements $\beta \in G$ with $\beta^n \in K$ is divisible by d.*

(It is well known that a slightly better result is obtained by applying the theorem to the centralizer in G of an element of K. If K consists of k elements,

it follows that N is divisible by (g, nk).)

Using Sylow's Theorem, we can obtain Frobenius's Theorem as an immediate consequence of Proposition 5. Indeed, if p^r is any prime power dividing $d=(g, n)$, we choose H as a subgroup of order $h=p^r$ of G. Since the set of elements β with $\beta^n \in K$ consists of full weak equivalence classes with regard to H, the number N is divisible by p^r and hence by d.

It is clear that the same method still applies to infinite groups G, provided that G possesses suitable finite subgroups. We have

THEOREM. *Let G be a group and let K be a conjugate class of G. Let n be an integer and assume that the number N of elements β of G with $\beta^n \in K$ is finite. Then N is divisible by every prime power p^r dividing n for which there exist subgroups H of order p^r of G.*

Far-reaching generalizations of Frobenius's Theorem have been given by P. Hall [5]. These cannot be obtained here.

6. We mention another consequence of the lemma in Section 2.

PROPOSITION 6. *Let G be a finite group of order g which has a normal subgroup H of order h. If $\beta \in G$ has order relatively prime to h and if C is the centralizer of β in G, then the centralizer $\bar{C}$ of βH in G/H is CH/H.*

Proof. The reciprocal image C^* of $\bar{C}$ in G consists of the elements $\gamma \in G$ for which

$$\gamma^{-1}\beta\gamma \in \beta H. \tag{6}$$

It is clear that $CH \subseteq C^*$. Conversely, if $\gamma \in C^*$ then by the lemma and the remark following it, there exists an element $\sigma \in H$ such that

$$\sigma^{-1}\beta^h\sigma = \gamma^{-1}\beta^h\gamma.$$

Since h and the order of β are coprime, then

$$\sigma^{-1}\beta\sigma = \gamma^{-1}\beta\gamma.$$

Hence $\gamma\sigma^{-1} \in C$, $\gamma \in CH$.

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FLYING IN A WIND FIELD, I

M. S. KLAMKIN, Ford Motor Company, Dearborn, Michigan, and D. J. NEWMAN, Ford Motor Company (and also Yeshiva University)

A usual problem in flying from a given point to another given point is to determine the course of minimum flight time. While the answer to this problem is not unexpected, an elementary solution is by no means obvious. Consequently, any interest in this paper should not be focused on the result but on the various elementary geometrical techniques which are employed and which should be understandable to an unsophisticated audience.

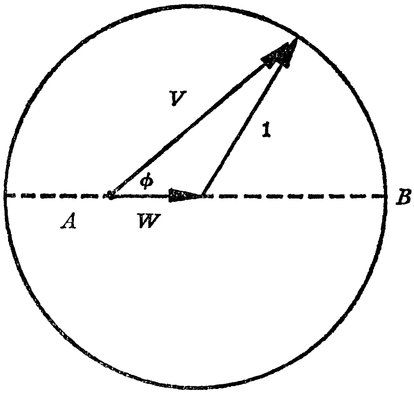
In our discussion, we will be assuming that the wind velocity W is constant and horizontal and that the airplane is flown in a horizontal plane with a constant relative speed of unity with respect to the wind. This reduces the problem to a two-dimensional one. We will say something about the corresponding three-dimensional problem at the end.

Many pilots, intuitively, fly the straight line course between the two given points. While it is obvious that this gives the path of least distance, it is not obvious, however, that this also gives the path of minimum flight time. E. Walters, (University of Michigan and also a pilot) to whom we are grateful for suggesting this problem to us, set up a calculus of variations approach to establish the desired path. Although it is easy to show that the corresponding Euler-Lagrange equation leads to a straight line path, one should avoid the latter approach if possible since the verification that the straight line path is both necessary and sufficient among the class of *all* possible flying courses is not easy. Also, since the solution is so simple, one should expect a simpler method of proof.

We will establish that the straight line course is the one of minimum flight time by several different methods. The first method will be to minimize an integral representing the time of flight without recourse to the calculus of variations (i.e., by elementary means). The second method will involve an elementary inequality concerning a circle. The third and last method, which is the most elegant, will use the world line of the airplane. Subsequently, we consider the more general problem of determining the path of minimum flight time from a given point to a given curve. The solution of this problem is obtained in an elementary manner by considering an appropriate family of homothetic circles.

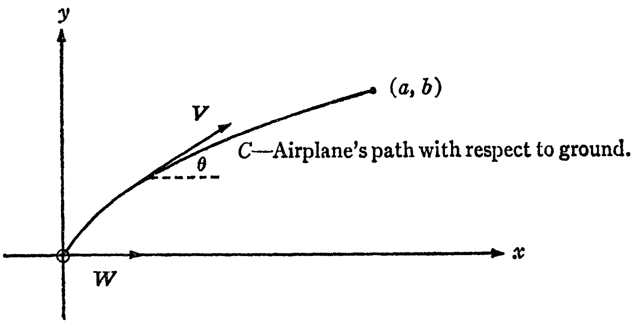
Solution 1. If V denotes the velocity of the airplane with respect to the ground then the following diagram indicates the relationship of the three velocities W, V, and 1. (It also follows from the diagram that the airplane starting from point A and flying a straight line course will reach any point on the circle in one unit of time. This, incidentally, provides the theoretical basis for the circular graphic slide rule used by pilots to determine proper headings to take in a wind and also their speed with respect to the ground.)

For the present, we are also assuming $w<1$ which is the usual physical case. Subsequently, we will also consider $w \geqq 1$.



Since $|V - W| = 1$,
$|V| = w \cos\phi + \sqrt{1 - w^2 \sin^2\phi}$
where $w = |W|$.
Circle of unit radius.

If we wish to travel from the point (0,0) to the point (a,b), it follows that



the time of flight t is given by

$$t = \int_C \frac{ds}{|V|} = \int_0^a \frac{\sec\theta dx}{w\cos\theta + \sqrt{1 - w^2\sin^2 6}}$$

or, rationalizing the integrand,

$$t = \int_0^a \frac{\sec\theta\sqrt{1 - w^2\sin^2\theta}\, dx}{1 - w^2} - \frac{wa}{1 - w^2}.$$

Since $\tan\theta = dy/dx$, the above reduces to

$$t = \int_0^a \frac{\sqrt{1 + (1 - w^2)y'^2}\, dx}{1 - w^2} - \frac{wa}{1 - w^2}, \tag{1}$$

subject to the boundary conditions $y(0) = 0$, $y(a) = b$.

Note that in the above, we have chosen the x-axis parallel to the wind velocity. If we chose the x-axis to contain the two given points, the boundary condi-

tions would be simpler (i.e., $y(0)=y(c)=0$). However, the integrand would be less simple.

Since the integrand of (1) is only an explicit function of y', it follows that the solution of the corresponding Euler-Lagrange equation

$$(\partial F/\partial y) - (d/dx)(\partial F/\partial y') = 0 \text{ is } y = xb/a.$$

However, since the justification of this solution among the class of all possible flight paths is too painful, we proceed in the following manner:

In the integral I of (1), we let $x=r(1-w^2)^{1/2}$. Then,

$$I\sqrt{1-w^2} = \int_0^{a/(1-w^2)^{\frac{1}{2}}} \sqrt{1+(dy/dr)^2}\, dr.$$

Thus the minimization of I is equivalent to finding the shortest distance between the two points (0,0) and $(a/(1-w^2)^{1/2}, b)$ which is achieved by the straight line segment connecting the points. This latter well-known result is usually given as an introductory problem in calculus of variations texts (e.g., C. Fox, An Introduction to the Calculus of Variations, Oxford University Press, London, 1954, p. 11). However, this result can be established very simply by an appropriate rotation of the coordinate system. Let

$$J = \int_0^a \sqrt{1+(dy_1/dx_1)^2}\, dx_1 = \int_0^a \sqrt{dx_1^2+dy_1^2}, \tag{2}$$

where $y_1(0)=0$, $y_1(a)=b$. Then under the rotation

$$x = x_1 \cos\theta + y_1 \sin\theta,$$
$$y = -x_1 \sin\theta + y_1 \cos\theta,$$

where $\tan\theta = b/a$, $dx_1^2+dy_1^2=dx^2+dy^2$ and (2) becomes

$$J = \int_0^{(a^2+b^2)^{\frac{1}{2}}} \sqrt{1+(dy/dx)^2}\, dx \tag{3}$$

subject to $y(0)=y(\sqrt{a^2+b^2})=0$. It now follows immediately that the minimum of (3) is taken on for $dy/dx=0$ or $y=0$.

Before we continue to the 2nd method it is of interest to consider a special case of (1), i.e., the given two points are (0,0) and (a,b), and the wind is blowing with a speed $w=1$ along the positive x-axis. Here, the integral to be minimized is

$$K = \int_0^a y'^2 dx \tag{4}$$

subject to $y(0)=0$, $y(a)=b$.

Note that due to the wind speed being equal to the relative air speed, the plane cannot head in all directions but is limited by

$$-\frac{\pi}{2} < \arctan y' < \frac{\pi}{2}\cdot$$

The minimization of (4) follows immediately by letting $y' = b/a + E(x)$. Since

$$\int_0^a y'dx = b, \qquad \int_0^a E(x)dx = 0.$$

Then,

$$\int_0^a y'^2dx = \int_0^a \left\{\frac{b^2}{a^2} + \frac{2b}{a}E + E^2\right\} dx = \frac{b^2}{a^2} + \int_0^a E^2dx.$$

This is clearly a minimum if $E = 0$, i.e., $y = bx/a$.

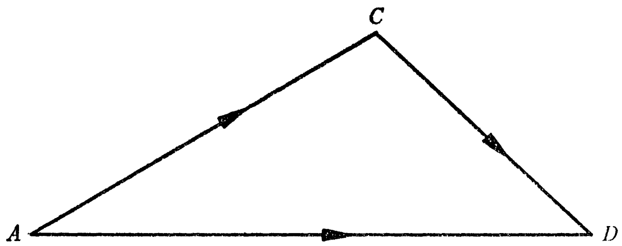
The more sophisticated reader will note that the solution here is a special case of the derivation of the Schwarz-Buniakowsky inequality, i.e.,

$$\int_n^m \phi(x)^2dx \cdot \int_n^m \psi(x)^2dx \geqq \left\{\int_n^m \phi(x)\psi(x)dx\right\}^2$$

with equality if and only if $\phi(x) = k\psi(x)$.

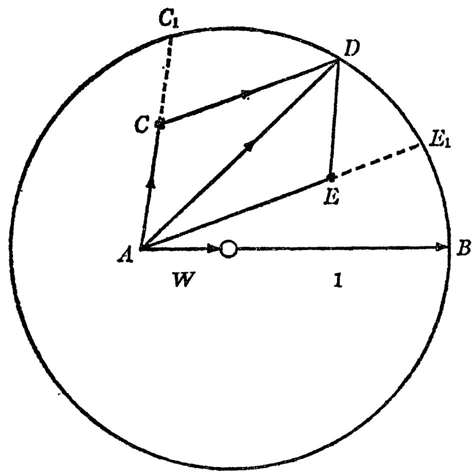
For examples of other variational problems which can be solved by integral inequalities see R. Courant, Differential and Integral Calculus, Vol. 2, Nordeman, New York, 1944, pp. 505, 655, Ex. 9, and G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, London, 1934, Chap. 7.

Solution 2. It follows that if we can show that the time of flight between any two points A and D is less for the straight line course AD than any arbitrary 2-legged flight path $AC+CD$, then the result will hold as well for arbitrary paths by using a limiting procedure.



Referring back to the circular velocity diagram on p. 17, this is equivalent to establishing the following elementary geometrical result:

$$\text{Given: } \mathbf{AC} + \mathbf{CD} = \mathbf{AD},$$
$$AE \| CD, \; ED \| AC.$$
$$\text{Prove: } \frac{AC}{AC_1} + \frac{AE}{AE_1} \geqq 1.$$



(Note that the time to fly from A to either C_1, D, or E_1 is one unit and the time to fly from A to C is then AC/AC_1. For any larger homothetic circle, the corresponding times will be proportional.)

Using complex numbers, the above is equivalent to showing $r+s\geqq 1$, where $r(z_1+w)+s(z_2+w)=z+w$(or $\boldsymbol{AC}+\boldsymbol{AE}=\boldsymbol{AD}$), and

$$|z_1| = |z_2| = |z| = 1.$$

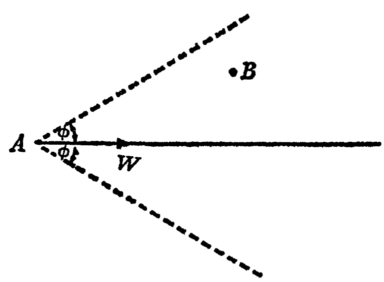
(Here, $w=\boldsymbol{AO}$, $z=\boldsymbol{OD}$, $z_1=\boldsymbol{OC_1}$, $z_2=\boldsymbol{OE_1}$, $r=AC/AC_1$, $S=AE/AE_1$.) Subtracting w from both sides and using the triangle inequality, we obtain

$$r+s+|w|\cdot|r+s-1| \geqq 1.$$

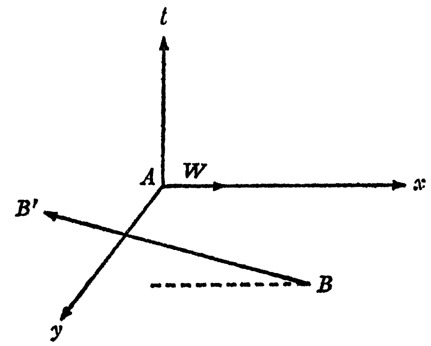
If we assume $r+s<1$, we obtain a contradiction. Thus, $r+s\geqq 1$ with equality only when C falls on AD.

Solution 3. Note that in all the previous proofs we have assumed $w<1$. We now give a proof without using this restriction and which, by the way, can be extended to give the same result for the case of a time-varying wind field.

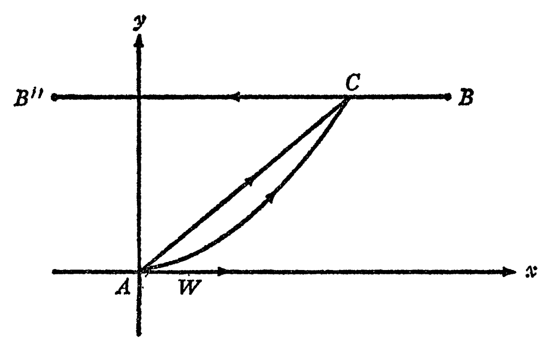
Since the wind speed can be greater than 1, we can only reach destinations (B) which lie in an angle 2ϕ centered at the starting point (A) where $1\geqq w\sin\phi$.



The simplicity in the proof here is achieved by noting that flying from point A to point B subject to a constant wind velocity $\boldsymbol{W}$ is equivalent to flying from A to B without a wind but with point B moving with a velocity of $-\boldsymbol{W}$.



Referring to the above diagram, the world line of B is a straight line parallel to the (x,t) plane. The minimum time flight path from A will be the one whose world line intersects the world line BB' at the lowest possible point (smallest t-value) subject to the condition that the tangent of the angle between the t-axis and any tangent line to the world line of the airplane is $\leqq 1$ (the relative speed of the airplane). It is now intuitive that the world line from A will also have to be a straight line. To see this more easily, consider the projections of the world lines on the (x,y) plane.



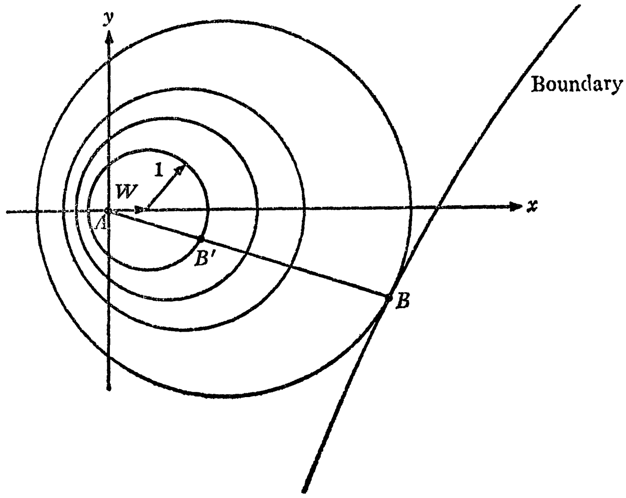
Let C denote the intersection of the minimum time path from A to B in both space and time. The minimum flight path must then correspond to the straight line segment AC and not an arc AC since the path AC is shorter than any arc AC and the plane travels at the same speed of unity.

The same conclusion holds even if the wind velocity varies in time. For then the only difference is that the path BB'' is no longer a straight line but some curve.

Another related problem would be to determine the minimum time path

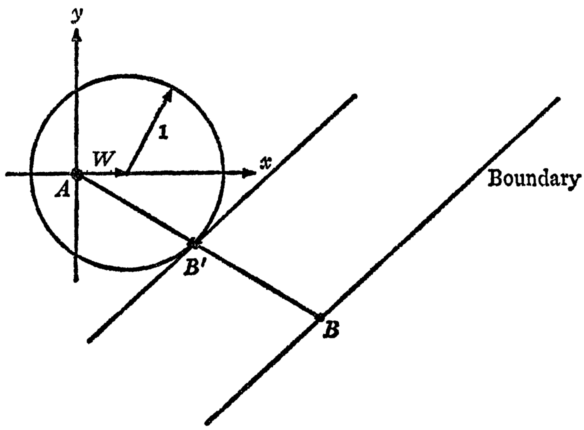
starting from a given path and terminating on some given curve rather than a point. This problem could arise if some military pilot accidentally strayed across a foreign boundary and wished to get back in a hurry.

It follows from our previous result that the desired course here is also a straight line. To determine the appropriate boundary point to head for, we again use the circular velocity diagram on p. 17.



The equivalent problem here is to determine the minimum t so that a member of the family of homothetic circles $(x-wt)^2+y^2=t^2$ is tangent to the boundary curve. This can be done graphically by drawing several circles of the family and interpolating or extrapolating. The point of tangency B will be the point to head for. The time of flight to point B will then be given by $t=AB/AB'$ units.

For the case when the boundary is a straight line, the solution is particularly easy.



Just draw a line tangent to the unit circle parallel to the boundary. The desired heading is then AB'.

For a space-varying wind field, we have the following known result (due to the authors) which appears as Problem 61-4, Flight in an Irrotational Wind Field, SIAM Review, April 1962, pp. 155–156.

If an aircraft travels at a constant speed relative to the wind and traverses any closed curve, the time taken is always less when there is no wind than when there is any irrotational wind field. (It would be of interest if this result would have some effect on track records. Apparently a track record is not official if the wind was blowing over four miles per hour. However, if one ran an integral number of revolutions around a track subject to any irrotational wind field, his record time should be counted all the more. Maybe in the future, as our approach to sports gets more and more scientific, the wind velocity will be recorded continuously at different positions around the track to see whether or not the wind actually helped.)

It is easy to show that if one is flying in an irrotational wind field, the minimum time path between two given points is not necessarily the segment joining the two given points. This suggests a new problem which will be considered in a subsequent paper, i.e.,

What is the most general wind field such that the minimum time path between any two given points is along the segment joining them?

Another problem to be considered in the subsequent paper is the minimum time path in three dimensions. Even in a gravitational field, if the air speed with respect to the wind were held constant (by continuously adjusting the throttle setting) the results would be the same as before. However, if we maintain a fixed throttle setting, the relative air speed will increase as one descends and we get (under certain simplifying assumptions), the classical brachistochrone problem but subject to a wind field.

For the latter two problems, it seemed necessary to use the calculus of variations.

HISTORY OF A FORMULA FOR PRIMES

UNDERWOOD DUDLEY, DePauw University

In 1947, Mills [7] published the following striking result:

THEOREM A. *There is a real number θ such that $[\theta^{3^n}]$ is a prime for all n, $n=1, 2, \cdots$.*

($[x]$ denotes the integral part of x.) This simple formula yielding prime values exclusively inspired several other papers; it is the purpose of this note to show the development of the idea used in the proof of Theorem A through these later contributions.

Proof of Theorem A. The proof depends on the following result of Ingham [5]: if p_n denotes the nth prime, then there is a constant k such that

$$p_{n+1} - p_n < kp_n^{5/8} \tag{1a}$$

for $n=1, 2, \cdots$. If $N>k^8$ and p_n is the largest prime $<N^3$, then from (1a),

$$p_n < N^3 < p_{n+1} < p_n + kp_n^{5/8} < N^3 + N^{1/8}N^{15/8} = N^3 + N^2 < (N+1)^3 - 1. \tag{2a}$$

Thus, given any integer $N>k^8$, we can find a prime between N^3 and $(N+1)^3-1$. This allows us to construct a sequence of primes $\{q_n\}$ as follows: let $q_1>k^8$ be a prime and let q_{n+1} be a prime such that

$$q_n^3 < q_{n+1} < (q_n + 1)^3 - 1; \tag{3a}$$

for definiteness, we could choose q_{n+1} to be the smallest prime in that range.

Let

$$u_n = q_n^{3^{-n}} \quad \text{and} \quad v_n = (q_n + 1)^{3^{-n}}, \tag{4a}$$

$n=1, 2, \cdots$. We see that $\{u_n\}$ is increasing and $\{v_n\}$ is decreasing:

$$u_{n+1} = q_{n+1}^{3^{-n-1}} > (q_n^3)^{3^{-n-1}} = q_n^{3^{-n}} = u_n,$$

$$v_{n+1} = (q_{n+1} + 1)^{3^{-n-1}} < ((q_n + 1)^3 - 1 + 1)^{3^{-n-1}} = (q_n + 1)^{3^{-n}} = v_n.$$

Also, from (4a), $u_n<v_n$. Hence $u_n<u_{n+1}<v_{n+1}<v_n$ for all $n=1, 2, \cdots$. It follows that both sequences are monotone and bounded and hence converge: let

$$\theta = \lim_{n\to\infty} u_n \quad \text{and} \quad \phi = \lim_{n\to\infty} v_n.$$

From $u_n<\theta\leqq\phi<v_n$ follows

$$u_n^{3^n} < \theta^{3^n} \leqq \phi^{3^n} < v_n^{3^n} \tag{5a}$$

or, from (4a),

$$q_n < \theta^{3^n} < q_n + 1. \tag{6a}$$

Thus $q_n=[\theta^{3^n}]$ and the theorem is proved.

After the proof, we see that the result is less astounding: to construct $\{u_n\}$ and hence θ, we need to be able to recognize arbitrarily large primes. If we could do that, we would already implicitly have a formula for primes. But the result is still picturesque, the idea pleasing, and the details neatly worked out.

The theorem is also susceptible to generalization. The first to do so was Kuipers [6] who proved, quite simply, that the 3 in Theorem A is not essential:

THEOREM B. *If $c\geqq 3$ is an integer, then there is a real number θ such that $[\theta^{c^n}]$ is a prime for all n, $n=1, 2, \cdots$.*

The modifications in the proof of Theorem A which are needed to prove this are as follows. If $c \geqq 3$ is an integer, let $a=3c-4$ and $b=3c-1$. Then $a/b \geqq 5/8$, so it follows from Ingham's result that there is a constant k_1 such that

$$\text{(1b)} \qquad p_{n+1} - p_n < k_1 p_n^{a/b}$$

for $n=1, 2, \cdots$. If $N > k_1^b$ and p_n is the smallest prime $< N^c$, then

$$\text{(2b)} \quad p_n < N^c < p_{n+1} < p_n + k_1 p_n^{a/b} < N^c + N^{1/b} N^{ca/b} = N^c + N^{c-1} < (N+1)^c - 1$$

(in the equality, we used the fact that $ca+1=(c-1)b$). So, given $N > k_1^b$, there is a prime between N^c and $(N+1)^c - 1$. Hence we can construct a sequence of primes $\{q_n\}$ such that

$$\text{(3b)} \qquad q_n^c < q_{n+1} < (q_n + 1)^c - 1,$$

and the proof proceeds as in Theorem A to show that $q_n = [\theta^{c^n}]$ for $n=1, 2, \cdots$.

Ansari [1] had the same idea as Kuipers and carried it out in the same way, except he noted that there is no need to require c to be an integer. He proved

THEOREM C. *If $c > 8/3$, then there is a real number θ such that $[\theta^{c^n}]$ is a prime for all n, $n=1, 2, \cdots$.*

To prove this, we note that $c > 8/3$ implies $\delta = 3c/8 - 1 > 0$. Then, if $N > k^{1/\delta}$ (the k in (1a)) and p_n is the smallest prime $< N^c$, we have

$$\text{(2c)} \quad p_n < N^c < p_{n+1} < p_n + k p_n^{5/8} < N^c + N^{\delta} N^{5c/8} = N^c + N^{c-1} < (N+1)^c - 1.$$

The proof then proceeds as before.

$c > 8/3$ can be improved further. Actually, Ansari proved Theorem C with $c > 77/29$, using an improvement of (1a) due to Titchmarsh (unpublished). Titchmarsh showed that 5/8 in (1a) could be replaced by any number larger than 48/77, which leads to the lower bound $c > 77/29$. This has since been improved. The exponent in (1a) is connected with the rate of growth of the Riemann ζ-function along the line $t=1/2$. In fact, if $\zeta(\frac{1}{2}+it) < k_2 t^{\alpha}$ ($k_2, k_3, \cdots$ are constants), then $p_{n+1} - p_n < k_3 p_n^{\beta}$ with $\beta = (1+4\alpha)/(2+4\alpha)+\epsilon$ for any $\epsilon > 0$. Currently, the best estimate of α is $6/37+\epsilon$, for any $\epsilon > 0$, by Haneke [3]. This gives $\beta = 61/98+\epsilon$ and lets us state Theorem C with $c > 98/37$. If the conjecture that $\zeta(\frac{1}{2}+it) < k_4 t^{\epsilon}$ for any $\epsilon > 0$ is true, then we could take $\beta = 1/2+\epsilon$ and c to be any real number > 2. The conjecture seems far from being verified though: Haneke's paper is 74 pages long.

All of the preceding results, though themselves elementary, have the esthetic defect of depending on (1a), which is far from elementary. Wright [14] remedied this by replacing (1a) with Bertrand's Postulate:

$$\text{(1d)} \qquad p_{n+1} - p_n < p_n,$$

which is true for all n, $n=1, 2, \cdots$, and has an elementary proof (see, for ex-

ample, [4] Theorem 418). Thus, if p_n is the largest prime $<N$,

(2d) $$p_n < N < p_{n+1} < 2p_n < 2N,$$

and there is a prime between N and $2N$ for any integer $N \geqq 2$. Hence we can construct the sequence $\{q_n\}$ by taking q_1 to be any prime and requiring that

(3d) $$2^{q_n} < q_{n+1} < 2^{q_n+1}$$

If we then let

(4d) $$u_n = \log^{(n)} q_n \quad \text{and} \quad v_n = \log^{(n)} (q_n + 1),$$

(where $\log^{(n)}$ denotes the n-times iterated logarithm to the base 2), then from (3d),

$$q_n < \log^{(1)} q_{n+1} < \log^{(1)} (q_{n+1} + 1) < q_n + 1.$$

If we take logarithms to the base 2 of the preceding inequalities n times, we have $u_n < u_{n+1} < v_{n+1} < v_n$. So, as before,

$$\theta = \lim_{n\to\infty} u_n \quad \text{and} \quad \phi = \lim_{n\to\infty} v_n$$

exist. If $\exp^{(n)}$ denotes the n-times iterated exponential to the base 2, we have

(5d) $$\exp^{(n)} u_n < \exp^{(n)} \theta < \exp^{(n)} v_n$$

or

(6d) $$q_n < \exp^{(n)} \theta < q_n + 1.$$

Restating (6d), we have proved

THEOREM D. *There exists a real number* θ *such that* $[2^{2^{\cdot^{\cdot^{\cdot^{2^\theta}}}}}]$ *is a prime for any number of iterations of the exponential.*

Of course, 2 could be replaced by any number >1. To balance the virtue of being completely elementary, this theorem has the small defect of being more awkward to state than Theorems A–C.

Ore [9] was able to put both Mills's and Wright's results in one theorem. Ore defines a selection function f for a sequence $\{p_n\}$ to be a function which is continuous, eventually increasing, and such that there exists a subsequence $\{q_n\}$ of $\{p_n\}$ with the property that for all n, $n = 1, 2, \cdots$,

(3e) $$f(q_n - \delta_n) < q_{n+1} - \delta_{n+1} \leqq q_{n+1} + \epsilon_{n+1} < f(q_n + \epsilon_n)$$

for some $\delta_n \geqq 0$ and $\epsilon_n \geqq 0$. If we let

(4e) $$u_n = f^{(-n)}(q_n - \delta_n) \quad \text{and} \quad v_n = f^{(-n)}(q_n + \epsilon_n)$$

(where $f^{(n)}$ and $f^{(-n)}$ denote the nth iterates of f and f^{-1} respectively), then, as in Theorems A–D, $\theta = \lim_{n\to\infty} u_n$ exists, and we can prove

THEOREM E. *With the above notation*, $q_n = [f^{(n)}(\theta)]$.

The details of the proof can be traced through as in Theorem A. Ore abstracted in (3e) the idea in (3d), (3b), and (3a); choosing $f(x)=x^3$ gives Mills's result, $f(x)=x^c$ gives Kuiper's and Ansari's, and $f(x)=2^x$ gives Wright's.

Niven [8] looked at θ^{c^n} in a different way and proved

THEOREM F. *Given any* $c>1$, *there exists a real number* θ *such that* $[c^{\theta^n}]$ *is a prime for all* n, $n=1, 2, \cdots$.

To prove this, choose q_1 to be a prime satisfying $q_1>c^8$ and $q_1>k$, where k is the constant in (1a). Then, using (1a), Niven shows, with great cleverness, that it is possible to find a prime satisfying

$$q_n^{(\log q_n)^{1/n}} < q_{n+1} < (1+q_n)^{(\log(1+q_n))^{1/n}} - 1 \tag{3f}$$

for $n=1, 2, \cdots$. The logarithms are to the base c. (3f) is established by induction. It is the same as

$$c^{(\log q_n)^{(n+1)/n}} < c^{\log q_{n+1}} < c^{(\log(1+q_n))^{(n+1)/n}} - 1,$$

whence

$$(\log q_n)^{1/n} < (\log q_{n+1})^{1/(n+1)} < (\log(1+q_{n+1}))^{1/(n+1)} < \log(1+q_n)^{1/n}.$$

If

$$u_n = (\log q_n)^{1/n} \quad \text{and} \quad v_n = (\log(1+q_n))^{1/n}, \tag{4f}$$

then the preceding inequalities show that

$$u_n < u_{n+1} < v_{n+1} < v_n.$$

Putting $\theta = \lim_{n\to\infty} u_n$ we have, as before, $q_n = [c^{\phi^n}]$.

Mills's original idea has three branches—Theorems C, D, and F—and these were brought back together in a satisfying paper by Wright [15]. In it he shows

THEOREM G. *The set of suitable* θ *in Theorems* C, D, *and* F *has cardinality* c, *measure* 0, *and is nowhere dense.*

The proof is not easy.

After Wright's effort, no further results of the preceding type have appeared, perhaps because they would seem anticlimactic. But formulas for primes, more or less interesting, continue to appear. The interested reader can find some recent ones in papers by Bang [2], Sierpinski [10], Srinivasan [11, 12], and Willans [13].

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RATIONAL EXPONENTIAL EXPRESSIONS AND A CONJECTURE CONCERNING π AND e

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Abstract. One of the most controversial and least well defined of mathematical problems is the problem of simplification. The recent upsurge of interest in mechanized mathematics has lent new urgency to this problem, but so far very little has been accomplished. This paper attempts to shed light on the situation by introducing the class of rational exponential expressions, defining simplification within this class, and showing constructively how to achieve it. It is shown that the only simplified rational exponential expression equivalent to 0 is 0 itself, provided that an easily stated conjecture is true. However the conjecture, if true, will surely be difficult to prove, since it asserts as a special case that π and e are algebraically independent, and no one has yet been able to prove even the much weaker conjecture that $\pi+e$ is irrational.

1. Introduction. Basically simplification means the application of mathematical identities to transform a given expression into an equivalent expression satisfying some desired criteria.

If the class of admissible expressions is too broad, many dilemmas, of which the following is typical, arise. A basic identity for simplification of expressions involving the logarithm function is

$$\log z_1 z_2 = \log z_1 + \log z_2 \tag{1}$$

which holds everywhere on the Riemann surface whose points have the form

$$z = re^{i\theta},\ r > 0,\ -\infty < \theta < \infty. \tag{2}$$

On this surface one must abandon the identity

$$e^{2i\pi} = 1 \tag{3}$$

which is fundamental for simplification of expressions involving the exponential function. This identity can be saved if we define the logarithm function over a cut plane (for example, by restricting θ in (2) to the interval $-\pi<\theta\leqq\pi$), but then (1) is lost.

Surprisingly, it is not even necessary to introduce complex numbers or multivalued functions in order to get into insurmountable difficulty. Consider the class of expressions generated from the integers, the real constants π and log 2, and the real indeterminate x by application of the rational operations, the sine function, the exponential function, the absolute value function, and substitution. Richardson [1] has shown that there is no algorithm to test whether or not a given expression in this class is identically zero. Caviness [2] has shown that the same result applies to the smaller class of expressions generated as above, but with log 2 and the exponential function omitted, provided that one assumes the unsolvability of Hilbert's tenth problem [3]. The two proofs are essentially the same except that Caviness begins with this assumption, while Richardson begins with a theorem of Davis, Putnam, and Robinson [4].

This paper attempts to shed light on the simplification problem by introducing the class of rational exponential (REX) expressions, defining simplification within this class, and showing constructively how to achieve it.

The definition is such that distinct simplified expressions may be equivalent. However, it is shown that the only simplified REX expression equivalent to 0 is 0 itself, provided that an easily stated conjecture is true. Unfortunately, the conjecture, if true, will surely be difficult to prove, since it asserts as a special case that π and e are algebraically independent, and no one has yet been able to prove even the much weaker conjecture that $\pi+e$ is irrational [5].

Section 2 defines the class of REX expressions, and defines simplification for expressions in this class. The fundamental conjecture is discussed in Section 3, the zero-equivalence theorem for simplified expressions in Section 4, and the simplification algorithm in Section 5.

2. Definitions. The *rational exponential* (*REX*) *expressions* are those which are obtained by addition, subtraction, multiplication, division, and substitution starting with the (rational) integers, the constants π and i, some indeterminates $z_1, \cdots, z_N$ (denoted collectively by z), and the exponential function. The *rational exponential functions* are those which can be represented by REX expressions.

Note that all roots of unity are REX. It is convenient to introduce the abbreviations

$$\omega_m = \exp(i\pi/2m) \qquad m \geqq 1. \tag{4}$$

In particular $\omega_1=\exp(i\pi/2)=i$.

A REX expression is *equivalent to* 0 if it is identically 0 when viewed as an analytic function of z. A REX expression is *equivalent to* U (that is, *undefined*) if it involves division by a subexpression equivalent to 0. For example, the expression $2(z_1-z_1)$ is equivalent to 0, while the expression $3+1/2(z_1-z_1)$ is equivalent to U. The symbol U is considered to be a REX expression, but the construction of more complicated expressions involving it is forbidden. Two REX expressions are *equivalent* if both are equivalent to U or if their difference is equivalent to 0.

Any REX expression can be transformed by a straightforward process [6] into an equivalent weakly simplified REX expression. A REX expression is said to be *weakly simplified* if it is 0 or U or if it has the form

$$\frac{f(\exp p_1, \cdots, \exp p_n, z, \pi, \omega_m)}{g(\exp p_1, \cdots, \exp p_n, z, \pi)}, \tag{5}$$

where

(a) f and g are relatively prime nonzero polynomials;
(If g is the polynomial ± 1, we shall permit, and indeed insist, that it be omitted, with the sign of f being reversed if necessary. Although we require that the polynomial f be nonzero, the numerator of (5) might nevertheless be equivalent to 0. For example, it might be the expression $\exp(z_1+z_2)-\exp(z_1)\exp(z_2)$. In this case f is the nonzero polynomial defined by $f(a, b, c) = a - bc$.)

(b) the degree of f in ω_m is less than the degree of the minimal polynomial of ω_m; and

(c) $p_1, \cdots, p_n (n \geqq 0)$ are distinct weakly simplified REX expressions other than 0 or U.

A weakly simplified REX expression is said to be *simplified* if it is 0 or U; *or* if it has the form (5), and

(d) $p_1, \cdots, p_n$ are simplified, and

(e) the set $\{p_1, \cdots, p_n, i\pi\}$ is linearly independent over the rationals.

The significance of this last condition will become clear in Section 4.

3. The fundamental conjecture. This section discusses the fundamental conjecture on which our zero-test algorithm depends. The simplification algorithm also depends on it, but only in that no use is made of the presently unknown identities whose existence would be implied by its falsity.

ROUGH STATEMENT. *Roughly speaking, the conjecture is that the only algebraic relations involving π and the z's and exponentials of rational exponential expressions are those which follow directly from the fact that roots of unity are algebraic numbers and from the identities*

$$
\begin{aligned}
\exp(0) &= 1 \\
\exp(i\pi) &= -1 \qquad (6) \\
\exp(z_1 + z_2) &= \exp(z_1)\exp(z_2).
\end{aligned}
$$

PRECISE STATEMENT. *Let $p_1, \cdots, p_n$ be nonzero rational exponential expressions such that the set $\{p_1, \cdots, p_n, i\pi\}$ is linearly independent over the rationals. Then the set $\{\exp p_1, \cdots, \exp p_n, z, \pi\}$ is algebraically independent over the rationals.*

Proof of the Converse. Suppose the set $\{p_1, \cdots, p_n, i\pi\}$ were linearly dependent over the rationals. Then there would exist integers $a_0, \cdots, a_n$ such that

$$a_0 i\pi + \sum_{i=1}^{n} a_i p_i = 0. \tag{7}$$

Using (6),

$$\prod_{i=1}^{n} (\exp p_i)^{a_i} = (-1)^{a_0}, \tag{8}$$

so the set $\{\exp p_1, \cdots, \exp p_n\}$, and *a fortiori* the set $\{\exp p_1, \cdots, \exp p_n, z, \pi\}$, would be algebraically dependent over the rationals.

COROLLARY. *Setting $n=1$ and $p_1=1$ in the precise statement of the fundamental conjecture, we obtain the corollary that π and e are algebraically independent.*

4. Zero-equivalence theorem for simplified REX expressions. *Let p be a simplified REX expression other than* 0 *or* U. *Then p is not equivalent to* 0 *or* U.

Proof. Since p is simplified, we can write it in the form (5) where (5a)–(5e) hold. The proof is by induction on the *depth* of p, defined as follows. If $n=0$ in (5), then the depth of p is 0. Otherwise the depth of p is

$$\max(d_1, \cdots, d_n) + 1$$

where $d_1, \cdots, d_n$ are the depths of $p_1, \cdots, p_n$ respectively.

Now let d be the depth of p. If $d=0$, then the proof is straightforward. In the case $d>0$, we assume inductively that the theorem is true for expressions of depth less than d. It follows, using (5c) and (5d), that none of the $p_1, \cdots, p_n$ is equivalent to U. Therefore neither the numerator nor the denominator of (5) is equivalent to U.

Viewed as a polynomial in the elements of the set

$$\{\exp p_1, \cdots, \exp p_n, z, \pi\}, \tag{9}$$

which is algebraically independent by (5e) and the fundamental conjecture, the numerator of (5) has, by (5a), at least one nonzero coefficient. This coefficient is a polynomial in ω_m (over the integers), whose degree, by (5b), is less than the degree of the minimal polynomial of ω_m. It follows that this polynomial is not equivalent to 0, and therefore the numerator of (5) is not equivalent to 0. A

similar argument shows that the denominator of (5) is not equivalent to 0, and it follows immediately that p is not equivalent to 0 or U.

5. Simplification algorithm. The purpose of this algorithm is to transform a weakly simplified REX expression, other than 0 or U, into a simplified REX expression. The given expression has the form (5), and conditions (5a) through (5c) are satisfied. We now define the *innermost exponentials* in a REX expression as those whose arguments do not involve the exponential function. The algorithm proceeds from the innermost exponentials outward.

Step 1 (*Initialize*). Set $k=0$, so that $\{q_1, \cdots, q_k\}$ and $\{r_1, \cdots, r_k\}$ denote the empty set. Now rewrite (5) in the form

$$\frac{f(\exp p_1, \cdots, \exp p_n, r_1, \cdots, r_k, z, \pi, \omega_m)}{g(\exp p_1, \cdots, \exp p_n, r_1, \cdots, r_k, z, \pi)}. \tag{10}$$

Step 2 (*Begin loop with test for completion*). At this point the expression (10), with $r_1, \cdots, r_k$ viewed as additional indeterminates, is weakly simplified. Furthermore, $r_1, \cdots, r_k$ are abbreviations for $\exp q_1, \cdots, \exp q_k$ respectively, and $q_1, \cdots, q_k$ are simplified expressions of the form

$$q_i = \frac{f_i(r_1, \cdots, r_{i-1}, z, \pi, \omega_{l_i})}{g_i(r_1, \cdots, r_{i-1}, z, \pi)} \qquad i = 1, \cdots, k. \tag{11}$$

Finally, the set $\{q_1, \cdots, q_k, i\pi\}$ is linearly independent over the rationals, and the set $\{r_1, \cdots, r_k, z, \pi\}$ is algebraically independent over the rationals. Note that the $p_1, \cdots, p_n$ may depend on the $r_1, \cdots, r_k$.

If n is zero in (10), then replace $r_1, \cdots, r_k$ by $\exp q_1, \cdots, \exp q_k$ respectively. Now (10) has the form (5) with $q_1, \cdots, q_k$ playing the role of $p_1, \cdots, p_n$, and (5a) through (5e) are satisfied. Therefore we are finished.

Otherwise (that is, if $n>0$), proceed to Step 3.

Step 3 (*Introduce* q_{k+1} *and* r_{k+1}). Let q_{k+1} be the argument of any innermost exponential in (10), and replace $\exp q_{k+1}$ by the abbreviation r_{k+1}. In general, q_{k+1} will be a subexpression of one of the $p_1, \cdots, p_n$. Clearly it has the form

$$q_{k+1} = \frac{f_{k+1}(r_1, \cdots, r_k, z, \pi, \omega_m)}{g_{k+1}(r_1, \cdots, r_k, z, \pi)}. \tag{12}$$

If we replace $r_1, \cdots, r_k$ by $\exp q_1, \cdots, \exp q_k$, then (12) has the form (5) with $q_1, \cdots, q_k$ playing the role of $p_1, \cdots, p_n$, and (5a) through (5e) are satisfied. Therefore q_{k+1} is simplified.

Step 4 (*Test for linear dependence*). Recall that the set $\{q_1, \cdots, q_k, i\pi\}$ is linearly independent over the rationals. If q_{k+1} can be expressed as a linear combination of $i\pi$ and $q_1, \cdots, q_k$, then we shall rewrite (10) accordingly (see Step 5a). Otherwise, we shall adjoin q_{k+1} to the set (see Step 5b).

Consider the linear dependence equation

$$a_0 i\pi + \sum_{i=1}^{k+1} a_i q_i = 0 \tag{13}$$

where $a_0, \cdots, a_{k+1}$ are undetermined rationals. Substituting (11) and (12) into (13), replacing all of the ω_{l_i} by powers of ω_l for some sufficiently large l, and rearranging terms, we obtain a polynomial in $r_1, \cdots, r_k, z, \pi$, and ω_l whose degree in ω_l is less than the degree of the minimal polynomial of ω_l and whose coefficients are integral linear combinations of $a_0, \cdots, a_{k+1}$. Because of the algebraic independence of the set $\{r_1, \cdots, r_k, z, \pi\}$, equation (13) implies that all of these coefficients vanish. Thus we obtain a set of homogeneous integral linear equations in the unknowns $a_0, \cdots, a_{k+1}$. Since the set $\{q_1, \cdots, q_k, i\pi\}$ is linearly independent, a_{k+1} must be nonzero in any nontrivial solution vector, and the solution space, if any, must be one dimensional. Using standard linear methods we can determine whether or not a solution exists, and if so, we can find it.

If a solution exists, go on to Step 5a. Otherwise, proceed to Step 5b.

Step 5a (*Linear dependence—replace* q_{k+1} *and* r_{k+1}). In this case we can write

$$q_{k+1} = \frac{b_0 i\pi}{2c_0} + \sum_{i=1}^{k} \frac{b_i q_i}{c_i}, \tag{14}$$

where b_i and c_i are relatively prime integers for $i=0, \cdots, k$. Letting

$$q_i' = \frac{q_i}{c_i}, \qquad r_i' = \exp q_i', \qquad i = 1, \cdots, k \tag{15}$$

we have

$$r_i = \exp(q_i) = \exp(c_i q_i') = (r_i')^{c_i}, \qquad i = 1, \cdots, k \tag{16}$$

and

$$r_{k+1} = \exp(q_{k+1}) = \omega_{c_0}^{b_0} \cdot \prod_{i=1}^{k} (r_i')^{b_i}. \tag{17}$$

Substitute (16) and (17) into (10), set $k' = k$, and proceed to Step 6.

Step 5b (*Linear independence—adjoin* q_{k+1}). In this case the set

$$\{q_1, \cdots, q_{k+1}, i\pi\}$$

is linearly independent. Now, in (10), replace r_i by r_i' for all $i=1, \cdots, k'=k+1$, and proceed to Step 6.

Step 6 (*Simplify* (10) *weakly*). Simplify (10) weakly, treating $r_1', \cdots, r_{k'}'$ as indeterminates. The result is an expression of the form

$$\frac{f(\exp p_1', \cdots, \exp p_{n'}', r_1', \cdots, r_{k'}', z, \pi, \omega_{m'})}{g(\exp p_1', \cdots, \exp p_{n'}', r_1', \cdots, r_{k'}', z, \pi)} \tag{10'}$$

where the $p_1', \cdots, p_{n'}'$ may depend on the $r_1', \cdots, r_{k'}'$. Note that the set $\{r_1', \cdots, r_{k'}', z, \pi\}$ is algebraically independent, by the fundamental conjecture.

Step 7 (*Iterate*). Drop all primes, so that (10′) replaces (10), and return to Step 2.

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MATHEMATICAL NOTES

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AN AREA-WIDTH INEQUALITY FOR CONVEX CURVES

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Theorem. *Let C be a closed convex plane curve, bounding a region of area A. Let $w(\theta)$ denote the width of C in the θ-direction. Then*

$$A \leqq \frac{1}{2}\int_0^{\pi/2} w(\theta)w\left(\theta+\frac{\pi}{2}\right)d\theta \tag{1}$$

with equality if and only if C is a circle.

Proof. We shall employ a method due to A. Hurwitz (see Courant [1] p. 213) to prove the isoperimetric inequality. It clearly suffices to establish the theorem for the case of C^2 curves such that for each θ, $0 \leq \theta < 2\pi$, there is exactly one point, say $(x(\theta), y(\theta))$, at which the normal to C makes an angle θ with the X-axis. Then, as Courant indicates, there is a C^2 periodic function $p(\theta)$ such that

$$\begin{aligned} x(\theta) &= p(\theta)\cos\theta - p'(\theta)\sin\theta \\ y(\theta) &= p(\theta)\sin\theta + p'(\theta)\cos\theta. \end{aligned} \tag{2}$$

In fact, if the origin 0 is chosen to be interior to C then $p(\theta)$ is the perpendicular distance from 0 to the tangent to C at $(x(\theta), y(\theta))$. We then have $w(\theta)=p(\theta)+p(\theta+\pi)$. Hence

$$I=\frac{1}{2}\int_0^{\pi/2} w(\theta)w\left(\theta+\frac{\pi}{2}\right)d\theta=\frac{1}{2}\int_0^{2\pi} p(\theta)p\left(\theta+\frac{\pi}{2}\right)d\theta,$$

while

$$A=\frac{1}{2}\int_0^{2\pi}(xy'-yx')d\theta=\frac{1}{2}\int_0^{2\pi}(p^2-p'^2)d\theta.$$

We now express I and A in terms of the Fourier coefficients of $p(\theta)$. Let us write

$$p(\theta)=\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$

where $\bar{a}_k=a_{-k}$ since p is real. We then have

$$A=\pi\sum_{k=-\infty}^{\infty}|a_k|^2(1-k^2)=\pi|a_0|^2+2\pi\sum_{k=2}^{\infty}|a_k|^2(1-k^2) \tag{3}$$

and

$$\begin{aligned} I &= \frac{1}{2}\sum_{k,l} a_k a_l \int_0^{2\pi} e^{ik\theta}e^{il(\theta+\pi/2)}d\theta \\ &= \pi\sum_{k=-\infty}^{\infty}|a_k|^2 e^{ik(\pi/2)}; \end{aligned}$$

that is,

$$I=\pi|a_0|^2+2\pi\sum_{k=2}^{\infty}|a_k|^2\cos k\frac{\pi}{2}\cdot \tag{4}$$

Comparing (3) and (4) we see that $A\leqq I$. Equality holds if and only if $a_k=0$ for $|k|>1$, i.e., $p(\theta)=a_0+2Re(a_1e^{i\theta})$, which corresponds to a circle.

As a corollary one has the well-known result that a convex region of area $>\pi/4$ must have width >1 in some direction. In particular, such a region must contain two interior points a unit distance apart. (It should perhaps be added that the discovery of (1) was inspired by the appearance of this latter statement as a problem in the 1967 Putnam competition.)

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(suggested by the referee) can be obtained as follows: If $f=\sum_0^m r_iX^{a_i}$ and $g=\sum_0^v s_iX^{b_i}$ are nonzero elements of $R[X^{\{G\}}]$, then f and g belong to $R[X^{\{H\}}]$, where H is the subgroup of G generated by $\{a_i\}\cup\{b_i\}$. Clearly H is a torsion-free finitely generated abelian group; hence H is a finite direct sum of infinite cyclic groups [3, page 48]. Thinking of H as $Z\oplus\cdots\oplus Z$, it follows that H can be totally ordered under the lexicographic ordering (that is, a nonzero element $(k_1, k_2, \cdots, k_n)$ is positive if its first nonzero coordinate is a positive integer). We can therefore assume that f and g are written so that r_0 and s_0 are nonzero, $a_0>a_1>\cdots$, and $b_0>b_1>\cdots$. Then the term $X^{a_0+b_0}$ appears as a monomial in the product fg with nonzero coefficient r_0s_0. Therefore $fg\neq 0$ and $R[X^{\{S\}}]$ is an integral domain, as we wished to show.

($\rightarrow$): We show that if (1), (2), or (3) fails, then $R[X^{\{S\}}]$ is not an integral domain. For (1) this is clear. Also, if r, s, and t are elements of S such that $r+s=r+t$ where $s\neq t$, then let c be a nonzero element of R. We have $cX^r\cdot cX^s=cX^r\cdot cX^t$, while $cX^s\neq cX^t$, and $R[X^{\{S\}}]$ is not an integral domain. Finally, if (3) fails for elements s, $t\in S$ and if n is minimal positive such that $ns=nt$, then

$$0=c^2X^{ns}-c^2X^{nt}=(cX^s-cX^t)(cX^{(n-1)s}+\cdots+cX^{is+(n-i-1)t}+\cdots+cX^{(n-1)t}).$$

But $cX^s-cX^t\neq 0$, and by our choice of n, the exponents of the terms in the second factor are distinct, so the second factor is also nonzero. Hence $R[X^{\{S\}}]$ is not an integral domain.

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FILLING BOXES WITH BRICKS

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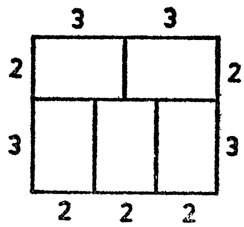
Brickwork is usually made of bricks whose measurements are $1\times 2\times 4$, so that they can fit together in many positions. It is very remarkable, however, that this flexibility does not help at all when it is required to fill a given rectangular box entirely with such bricks. It can be proved (and a more general result will be proved in this note) that if the box cannot be filled trivially (i.e., with all bricks in parallel position), then it cannot be filled at all. For instance, if the box is $10\times 10\times 10$, it cannot be filled trivially with $1\times 2\times 4$ bricks, since 4 does not divide 10. By the theorem just mentioned, it cannot be filled at all without breaking the bricks, although the number of bricks that is needed is an integer (viz., 125).

It has to be remarked that the situation is different with bricks with mea-

surements $1 \times 2 \times 3$. For, the flat box $1 \times 5 \times 6$ can be filled (see figure), but it cannot be filled trivially.

In this note we shall determine all bricks which share the above-mentioned property of the $1 \times 2 \times 4$. The problem will be solved for an arbitrary number of dimensions, and the answer will be quite simple.



The decisive idea of this paper is a generalization of a trick occurring in the solution of the following well-known problem. Cut out two diagonally opposite squares of an 8×8 checker-board, and show that it is impossible to cover the remaining 62 squares by 31 dominoes of size 1×2. The impossibility is shown as follows. Each domino will cover both a black and a white square. On the other hand, the region to be covered does not have the same number of squares of each colour since the two removed squares had the same colour.

Measurements of n-dimensional bricks and boxes will be expressed by integers. We consider bricks $a_1 \times \cdots \times a_n$ and a box $A_1 \times \cdots \times A_n$. The box $A_1 \times \cdots \times A_n$ is called a *multiple* of the brick $a_1 \times \cdots \times a_n$, if there are integers $q_1, \cdots, q_n$ such that the numbers $q_1 a_1, \cdots, q_n a_n$ form a rearrangement of $A_1, \cdots, A_n$. (Example: $10 \times 16 \times 17$ is a multiple of $1 \times 2 \times 4$.) It is easily seen that the box can be filled trivially if and only if it is a multiple of the brick.

The brick $a_1 \times \cdots \times a_n$ is called *harmonic* if the numbers $a_1, \cdots, a_n$ can be rearranged into $a_1', \cdots, a_n'$ with $a_1' | a_2', a_2' | a_3', \cdots, a_{n-1}' | a_n'$. (Example: the brick $2 \times 1 \times 6 \times 6$ is harmonic.)

The purpose of this note is to show that the harmonic bricks share the above-mentioned property of the $1 \times 2 \times 4$ bricks (Theorem 2), and that the nonharmonic bricks do not have this property (Theorem 3). In other words, if the brick is not harmonic, then there are boxes which can be filled, but cannot be filled trivially. If the brick is harmonic, then every box that can be filled, can already be filled trivially.

We first prove

THEOREM 1. *If the box $A_1 \times \cdots \times A_n$ can be filled with bricks $a_1 \times \cdots \times a_n$, then at least one of the A_i is a multiple of a_1, at least one of the A_i is a multiple of a_2, etc. (this does not necessarily imply that the box is a multiple of the brick; cf. the case of box $1 \times 5 \times 6$ and brick $1 \times 2 \times 3$).*

Proof. It suffices to consider a_1. As any brick $a_1 \times \cdots \times a_n$ can be subdivided into bricks $a_1 \times 1 \times \cdots \times 1$, we may start from the assumption that the box is filled, in some way or another, with such bricks $a_1 \times 1 \cdots \times 1$. We divide each

one of them into a_1 cubes $1\times\cdots\times 1$. The box now contains $A_1\cdots A_n$ cubes. We give these cubes coordinates $(k_1,\cdots,k_n)(1\leqq k_1\leqq A_1,\cdots,1\leqq k_n\leqq A_n)$. We consider the sum

$$S(A)=\sum_{k=1}^{A} e^{2\pi ik/a_1},$$

and the multiple sum

$$\sum_{k_1=1}^{A_1}\cdots\sum_{k_n=1}^{A_n} e^{2\pi i(k_1+\cdots+k_n)/a_1}=S(A_1)\cdots S(A_n).$$

Each term in this multiple sum corresponds to a cube in the box. These terms can be grouped together in blocks of a_1 terms each, combining terms corresponding to cubes belonging to one and the same brick. In such a group of terms, one of the indices runs through a set of a_1 consecutive integers, and the other indices remain constant. It follows that the contribution of such a group is zero, irrespective of the orientation of the brick.

We infer that the multiple sum over all cubes vanishes and therefore one of the $S(A_j)$ equals zero. Since

$$S(A_j)=x+x^2+\cdots+x^{A_j}=x(x^{A_j}-1)/(x-1)$$

with $x=e^{2\pi i/a_1}$, we have $e^{2\pi iA_j/a_1}=1$. It follows that A_j is a multiple of a_1.

THEOREM 2. *If a box is filled with harmonic bricks $a_1\times\cdots\times a_n$, then the box is a multiple of the brick.*

Proof. We use induction with respect to n. If $n=1$ we have a triviality. Assuming that the assertion holds in the $(n-1)$-dimensional case, we shall tackle the case of n dimensions.

Without loss of generality we may assume that $a_1|a_2$, $a_2|a_3,\cdots,a_{n-1}|a_n$. If the box is $A_1\times\cdots\times A_n$, we have, by the previous theorem, that one of the A_i is a multiple of a_n. Assume that A_n is a multiple of a_n.

We consider an $(n-1)$-dimensional face $A_1\times\cdots\times A_{n-1}$. This is entirely filled with $(n-1)$-dimensional bricks of various sizes, viz., $a_2\times\cdots\times a_n$, or $a_1\times a_3\times\cdots\times a_n,\cdots$, or $a_1\times a_2\times\cdots\times a_{n-1}$. Each one of these can be subdivided into $(n-1)$-dimensional bricks $a_1\times\cdots\times a_{n-1}$, by virtue of the divisibility property of $a_1,\cdots,a_n$. The brick $a_1\times\cdots\times a_{n-1}$ being harmonic, we observe that $A_1\times\cdots\times A_{n-1}$ is a multiple of $a_1\times\cdots\times a_{n-1}$, by virtue of the induction hypothesis. Since we know that $a_n|A_n$, we infer that $A_1\times\cdots\times A_n$ is a multiple of $a_1\times\cdots\times a_n$.

THEOREM 3. *If the brick $a_1\times\cdots\times a_n$ is not harmonic, then there is a box which can be filled without being a multiple of the brick.*

Proof. We may assume that $n>1$, $a_1\leqq a_2\leqq\cdots\leqq a_n$, and that k is the largest integer for which a_{k-1} does not divide a_k (so $2\leqq k\leqq n$).

A box $(a+b)\times ab$ can be filled with bricks $a\times b$ (see figure, where $a=2$, $b=3$). Therefore the box

$$a_1\times\cdots\times a_{k-2}\times(a_{k-1}+a_k)\times a_{k-1}a_k\times a_{k+1}\times\cdots\times a_n$$

can be filled with bricks $a_1\times\cdots\times a_n$. We show that this box is not a multiple of the brick. Let j be the smallest integer with $a_j=a_{k-1}$. Then no one of the numbers $a_1,\cdots,a_{j-1},a_{k-1}+a_k$ is divisible by any of the numbers $a_j,\cdots,a_{k-1},a_k,\cdots,a_n$. (Note that $a_j=\cdots=a_{k-1}$, that $a_{k-1}+a_k$ is not divisible by a_{k-1} or a_k, nor by any multiple of a_k. And $a_{k+1},\cdots,a_n$ are multiples of a_k, according to the way k was chosen.)

If the box $A_1\times\cdots\times A_n$ is a multiple of the brick $a_1\times\cdots\times a_n$, there can be at most $j-1$ i's such that A_i is no multiple of any of the numbers $a_j,\cdots,a_n$. Therefore, the box constructed above is not a multiple of the brick. This completes the proof.

The material of Theorems 1 and 2 was presented in the form of problems in Hungarian in Matematikai Lapok 12, problem 109, pp. 110–112 and 13, problem 119, pp. 314–317. Problem 109, dealing with bricks $1\times2\times4$, was solved by G. Hajós, G. Katona, D. Szasz, I. Thiry and the proposer. Problem 119 dealt with the n-dimensional case, and was solved by G. Hajós, G. Katona, D. Szasz, and the proposer.

The questions of the bricks $1\times2\times4$ arose from a remark by the proposer's son F. W. de Bruijn who discovered, at the age of 7, that he was unable to fill his $6\times6\times6$ box by bricks $1\times2\times4$.

A HILBERT SPACE PROOF OF THE BANACH-STEINHAUS THEOREM

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The Banach-Steinhaus theorem, or "Uniform Boundedness Principle," for Hilbert space is this statement:

THEOREM. *Let H denote a Hilbert space and let $(y_\alpha;\ \alpha\in A)$ be a family of vectors in H indexed by a nonempty (but otherwise arbitrary) set A. If for every $x\in H$ there is a positive number $M(x)$ independent of α such that $|(x, y_\alpha)|\leqq M(x)$ for all $\alpha\in A$, then there is an M such that $\|y_\alpha\|\leqq M$ for all $\alpha\in A$.*

We shall give a proof that uses only elementary facts about Hilbert space; everything we need is contained in the first three sections of Chapter 10 of Simmons' book [2], for example. In particular we do not use any criteria for boundedness of linear functionals, or, for that matter, even the concept of linear functional. And, of course, no category arguments.

Our proof seems to be more "natural" than Sarason's elementary proof as presented in Halmos [1, Solution 20], and involves about the same computational burden.

Here is the proof. Elementary reductions show that it is enough to prove this statement:

If $(y_n;\ n=1,2,\cdots)$ *is linearly independent and* $|(x, y_n)| \leqq M(x)$, *where* $n=1, 2, \cdots$ *for every* $x\in H$, *then* $\|y_n\| \leqq M$, $n=1, 2, \cdots$.

Now suppose for the moment that (y_n) were an orthogonal family, rather than just linearly independent. Were it not true that $\|y_n\| \leqq M$, $n=1, 2, \cdots$ for some M then we could select a subsequence, say (z_n), such that $\|z_n\| \geqq n^2$, $n=1, 2, \cdots$. As a subsequence of an orthogonal sequence, (z_n) would also be orthogonal. Then $e_n = L_n/\|L_n\|$ would be orthonormal, and so $x=\sum_{k=1}^{\infty}(1/k)e_k \in H$. But

$$(x, z_n) = \left(\sum_{k=1}^{\infty}(1/k)e_k, z_n\right) = \sum_{k=1}^{\infty}(1/k)(e_k, \|z_n\|e_n)$$
$$= \frac{1}{n}\|z_n\| \geqq \frac{n^2}{n} = n$$

so that $|(x, z_n)| \to \infty$ as $n\to\infty$. Since (z_n) is a subsequence of (y_n), this means that the sequence $|(x, y_n)|$, $n=1, 2, \cdots$ would be unbounded, a contradiction.

The basic idea of our proof is to select an "almost orthogonal" subsequence of (y_n) to which the above argument will apply. We go about this task as follows.

Suppose that the sequence of norms $(\|y_n\|;\ n=1, 2, \cdots)$ is unbounded. We select an integer $n(1)$ so that $\|y_{n(1)}\| \geqq 1^2$, and then set $z_1 = y_{n(1)}$ and $e_1 = (1/\|z_1\|)z_1$. Now by assumption $|(e_1, y_n)| \leqq M(e_1)$, $n=1, 2, \cdots$ so that, $\|y_n\|$ being unbounded, we can find an $n(2) > n(1)$ such that

$$(1/\|y_{n(2)}\|)\,|(e_1, y_{n(2)})| \leqq 2^{-2} \quad\text{and}\quad \|y_{n(2)}\| \geqq 2^2.$$

Then we set $z_2 = y_{n(2)}$ and $e_2 = (1/\|z_2\|)z_2$. Next we select $n(3) > n(2)$ so that

$$(1/\|y_{n(3)}\|)(|(e_1, y_{n(3)})| + |(e_2, y_{n(3)})|) \leqq 2^{-3} \quad\text{and}\quad \|y_{n(3)}\| \geqq 3^2.$$

Continuing by induction, we obtain a subsequence (z_n) of (y_n) such that $\|z_n\| \geqq n^2$, $n=1, 2, \cdots$, and for the corresponding unit vectors $e_n = z_n/\|z_n\|$

$$(*)\quad |(e_1, e_{n+1})| + |(e_2, e_{n+1})| + \cdots + |(e_n, e_{n+1})| \leqq 2^{-(n+1)}, \quad n = 1, 2, \cdots.$$

The point of what we have done in $(*)$ is to force the off-diagonal inner products (e_m, e_n), $m\neq n$, to be small, so that e_n is "almost" orthonormal.

The remainder of the proof consists in mimicking the argument used in the orthogonal case; first verify that $\sum_{k=1}^{\infty}(1/k)e_k$ converges to a vector x in H and then compute to get

$$|(x, z_n)| \geqq n\left(1 - \frac{n}{2^{n-1}}\right) \to \infty.$$

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A CLASS OF SPACES IN WHICH COMPACT SETS ARE FINITE

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In a recent note [3] in this journal, Norman Levine introduced the term *cf space* for a topological space in which each compact set is finite. The term *pseudo-finite* has also been used to describe such spaces [4]. Levine has shown that nice spaces (e.g., first countable T_1 spaces) which are pseudo-finite must be discrete. In this note we show that there exists a wide variety of nondiscrete pseudo-finite spaces. Another example is given by A. Wilansky in [4, p. 265].

A topological space is said to be an *MI-space* if it possesses no isolated point and if each dense set is open. D. R. Anderson [1] has shown that connected Hausdorff MI-spaces which possess arbitrary infinite dispersion character exist in abundance (the dispersion character is the least cardinality of a nonempty open set). Our main result states that each MI-space is pseudo-finite. First we present a sequence of lemmas. We omit the easy proof of the first lemma; the next two are due to Hewitt [2, p. 325].

LEMMA 1. *Let X be an infinite space with no isolated point. Then X contains an infinite subset with empty interior.*

LEMMA 2. *In an MI-space each set with empty interior is totally isolated (i.e., closed and discrete).*

LEMMA 3. *A subspace of an MI-space is also an MI-space provided it has no isolated points.*

LEMMA 4. *Each MI-space is noncompact.*

Proof. Let X be an MI-space and let S be an infinite subset of X with $\operatorname{Int} S = \emptyset$. Hence S is an infinite totally isolated subset of X, so S is not compact. Since S is closed, X is not compact.

THEOREM. *Each MI-space is pseudo-finite.*

Proof. Let K be a compact subset of the MI-space X. Then $K - \operatorname{Int} K$ is compact and has empty interior. Thus $K - \operatorname{Int} K$ is compact, totally isolated, and hence finite. Let I be the set of isolated points in K. Then I is contained in $K - \operatorname{Int} K$ since $\operatorname{Int} K$ is open and hence contains no isolated points. Thus I is finite so $K - I$ is a compact subset of X with no isolated points. By Lemmas 3 and 4, $K - I$ is empty. Therefore $K = I$, a finite set.

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HYPERSPHERES IN SPACES OF CONSTANT CURVATURE

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In this paper we determine the surface area Ω_n and the volume V_m of a hypersphere in m-dimensional space of (positive or negative) constant curvature. The special values of Ω_1 and V_2 [3, p. 250] as well as V_3 [1, p. 186] are in the permanent literature. Also, V_m appears in [5]. Our main purpose is to establish suitable coordinate systems to obviate most of the computation.

In our notation Greek indices take the values 1 to $m=n+1$ and Latin indices the values 1 to n. The distance Δs between the points y^α and z^α is given by

$$\cos(K^{1/2}\Delta s) = (1 - K\Sigma y^\alpha y^\alpha)^{1/2}(1 - K\Sigma z^\beta z^\beta)^{1/2} + K\Sigma y^\alpha z^\alpha. \tag{1}$$

From this one derives the element of arc setting $z^\alpha = y^\alpha + \Delta y^\alpha$ and $p = (1 - K\Sigma yy)^{1/2}$. Thus, (1) may be changed to $4 \sin^2 (K^{1/2}\Delta s/2) = K\Sigma\Delta y\Delta y + (\Delta p)^2$ and hence the metric tensor [4, p. 136] of the m-space becomes

$$a_{\alpha\beta} = \delta_{\alpha\beta} + Ky^\alpha y^\beta/(1 - K\Sigma yy). \tag{2}$$

Setting $r = (\Sigma yy)^{1/2}$ and using (1) we find that the distance R from the origin of a point y is given by $\cos (K^{1/2}R) = (1 - Kr^2)^{1/2}$ or equivalently

$$r = K^{-1/2} \sin(K^{1/2}R). \tag{3}$$

It follows that $r=\text{const}$ represents a hypersphere centered at the origin. If we now think of such a hypersphere as given by equations of the form $y^\alpha = y^\alpha(x^1, \cdots, x^n)$ we see that $\sum y^\alpha y^\alpha{}_{,i} = 0$. Then, with the aid of (2) we find the metric tensor of the hypersphere, namely

$$g_{ij} = a_{\alpha\beta} y^\alpha{}_{,i} y^\beta{}_{,j} = \delta_{\alpha\beta} y^\alpha{}_{,i} y^\beta{}_{,j}.$$

But this is the metric tensor of a hypersurface imbedded in a Euclidean space referred to rectangular Cartesian coordinates. Hence for measurements on the surface of the hypersphere Euclidean results become immediately applicable. It is known that the area of a hypersphere of radius r immersed in Euclidean space is $\omega_n r^n$ [2, p. 303], where ω_n denotes the surface area of the Euclidean unit sphere, that is $\omega_n = 2\pi^{m/2}/\Gamma(m/2)$. Therefore $\Omega_n(R)$, the area of a hypersphere of radius R in non-Euclidean space, is $\omega_n r^n$ where relation (3) provides the value of r in terms of R.

In order to evaluate V_m we realize that the orthogonal trajectories of a family of concentric hyperspheres are geodesics. Thus we may refer the m-space to geodesic normal coordinates [7, p. 71; 8, p. 81] for which these hyperspheres constitute a system of parallels. The metric form is $a_{ij}dy^i dy^j + (dy^m)^2$ and clearly a_{ij} is the metric tensor on $y^m = \text{const}$. Letting a designate the determinant of this tensor we have

$$V_m = \int \cdots \int a^{1/2} dy^1 \cdots dy^m = \int_0^R \Omega_n(y^m) dy^m.$$

ADDENDUM. In the notation of classical non-Euclidean geometry, with the natural unit of measurement so that $K=1$ or -1 according as the space is elliptic or hyperbolic, we have (dropping the superscript m)

$$V_m = \omega_n \int_0^R \sin^n y dy \quad \text{or} \quad \omega_n \int_0^R \sinh^n y dy$$

for $K=1$ or -1, respectively. Since the m-dimensional content of the whole elliptic m-space, being one half of a spherical m-space, is $\omega_m/2 = V_m(\pi/2)$, we obtain, as a corollary of the former result, the "beta function" identity

$$\int_0^{\pi/2} \sin^n y dy = \omega_m/2\omega_n$$

[6, p. 136].

For a final observation let us take $ds^2 = \sum dy^\alpha dy^\alpha - (dy^0)^2$ as the metric form of Minkowski $(m+1)$-space. In such a space, the "m-sphere"

$$(y^0)^2 - \Sigma yy = 1,$$

of time-like radius 1, being one sheet of a two-sheeted hyperboloid, has an induced metric which makes it isometric to the hyperbolic m-space [4, p. 135]. This "m-sphere" intersects the Euclidean hyperplane $y^0 = \cosh R$ in the n-sphere

$$y^0 = \cosh R, \qquad \Sigma yy = \sinh^2 R,$$

which has Euclidean radius $\sinh R$ and hyperbolic radius R. The latter n-sphere has the same surface area Ω_n as an n-sphere of radius R in hyperbolic m-space (with $K=-1$). Similar relations may be established for an n-sphere imbedded in spherical (or elliptic) m-space (with $K=1$).

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SOME COROLLARIES TO THE METRIZATION LEMMA

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1. Introduction. In [7], M. Katětov proved that every bounded uniformly continuous real-valued function on a uniform subspace of a uniform space X can always be extended to a bounded uniformly continuous real-valued function on X. This remarkable theorem, by comparison to the classical Tietze extension theorem (cf. [8] p. 242), serves to point out the strength of the concept of uniform continuity. Several proofs of this result are now known ([1], [4]), but here a simpler one is presented. We deduce Katětov's theorem from Isbell's theorem, which asserts that every bounded uniformly continuous pseudometric on a uniform subspace of a uniform space X has a bounded uniformly continuous pseudometric extension to all of X (see [5] or [6]), and we deduce Isbell's theorem from the metrization lemma [8, 6.12]. Thus, these results could be easily included in a one-year course in general topology.

We refer the reader to [2] or [8] for the elementary theory of uniform spaces, and we assume that the set $\mathbf{R}$ of real numbers is equipped with the usual uniformity. If $\alpha \in \mathbf{R}$, then $\boldsymbol{\alpha}$ denotes the constant function on a set X with value α. If f, g are real-valued functions on a set X, then by $f+g$, $-f$, fg, $|f|$, $f\vee g$, and $f\wedge g$, we mean the functions on X whose values are, respectively $f(x)+g(x)$, $-f(x)$, $f(x)g(x)$, $|f(x)|$, $\max\{f(x), g(x)\}$, and $\min\{f(x), g(x)\}$. Finally, if f and g are real-valued functions on a set X, then $f \geqq g$ means that $f(x) \geqq g(x)$ for all $x \in X$.

2. Isbell's Theorem. We begin this section by stating the uniform version of the metrization lemma.

LEMMA 1. *If $(X, \mathcal{U})$ is a uniform space and if $V \in \mathcal{U}$ is symmetric, then there exists a uniformly continuous pseudometric r on X such that $r \leqq 1$ and*

$$\{(x, y) \in X \times X : r(x, y) < 1/2\} \subset V.$$

This result follows immediately by combining 6.11 and 6.12 of [8]; another proof appears in [3, Section 1, No. 4, Prop. 2].

ISBELL'S THEOREM. *Let $(X, \mathcal{U})$ be a uniform space and let S be a uniform subspace of X. Then every bounded uniformly continuous pseudometric on S has a bounded uniformly continuous pseudometric extension to X.*

The proof given here roughly follows that given in [5], but is given in the context of the Bourbaki-Weil approach to uniform space theory—the approach normally taken in introductory topology courses.

Proof. Suppose that p is a bounded uniformly continuous pseudometric on S. For each integer n, set

$$U_n = \{(x, y) \in S \times S : p(x, y) < 2^n\},$$

so that $U_n \in \mathcal{U}_S$, where $\mathcal{U}_S$ denotes the relative uniformity on S, and let $V_n \in \mathcal{U}$

such that V_n is symmetric and $V_n \cap (S \times S) = U_n$. As a corollary to Lemma 1, we obtain:

LEMMA A. *For each integer n, there exists a uniformly continuous pseudometric r_n on X such that $r_n \leqq 1$ and*

$$\{(x, y) \in X \times X: r_n(x, y) < 1/2\} \subset V_n.$$

Next, for each integer n, we set $d_n = 2^{n+2} r_n$, so that d_n is a uniformly continuous pseudometric on X with $d_n \leqq 2^{n+2}$ and

$$\{(x, y) \in X \times X: d_n(x, y) < 2^{n+1}\} \subset V_n.$$

Since p is bounded, we may choose an integer k so that $p(x, y) \leqq 2^k$ for all $(x, y) \in S \times S$. Then set

$$d(x, y) = \sum_{n=-\infty}^{k} d_n(x, y), \qquad \text{for all } (x, y) \in X \times X.$$

As a uniformly convergent series of bounded uniformly continuous pseudometrics, d is a bounded uniformly continuous pseudometric.

LEMMA B. $d \mid S \times S \geqq p$.

In fact, if $(x, y) \in S \times S$ and $d(x, y) = 0$, then $d_n(x, y) = 0$ for each n, so $(x, y) \in U_n$; hence $p(x, y) < 2^n$ for each n. But $2^n \to 0$ as $n \to -\infty$, so $p(x, y) = 0$. On the other hand, if $d(x, y) > 0$, then there exists an integer n such that $2^n \leqq d(x, y) < 2^{n+1}$. We may assume $n < k$. Then $d_n(x, y) < 2^{n+1}$, so $(x, y) \in V_n \cap (S \times S) = U_n$, whence $p(x, y) < 2^n \leqq d(x, y)$.

Finally, for each $(x, y) \in X \times X$, we set

$$f(x, y) = \inf\{d(x, a) + p(a, b) + d(b, y): a, b \in S\},$$

and

$$r(x, y) = \min\{d(x, y), f(x, y)\}.$$

LEMMA C. *r is a bounded uniformly continuous pseudometric on X such that $r \mid S \times S = p$.*

In fact, since $d \geqq r$, it follows that r is bounded, uniformly continuous (use [8] 6.11), and $r(x, x) = 0$ for all $x \in X$. The function f is symmetric and nonnegative, and so it follows that r is symmetric and nonnegative. It is a routine matter to verify that r satisfies the triangle inequality and that, using Lemma B, $r \mid S \times S = p$.

3. Katětov's Theorem. Let us recall that if $(X, \mathcal{U})$ is a uniform space, if $\alpha \in \mathbf{R}$, and if f and g are uniformly continuous real-valued functions on X, then $f+g$, $f-g$, $f \vee \alpha$, and $f \wedge \alpha$ are also uniformly continuous real-valued functions on X.

The next lemma is the crucial step in deriving Katětov's Theorem from

Isbell's Theorem. We recall that if f is a real-valued function on a set X, then the *zero-set* of f is the set $Z_X(f)=\{x\in X: f(x)=0\}$.

LEMMA 2. *Let $(X, \mathcal{U})$ be a uniform space and let S be a uniform subspace of X. If f is a nonnegative bounded uniformly continuous real-valued function on S such that $Z=Z_S(f)\neq\emptyset$, then there exists a bounded uniformly continuous real-valued function g on X such that $g|S=f$.*

Proof. For each $(x, y)\in S$, we set

$$\psi_f(x, y) = |f(x) - f(y)|.$$

Then ψ_f is a bounded uniformly continuous pseudometric on S, so by Isbell's Theorem, there exists a bounded uniformly continuous pseudometric p on X such that $p|S\times S=\psi_f$. Define a function $g: X\to \mathbf{R}$ by setting, for each $x\in X$,

$$g(x) = p(x, Z) = \inf\{p(x, y): y \in Z\}.$$

Since $|g(x)-g(y)|=|p(x, Z)-p(y, Z)|\leqq p(x, y)$, it follows that g is a bounded uniformly continuous real-valued function on X. Moreover, if $x\in S$, then

$$g(x) = p(x, Z) = \psi_f(x, Z) = \inf\{|f(x) - f(y)| : y \in Z\} = f(x),$$

since $f\geqq 0$, and so $g|S=f$.

KATĚTOV'S THEOREM. *Let $(X, \mathcal{U})$ be a uniform space and let S be a uniform subspace of X. Then every bounded uniformly continuous real-valued function on S has a bounded uniformly continuous real-valued extension to X.*

Proof. The result is trivial if $S=\emptyset$. Thus assume that $S\neq\emptyset$, and let f be a bounded uniformly continuous real-valued function on S. Choose $a\in S$ and set $f(a)=\alpha$. Then $f\vee\alpha$ and $f\wedge\alpha$ are uniformly continuous, and hence $g=(f\vee\alpha)-\alpha$ and $h=-((f\wedge\alpha)-\alpha)$ are uniformly continuous bounded real-valued functions on S. Moreover, $g\geqq 0$, $h\geqq 0$, and $a\in Z_S(g)\cap Z_S(h)$. Therefore, by Lemma 2, there exist bounded uniformly continuous real-valued functions g_1 and h_1 on X such that $g_1|S=g$ and $h_1|S=h$. Now set $f_1=(g_1-h_1)+\alpha$. Then f_1 is a bounded uniformly continuous real-valued function on X such that $f_1|S=f$. This completes the proof.

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ON BLOCKS OF N CONSECUTIVE INTEGERS

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S. Pillai [3] proved that if $N \leqq 16$, every block B_N of N consecutive integers contains at least one integer which is relatively prime to all the others. He moreover conjectured that if $N \geqq 17$, there exist blocks B_N containing no such integer. A. Brauer [1] found a proof of this conjecture for $N \geqq 300$ and Pillai [3] took care of the cases $17 \leqq N \leqq 299$ using a dissimilar method. In the following I present a new and somewhat simpler proof which first covers the cases $N=17$ and $N=18$ and then treats systematically all $N \geqq 19$.

Let x be a solution of the system

$$x \equiv 0 \pmod{2\cdot 5\cdot 11\cdot 17} \qquad x \equiv -16 \pmod{3\cdot 7\cdot 13};$$

then it is easily seen that for $N=17$ and $N=18$, the block $\{x, x+1, \cdots, x+N-1\}$ contains no integer relatively prime to each of the others.

Now let $N \geqq 19$. Let $p_1 < p_2 < p_3$ be the three smallest consecutive primes $\geqq N/2$. It is known [2] that if $k \geqq 25$, there is at least one prime between k and $6k/5$. Hence, for $N \geqq 50$,

$$p_2 + p_3 - p_1 \leqq p_1(6/5) + p_1(6/5)^2 - p_1 \leqq (6/5 + 36/25 - 1)(6/5)(N/2) < N.$$

It is easily verified that

$$p_2 + p_3 - p_1 \leqq N \tag{1}$$

in fact holds for all $N \geqq 19$.

Now let x be a solution of the system

$$x \equiv 0 \pmod{q} \qquad \text{for each prime } q < p_1 \tag{2}$$

$$x + 1 \equiv 0 \pmod{p_1} \tag{3}$$

$$x - 1 \equiv 0 \pmod{p_2} \tag{4}$$

$$x + p_1 \equiv 0 \pmod{p_3}. \tag{5}$$

I claim that the block $B_N = \{x-(N-p_2), \cdots, x+(p_2-1)\}$ contains no integer relatively prime to each of the others. I will show this by producing, for each $r \in B_N$, a corresponding $s \in B_N$ such that $r \neq s$ and $(r, s) > 1$.

Since, by (1), $2 < p_1 < p_2 < p_3 < N$, B_N looks as follows: $x-(N-p_2), \cdots, x-1, x, x+1, \cdots, x+p_1, \cdots, x+(p_2-1)$. If $r=x$, we may, by (2), choose $s=x+2$. If $r=x+1$, we may, by (3), choose $s=x+(p_1+1)$. If $r=x-1$, we may, by (4), choose $s=x+(p_2-1)$. If $r=x+p_1$, we may, by (5), choose $s=x-(p_3-p_1)$, since (1) guarantees that $x-(p_3-p_1) \in B_N$. Every other $r \in B_N$ has the form $x \pm i$, where i is divisible by a prime $< p_1$. Hence, by (2), we may choose $s=x$ for each of these r.

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ON FERMAT'S LAST THEOREM

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In this note we shall prove two theorems pertaining to Fermat's last theorem, which shall be referred to as FLT.

THEOREM 1. *If $p \neq 3$ and is an odd prime and*

$$x^{p-1} + y^{p-1} = z^{p-1} \tag{1}$$

holds with positive, pairwise prime integers, then z is a quadratic residue mod p. *For $p = 3$ the statement is false.*

Proof. Assume that $(xyz, p) = 1$; then by Fermat's theorem we have

$$x^{p-1} \equiv y^{p-1} \equiv z^{p-1} \equiv 1 \pmod{p}. \tag{2}$$

From (1) and (2) we get $2 \equiv 1 \pmod{p}$ which is impossible. Thus, since the integers x, y, z are pairwise prime, exactly one of them must be divisible by p.

Suppose $p \mid z$; then from (1) and (2) we have $2 \equiv 0 \pmod{p}$. Hence z must be coprime to p; also either of x and y must be divisible by p and the other one must be coprime to it.

If x and y are both odd, $x^{p-1} + y^{p-1} \equiv 2 \not\equiv z^{p-1} \pmod{4}$. Hence z must be odd and, of x and y, one must be odd and the other even.

Let us consider that x is odd and is divisible by p, so that y is even and coprime to p.

Rewriting (1) as

$$x^{p-1} = (z^{(p-1)/2} - y^{(p-1)/2})(z^{(p-1)/2} + y^{(p-1)/2}), \tag{3}$$

since x is odd and z prime to y, the two factors on the right hand side of (3) must be coprime and thus each must be a perfect $(p-1)$th power. Hence

$$\tag{4} \begin{cases} z^{(p-1)/2} + y^{(p-1)/2} = a^{p-1} & \text{(a)} \\ z^{(p-1)/2} - y^{(p-1)/2} = b^{p-1} & \text{(b)} \end{cases}$$

where $x = ab$.

Adding (4a) and (4b) we get

$$2z^{(p-1)/2} = a^{(p-1)} + b^{(p-1)}. \tag{5}$$

Since $p \mid x$ and $(a, b) = 1$, one of a and b is divisible by p and the other coprime to it. Thus from (5) we have

$$2z^{(p-1)/2} \equiv 1 \pmod{p}.$$

Squaring and using (2) we get

$$4 \equiv 1 \pmod{p}.$$

Since $p \neq 3$, this congruence cannot be satisfied. Hence y, which is even, must be divisible by p.

Let us rewrite (1) as

$$(x^{(p-1)/2})^2 + (y^{(p-1)/2})^2 = (z^{(p-1)/2})^2.$$

It follows [1, p. 190] that we must have

$$(6) \qquad \begin{cases} x^{(p-1)/2} = m^2 - n^2 & \text{(a)} \\ y^{(p-1)/2} = 2mn & \text{(b)} \\ z^{(p-1)/2} = m^2 + n^2 & \text{(c)}, \end{cases}$$

where $(m, n) = 1$, $m > n$ and with opposite parity. Since $p \mid y$, p divides either m or n; let us assume that p divides the odd factor, say n, so that m is even and coprime to p. As $(2m, n) = 1$, then from (6b) each of them must be a perfect $(p-1)/2$th power. Hence

$$(7) \qquad \begin{cases} 2m = a^{(p-1)/2} & \text{(a)} \\ \ \ n = b^{(p-1)/2} & \text{(b)}. \end{cases}$$

Thus using (7) in (6c) we get

$$(8) \qquad 4z^{(p-1)/2} = a^{(p-1)} + 4b^{(p-1)}$$

$$p \mid b \Rightarrow 4z^{(p-1)/2} \equiv a^{(p-1)} \equiv 1 \pmod{p}.$$

Squaring this congruence and using (2) we get

$$16 \equiv 1 \pmod{p}.$$

Indeed, for $p = 5$, $x^4 + y^4 = z^4$ has no nontrivial solutions [1, p. 191], therefore $p \neq 5$. Hence, as also $p \neq 3$, the above congruence is impossible. Hence p cannot divide the odd factor. Therefore, out of m and n, the one which is even, say m, must be divisible by p; so that n is odd and thus coprime to p. Hence from (7a) we get

$$(9) \qquad a^{(p-1)} = 4m^2 \equiv 0 \pmod{16p}.$$

Thus using (9) in (8) we obtain

$$(10) \qquad z^{(p-1)/2} \equiv b^{(p-1)} \pmod{4p}.$$

Since b is odd and coprime to p, therefore, $b^{(p-1)} \equiv 1 \pmod{4p}$.

Hence from (10) we have

$$(A) \qquad z^{(p-1)/2} \equiv 1 \pmod{4p}.$$

It can be readily seen that this congruence also holds if, instead of m, n is even and thus divisible by p so that m is odd and coprime to p. Consequently,

it is obvious from (A) that

$$z^{(p-1)/2} \equiv 1 \pmod{p}.$$

Hence z must be a quadratic residue mod p, which was to be shown.

THEOREM 2. *Under the same hypothesis as in Theorem* 1, (p/z) *must be equal to* $+1$ [*where* (p/z) *denotes the Jacobi's Symbol*] *and hence* p *must be a quadratic residue* mod z *if* z *is prime.*

Proof. Evidently if $p\equiv 1 \pmod 4$, then (1) is impossible because $x^{4m}+y^{4m}=z^{4m}$ has no nontrivial solutions for all integral and positive values of m. Hence

$$p \not\equiv 1 \pmod 4, \quad \text{i.e.,} \quad (p-1)/2 \text{ is odd.} \tag{11}$$

It is obvious from congruence (A) of Theorem (1) that

$$z^{(p-1)/2} \equiv 1 \pmod 4 \qquad \text{which implies that } z \equiv 1 \pmod 4, \tag{12}$$

for if $z\equiv -1 \pmod 4$, then in view of (11) we have $z^{(p-1)/2}\equiv -1 \pmod 4$ which contradicts (12); hence

$$(z-1)/2 \text{ is even.} \tag{13}$$

Now, since p and z are two distinct, positive, odd, relatively prime integers, therefore, from the generalized reciprocity theorem [2], we get

$$\left(\frac{p}{z}\right)\cdot\left(\frac{z}{p}\right) = (-1)^{(p-1)/2\cdot(z-1)/2}. \tag{14}$$

But z is a quadratic residue mod p. Thus $(z/p)=1$, and hence using (13) in (14) we get

$$\left(\frac{p}{z}\right) = 1, \quad \text{i.e.,} \quad \left(\frac{p}{z}\right) = \left(\frac{p}{p_1}\right)\cdot\left(\frac{p}{p_2}\right)\cdot\left(\frac{p}{p_3}\right)\cdots\left(\frac{p}{p_r}\right) = +1.$$

Evidently if z is prime, p must be a quadratic residue mod z, which was to be shown.

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ON THE DIFFERENTIAL EQUATION $xy''-(x+n)y'+ny=0$

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In a recent article [1] in this MONTHLY, J. P. Hoyt reminded readers that if n is a nonnegative integer, the differential equation

$$xy'' - (x+n)y' + ny = 0 \tag{1}$$

has the solutions e^x and $P_n(x)=1+x/1!+x^2/2!+\cdots+x^n/n!$ (see [3] p. 430). The purpose of this note is to develop known properties of e^x and its Maclaurin approximant $P_n(x)$ by applying elementary differential equation theory to equation (1).

To this end, let $e(x)$ by any function $\not\equiv 0$ with the property that $[e(x)]'-e(x)\equiv 0(-\infty<x<\infty)$. It follows that $e(x)$ is a solution of the differential equation

$$y'' - y' = 0; \tag{2}$$

that is, of equation (1) with $n=0$. A second solution of (2) is the constant 1. Inasmuch as the Wronskian of $e(x)$ and 1,

$$\begin{vmatrix} 1 & e(x) \\ 0 & e(x) \end{vmatrix},$$

equals $e(x)$, it follows that these two solutions are linearly independent. It also follows that the function $e(x)$ has no zero on the interval $(-\infty, \infty)$.

That $e(x)$ and $P_n(x)$ are linearly independent follows from the fact that the Wronskian

$$\begin{vmatrix} P_n(x) & e(x) \\ P_{n-1}(x) & e(x) \end{vmatrix} = \frac{e(x)x^n}{n!}$$

of these solutions is not identically zero. This can also readily be shown directly: the assumption that there exist constants a and b, not both zero, such that $aP_n(x)+be(x)\equiv 0$ on any interval I containing the origin implies that $a=-be(0)\neq 0$ and, accordingly, that $P_n(x)\equiv P_{n-1}(x)$. It would then follow that $x^n\equiv 0$ on I.

Next, we employ the Sturm separation theorem to prove that the polynomials $P_n(x)$ can have at most one real zero. Recall that $e(x)$ never vanishes. If $P_n(x)$ had two distinct real zeros, both would be negative, and $e(x)$ would necessarily vanish at a point between these zeros; a contradiction. $P_n(x)$ cannot have a multiple real zero because first, this zero would also be negative; second, this point would be a nonsingular point of the differential equation, and the only solution of the equation that vanishes with its derivative at a nonsingular point is the null solution.

It follows, of course, that $P_n(x)$ has precisely one (negative) zero when n is odd and no real zeros when n is even [cf. 2, p. 444].

Henceforth, we assume that $e(0)=1$ and write $e(x)$ as e^x.

It may now be shown that the difference $D_n(x) = e^x - P_n(x)$ has no real zeros except those at the origin. It is clear immediately that $D_n(x)$ cannot have two such zeros, because $D_n(x)$ is also a solution of (1). Suppose then that $D_n(x)$ has a zero at $x = x_0 \neq 0$. Without loss of generality, assume that $x_0 < 0$. Then $D_n(x)$ is of one sign on $x_0 < x < 0$. We may suppose that $D_n(x) > 0$ on this interval [otherwise, consider $-D_n(x)$].

Then, for $k > 0$ and sufficiently small, the curve $y = ke^x$ will intersect the curve $y = D_n(x)$ at least twice on the interval $x_0 < x < 0$—say, at $x = x_1$ and $x = x_2$. The difference $D_n(x) - ke^x$ is also a solution, and it would vanish at $x = x_1$ and $x = x_2$. Again, the Sturm separation theorem yields a contradiction.

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AN ALTERNATIVE FORMULATION OF AN UNSOLVED PROBLEM OF SET THEORY

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If $P(S)$ denotes the family of all subsets of the set S, then the assertion: $P(A)$ equivalent to $P(B)$ implies A equivalent to B is an unproved and undisproved conjecture of set theory [1]. In this note we offer an alternative formulation of this assertion. We will prove the

THEOREM. *$P(A)$ equivalent to $P(B)$ implies A equivalent to B if and only if $P(A)$ equivalent to $P(B)$ implies $P(A)$ order-isomorphic to $P(B)$, where, as usual, $P(A)$ and $P(B)$ are partially ordered by set inclusion.*

Proof. The "only if" is clear. For the "if," let f be an order-isomorphism of $P(A)$ onto $P(B)$. If $\emptyset$ denotes the empty set, then $f(\emptyset) = \emptyset$ because $\emptyset \subseteq f^{-1}(\emptyset)$ implies $f(\emptyset) \subseteq \emptyset$. Let a belong to A. Then $f(\{a\}) \neq \emptyset$. If $f(\{a\}) = \{b, c, \cdots\}$, a set of at least two elements of B, then $\{b\} \subset \{b, c, \cdots\}$ implies $f^{-1}(\{b\}) \subset f^{-1}(\{b, c, \cdots\}) = \{a\}$, hence $\{b\} = \emptyset$, a contradiction. Thus $f(\{a\}) = \{b\}$, for some b of B. If b of B is given, there exists some X of $P(A)$ such that $f(X) = \{b\}$. Then $X \neq \emptyset$. If $X = \{a, c, \cdots\}$, a set of at least two elements of A, then $\{a\} \subset X$ implies $f(\{a\}) \subset f(X) = \{b\}$, hence $\{a\} = \emptyset$, a contradiction. Thus f is a one-one mapping of A onto B.

Reference

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RESEARCH PROBLEMS*

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

CAN A PLANE CONVEX BODY HAVE TWO EQUICHORDAL POINTS?

VICTOR KLEE, University of Washington

Dedicated to Hugo Hadwiger on his sixtieth birthday*

As the term is used here, a plane convex body is a subset B of the Euclidean plane which is bounded, closed, convex, and has nonempty interior. A *chord* of B is a line segment which joins two boundary points and passes through an interior point p of B. The point p is called an *equichordal point* of B provided that all of B's chords through p are of the same length. The center of a circular disk D is an equichordal point of D. Noncircular bodies with equichordal points are easily constructed and have been studied by several authors (Fujiwara [6], P. J. Kelly [9], and others).

The problem of the title was first raised by Fujiwara [6] in 1916 and independently by Blaschke, Rothe, and Weitzenböck [1] in 1917. Fujiwara proved that no convex body has three equichordal points. Plane convex bodies K with two equichordal points p and q have been studied (assuming their existence) by several authors. Süss [13] showed that K is symmetric relative to the line through p and q and relative to the midpoint of the segment pq. Dirac [2] obtained quantitative results on K's chord containing pq and a related quantitative study was made by Ehrhart [5]. Dirac [2] showed that K's boundary curve C is differentiable and C's tangents were studied by Dulmage [4]. Wirsing [14] proved that C is analytic. Previously, Helfenstein [7] had claimed to prove that C is not six times differentiable, Linis [10] that C is not twice differentiable. In conjunction with Wirsing's result, these would solve the problem by showing that K does not exist. However, it appears that there are mistakes in the reasoning of Helfenstein and Linis (see Wirsing [14], Dirac [3]), and the problem is still open. An exposition of the problem was given by Sancho de San Román [12].

Note that an interior point p of B is an equichordal point if and only if the sum $\|x-p\|+\|y-p\|$ is constant for all chords xy through p. Analogously, let us call p an *equireciprocal point* [resp. an *equiproduct point*] provided that the sum $\|x-p\|^{-1}+\|y-p\|^{-1}$ [resp. the product $\|x-p\|\,\|y-p\|$] is constant for all such chords. Apparently the following problem is open:

Can a plane convex body have two equireciprocal points?

* This new Department of the MONTHLY was inspired by H. Hadwiger's series of articles on unsolved problems in *Elemente der Mathematik.*

The results of Süss [13] imply certain symmetry properties for a convex body with two equireciprocal points and show that no convex body has three such points.

Convex bodies with equiproduct points have been studied by Yanagihara [15, 16], Rosenbaum [8, 11], and J. B. Kelly [8]. Each interior point of a spherical body is an equiproduct point. The set P_B of equiproduct points of a convex body B is the intersection of a flat with the interior of B. If $\dim P_B \geqq n-1$, or if B is smooth and $\dim P_B \geqq 1$, then B is spherical; however, for $0 \leqq k \leqq n-2$ there are nonspherical n-dimensional convex bodies with $\dim P_B = k$.

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CLASSROOM NOTES

Edited by David Drasin

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UNIFORM CONVERGENCE OF FOURIER SERIES

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We present here a proof of the theorem below which is of a sufficiently elementary nature to be included in any standard "Advanced Calculus" course. Surprisingly, the theorem is not proved in any of the "standard" advanced calculus texts although it is often quoted. The proof depends only on:

(1) Convergence in the mean square of Fourier series.
(2) The Cauchy-Schwarz inequality on finite-dimensional spaces.
(3) The Cauchy criterion for uniform convergence.
(4) The fact that a uniform limit of continuous functions is continuous.

THEOREM. *Let f be a continuous function on $[0, 2\pi]$ with $f(0)=f(2\pi)$. Suppose f has a bounded piecewise continuous derivative. Then the Fourier series for f converges uniformly.*

Proof. Suppose $f=\sum_{-\infty}^{\infty} a_n e^{+inx}$ is the Fourier series for f. Since f' is piecewise continuous, its Fourier coefficients

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-inx}dx \\ &= -\frac{1}{2\pi}\int_0^{2\pi} f(x)\,\frac{d}{dx}\,(e^{-inx}) + \frac{e^{-inx}}{2\pi} f(x)\,\Big|_0^{2\pi} \\ &= ina_n \end{aligned}$$

have the property that $\sum |b_n|^2 < \infty$. Thus:

$$\sum_{n\neq 0} n^2 \,|\, a_n|^2 < \infty. \tag{1}$$

The standard Cauchy-Schwarz inequality for vector spaces shows immediately that $\sum |x_n|^2 < \infty$ and $\sum |y_n|^2 < \infty$ imply $\sum |x_n y_n| < \infty$. Thus $\sum_{n\neq 0} (1/n^2) < \infty$ and (1) imply $\sum |a_n| < \infty$. Therefore, by the Cauchy criterion and the fact that $\sup |e^{inx}| = 1$, $\sum_{-\infty}^{\infty} a_n e^{inx}$ converges uniformly to some continuous function, g. But $\int_0^{2\pi} |g-f|^2\, dx = 0$ because $\sum_{-\infty}^{\infty} a_n e^{inx}$ converges in the mean to both f and g. Since g and f are both continuous, $f=g$.

We remark that the proof only requires f' to be square integrable, so, for example, f' need not be bounded and could behave like $(x-x_0)^{-a}$ with $a<\frac{1}{2}$ at some point x_0.

AN APPLICATION OF THE WRONSKIAN

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Let $y(x), y_1(x), y_2(x), \cdots, y_n(x)$ be $(n-1)$-times differentiable functions on an interval (a,b). If W denotes the Wronskian of n functions, then

$$W(yy_1, yy_2, \cdots, yy_n) = y^n W(y_1, y_2, \cdots, y_n)$$

on (a, b).

This theorem is attributed to Christoffel (1857) by Thomas Muir [1] who also outlines a proof of the theorem due to Frobenius (1873). The reader does not need these references, however, since he may verify that the theorem is, if

properly viewed, an elementary exercise. Simply form $W(yy_1, yy_2, \cdots, yy_n)$ using the product rule for differentiation. Each entry of the ith row contains i terms and so $W(yy_1, yy_2, \cdots, yy_n)$ may be written as the sum of $n!$ determinants each of which is zero, because of proportional rows, except for one of the $n!$ determinants and it is $y^n W(y_1, y_2, \cdots, y_n)$.

Besides its value as an exercise in determinants, I've found the theorem useful in evaluating certain determinants with constant entries. A good example of an application of the theorem came from a question of a student, Robert Wheeler, who needed the value of the persymmetric determinant

$$\Delta = \begin{vmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{s!} \\ \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(s+1)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{s!} & \frac{1}{(s+1)!} & \frac{1}{(s+2)!} & \cdots & \frac{1}{(2s-1)!} \end{vmatrix}$$

in order to prove a theorem in differential equations.

This determinant may be evaluated in other ways but I noticed that Δ is

$$(-1)^{s(s-1)/2} W\left(\frac{x^s}{s!}, \frac{x^{s+1}}{(s+1)!}, \cdots, \frac{x^{2s-1}}{(2s-1)!}\right)$$

for $x=1$. According to the Christoffel theorem this is

$$(-1)^{s(s-1)/2} x^s W\left(\frac{1}{s!}, \frac{x}{(s+1)!}, \cdots, \frac{x^{s-1}}{(2s-1)!}\right).$$

evaluated at $x=1$. This Wronskian is a constant and the value of Δ is easily seen to be

$$(-1)^{s(s-1)/2} \frac{1}{s!} \frac{1}{(s+1)!} \frac{2!}{(s+2)!} \cdots \frac{(s-1)!}{(2s-1)!}.$$

This calculation is especially rapid because the factor

$$W(c_0, c_1 x, \cdots, c_{s-1} x^{s-1})$$

is constant. In cases where the functions y_i do not give a constant Wronskian, applications of the theorem to the evaluation of determinants are less trivial.

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UNUSUAL BINARY OPERATIONS FOR DEFINING LINES

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In geometric algebra [2, 3, 4, 5] one studies (1) the *coordinatization* of affine planes by not only naming points, but by using ternary rings to name lines, and (2) the *correspondence* between the geometric properties of an affine plane and the algebraic properties of the corresponding ternary ring. The geometric properties include the well-known (universal) configurations of Pappus and Desargues and the condition (see [1]) that for every pair of points P, Q there is a translation T such that $T(P)=Q$. An affine plane satisfying the condition stated is a translation plane. Associated with every ternary ring is an algebraic system $\mathcal{S}(S, +, \cdot)$. If we have a translation plane, then we can define the lines by the equations $x=k$ and $y=xm+b$. Two well-known results are: (1) Pappus configuration holds universally (Pappian plane) iff $\mathcal{S}$ is a field; Desargues configuration holds universally (Desarguesian plane) iff $\mathcal{S}$ is a division ring; each affine plane obtained from the projective plane—the line extension of the given affine plane—is a translation plane (Moufang plane) iff $\mathcal{S}$ is an alternative division ring. (2) Pappian implies Desarguesian, Desarguesian implies Moufang. Our purpose in this note is to show the importance of including the ternary ring approach to naming lines.

In light of the above known results, consider the following startling system $\mathcal{S}'(S, \oplus, *)$ where S is the set of complex numbers, and the conjugate of a is denoted by a'. Using the usual operations of the complex numbers we define $\oplus$, $*$ as follows: $a\oplus b=ab-b$ for $b\neq 0$, and $a\oplus 0=a$, $a*b=ab'$. Note that in $\mathcal{S}'$ both operations are noncommutative, nonassociative, lack two-sided identities, $*$ does not left distribute over $\oplus$, and $\oplus$ is not solvable for $a=1$. Nevertheless $y=xm+b=x(-(m/b))(-b)+b=(x*(-m'/+b'))(-b)-(-b)=x*m''\oplus b''$, where $m''=-m'/+b'$ and $b''=-b$, for $b\neq 0$, and $y=xm=x*m'\oplus 0$ for $b=0$. Thus the lines of a Pappian plane can be defined by equations using the system $\mathcal{S}'$.

Thus a "nice" geometry is definable by a "wild" algebraic system $\mathcal{S}'$ and equations $x=k$ and $y=x*m\oplus b$.

THEOREM. *Let a Pappian (Desarguesian, Moufang) plane π be coordinatized by a field (division ring, alternative ring) $\mathcal{S}(S, +, \cdot)$ and if the lines of the plane are also defined by $y=x*m\oplus b$ and $x=k$ where $\mathcal{S}'$ has properties:*

(1) $a\oplus 0=0\oplus a=a$ *for all* a (*where* 0 *is identity under* $+$),
(2) $0*a=0$ *for all* a,
(3) $1*a=a$ *for all* a (*where* 1 *is identity under* $\cdot$),
(4) $a*(b\oplus c)=a*b\oplus a*c$ *for all* a, b, c,
(5) $1\oplus -1=0$ (*where* -1 *is inverse under* $+$),

then $\mathcal{S}=\mathcal{S}'$. *Moreover, if* $1+1\neq 0$ *and* $\mathcal{S}$ *is not the integers modulo* 3, *then the statement above is false if any of* (1)–(5) *is removed for a field* (*div.ring*).

Proof. Necessity. For (1), consider $\cdot$ and $*$ the same, $a\oplus b=-a-b$, and $\mathcal{S}$ not characteristic 2; then $y=xm+b=x*(-m)\oplus(-b)$. For (2), consider $+$ and $\oplus$ the same, $a*b=(2-a)b$; then $y=xm+b=x*(-m)\oplus(b+2m)$. For (3), consider $+$ and $\oplus$ the same, $a*b=-ab$, then $y=xm+b=x*(-m)\oplus b$. For (4), consider $\cdot$ and $*$ the same, $a\oplus b=ab^2+b$, $b\neq 0$, and $a\oplus 0=0$, and $\mathcal{S}$ not integers modulo 3; then $y=xm+b=x*(m/b^2)\oplus b$ for $b\neq 0$, and $y=xm=x*m\oplus b$ for $b=0$. For (5), consider $\cdot$ and $*$ the same, $a\oplus b=-a+b$, $b\neq 0$, and $a\oplus 0=0$ otherwise; then $y=xm+b=x*(-m)\oplus b$ for $b\neq 0$, $y=xm=x*m\oplus b$ for $b=0$.

Sufficiency. Roughly speaking we want $m=m'$, $b=b'$ where $y=xm+b$ $=x*m'\oplus b'$. From (2) and left identity 0 for $\oplus$ we obtain $b=b'$. If we use also right identity 0 for $\oplus$ we obtain $y=xm=x*m'$. Then using (3) we obtain $m=m'$ (*for* $b=0$). Then $*$ and $\cdot$ are the same. There is a line $y=x*1\oplus b=x\oplus b$, therefore $\oplus$ is right unique solvable (i.e., we mean the lines are defined by the equations provided that each line has an equation, and every equation defines a line). Now $am+b=am'(m,\ b)\ \oplus\ b$. For $a\neq 0$, using (4) $a(1\cdot m+b/a)$ $=a(1\cdot m'(m,\ b/a)\oplus b/a)=am'(m,\ b/a)\oplus b$. Therefore $am'(m,\ b)=am'(m,\ b/a)$. But $a\neq 0$, so $m'(m,b)=m'(m,b/a)$. Thus $m'(m,b)=m'$ for all $b\neq 0$, and $m'(m,0)$ $=m$. Let $u=m'(1,b)$ for $b\neq 0$. Thus $y=x+b=xu\oplus b$. Now $0=1+(-1)$ $=u\oplus(-1)$, but using (5) and right unique solvability $u=1$, but then $\oplus$ and $+$ are the same.

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A PARADOX OF SET THEORY

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The following paradox of classical set theory may have some pedagogical advantages over Russell's Paradox. Its construction is comparable in difficulty, and a good deal more natural than the process in which one asks if sets can be elements of themselves. To my knowledge, this paradox has first appeared in an article which I wrote [1] on the teaching of set theory in secondary schools.

It is assumed that some elementary properties of sets have already been deduced, among which the inequality

$$\#X < \#\mathcal{P}(X) \tag{1}$$

for the cardinality of a set X and that of its set of subsets $\mathcal{P}(X)=2^X$.

Assuming that "everything" is a set, the collection **1** of all singletons:

$$\mathbf{1} = \{\{a\} \mid \text{all } a\}$$

will be a set. Let now V be any subset of **1**; then $\phi(V)$ shall denote $\{V\}$, which is a singleton. Hence we have a map $\phi: \mathcal{P}(\mathbf{1}) \to \mathbf{1}$, with the obvious property $V_1 \neq V_2 \Rightarrow \phi(V_1) \neq \phi(V_2)$. This implies that $\mathcal{P}(\mathbf{1})$ is equivalent to a subset of **1**, so

$$\#\mathcal{P}(\mathbf{1}) \leqq \#\mathbf{1}. \tag{2}$$

Result (2) contradicts (1); apparently, **1** is *not* a set.

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A PROOF OF THE STRUCTURE THEOREM OF FINITE ABELIAN GROUPS

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Let A be a finite abelian group. We denote its composition law multiplicatively. By using elementary arithmetic we know that A can be represented as a direct product of its p-primary components. So we restrict ourselves to the case where A is a finite abelian p-group of order p^k, p denoting a prime. For any $i \in N$, the set of nonnegative integers, we denote by G_i the group of p^ith roots of 1 as contained in the complex numbers C. With G_∞ we denote the union of G_i, $i \in N$, also in C. We will use the following elementary properties of G_∞:

(s) G_i and G_∞ are the only subgroups of G_∞,

(d) G_∞ is a divisible group, that is, for any $x \in G_\infty$ and any $m \in N$ there is $z \in G_\infty$ satisfying $x = z^m$.

Property (d) follows easily from property (s).

In this note we give a very elementary proof of the following well-known result:

THEOREM. *A is a direct product of cyclic groups.*

We start with the following:

LEMMA. *Let A be a finite abelian group and H a subgroup. Then every homomorphism $\mathcal{G}: H \to G_\infty$ can be extended to A.*

Proof. Assume $H \neq A$ and let $x \in A - H$. Let $h_0 = \min\{h / x^h \in H\}$. Clearly $x^h \in H$, $h \in N \Rightarrow h_0 \mid h$. Since $\mathcal{G}(H)$ is a finite subgroup of G_∞ we have $\mathcal{G}(H) = G_m$ for some $m \in N$. Let $y \in H$ satisfy $\mathcal{G}(y) = w_m =$ a primitive p^mth root of 1. Let $w \in G_\infty$ satisfy $w^{h_0} = w_m^s = \mathcal{G}(x^{h_0})$; we need here property (d). Let $(x) \cdot H$ be the subgroup of A generated by x and H. We set

$$x^i \cdot h \to w^i \cdot \mathcal{G}(h) \qquad i \in Z,\ h \in H$$

and we claim that this gives a well defined homomorphism of $(x)\cdot H$ into G_∞. In fact, if $x^i \in H$ we have that $h_0 | i$, say $i = t\cdot h_0$. Therefore

$$w^i = (w^{h_0})^t = (w_m^s)^t = (\mathcal{G}(x^{h_0}))^t = \mathcal{G}(x^i)$$

and consequently if $e = x^i \cdot h =$ the identity element of A, we have

$$w^i \cdot \mathcal{G}(h) = \mathcal{G}(x^i)\cdot \mathcal{G}(x^{-i}) = \mathcal{G}(x^i)\cdot \mathcal{G}(x^i)^{-1} = 1.$$

From the finiteness of A the Lemma follows immediately.

Proof of theorem. We proceed to prove the theorem by induction on the exponent k of the order p^k of A. For $k=1$, A is isomorphic to G_1 and nothing has to be proved. Let $1<k$. Choose an element $a\in A$ with maximal order p^h in A. Then we have an isomorphism $(a)\to G_h$ of the subgroup (a) generated by a in A, into G_h. This isomorphism can be extended to a homomorphism $\mathcal{G}: A\to G_\infty$. We have $\mathcal{G}(A) = G_t$ for some t and clearly $G_h \subset G_t$. Let $b\in A$ satisfy $\mathcal{G}(b) = w_t =$ a primitive p^tth root of 1. Then b has order at least p^t, therefore we must have $t=h$ and so $\mathcal{G}(A) = G_h = \mathcal{G}((a))$. Let $x\in A$, then $\mathcal{G}(x) = w_h^m$ and hence $x\cdot a^{-m} \in \ker(\mathcal{G})$ which proves that $A = (a)\cdot \ker(\mathcal{G})$. Since the restriction of $\mathcal{G}$ to (a) is an isomorphism we can conclude that $A = (a)\times \ker(\mathcal{G})$. On applying the inductive argument to $\ker(\mathcal{G})$, the theorem follows.

REMARK. Incidentally we have proved that every element of maximal order in A generates a direct summand of A.

TEACHING USING EXPERIMENTAL MATHEMATICS

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One of the delights in teaching is the reward that comes from helping students discover mathematical results for themselves. Most often this pleasure has been reserved for those who supervise graduate research. The opportunities for student discovery that do arise in undergraduate studies are rather few; more should be sought. Fortunately, numerical analysis courses are admirably suited for sojourns into such experimental mathematics.

It is the purpose of this paper to study a prototype problem which not only encourages intuitional discovery, but which also displays the interplay that exists between numerical analysis and other branches of mathematics. The author developed this problem while seeking ways to present to sophomores and juniors suitable concrete applications of numerical analysis.

During the study of Lagrange interpolating polynomials, it is usual to analyze the error of interpolation. What is not so usual is to discuss the control of this error. This essential, natural problem can be approached by considering the following question:

PROBLEM 1. *Given $K\in(0, \pi/2]$, let $P(x)$ be the (unique) linear polynomial interpolating* $\sin x$ *at $x=0$ and at $x=t$ for some (as yet unspecified) $t\in(0, K]$; that*

is, $P(0)=0$ *and* $P(t)=\sin t$. *Find a value of t which minimizes the quantity*

$$\max_{0\leqq x\leqq K} \left| \sin x - P(x) \right|.$$

SOLUTION. Clearly $P(x)=Ax$ where $At=\sin t$ $(0\leqq A\leqq 1)$. Let $e(x)=\sin x-Ax$. It is not too difficult for the student to apply calculus to show that if $\cos\alpha=A$ and $0\leqq\alpha\leqq K$, then (see Figure 1)

$$\max_{0\leqq x\leqq K} \left| e(x) \right| = \max(\left| e(K) \right|, \left| e(\alpha) \right|).$$

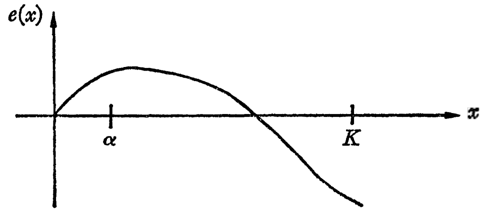

FIG. 1

Let

$$\phi_1(\alpha) = e(\alpha) = \sin\alpha - \alpha\cos\alpha$$

$$\phi_2(\alpha) = -e(K) = K\cos\alpha - \sin K.$$

A simple graphical monotonicity argument (which can serve as an exercise for the student) shows that to solve this problem, one wishes to choose the unique $\alpha\in[0, K]$ such that $\phi_1(\alpha)=\phi_2(\alpha)$. (See Figure 2.)

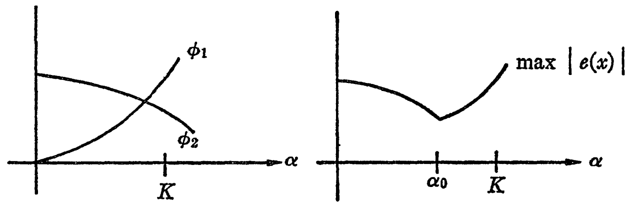

FIG. 2

Let α_0 be this optimal α. Then α_0 satisfies $g(\alpha_0)=0$ where

$$g(\alpha) = \phi_1(\alpha) - \phi_2(\alpha) = \sin\alpha - (\alpha + K)\cos\alpha + \sin K.$$

(Clearly $K=0$ implies $\alpha_0=0$. This case is omitted.)

Computing an approximate zero for the transcendental function g is possible by the application of some standard iteration algorithm and convergence theorem. The problem is one that is readily programmed on a digital computer. In

numerical analysis courses where iteration concepts have yet to be introduced, the problem of seeking a zero for g may well serve as suitable motivation for studying iteration.

After approximating α_0, one must still compute the desired value of t from $t \cos \alpha_0 = \sin t$, another transcendental equation whose nonzero approximate solution may also be programmed. Let t_0 denote this unique (nonzero when $K \neq 0$) solution in $(0, \pi/2)$. There now remains for the student to prove that $t_0 \in [0, K]$. Table 1 lists K, α_0, t_0, $\max|e(x)|$, and $P(x)$.

TABLE 1

K(radians)	α_0	t_0	$10^3 \times \max\|e(x)\|$	$P(x) = Ax$
.1	.050	.087	.04	$A = .999$
.2	.100	.173	.33	.995
.3	.150	.260	1.1	.989
.4	.200	.346	2.6	.980
.5	.249	.433	5.1	.969
.6	.299	.519	8.8	.956
.7	.348	.605	14	.940
.8	.397	.691	20	.922
.9	.445	.777	29	.902
1.0	.494	.862	39	.881
$\frac{\pi}{2} = 1.57 \cdots$	.760	1.345	138	.725

The lecturer who enjoys opportunities for experimental mathematics can use Table 1 to point out to the student that $\max|e(x)|$ seems to be multiplied approximately by 2^3 each time K is doubled. This suggests the conjecture that $\max|e(x)|$ is of order K^3 as $K \to 0$. For $\alpha = \alpha_0$, $\max|e(x)| = |e(K)| = |\sin K - K \cos \alpha_0|$. Since $\alpha_0 \to 0$ as $K \to 0$, then $\max|e(K)| \to |\sin K - K|$. Thus, the above conjecture is equivalent to the conjecture that $\sin K = K + O(K^3)$, a conjecture which the student can readily prove. By repeated application of L'Hospital's rule one shows that

$$\lim_{K \to 0} \frac{\sin K - K}{K^3} = -\frac{1}{6}.$$

For those who wish to go deeper, the opportunity is now present for introducing some facts about power series.

Many students will use Table 1 to conjecture that $\lim_{K \to 0} 2\alpha_0/K = 1$. This can be proved using the (relatively sophisticated) fact that the zeros of a cubic are continuous functions of the coefficients.

Problem 1 can be modified in an interesting way by considering the relative error $u(x)$, defined as follows:

$$u(x) = \frac{\sin x - P(x)}{\sin x}, \qquad x > 0$$

$$u(0) = 1 - A \text{ (for continuity).}$$

Recall $0 \leqq A \leqq 1$.

PROBLEM 2. *Given $K \in (0, \pi/2]$ and $P(x)$ as in Problem 1, select $t \in [0, K]$ which minimizes the quantity* $\max_{0 \leq x \leq K} |u(x)|$.

SOLUTION. The student usually finds it challenging (though not difficult) to apply calculus to show that

$$\begin{aligned}\max_{0 \leqq x \leqq K} |u(x)| &= \max(|u(0)|, |u(K)|) \\ &= \max\left(1 - A, \left|A \frac{K}{\sin K} - 1\right|\right).\end{aligned}$$

Again, a simple argument shows (see Figure 3) that the max is minimized by picking $A > 0$ such that $1 - A = -1 + AK/\sin K$; that is, $A = 2 \sin K/(K + \sin K)$.

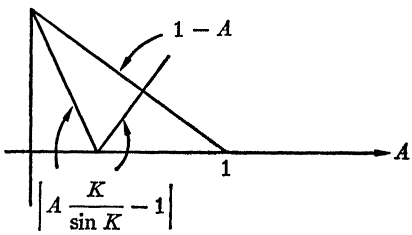

FIG. 3

Since A satisfies $At = \sin t$, then t may be approximated by solving numerically the transcendental equation

$$\sin t = \frac{2 \sin K}{K + \sin K} t.$$

We shall now apply Problem 2 to the following fundamental numerical analysis question: It is desired to use $P(x) = Ax$ to generate a table of approximations to $\sin x$ for $x \in [0, K]$ subject to the requirement that the relative error not exceed ϵ; that is, there must be at least $1/\epsilon$ significant digits of accuracy. How large may the interval $[0, K]$ be made?

Clearly $\epsilon = 1 - A$. Then K is the (unique) positive solution to

$$\sin K = \frac{1 - \epsilon}{1 + \epsilon} K; \quad \text{that is,} \quad K = \frac{1 + \epsilon}{1 - \epsilon} \sin K.$$

Again, we solve numerically. A precaution is needed, though. The factor $(1 + \epsilon)/(1 - \epsilon)$ is subject to rounding errors for small ϵ. It is well to note that

$$\frac{1+\epsilon}{1-\epsilon} = 1 + 2\epsilon + 2\epsilon^2 + \cdots + 2\epsilon^n + \frac{2\epsilon^{n+1}}{1-\epsilon} \cdot$$

Truncate after the nth term, picking n large enough so that the remainder $2\epsilon^{n+1}/(1-\epsilon)$ is suitably small.

Let

$$\epsilon = \max_{x\in[0,K]} \left| \frac{\sin x - P(x)}{\sin x} \right| .$$

Then $A=1-\epsilon$ and $P(x)=Ax$. Table 2 lists ϵ, K, and $P(x)$.

TABLE 2

$\epsilon=10^{-\nu}$	Approx. K	$P(x)=Ax$
$\nu=1$	1.0751	$A=.9$
2	.3457	.99
3	.1095	.999
4	.0346	.9999
5	.0110	.99999

Table 2 can be used to present another opportunity for experimental mathematics. Observe: as ϵ is divided by 10^2, then K seems to be divided approximately by 10. One can formalize this observation by conjecturing that $\lim_{\epsilon\to 0} K^2/\epsilon$ exists, where $(1+\epsilon)\sin K=(1-\epsilon)K$. The information gained from the proof of the conjecture about $\sin x$ in Problem 1 can now be used to prove this present conjecture. Indeed, for some $r\in(0, K)$

$$K = \frac{1+\epsilon}{1-\epsilon}\sin K = \frac{1+\epsilon}{1-\epsilon}\left(K - \frac{K^3}{6}\cos r\right).$$

Since $K\neq 0$, divide by K and rearrange terms to obtain

$$\frac{K^2}{\epsilon} = \frac{12}{(1+\epsilon)\cos r} \cdot$$

Since $\epsilon\to 0$ implies $K\to 0$ implies $r\to 0$, then $\lim_{\epsilon\to 0} K^2/\epsilon=12$; that is $K\sim\sqrt{12\epsilon} \cong 3.46\sqrt{\epsilon}$. (Compare with Table 2.) This is now an interesting, and possibly practical, approximation theory result. (Educational value from experimental mathematics is not limited to the student. The author made the above conjecture and proof exactly in the manner outlined. To him this result is new. He does not know if, perhaps, it is well-known at most computation laboratories.) In the interval $[0, \sqrt{12\epsilon}]$ one can approximate $\sin x$ by $(1-\epsilon)x$ with about $1/\epsilon$ significant digits of accuracy.

The approach and problems presented here use the student's intuition and theorem proving ability to extend his knowledge of elementary analysis. The

glue in this package of discovery is the judicious use of numerical analysis and the computer. The author has had excellent results using aspects of these problems as an educational tool for sophomores and juniors at Brown University. He welcomes correspondence from those who have had similar success with other approaches.

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ON RADIX REPRESENTATION AND THE EUCLIDEAN ALGORITHM

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1. Introduction. If N denotes the natural numbers (including zero), it is well known [1, p. 13, Th. 8] that for any $k(>1)$ and j in N, there exist unique natural numbers $s, a_s, a_{s-1}, \cdots, a_0$ such that

$$j = a_s k^s + a_{s-1}k^{s-1} + \cdots + a_0, \qquad 0 \leqq a_i \leqq k-1, \qquad (a_s = 0 \text{ only if } j = 0; \text{ take } s = 0 \text{ if } j = 0). \tag{1}$$

The above is called the radix representation of j (with respect to k).

It is standard procedure in books on elementary number theory to prove (1) from the division lemma for the Euclidean algorithm [1, p. 12, Th. 7]. Namely, if $k(>0)$ and j are in N, then there exist unique natural numbers r and q such that

$$j = kq + r, \qquad 0 \leqq r \leqq k-1. \tag{2}$$

The proof of (1) is done by a tedious but straightforward application of mathematical induction utilizing (2).

We first remark that (2) is an obvious corollary of (1): if $k=1$ take $q=j$, $r=0$; if $k>1$ take $q=a_sk^{s-1}+\cdots+a_1$, $r=a_0$; uniqueness is trivial. In Section 2 we show that (1) may be derived in a relatively easy manner from the pigeon-hole principle [1, p. 42]. In Section 3, we derive (1) from a simple combinatorial argument.

2. Pigeon-hole Proof. Let $R_k(n)$ denote the set of all natural numbers of the form $a_nk^n+a_{n-1}k^{n-1}+\cdots+a_0$, $0\leqq a_i\leqq k-1$. Let $I_k(n)$ denote all the natural numbers m, with $0\leqq m<k^{n+1}$.

Note that we cannot have two different radix representations of the same number. For suppose one representation for the number has as the maximum

exponent of k, s_1, and the other representation has as maximum exponent of k, s_2. Then let $n=\max(s_1, s_2)$, and assume

$$a_nk^n + a_{n-1}k^{n-1} + \cdots + a_0 = b_nk^n + b_{n-1}k^{n-1} + \cdots + b_0, \qquad 0 \leqq a_i \leqq k-1,\ 0 \leqq b_i \leqq k-1. \tag{3}$$

(Some of the a_i and some of the b_i may well be zero.) If there exists i such that $a_i \neq b_i$, let m be the largest such i and suppose $a_m > b_m$. Then

$$a_mk^m + a_{m-1}k^{m-1} + \cdots + a_0 = b_mk^m + b_{m-1}k^{m-1} + \cdots + b_0,$$

and therefore

$$\begin{aligned} k^m \leqq (a_m - b_m)k^m &= (b_{m-1} - a_{m-1})k^{m-1} + \cdots + (b_0 - a_0) \\ &\leqq (k-1)k^{m-1} + \cdots + (k-1) = k^m - 1, \end{aligned}$$

which is impossible.

Next note that $R_k(n) \subseteq I_k(n)$ because if $s \in R_k(n)$,

$$\begin{aligned} 0 \leqq s = a_nk^n + \cdots + a_0 &\leqq (k-1)k^n + (k-1)k^{n-1} + \cdots + (k-1) \\ &= k^{n+1} - 1. \end{aligned}$$

Finally $R_k(n)$ has k^{n+1} elements since each different representation gives a different natural number and since there are k independent ways of assigning each of the $n+1$ coefficients.

Thus $R_k(n)$ is a subset of $I_k(n)$ with the same number of elements. Therefore $R_k(n)=I_k(n)$ by the pigeon-hole principle.

Consequently for any k (>1) and j in N, clearly $j<2^j \leqq k^j$, and hence $j \in I_k(j-1)=R_k(j-1)$. Thus j has a radix representation with respect to k, and by the remark preceding (3) it is unique.

3. Combinatorial Proof. Let $r_k(n)$ denote the number of partitions of n into powers of k where no part is repeated more than $k-1$ times. The existence and uniqueness of the radix representation is clearly equivalent to the following equation:

$$r_k(n) = 1, \qquad \text{for all } k > 1, n > 0. \tag{4}$$

Proof of (4). We fix k throughout our discussion. If

$$n = a_1k^{r_1} + a_2k^{r_2} + \cdots + a_{s-1}k^{r_{s-1}} + a_sk^{r_s}, \qquad r_i > r_{i+1} \geqq 0,\ 0 < a_i < k,$$

then

$$n - 1 = a_1k^{r_1} + a_2k^{r_2} + \cdots + a_{s-1}k^{r_{s-1}} + (a_s - 1)k^{r_s} + \sum_{j=0}^{r_s-1} (k-1)k^j$$

by the summation formula for a finite geometric series with the convention that empty sums $=0$. Thus we see that to each different representation of n there corresponds a different representation of $n-1$. Consequently $r_k(n) \leqq r_k(n-1)$.

This implies that if $m \geqq n$, then $r_k(n) \geqq r_k(m)$. Clearly then

$$1 \leqq r_k(k^n) \leqq r_k(n) \leqq r_k(1) = 1.$$

Hence equality holds everywhere, and $r_k(n) = 1$.

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THE INADEQUACY OF SEQUENCES

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The purpose of this note is to give an example of a set with two distinct topologies, both of which yield the same collection of convergent sequences. Through this example the inadequacy of sequential convergence for describing a topology becomes apparent.

This result is not new and is known to many functional analysts as a theorem of Schur [1, p. 137]. Nevertheless, the proof will be new to most. But even here we cannot take credit for originality, as our proof is an adaptation of a proof by Le Cam of a theorem on the sequential convergence of measures [2, p. 218]. Our only contribution is to simplify his proof in our particular case and thereby bring this interesting (and overlooked) result to an audience unfamiliar with either the theorem or the method of proof.

The only prerequisites for this note are a knowledge of the basics of point set topology (including the Baire Category Theorem) and linear algebra.

Let l^1 be the space of all summable real sequences; i.e., α is in l^1 iff $\alpha = \{a_j\}_{j=1}^{\infty}$ with

$$\|\alpha\| = \sum_{j=1}^{\infty} |a_j| \tag{1}$$

finite. Then (1) defines a norm on l^1 with respect to which l^1 is a complete metric space. Now let l^∞ be the space of all real sequences $\xi = \{x_j\}_{j=1}^{\infty}$ with $\sup\{|x_j| : j \geqq 1\} < \infty$. For ξ in l^∞ and α in l^1 define

$$\langle \xi, \alpha \rangle = \sum_{j=1}^{\infty} x_j a_j, \tag{2}$$

which converges. We define ω to be the weak topology on l^1. Hence, a subbasic ω neighborhood for α in l^1 is of the form $\{\beta \in l^1 : |\langle \xi, \alpha - \beta \rangle| < \epsilon\}$ where ξ is in l^∞ and $\epsilon > 0$.

To see that the ω topology differs from the metric topology on l^1 (a fact true of all infinite dimensional normed linear spaces), observe that $S = \{\alpha \in l^1 : \|\alpha\| = 1\}$, while closed in the metric topology, is an ω dense subset of $\{\alpha \in l^1 : \|\alpha\| \leqq 1\}$. In

fact, that zero is an ω limit point of S can be seen as follows: If $0 \in U \in \omega$ then there are $\xi_1, \cdots, \xi_n$ in l^∞ and $\epsilon > 0$ such that

$$\bigcap_{k=1}^{n} \{\beta \in l^1 : |\langle \xi_k, \beta \rangle| < \epsilon\} \subset U.$$

But then $\{\beta: \langle \xi_k, \beta\rangle = 0, 1 \leqq k \leqq n\}$ is an infinite dimensional vector subspace of l^1 which is contained in U. In particular, U contains a β with $\|\beta\| = 1$.

It is clear that the metric topology is finer than ω; so to complete this exposition we need only prove the following

THEOREM (Schur). *If a sequence* $\{\alpha^{(n)}\}$ *in* l^1 *converges to zero in* ω *then it converges in norm.*

Proof. Let X be the collection of all ξ in l^∞ with $\sup\{|x_j| : j \geqq 1\} \leqq 1$. Define a metric d on X by letting

$$d(\xi, \xi') = \sum_{j=1}^{\infty} 2^{-j} | x_j - x_j' |$$

for all ξ and ξ' in X. It is routine to show that (X, d) is a complete metric space, and that all the sets

$$S(\xi; J, \delta) \equiv \{\xi' \in X : | x_j - x_j' | < \delta \text{ for } 1 \leqq j \leqq J\},$$

where $\delta > 0$ and $J \geqq 1$, form a basis for the neighborhood system of ξ in (X, d).

Let $\epsilon > 0$ and define

$$F_m = \{\xi \in X : | \langle \xi, \alpha^{(n)} \rangle | \leqq \epsilon/3 \text{ for all } n \geqq m\}.$$

It follows that for α in l^1 the function $\xi \to \langle \xi, \alpha \rangle$ is continuous on (X, d). (If $\{\xi^{(n)}\}$ converges to ξ in (X, d) and $\delta > 0$, let J be such that $\sum_{j=J+1}^{\infty} |a_j| < \frac{1}{4}\delta$. Then for n sufficiently large that $|x_j^{(n)} - x_j| < \delta(2\|\alpha\| J)^{-1}$ for $1 \leqq j \leqq n$, we have that $|\langle \xi^{(n)}, \alpha\rangle - \langle \xi, \alpha \rangle| < \delta$.) Hence each F_m is closed in (X, d) and, since $\lim \langle \xi, \alpha^{(n)} \rangle = 0$ for every ξ in X, we have that $X = \bigcup_{m=1}^{\infty} F_m$. By the Baire Category Theorem there is a F_m with nonvoid interior. That is, there is a ξ, a $\delta > 0$, and a $J \geqq 1$ such that $S(\xi; J, \delta) \subset F_m$. If $n \geqq m$ is fixed then let $\xi' \in X$ be defined by $x_j' = x_j$ for $1 \leqq j \leqq J$ and $x_j' = \operatorname{sign}(a_j^{(n)})$ for $j \geqq J+1$. Then ξ' is in $S(\xi; J, \delta)$ and so

$$\sum_{j=J+1}^{\infty} | a_j^{(n)} | \leqq \epsilon/3 + \sum_{j=1}^{J} | a_j^{(n)} |.$$

The result now follows by choosing $m_1 \geqq m$ so that $|\alpha_j^{(n)}| < \epsilon(3J)^{-1}$ for $1 \leqq j \leqq J$ whenever $n \geqq m_1$.

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ON MEAN VALUE THEOREMS

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In this note we obtain generalizations and extensions of the following mean value theorem, due to Flett [3]: *If $f(x)$ is differentiable in $[a, b]$ with $f'(a) = f'(b)$, then there exists z in (a, b) such that* $f'(z) = [f(z) - f(a)]/(z-a)$. These generalizations and extensions were motivated by [4] and by [5]. They deal with Dini derivatives and with the symmetric derivative and appear in Sections 1 and 3. In Sections 2 and 4 we present another set of mean value theorems which has some resemblance to the set arising from Flett's theorem.

1. LEMMA 1. *Let f satisfy the following conditions*: (1) *f is continuous in $[a, b]$*, (2) *f is differentiable at b*, (3) *the four Dini derivatives of f are finite in (a, b)*, (4) $f'(b)[f(b)-f(a)] \leqq 0$. *Then either* (a) *there exists c in $(a, b]$ such that $f^+(c) \leqq 0$ and $f_-(c) \geqq 0$, or* (b) *there exists d in $(a, b]$ such that $f_+(d) \geqq 0$ and $f^-(d) \leqq 0$.*

Proof. If $f'(b) = 0$, let $c = d = b$. If $f(b) = f(a)$, then either $f(x) = f(a)$ for every x in $[a, b]$ and then f' exists and c, d can be any x in (a, b), or f attains (either) its maximum (or its minimum) at c (at d) in (a, b). By [2], (a) $((b))$ holds. If $f'(b)[f(b)-f(a)] < 0$, then again f has a maximum (minimum) at $c(d)$ in (a, b) and accordingly $(a)((b))$ holds.

LEMMA 2. *If f satisfies the conditions* (1)–(3) *of Lemma* 1 *with* (4) *replaced by* (4′) $f'(b)[f(b)-f(a)] < 0$, *then the c and the d of* (a), (b) *of Lemma* 1 *are in* (a, b).

Proof. The proof of this lemma is contained in the proof of Lemma 1.

THEOREM 1. *Let f be a function with the following properties*: (1) *f is continuous in $[a, b]$*, (2) *f is differentiable at a and at b*, (3) *the four Dini derivatives of f are finite in (a, b)*, (4) $[f'(b) - [f(b)-f(a)]/(b-a)][f'(a) - [f(b)-f(a)]/(b-a)] \geqq 0$. *Then either* (a) *there exists c in $(a, b]$ such that* $f^+(c) \leqq [f(c)-f(a)]/(c-a) \leqq f_-(c)$, *or* (b) *there exists d in $(a, b]$ such that* $f^-(d) \leqq [f(d)-f(a)]/(d-a) \leqq f_+(d)$.

Proof. On $(a, b]$ let $h(x) = [f(x)-f(a)]/(x-a)$ and let $h(a) = f'(a)$. Then h satisfies conditions (1)–(4) of Lemma 1. Therefore, either (a) there exists c in $(a, b]$ such that

$$h^+(c) = [1/(c-a)][f^+(c) - [f(c) - f(a)]/(c-a)] \leqq 0$$

and

$$h_-(c) = [1/(c-a)][f_-(c) - [f(c) - f(a)]/(c-a)] \geqq 0,$$

or (b) there exists d in $(a, b]$ such that

$$h_+(d) = [1/(d-a)][f_+(d) - [f(d) - f(a)]/(d-a)] \geqq 0$$

and

$$h^-(d) = [1/(d-a)][f^-(d) - [f(d) - f(a)]/(d-a)] \leqq 0.$$

Since $c-a$ and $d-a$ are positive, the result follows.

COROLLARY 1. *If f satisfies conditions* (1)–(3) *of Theorem* 1 *with* (4) *changed into* (4′) $f'(a)=f'(b)$, *the conclusion of Theorem* 1 *holds with c and d in* (a, b).

Proof. If $h(a)=h(b)$ where h is the h of the proof of Theorem 1, that is, if $f'(b)=[f(b)-f(a)]/(b-a)$, h attains either its maximum (or its minimum) in (a, b), or is constant throughout $[a, b]$, and the result follows as in the proof of Lemma 1. If $h(a)\neq h(b)$, then h satisfies the conditions of Lemma 2 and the result follows.

THEOREM 2. *Let f and g satisfy the following conditions*: (1) *f is continuous in* $[a, b]$ *and differentiable at a and at b*, (2) *the four Dini derivatives of f are finite in* (a, b), (3) *g is differentiable in* $[a, b]$, (4) $g'(a)\neq 0$ *and* $g(x)\neq g(a)$ *for all x in* $(a, b]$,

(5) $[f'(a)/g'(a)-[f(b)-f(a)]/[g(b)-g(a)]][[g(b)-g(a)]f'(b)-[f(b)-f(a)]g'(b)]\geqq 0.$

Then either (a) *there exists c in* $(a, b]$ *such that* $[g(c)-g(a)]f^+(c)\leqq[f(c)-f(a)]g'(c)$ *and* $[g(c)-g(a)]f_-(c)\geqq[f(c)-f(a)]g'(c)$, *or* (b) *there exists d in* $(a, b]$ *such that* $[g(d)-g(a)]f_+(d)\geqq[f(d)-f(a)]g'(d)$ *and* $[g(d)-g(a)]f^-(d)\leqq[f(d)-f(a)]g'(d)$.

Proof. Let $h(x)=[f(x)-f(a)]/[g(x)-g(a)]$ for $x\neq a$ and let $h(a)=f'(a)/g'(a)$. Then $[g(x)-g(a)]^2h^+(x)=[g(x)-g(a)]f^+(x)-[f(x)-f(a)]g'(x)$ and similar formulas hold for the other three Dini derivatives. Using Lemma 1 for h and noting that $[g(x)-g(a)]^2>0$ for all x in $(a, b]$, we arrive at the desired conclusion.

These were the extensions of Theorems 1, 2 and of Corollary 1 of [5] (our Corollary 1 is also the theorem of [4]). For the sake of brevity and in order to avoid unnecessary repetitions, we leave the extensions of Theorems 3, 4, 5, of their analogues arising from Lemma 2, and of Corollaries 2, 3—all of them appear in [5]—to the reader. The extensions, as well as their proofs, are on the same lines as those discussed above.

2. In this section we introduce the second set of mean value theorems. It will be based on the following lemma:

LEMMA 3. *Let f be a function with the following properties*: (1) *f is continuous in* $[a, b]$, (2) *the four Dini derivatives of f are finite in* (a, b), (3) $f(a)=f(b)$. *Then either* (a) *there exists c in* (a, b) *such that* $f^+(c)\leqq 0$ *and* $f_-(c)\geqq 0$, *or* (b) *there exists d in* (a, b) *such that* $f_+(d)\geqq 0$ *and* $f^-(d)\leqq 0$.

Proof. See [2].

THEOREM 3. *Let f satisfy conditions* (1), (2) *of Lemma* 3. *Then either* (a) *there exists c in* (a, b) *such that* $f^+(c)\leqq[f(c)-f(a)]/(b-c)\leqq f_-(c)$ *or* (b) *there exists d in* (a, b) *such that* $f^-(d)\leqq[f(d)-f(a)]/(b-d)\leqq f_+(d)$.

Proof. Let $h(x)=[f(x)-f(a)](b-x)$. Then $h^+(x)=(b-x)f^+(x)-[f(x)-f(a)]$ and similar formulas hold for the other three Dini derivatives of h. Since $h(a)=h(b)$, we have, by Lemma 3, either $h^+(c)\leqq 0$, $h_-(c)\geqq 0$ or $h_+(d)\geqq 0$, $h^-(d)\leqq 0$. The result follows because $b-x>0$ for all x in (a, b).

Similarly one can obtain Cauchy-like and Taylor-like theorems, analogous to those of Section 1 and to those of [5]. By assuming that the ordinary derivative exists one can obtain corollaries too (cf. Section 4). Again we leave all this to the reader.

3. We turn now to the symmetric derivative [1] defined by the following limit: $f^s(x) = \lim_{h\to 0} [f(x+h) - f(x-h)]/(2h)$. The phrase "$f^s$ exists" will mean that this limit exists and is finite.

LEMMA 4. *Let f satisfy conditions* (1), (2), (4) *of Lemma* 1 *with* (3) *changed into* (3′) *f^s exists in (a, b). Then there exist c and d in $(a, b]$ such that $f^s(c) \geqq 0$ and $f^s(d) \leqq 0$.*

Proof. If $f'(b) = 0$, let $c = d = b$. If $f(a) = f(b)$, either f is constant throughout $[a, b]$ and then f' exists and c, d can be any x in (a, b), or f attains its maximum M or its minimum m or both in (a, b). Either $M \neq f(a)$ or $m \neq f(a)$. Suppose $M \neq f(a)$ (the proof is similar if only $m \neq f(a)$), and let f attain M at y. Then $f(a) < f(y)$ and $f(y) > f(b)$. By [1] there exist $c, d, a < c < y < d < b$, such that $f^s(c) \geqq 0$ and $f^s(d) \leqq 0$. If $f'(b)[f(b) - f(a)] < 0$, suppose that $f(b) > f(a)$ (the proof in the case when $f(b) < f(a)$ is similar). Then $f'(b) < 0$ and there is a point y, $a < y < b$ such that $f(y) > f(b) > f(a)$. Now we repeat the argument which was used above.

THEOREM 4. *Let f satisfy conditions* (1), (5) *of Theorem* 2 *with* (2) *changed into* (2′) *f^s exists in (a, b), and let g satisfy conditions* (4), (5) *of Theorem* 2 *with* (3) *changed into* (3′) *g is continuous in $[a, b]$, g is differentiable at a and at b, g^s exists in (a, b). Then there exist c, d in $(a, b]$ such that $[g(c) - g(a)]f^s(c) \geqq [f(c) - f(a)]g^s(c)$ and $[g(d) - g(a)]f^s(d) \leqq [f(d) - f(a)]g^s(d)$.*

Proof. Define h as in the proof of Theorem 2. By applying Lemma 4 to h and by using the formula $[g(x) - g(a)]^2 h^s(x) = [g(x) - g(a)]f^s(x) - [f(x) - f(a)]g^s(x)$, we arrive at the desired result.

COROLLARY 2. *If f is differentiable in $[a, b]$ with $f'(a) = f'(b)$, then there exists z in (a, b) such that $f'(z) = [f(z) - f(a)]/(z - a)$.*

As mentioned above, this is Flett's original theorem and we give here a new proof of it.

Proof. In Theorem 4 let $g(x) = x$, let f' exist in $[a, b]$ and let h be the h defined in the proof of this theorem. If $f'(b) = [f(b) - f(a)]/(b - a)$, we have $h(a) = h(b)$ and by Rolle's Theorem $h'(z) = 0$ for z in (a, b). The result follows. If $f'(b) \neq [f(b) - f(a)]/(b - a)$, we have $f'(c) \geqq [f(c) - f(a)]/(c - a)$ and $f'(d) \leqq [f(d) - f(a)]/(d - a)$ with c, d in (a, b) by the proof of Lemma 4. If equality holds, there is nothing else to prove. If inequality holds for both c and d, let $j(x) = f'(x) - [f(x) - f(a)]/(x - a)$ for x in $[c, d]$ (or in $[d, c]$). Clearly there is a function $J(x)$ such that $J'(x) = j(x)$ in $[c, d]$. That is, $j(x)$ is a Darboux function, and since $j(c) > 0$, $j(d) < 0$, we must have $j(z) = 0$ for some z, $a < c < z < d < b$.

THEOREM 5. (*This is an example of an extension of a Taylor-like theorem.*) *Let f satisfy the following conditions*:

(1) *f is continuous in $[a, b]$,*

(2) *$f'(b)$ and $f^{(n)}(a)$ exist,*

(3) *f^s exists in (a, b),*

(4) $[f(b)-\sum_{k=0}^{n}[f^{(k)}(a)[b-a]^k]/k!][f'(b)-\sum_{k=1}^{n}[f^{(k)}(a)[b-a]^{k-1}]/(k-1)!]$
$\leqq 0$.

Then there exist c, d in $(a, b]$ such that

$$f^s(c) \geqq \sum_{k=1}^{n} [f^{(k)}(a)[c-a]^{k-1}]/(k-1), \text{ and } f^s(d) \leqq \sum_{k=1}^{n} [f^{(k)}(a)[d-a]^{k-1}]/(k-1)!$$

Proof. Apply Lemma 4 to $h(x)=f(x)-\sum_{k=0}^{n}[f^{(k)}(a)[x-a]^k]/k!$.

Once more, the reader can in a similar manner formulate and prove extensions of other theorems appearing in [5].

4. This section is an analogue of Section 2. Consequently, we will state and prove only one theorem. As usual, we leave the rest to the interested reader.

LEMMA 5. *Let f satisfy the following conditions*: (1) *f is continuous in $[a, b]$,* (2) *f^s exists in (a, b),* (3) *$f(a)=f(b)$. Then there exist c, d in (a, b) such that $f^s(c)\geqq 0$ and $f^s(d)\leqq 0$.*

Proof. See [1].

THEOREM 6. *Let f be continuous in $[a, b]$ and let f^s exist in (a, b). Then there exist c, d in (a, b) such that $f^s(c)\geqq[f(c)-f(a)]/(b-c)$ and $f^s(d)\leqq[f(d)-f(a)]/(b-d)$.*

Proof. Apply Lemma 5 to $h(x)=[f(x)-f(a)](b-x)$.

COROLLARY 3. *Let f be continuous in $[a, b]$ and let f' exist in (a, b). Then there exists z in (a, b) such that $f'(z)=[f(z)-f(a)]/(b-z)$.*

Proof. By Theorem 6 we have $f'(c)\geqq[f(c)-f(a)]/(b-c)$ and $f'(d)\leqq[f(d)-f(a)]/(b-d)$ with c, d in (a, b). If equality holds, we are through. If inequality holds for both c and d, let $i(x)=f'(x)-[f(x)-f(a)]/(b-x)$ for x in $[c, d]$ (or in $[d, c]$). Now proceed as in the proof of Corollary 2 to obtain $i(z)=0$ for some z, $a<c<z<d<b$.

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JOINT CONTINUITY OF MONOTONIC FUNCTIONS

R. L. KRUSE and J. J. DEELY, Sandia Laboratory, Albuquerque

The following might be an instructive exercise for an advanced calculus class:

PROPOSITION 1. *Let $f(x, y)$ be a real valued function defined on an open set G in the plane. Suppose that $f(x, y)$ is continuous in x and y separately and is monotone in x for each y. Then $f(x, y)$ is (jointly) continuous on the set G.*

Proof. Choose $(x_0, y_0) \in G$. Given any $\epsilon > 0$ there exist, by separate continuity, positive numbers δ, η_1, and η_2 such that

$$|f(x, y_0) - f(x_0, y_0)| < \epsilon/2 \quad \text{for } |x - x_0| \leqq \delta, \tag{1}$$

$$|f(x_0 + \delta, y) - f(x_0 + \delta, y_0)| < \epsilon/2 \quad \text{for } |y - y_0| \leqq \eta_1, \tag{2}$$

$$|f(x_0 - \delta, y) - f(x_0 - \delta, y_0)| < \epsilon/2 \quad \text{for } |y - y_0| \leqq \eta_2; \tag{3}$$

and the obvious products of intervals are contained in G. Let $\eta = \min(\eta_1, \eta_2)$, and let (x, y) be any point in the rectangle

$$[x_0 - \delta, x_0 + \delta] \times [y_0 - \eta, y_0 + \eta].$$

Note that the direction of monotonicity of f in x may depend on the value of y. We assume f is nondecreasing in x for the given y. If f is nonincreasing the reverse inequalities will hold. Thus

$$\begin{aligned}
&[f(x_0 - \delta, y) - f(x_0 - \delta, y_0)] + [f(x_0 - \delta, y_0) - f(x_0, y_0)] \\
&\qquad \leqq f(x, y) - f(x_0, y_0) \\
&\qquad \leqq [f(x_0 + \delta, y) - f(x_0 + \delta, y_0)] + [f(x_0 + \delta, y_0) - f(x_0, y_0)]
\end{aligned}$$

and so by (1), (2), and (3)

$$-\epsilon/2 - \epsilon/2 < f(x, y) - f(x_0, y_0) < \epsilon/2 + \epsilon/2.$$

Thus f is continuous at (x_0, y_0).

Proposition 1 may be extended to a function of n variables as follows:

PROPOSITION 2. *Let $f(x_1, \cdots, x_{n-1}, y)$ be a real valued function defined on an open set G in E^n, $n \geqq 2$. Suppose f is continuous in each variable separately and is monotone in each x_i separately, $1 \leqq i \leqq n-1$. Then f is continuous on G.*

REMARKS. (i) By separate monotonicity of f we mean that, for each permissible fixed value of the $n-1$ remaining variables, f is monotonic in x_i. The direction of monotonicity may depend on the values of the remaining variables.

(ii) Application of a homeomorphism allows the directions of monotonicity and continuity of f to be chosen along curves other than the coordinate axes.

Proof. For convenience we introduce the vector notation

$$\mathbf{x} = (x_1, \cdots, x_{n-1})$$

for points of E^{n-1}, $n \geqq 2$. For $\delta > 0$ and $\mathbf{x}' \in E^{n-1}$ we define an $(n-1)$-dimensional closed cube by

$$C(\mathbf{x}', \delta) = \{\mathbf{x} \,|\, |x_i - x_i'| \leqq \delta, 1 \leqq i \leqq n-1\}.$$

The *corners* of $C(\mathbf{x}', \delta)$ are the 2^{n-1} vectors $\mathbf{x}$ at which $|x_i - x_i'| = \delta$ for all i, $1 \leqq i \leqq n-1$. We shall need the following

LEMMA. *Let $f(\mathbf{x}, y)$ be a real valued function defined on a closed cube $H = C(\mathbf{x}', \delta) \times [y'-\eta, y'+\eta]$ in E^n, with $\eta \geqq 0$ and y' a real number. Suppose f is continuous in y for each $\mathbf{x}$ in $C(\mathbf{x}', \delta)$, and is monotonic in each x_i separately. Then $f(\mathbf{x}, y)$ takes on its absolute maximum on H at a point $(\mathbf{x}^*, y^*)$, where $\mathbf{x}^*$ is a corner of $C(\mathbf{x}', \delta)$.*

Proof of Lemma. Let $\{(\mathbf{x}_i, y_i) | i = 1, 2, \cdots\}$ be a sequence of points in H for which the sequence $\{f(\mathbf{x}_i, y_i)\}$ converges to the least upper bound of f on H if the l.u.b. exists, or diverges to infinity otherwise. We may assume that each $\mathbf{x}_i$ is a corner of $C(\mathbf{x}', \delta)$, since for any $(\mathbf{x}, y) \in H$, separate monotonicity implies there exists a corner $\mathbf{x}_c$ of $C(\mathbf{x}', \delta)$ for which $f(\mathbf{x}_c, y) \geqq f(\mathbf{x}, y)$. The corner $\mathbf{x}_c$ need not be unique.

Let $\mathbf{x}^*$ be a corner which occurs infinitely often in the sequence $\{\mathbf{x}_i\}$. Let $y_{i_1}, y_{i_2}, \cdots$ be a convergent subsequence of the sequence of those y_i for which $\mathbf{x}_i = \mathbf{x}^*$, and let $y^* = \lim_{j\to\infty} y_{i_j}$. By continuity of f in y,

$$f(\mathbf{x}^*, y^*) = \lim_{j\to\infty} f(\mathbf{x}^*, y_{i_j}) = \lim_{j\to\infty} f(\mathbf{x}_{i_j}, y_{i_j}),$$

which is finite and is the l.u.b. of f on H.

Proof of Proposition 2. For $n=2$ the desired result is Proposition 1. For $n > 2$ we proceed by induction. Since f is separately continuous in $x_1, \cdots, x_{n-1}$ and, for each fixed value of x_{n-1} and y, is separately monotone in $x_1, \cdots, x_{n-2}$, the induction hypothesis implies that f is jointly continuous in $x_1, \cdots, x_{n-1}$ for each fixed y.

Choose $(\mathbf{x}', y') \in G$. Choose $\epsilon > 0$. By the above joint continuity there exists $\delta > 0$ such that $C(\mathbf{x}', \delta) \times \{y'\} \subset G$ and

$$|f(\mathbf{x}, y') - f(\mathbf{x}', y')| < \epsilon/2$$

for $\mathbf{x} \in C(\mathbf{x}', \delta)$. Let $\{\mathbf{x}_i | 1 \leqq i \leqq 2^{n-1}\}$ be the corners of $C(\mathbf{x}', \delta)$. By continuity of f in y there exist $\eta_i > 0$, $1 \leqq i \leqq 2^{n-1}$, such that $C(\mathbf{x}', \delta) \times [y'-\eta_i, y'+\eta_i] \subset G$, and

$$|f(\mathbf{x}_i, y) - f(\mathbf{x}_i, y')| < \epsilon/2$$

for $|y - y'| \leqq \eta_i$. Let $\eta = \min\{\eta_i | 1 \leqq i \leqq 2^{n-1}\}$ and define $H = C(\mathbf{x}', \delta) \times [y'-\eta, y'+\eta]$. Then for each point $(\mathbf{x}_i, y) \in H$, $1 \leqq i \leqq 2^{n-1}$,

$$|f(\mathbf{x}_i, y) - f(\mathbf{x}', y')| \leqq |f(\mathbf{x}_i, y) - f(\mathbf{x}_i, y')| + |f(\mathbf{x}_i, y') - f(\mathbf{x}', y')| < \epsilon.$$

By the lemma the maximum of f on H occurs at $(\mathbf{x}^*, y^*)$, where $\mathbf{x}^*$ is a corner of $C(\mathbf{x}', \delta)$. Dually the minimum of f on H occurs at some point $(\mathbf{x}^{**}, y^{**})$ where

$\mathbf{x}^{**}$ is a corner of $C(\mathbf{x}', \delta)$. Thus, for any $(\mathbf{x}, y) \in H$,

$$|f(\mathbf{x}, y) - f(\mathbf{x}', y')| < \epsilon,$$

and hence f is continuous at (x', y').

This work was supported by the United States Atomic Energy Commission.

THE MIXED PARTIAL DERIVATIVES AND THE DOUBLE DERIVATIVE

DONALD H. TRAHAN, Naval Postgraduate School

1. Introduction. In this note, we will briefly consider H. A. Schwarz's theorem on the equality of the mixed partial derivatives. The main aim of this paper is to point out that this theorem is directly connected to a concept which the author calls "the double derivative."

Consider a function f from E_2 to E_1, and let

$$F(h, k) = f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b). \tag{1}$$

It is easy to show that

$$f_{12}(a, b) = \lim_{k\to 0} \lim_{h\to 0} \frac{F(h, k)}{hk} \quad \text{and} \quad f_{21}(a, b) = \lim_{h\to 0} \lim_{k\to 0} \frac{F(h, k)}{hk}\cdot$$

Thus it seems natural to make the following definition:

DEFINITION. *Given a function f from E_2 to E_1, if $F(h, k)$ is given by equation* (1), *then* $\lim_{(0,0)} (F(h, k)/hk)$ *is called the double derivative of f at (a, b). The notation that we will use is $Df(a, b)$.*

The reader can easily verify that $f_{12}(a, b) = F_{12}(0, 0)$, $f_{21}(a, b) = F_{21}(0, 0)$, and $Df(a, b) = DF(0, 0)$.

2. The results. The two theorems of this section are a mean value theorem and Schwarz's theorem. We will first state a lemma which is a well-known result.

LEMMA. *Given a function f from E_2 to E_1, if the double limit of f exists at (a, b) and $\phi(y) = \lim_{x\to a} f(x, y)$ exists in a deleted neighborhood of $y = b$, then the iterated limit,* $\lim_b \lim_a f$ *exists and equals the double limit.*

THEOREM 1. *Given a function f from E_2 to E_1, let $F(h, k)$ be given by equation* (1). *If f_{12} exists in a neighborhood of (a, b), then $F(h, k) = hkF_{12}(\theta h, \phi k)$ where* $0 < \theta < 1$, $0 < \phi < 1$.

Proof. It follows that F_{12} exists in a neighborhood of $(0, 0)$. By the mean value theorem for functions of one variable, $F(h, k) = hF_1(\theta h, k)$ where $0 < \theta < 1$, and $F_1(\theta h, k) = kF_{12}(\theta h, \phi k)$ where $0 < \phi < 1$.

THEOREM 2. *Given a function f from E_2 to E_1, if f_2, f_{12} exist in a neighborhood of (a, b) and f_{12} is continuous there, then $f_{12}=f_{21}=Df$ at (a, b).*

Proof. By Theorem 1, $DF(0, 0)=F_{12}(0, 0)$. Now $\lim_{k\to 0}(F(h, k)/k) = f_2(a+h, b)-f_2(a, b)$. Therefore, by the lemma $F_{21}(0, 0)=DF(0, 0)$.

MATHEMATICAL EDUCATION

EDITED BY J. G. HARVEY AND M. W. POWNALL

Material for this Department should be sent to either of the editors: J. G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706; M. W. Pownall, Department of Mathematics, Colgate University, Hamilton, NY 13346.

INTERNATIONAL HIGH SCHOOL COMPETITION

A subcommittee of the M.A.A. Committee on High School Contests is studying the desirability and the feasibility of an international competition in mathematics for high school students. We also want suggestions for selecting the United States team to compete with one or more European teams (pending approval of the idea).

If you have an opinion on this question, strong or mild, please write to: W. H. Fagerstrom, Pan American College, Edinburg, TX 78539.

REPORT OF THE UPSTATE NEW YORK MAA CONTEST SECTION—BRITISH MATHEMATICAL OLYMPIAD COMPETITION

NURA D. TURNER, State University of New York at Albany

The May issue of the Monthly carried an article on the Upstate New York MAA Contest Section—British Mathematical Olympiad to be held in London May 20th. The competition was to be the first attempt at an international meet in secondary school mathematics in the "western" world. (The contest problems are listed below.)

Details of the Competition. The simple story of the results is that the team of Upstaters went to London, wrote the paper of the Fourth British Mathematical Olympiad, and came out second to the British team. One of our six team members ranked fifth among the twelve participants. In terms of the training provided in our schools in comparison with that provided by the British, our team members performed well.

The two teams met on as equal a basis as could be determined. Upstate team members selected were those who ranked in the top one percent in the Upstate New York MAA Contest Section in the examination of the 1967 Annual High School Contest, and who were of sophomore or junior status at the time of

that Contest. Mrs. Margaret Hayman, my counterpart of the British team, made an attempt to select as members of that team students who came as close as possible to matching our students. How well that matching was accomplished is to be noted in several facts.

The average age for the Upstaters: 16 years 10 months; for the British: 16 years 8 months.

The number of years of academic training was the same for members of both teams, since the British as well as the Upstaters started school at age 6.

The average scores of members of both teams on the examination of the 1968 Annual High School Mathematics Contest, the "American paper," as the British call it, were approximately the same. The average score for the British was 111.45; that for the Upstaters was 110.63. The top and bottom scores for the British were 125 and 95, respectively; those for the Upstaters were 132.75 and 93.75, respectively.

The idea of the competition originated at the International Congress of Mathematicians in Moscow, 1966. There I met Mrs. Hayman and suggested that the British and the Upstaters meet. Plans began developing. More definite plans were made when I attended the Fourth Guinness Awards Ceremony in London in April 1967. At that ceremony, winners of the Second British Mathematical Olympiad were honored and awarded prizes. The go ahead came when the Upper New York State Section of the MAA and the members of the Executive and Finance Committees of the national organization gave their approval for my soliciting funds from industry to make the trip to London possible. From then on, a good deal of work on both sides of the Atlantic was involved in organizing the competition and events related to it. During the year Elmer E. Haskins, Professor of Mathematics, State University College, Potsdam, New York, gave assistance by advising team members on their preparation; he set some of the twelve questions for the Olympiad, accompanied the team and me to London, and assisted Professor and Mrs. Walter K. Hayman in the grading of the papers. Professor and Mrs. Hayman originated the British Mathematical Olympiad.

The examination was difficult. No one attempted all twelve problems during the three hours allowed. Two students attempted eleven; both were British. The average number attempted by the British was ten; by the Upstaters eight.

What was learned from this Competition. The British provide a curriculum that challenges the creative power of students gifted with mathematical ability; they provide it in schools both public (private to us) and State (public to us). The curriculum allows students to concentrate on the study of mathematics at the secondary school level. Some British team members were taking as many as fourteen mathematics classes a week with periods the same length as our class periods. They were proficient in calculus—having taken at least three terms. They had a background in statistics and vector analysis and some had a conception of complex variables. One of our team members summed up the situation pretty well when he said that secondary school students in England concentrate

on subjects in the same way as our graduate students do. The British provide the opportunity for acceleration in their secondary school preparation. Two of the British team members did in eleven years the work of secondary school preparation normally done in thirteen years. Let me explain here that while the British provide thirteen years of precollege training to our twelve, no member of the British team had thirteen years of such training. The British provide through their confirmed system of subjective testing the challenging experience of thinking a problem through, of organizing a proof, and of expressing that organization in the written word.

The Russians have still more ambitious schemes. In the USSR mathematical excellence has been accomplished on a grand scale. The Russian academician has long been aware of the nonsense typical in the USA that one is not fully trained until his late twenties, and that it does not matter if he has years to make up after entering the university. Kolmogorov has pioneered by establishing a boarding school attached to the University of Moscow where children gifted with mathematical ability study together regardless of age and sex. While education is general, there is strong emphasis on mathematical studies. The school has been the model for others at Novosibirsk, Kiev, and Tbilisi. With modifications, similar schools are being established all over Eastern Europe, and with great success academically. While there have been a few research projects in our country, Albert Wilansky's at Lehigh University, Prenowitz' at Brooklyn College, and an attempt at Harvard to find out the capacities of children, I think I am correct in saying there has been no enduring program in our country similar to the Kolmogorov approach. C. P. Snow (recently Lord Snow, well known novelist and educator and parliamentary secretary to the British Minister of Education in 1964) has said, "I don't think anyone familiar with the field doubts that, say, the top 50 Soviet mathematicians under 18 would be able, in any kind of open competition, to take on the rest of the world. Why is this? The teaching is good, but teaching is only part of the answer. The competitive pressure is considerable, and so are the rewards for success, but that too is only part of the answer. Most people who have studied the problem believe the real secret lies in what physicists call 'critical mass.' That is, if you assemble enough bright pupils and enough good teachers, you produce a level of excellence which is far higher than if the bright students and good teachers were split up and scattered in penny packets. Put in a crude sentence, 'Bright people teach each other.'" (From the text of the speech given by The Rt. Hon. Lord Snow, C. B. E., on the occasion of the Fifth Guinness Awards Ceremony held in London, May 22, 1968.)

What can be done by the USA. We should recognize that not only underprivileged children need special treatment but also those children who are highly talented. We must face the fact that the talented child gets bored by the normal curriculum and the dissipation of his energies in various fringe subjects. He needs stimulation and challenge. He needs the time and the encouragement to sit down and think a problem through.

What can we do to see that the talented child gets what he needs? The following are some suggestions we might try:

- We can put bright students in mathematics together in some school in a community; we can bus the lot of them from schools in the surrounding area to that school. There, they would have the benefit of a special program and the stimulation of being with their peers.
- We can cut down on the amount of multiple-choice testing we use. Multiple-choice testing is no encouragement to sustained concentration and for some students is an invitation to mathematical illiteracy.
- We can organize national and international competitions in mathematics in order to obtain some sort of objective measure of our provision for talented children in mathematics.

I have provided the reader with background information on the competition, with information in the way of explanation as to why the British performance was better, and with suggestions as to what we in the USA might do to see that the child talented with mathematical ability obtains the attention he needs. Procrustes would undoubtedly have approved of our present school system. But should we?

EXAMINATION FOR THE FOURTH BRITISH MATHEMATICAL OLYMPIAD

MAY 20, 1968—9:30–12:30.

1. A circle C of unit radius rolls without slipping along the outside of the circle with center the origin and radius 2 in the (x, y) plane. A fixed point P of C is originally at the position $(2, 0)$. Find equations, giving the coordinates (x, y) of P in terms of a suitable parameter θ, as C rotates. Sketch the locus S described by P, showing all tangents parallel to the x and y axes.

2. Cows are put out to pasture when the grass has reached a certain height. Thereafter, as the cows eat the grass, the grass continues to grow as the pasture is consumed. If 15 cows can consume the grass in 3 acres of pasture in 4 days, while 32 cows can consume the grass in 4 acres in 2 days, how many cows will be required to consume the grass in 6 acres in 3 days?

3. A "distance" between two points (x_1, y_1) and (x_2, y_2) in the (x, y) plane is defined by

$$d = |x_2 - x_1| + |y_2 - y_1|.$$

Using this notion of distance find the locus of all points (x, y) for which $x \geqq 0$, $y \geqq 0$ and which are equidistant from the origin and a fixed point (a, b) in the plane, where $a > b$. Distinguish the cases according to the signs of a and b.

4. Two spheres of radii a, b are tangent to each other and a plane is tangent to these spheres at different points. Find the radius of the largest sphere which can pass between the first two spheres and the plane.

5. Given x, y, z such that

$$\sin x + \sin y + \sin z = 0$$

and

$$\cos x + \cos y + \cos z = 0,$$

prove that

$$\sin 2x + \sin 2y + \sin 2z = 0$$

and

$$\cos 2x + \cos 2y + \cos 2z = 0.$$

6. If $a_1, a_2, \cdots, a_7$ are integers and $b_1, b_2, \cdots, b_7$ are the same integers rearranged show that

$$(a_1 - b_1)(a_2 - b_2) \cdots (a_7 - b_7)$$

is even.

7. A knock-out ping pong tournament is arranged among n people and a new ball is used for each game. How many balls are needed?

8. A chord of length l divides the interior of a circle of radius r into two regions D_1 and D_2. A circle S of maximal radius is inscribed in D_1 and the area of the part of D_1 outside S is A. Show that A is greatest when

$$l = \frac{16\pi r}{16 + \pi^2},$$

and that area of D_1 exceeds that of D_2.

9. Find the lengths of the sides of a triangle with altitudes 3, 4 and 6 inches.

10. The faces of a tetrahedron are formed by 4 congruent triangles. If α is the angle between a pair of opposite edges of the tetrahedron show that

$$\cos \alpha = \frac{\sin (B - C)}{\sin (B + C)}$$

where B, C are the angles adjacent to one of these edges in a face of the tetrahedron.

11. The sum of the reciprocals of a set of n different positive integers is equal to one. If $n = 3$ show that there is only one such set and find it. Find also such a set for $n = 4$, 5 and more generally any value of $n > 3$.

12. Find the maximum number of points which can be placed on the surface of a sphere of unit radius such that the distance between any two of the points is

a) at least $\sqrt{2}$; b) greater than $\sqrt{2}$.

Justify your answer.

E.E.H., M.R.H., W.K.H.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before May 31, 1969. Contributors (in North America) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2141. *Proposed by H. S. Hahn, West Georgia College*

(1) Find convex nonregular equilateral (equal-edged) polyhedra with their n $(4<n<16)$ vertices on a sphere.

(2) Prove or disprove that there is no convex nonregular equilateral polyhedron with an odd number n of vertices all on a sphere, except for $n=5, 9, 11, 15$, and 55.

E 2142. *Proposed by Erwin Just, Bronx Community College*

Let k, b, and r be fixed integers. Call an integer n "special" if each member of $\{kb^n+i\}$, $i=1, 2, \cdots, r$, is composite. Prove that the number of "special" integers is infinite.

E 2143. *Proposed by Peter Kornya, University of British Columbia*

In a triangle with sides a, b, c, the line joining the centroid and the orthocenter is perpendicular to the bisector of the angle opposite side c. Show that the arithmetic mean of a, b, c is twice the harmonic mean of a and b.

E 2144. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha*

Solve the equation

$$\frac{x}{y}=\frac{(x^2-y^2)^{y/x}+1}{(x^2-y^2)^{y/x}-1}$$

for positive integral values of x and y.

E 2145. *Proposed by V. F. Ivanoff, San Carlos, California*

In an arbitrary quadrangle $ABCD$, let line CE parallel to DA cut AB in E and line CF parallel to BA cut AD in F. Denote by K the point of intersection of BF and ED. Show that the quadrangles $AEKF$ and $KBCD$ have equal directed areas.

E 2146. *Proposed by A. M. Kirch, University of Missouri*

Find, for each positive integer m, the last three digits of m^{100}.

E 2147. *Proposed by R. Shantaram, State University of New York at Stony Brook*

Let a and b be complex numbers and let $r \geqq 0$. Show that

$$|a+b|^r \leqq k_r(|a|^r + |b|^r),$$

where $k_r = 1$ if $r \leqq 1$ and $k_r = 2^{r-1}$ if $r \geqq 1$.

E 2148. *Proposed by G. R. MacLane, Purdue University*

Let

$$\sum_{\nu=1}^{\infty} a_\nu \tag{1}$$

be a conditionally convergent series with real constant terms. The familiar examples (those that satisfy the Leibniz convergence criterion, for example) are such that

$$\sum_{\nu=1}^{\infty} \operatorname{sgn}(a_\nu)\,|a_\nu|^\lambda \tag{2}$$

is convergent for each $\lambda > 0$. Find a series (1) which converges, but such that (2) is divergent for each λ, $0<\lambda$, $\lambda \neq 1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Property of 1093

E 2033 [1967, 1134]. *Proposed by D. R. Rao, Secunderabad, India*

Show that $p = 1093$ simultaneously satisfies

$$2^{(p-1)/2} + 1 \equiv 0 \pmod{p^2},$$
$$2^j - 2^{j-k} + 2^{j-2k} - \cdots + 1 \equiv 0 \pmod{p^2},$$

where $j = \frac{1}{2}[p-(2k+1)]$, $k = 2+4s_n$, $(n = 0, 1, 2, 3, 4)$, and s_n is the sum of the first n natural numbers.

Solution by J. B. Muskat, University of Pittsburgh. It is well known (see, e.g., Hardy & Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford

University Press, 1960, Theorem 91, p. 73) that the congruence

$$2^{p-1} \equiv 1 \pmod{p^2}$$

is satisfied by $p=1093$. Then

$$(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1) \equiv 0 \pmod{p^2}.$$

2 is a quadratic nonresidue (mod p), since $p \equiv 5$ (mod 8). Hence $2^{(p-1)/2}-1 \not\equiv 0$ (mod p), so $2^{(p-1)/2}+1 \equiv 0$ (mod p^2), $p=1093$.

Now $k=2(n^2+n+1)$, $n=0, 1, 2, 3, 4$; $k=2, 6, 14, 26, 42$; $j=\frac{1}{2}(p-1)-k=546-k$. Since 546 is an odd multiple of each k,

$$2^j - 2^{j-k} + 2^{j-2k} - \cdots + 1 = \frac{2^{546}+1}{2^k+1}.$$

$2^{546}+1 \equiv 0$ (mod 1093^2). By direct calculation, one finds (mod 1093)

$$2^2 \equiv 4, \quad 2^6 \equiv 64, \quad 2^{14} \equiv 1082, \quad 2^{26} \equiv 850, \quad 2^{42} \equiv 855.$$

So $2^k+1 \not\equiv 0$ (mod 1093). Hence

$$2^j - 2^{j-k} + 2^{j-2k} - \cdots + 1 \equiv 0 \pmod{1093^2}.$$

Also solved by L. Carlitz, M. G. Greening (Australia), Donald Jeffords, E. S. Langford, Simeon Reich (Israel), and the proposer.

A Diophantine System

E 2034 [1967, 1134]. *Proposed by Merrill Barnebey, Wisconsin State University*

Show that there exist solutions in positive integers of the system:

$$a+b+c = x+y, \qquad a^3+b^3+c^3 = x^3+y^3.$$

In particular, show that there are infinitely many in which a, b, c form an arithmetic progression.

Solution by W. J. Blundon, Memorial University of Newfoundland. If we set $a=3d$, $c=2b-3d$, we have $x+y=3b$ and the second equation may be reduced to the form

$$(x-y)^2 = (b-8d)^2 - 40d^2$$

one solution of which is

$$x-y = p^2-10q^2, \quad b-8d = p^2+10q^2, \quad d = pq.$$

Thus we have the two-parameter family of solutions with a, b, c in arithmetic progression:

$$a = 3pq, \quad b = p^2+8pq+10q^2, \quad c = 2p^2+13pq+20q^2;$$
$$x = 2p^2+12pq+10q^2, \quad y = p^2+12pq+20q^2.$$

Also solved by D. D. Adamović (Yugoslavia), Joseph Arkin, Anders Bager (Denmark), M. S. Demos, Michael Goldberg, M. G. Greening (Australia), Robert Heller, C. V. Heuer & G. A. Heuer, J. A. H. Hunter, Donald Jeffords, Geoffrey Kandall, Lew Kowarski, D. C. B. Marsh, J. J. Martinez, Norman Miller, Margaret B. Seay, Gregory Wulczyn, K. L. Yocom, Alexander Žujus, and the proposer.

Wulczyn notes that a two-parameter solution (not in A.P.) is given on p. 42 of A. Gloden, *Mehrgradige Gleichungen:*

$$a = 2m^2 + 2n^2, \qquad b = 3mn - n^2, \qquad c = m^2 - mn - 2n^2;$$
$$x = m^2 + 3mn, \qquad y = 2m^2 - mn - n^2.$$

A Condition for a Triangle to be Isosceles

E 2035 [1967, 1261]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Can the Euler line of a nonisosceles triangle pass through the Fermat point of the triangle? (Lines to the vertices from the Fermat point make angles of 120° with each other.)

I. *Solution by Leon Bankoff, Los Angeles, California.* The answer is "no." If D and E denote the remote vertices of the equilateral triangles BDC and CEA described externally on sides BC and CA of triangle ABC, the Fermat point F lies on the intersection of AD and BE. If F also lies on the line OH connecting the circumcenter and the orthocenter, we have by the similarity of triangles ODF and HAF,

$$\frac{OF}{FH} = \frac{OD}{AH} = \frac{R\cos A + R\sqrt{3}\sin A}{2R\cos A} = \frac{1 + \sqrt{3}\tan A}{2}.$$

Also, in triangles BHF and EOF,

$$\frac{OF}{FH} = \frac{OE}{BH} = \frac{R\cos B + R\sqrt{3}\sin B}{2R\cos B} = \frac{1 + \sqrt{3}\tan B}{2}.$$

Hence F lies on the Euler line if and only if $\tan B = \tan A$, i.e., if and only if $BC = AC$.

II. *Solution by A. Vendeghen, Liege, Belgium.* The trilinear coordinates of the centroid, circumcenter, and the Fermat point are respectively proportional to $\sin B \sin C, \cdots$; $\cos A, \cdots$; and $\sin(B+60°)\sin(C+60°), \cdots$. These points are collinear iff the determinant of their coordinates is zero. Developing the determinant and making a few transformations, we obtain

$$\sin(A - B)\sin(B - C)\sin(C - A) = 0,$$

which is satisfied iff the triangle is isosceles.

Also solved by M. G. Beumer (Netherlands), M. G. Greening (Australia), J. M. Quoniam (France), Simeon Reich (Israel), Marlow Sholander, and Sister Stephanie Sloyan.

A Telescoping Series

E 2036 [1967, 1262]. *Proposed by L. C. Grove, University of Oregon*

If n is any fixed positive integer, set $x_0=1/n$ and $x_j=(n-j)^{-1}\sum_{i=0}^{j-1} x_i$, for $j=1, 2, \cdots, n-1$. Compute $\sum_{j=0}^{n-1} x_j$.

Solution by E. S. Langford, Naval Postgraduate School, Monterey, Cal. We shall show by induction that $\sum_{j=0}^{k} x_j=1/(n-k)$, for $k=0, 1, \cdots, n-1$. From this it follows that $\sum_{j=0}^{n-1} x_j=1$. The case $k=0$ is true by definition, so suppose that $\sum_{j=0}^{k} x_j=1/(n-k)$ for some k, $0\leqq k\leqq n-2$. Then

$$\sum_{j=0}^{k+1} x_j = x_{k+1} + \sum_{j=0}^{k} x_j = \sum_{j=0}^{k} x_j/(n-k-1) + \sum_{j=0}^{k} x_j$$

by definition of x_{k+1}. Using the inductive assumption, we have

$$\sum_{j=0}^{k+1} x_j = \frac{1}{(n-k-1)(n-k)} + \frac{1}{(n-k)} = \frac{1}{n-k-1}.$$

This completes the proof.

Also solved by 97 other readers.

A Triangle Inequality

E 2037 [1967, 1262]. *Proposed by J. Garfunkel, Forest Hills (N. Y.) High School*

If p_a, p_b, p_c denote the lengths of the perpendiculars from the vertices B, C, A to the medians m_a, m_b, m_c, respectively, of an acute triangle ABC, then $p_a+p_b+p_c$ is not less than the sum of the sides of the triangle of minimum perimeter which can be inscribed in ABC.

Solution by Leon Bankoff, Los Angeles, California. Using conventional notation, we are required to show that

$$\sum p_a \equiv \frac{2\Delta}{3}\sum \frac{1}{AG} \geqq \frac{2\Delta}{R},$$

the right member representing the perimeter of the orthic triangle of ABC. This is equivalent to showing that R is not less than the harmonic mean of AG, BG, and CG. By applying the quadratic-arithmetic-geometric-harmonic mean inequalities to the relation $AG^2+BG^2+CG^2+OG^2=3R^2$, we obtain

$$R^2 \geqq \frac{AG^2+BG^2+CG^2}{3} \geqq \left(\frac{AG+BG+CG}{3}\right)^2 \geqq \left(\sqrt[3]{AG\cdot BG\cdot CG}\right)^2$$

from which we find that $R\geqq$ the harmonic mean of AG, BG, CG, the desired result.

Also solved by M. G. Greening (Australia), Simeon Reich (Israel), and the proposer.

Another Triangle Inequality

E 2038 [1967, 1262]. *Proposed by J. Garfunkel, Forest Hills (N. Y.) High School*

If P is an interior point of an acute triangle ABC, then $PA+PB+PC \geqq 2/\sqrt{3}$ times the perimeter of the pedal triangle of ABC.

Solution by Leon Bankoff, Los Angeles, California. It is known that $PA+PB+PC \geqq 6r$ and that the triangle of maximum perimeter inscribed in a circle is equilateral. Hence if D, E, F are the incircle contacts with the sides of the triangle, we have

$$PA + PB + PC \geqq 6r \equiv \frac{2}{\sqrt{3}}(3r\sqrt{3}) \geqq \frac{2}{\sqrt{3}}(DE + EF + FD),$$

equality holding if and only if triangle ABC is equilateral and P is its centroid.

This inequality is stronger than the one proposed because the orthic triangle (otherwise known as the pedal triangle of the orthocenter) of triangle ABC is the triangle of minimum perimeter inscribed in ABC.

Also solved by L. Carlitz, M. G. Greening (Australia), Charles McCracken, Simeon Reich (Israel), Jonathan Ryshpan, and the proposer.

A Number-Theoretic Function

E 2039 [1967, 1262]. *Proposed by Irving Katz, George Washington University*

Let $n = \prod_{i=1}^{t} p_i^{\alpha_i}$ be the canonical factorization of the positive integer n, and define $F(n) = \sum_{i=1}^{n} (i, n)$. Evaluate $F(n)$ in terms of p_i and α_i.

Solution by R. P. Kelisky, IBM Research, Yorktown Heights, N. Y. Suppose d is a divisor of n. To count the number of solutions of $(i, n) = d$ observe that if $i = td$, there is exactly one solution for each t such that $(t, n/d) = 1$. There are $\phi(n/d)$ such t's for each d. Hence

$$(*) \qquad F(n) = \sum_{d|n} d\phi(n/d) = n \sum_{d|n} \phi(d)/d.$$

Since $\phi(d)/d$ is multiplicative, $F(n) = \prod F(p_i^{\alpha_i})$. But

$$(**) \qquad F(p^k) = p^k \sum_{=1}^{k} \phi(p^j)/p^j = p^k[1 + (p-1)k/p].$$

Therefore

$$F(n) = \prod_{i=1}^{t} [(\alpha_i + 1)p_i^{\alpha_i} - \alpha_i p_i^{\alpha_i - 1}].$$

The result is known. See L. E. Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. 1, p. 127. E. Cesaro (1883) found (*), and

E. Catalan (1888), *loc. cit.* p. 130, obtained (**) for the right side of (*).

Also solved by M. G. Beumer (Netherlands), M. C. Bhandari (India), D. Ž. Djoković, R. C. Entringer, G. K. Goff, Jerry Goodman, M. G. Greening (Australia), Heiko Harborth (Germany), John Kieffer, Peter Kornya, E. S. Langford, D. C. B. Marsh, M. J. Merscher, C. B. A. Peck, Simeon Reich (Israel), Jonathan Ryshpan, D. P. Sumner, C. S. Venkataraman (India), R. L. Vogt, and the proposer. Venkataraman notes that the result, in the form

$$F(n) = n \prod_{i=1}^{t} (\alpha_i + 1 - \alpha_i/p_i),$$

was given by S. S. Pillai *On an arithmetic function*, J. Annamalai Univ. (India), 2, pp. 243–248. A. C. Vasu generalized Pillai's results *On a certain arithmetic function*, Math. Student, 34, 2 (1966) pp. 93–95. Further extensions will appear in a paper by R. Sivaramakrishnan.

Sequences with the Greatest Common Divisor Property

E 2040 [1967, 1262]. *Proposed by Douglas Lind, University of Virginia*

A sequence of nonzero integers $\{a_n\}$ has the greatest common divisor property if $(a_k, a_n) = a_{(k,n)}$. For example, the Fibonacci sequence has this property. Resolve the conjecture that all sequences with this property obey a second order linear difference relation.

Solution by L. Carlitz, Duke University. Presumably the question is whether $\{a_n\}$ must satisfy a (homogeneous) linear difference equation of the second order with constant coefficients. The answer to this question is no. Indeed, consider $a_n = n^2$. Clearly

$$(a_m, a_n) = (m^2, n^2) = (m, n)^2 = a_{(m,n)}.$$

On the other hand if $A(n+2)^2+B(n+1)^2+Cn^2=0$, $(n=1, 2, \cdots)$, where A, B, C are constants, then

$$A + B + C = 0, \quad 4A + 2B = 0, \quad 4A + B = 0,$$

which implies $A = B = C = 0$.

More generally, the example $a_n = n^k$, where $k \geqq 2$ and independent of n, shows that $\{a_n\}$ need not satisfy a recurrence (with constant coefficients) of order k.

Also solved by R. C. Entringer, E. S. Langford, D. C. B. Marsh, E. J. F. Primrose (England), and the proposer.

Langford's solution exhibits a function $a(n)$ with the greatest common divisor property such that $a(n)$ does not satisfy a linear differential equation of any order when the coefficients are restricted to be polynomials.

Packing a Set of Squares

E 2041 [1967, 1262]. *Proposed by D. J. Newman, Yeshiva University*

Given a collection of squares whose total area is 1, prove that they can be placed in a nonoverlapping manner into a square of side $\sqrt{2}$. ($\sqrt{2}$ is best possible.)

Solution by the proposer. Arrange the given squares in nonincreasing order of size. Now place the first square in the left lower corner of the square S of side $\sqrt{2}$, place the second square next to it on the bottom of S, continuing until no more fit along the bottom. Now start a new row along the top of the first square extended, and line up as many as possible of the next squares along this. Then comes a third row, and so on. The claim is that all of these rows will fit into our square S.

To prove this, suppose that the sides of the squares are $s_1 \geqq s_2 \geqq s_3 \geqq \cdots$, and that the first n_1 squares fit into the first row, the next n_2 squares into the second row, and so on. Thus we have

$$\sqrt{2} - s_{n_1+1} \leqq s_1 + s_2 + \cdots + s_{n_1} \leqq \sqrt{2},$$
$$\sqrt{2} - s_{n_1+n_2+1} \leqq s_{n_1+1} + \cdots + s_{n_1+n_2} \leqq \sqrt{2},$$
$$\sqrt{2} - s_{n_1+n_2+n_3+1} \leqq s_{n_1+n_2+1} + \cdots + s_{n_1+n_2+n_3} \leqq \sqrt{2},$$

and so on, and we must show that $s_1+s_{n_1+1}+s_{n_1+n_2+1}+ \cdots \leqq \sqrt{2}$. But

$$s_2 + s_3 + \cdots + s_{n_1+1} \geqq \sqrt{2} - s_1$$

so that

$$s_2^2 + s_3^2 + \cdots + s_{n_1+1}^2 \geqq (\sqrt{2} - s_1)s_{n_1+1}.$$

Similarly,

$$s_{n_1+2}^2 + \cdots + s_{n_1+n_2+1}^2 \geqq (\sqrt{2} - s_1)s_{n_1+n_2+1},$$

and so on. Adding these inequalities gives

$$1 - s_1^2 = s_2^2 + s_3^2 + \cdots \geqq (\sqrt{2} - s_1)(s_{n_1+1} + s_{n_1+n_2+1} + \cdots),$$

so that

$$s_1 + s_{n_1+1} + s_{n_1+n_2+1} + \cdots \leqq \frac{1 - s_1^2}{\sqrt{2} - s_1} + s_1.$$

The function on the left, however, is equal to

$$\sqrt{2} - \frac{(1 - \sqrt{2}\, s_1)^2}{\sqrt{2} - s_1} \leqq \sqrt{2},$$

and the proof is complete.

Also solved by R. B. Eggleton (Australia), Michael Goldberg, J. G. Mauldon (England), and Jonathan Ryshpan.

Range of the Function, Degree of a Root of Unity

E 2042 [1967, 1262]. *Proposed by H. M. Edgar, San Jose State College*

Let α be an arbitrary root of unity and let $d(\alpha)$ denote the degree of α (as an algebraic number). Determine the range of the function $d(\alpha)$.

Solution by Erwin Just, Bronx Community College, New York. If α is a primitive kth root of unity, then $d(\alpha)=\phi(k)$ where ϕ is the Euler ϕ-function. Thus the range of the given function d is contained in the range of ϕ. On the other hand, since there exists a primitive nth root of unity for each positive integer n, namely $e^{i(2\pi/n)}$, the range of ϕ is contained in the range of d. It follows that the range of d is precisely the range of ϕ.

Also solved by Anders Bager (Denmark), D. Ž. Djoković, John Kieffer, E. S. Langford, Douglas Lind, Simeon Reich (Israel), D. P. Sumner, and the proposer.

Our contributors believe that the range of $\phi(n)$ is still undetermined. Langford refers to Amer. J. Math., 30 (1908) 394–400, where Carmichael has tabulated all solutions (if any) of the equation $\phi(x)=n$ for $1\leqq n\leqq 1000$. Lind refers to Problem 601, Math. Mag., 39 (1966) 190, where it is shown that there are infinitely many even integers not appearing in the range of $\phi(n)$, the smallest being 14. In particular, C. L. Klee proved that $\phi(x)=2m$ has no solution if m has no divisor $d>1$ for which $2d+1$ is prime (*On the equation* $\phi(x)=2m$, this MONTHLY, 53 (1946) 327).

Postulates for a Group

E 2043 [1967, 1262; 1968, 669]. *Proposed by P. Baxandall, University of Keele, Staffordshire, England*

Let S be a semigroup with a right identity e (so $xe=x$ for all $x\in S$) such that given any $y\in S$ there exists $\bar{y}\in S$ with $\bar{y}y=e$. It is well known that S may not be a group. Is this still true if S has only one right identity?

Solution by Simeon Reich, Israel Institute of Technology, Haifa. Clearly we have only to prove that $y\bar{y}=e$. Let $y\bar{y}=b$. Then

$$b^2 = y\bar{y}y\bar{y} = (ye)\bar{y} = y\bar{y} = b \Rightarrow \bar{b}bb = \bar{b}b \Rightarrow eb = e.$$

It follows that for every $s\in S$, $sb=(se)b=s(eb)=se=s$. Thus b is a right identity and since S has only one right identity, $b=e$, as required.

Also solved by Robert Baumel, Lynne Ellen Bliss, W. E. Bodden, F. D. Cheek, Jr., D. F. Dawson, D. Ž. Djoković, B. L. Eave, Charles Green, Robert Heller, Geoffrey Kandall, G. A. Kraus, E. S. Langford, J. F. Leetch, W. G. McArthur, S. J. Milles & Al Somayajulu, J. H. Osborn, W. M. Patterson III, Stan Pauli, D. E. Penney, David Promislow, S. N. Rao, Avrum Rosner, John Shafer, Stephen Spindler, D. P. Sumner, E. J. Taft, W. A. Thrash, Jr., Charles Wells, Albert White, J. A. Winthrop, and the proposer.

Editorial Note. We offer apologies for the misprint ($y\bar{y}$ instead of $\bar{y}y$) corrected in a later issue, to the many readers who wrote us pointing out that under the hypothesis as originally printed it is well known that S *must be* a group.

Leetch notes that the assumed structure is termed a *multiple group* by A. H. Clifford and the number of right identities is called the index of S. From Clifford's several characterizations of a multiple group it is evident that a multiple group of index 1 is a group. See *A system arising from a weakened set of group postulates*, Ann. of Math., 34 (1933) 965–971. See also H. B. Mann, *On certain systems which are almost groups*, Bull. Amer. Math. Soc., 50 (1944) 879–881.

Determination of an Ellipse

E 2044 [1967, 1262]. *Proposed by N. X. Vinh, University of Colorado*

Prove: an ellipse is determined by three tangents and its center.

Solution by Michael Goldberg, Washington, D. C. It is a trivial exercise to find three lines and a point O such that no ellipse with center O can be tangent to the lines. We will prove: if three given lines, no two of them parallel, and a given point O are such that there exists an ellipse with center O and tangent to the lines, the ellipse is unique.

Reflect the three given tangents across the center to obtain a total of six distinct tangents. Since only five tangents are needed to determine an ellipse, the solution is unique.

However, if two of the given tangents are parallel, then a parallelogram consisting of only four distinct tangents is obtained. A continuous infinity of distinct ellipses can be inscribed in this parallelogram.

Also solved by Donald Batman, J. C. Egsgard, Gerald Goertzel & Fred Gustavson, M. G. Greening (Australia), Lew Kowarski, W. Liniger, D. C. B. Marsh, Charles McCracken, Hugh Noland, J. M. Quoniam (France), Simeon Reich (Israel), Robin Robinson, Marlow Sholander, P. D. Thomas, Dimitrios Vathis (Greece), and the proposer.

F. D. Pederson (Denmark) locates the problem as Problem 71, p. 122, of T. H. Eagles, *Constructive Geometry of Plane Curves*, (Macmillan 1885).

Complex Numbers Satisfying a Given Inequality

E 2045 [1968, 75]. *Proposed by Zalman Rubinstein, Clark University*

Given a set of n distinct complex numbers z_i, $i=1, 2, \cdots, n$, which satisfy $\min_{i\neq j}|z_i-z_j| \geqq \max_i|z_i|$, find the upper bound for n and classify all the maximal sets. What can be said in the case of arbitrary Euclidean spaces?

Solution by Peter Kornya, Student, University of British Columbia. Let $|z_m| = \max_i|z_i|$. Then, in the complex plane, all the numbers z_i are on or inside the circle R with radius $|z_m|$ and center the origin. Clearly a set of six complex numbers on the circumference of R, defining a hexagon, together with $z_1=0$, makes $n=7$. Also, if $n>7$, or $n=7$ and some point $z\neq 0$ is not on the circumference of R, then, invoking the law of cosines we deduce that $|z_i-z_j|<|z_m|$ for some z_i, z_j. Hence 7 is an upper bound for n.

The maximal sets may be classified as:

$$S_{\delta r} = \{z_i \mid z_i = 0 \quad \text{or} \quad z_i = r(\cos(\tfrac{1}{3}k\pi + \delta) + i\sin(\tfrac{1}{3}k\pi + \delta)), k=0, 1, \cdots, 5\}$$

where δ, r are arbitrary real numbers such that $r>0$ and $0\leqq\delta<\pi/3$.

For arbitrary Euclidean spaces the problem reduces to finding the maximum number of points inside the unit hypersphere such that the distance between any two points is at least 1.

Also solved by R. B. Eggleton (Australia), Michael Goldberg, Steven Russ, G. J. Simmons, and the proposer.

Eggleton observes that the solution of the first part of the problem follows from Ron Graham's solution to E 1921 [1968, 192–193]. J. L. Brenner points out that the problem is really due to Faddeev-Sominskii, and is Problem 155 in his translation of their *Problems in Higher Algebra* (W. H. Freeman, 1965). Simmons and Goldberg refer to related problems; in particular, solutions are to be found in Coxeter, *Regular Polytopes* (Macmillan, 1963), pp. 292–293, and John Leech, Canad. J. Math., 19 (1967) 251–267.

Divisor of a Determinant

E 2046 [1968, 75]. *Proposed by L. P. Zukowski, University of Michigan*

Each element of the set $S=\{112, 518, 322\}$ is divisible by 14. The 3×3 determinant formed from S by associating each number with a distinct row and each digit with a definite column, viz.,

$$\begin{vmatrix} 1 & 1 & 2 \\ 5 & 1 & 8 \\ 3 & 2 & 2 \end{vmatrix} = 14$$

is also divisible by 14. Show that this generalizes to the result that any common factor d, of n arbitrary n-digit integers, is also a factor of the analogously formed $n\times n$ determinant.

I. *Solution by R. B. Eggleton, Avondale College, Cooranbong, N.S.W., Australia.* Let Δ be the given $n\times n$ determinant. Since the value of a determinant is unaltered when any column is changed by adding to it any multiple of any other column, to the nth column of Δ we may add 10^{n-j} times the jth column, successively for $j=1, 2, \cdots, n-1$. The nth column of the new determinant consists of the given n arbitrary n-digit integers, so has a common factor d, and when this is divided out all the remaining entries are integers, so the value of Δ is some integer multiple of d. (This construction does not apply if $n=1$, but the desired result is trivial in this case.)

II. *Solution by P. M. Berry, Mobil Research, Dallas, Texas.* Let the (distinct) elements of S be b_i, $i=1, 2, \cdots, n$, and let $b_i=a_{i1}10^{n-1}+a_{i2}10^{n-2}+\cdots+a_{in}$. The linear system of equations

$$a_{i1}x_n + a_{i2}x_{n-1} + \cdots + a_{in}x_1 = b_i, \qquad i = 1, 2, \cdots, n$$

obviously has as solution $x_j=10^{n-j}$, $j=1, 2, \cdots, n$. Applying Cramer's rule to solve for $x_n(=1)$, we have $\det(a_{i1}, a_{i2}, \cdots, a_{in}) = \det(a_{i1}, a_{i2}, \cdots, b_i)$.

If d is a factor of each b_i, then d is a factor of the left hand side of the above.

Also solved by 65 other readers.

Several solvers point out that the n numbers could be written in any base whatever and that the numbers need not all have the same number of digits. Langford notes that a modified proof would allow a given column of the determinant to contain $k\geqq 1$ digits of each number. For example, using $N_1=1120$, $N_2=518$, $N_3=28$, and using 1, 2, 1 digits per column, obtain the determinant

$$\begin{vmatrix} 1 & 12 & 0 \\ 0 & 51 & 8 \\ 0 & 2 & 8 \end{vmatrix}.$$

Order of Certain Elements of a Group

E 2047 [1968, 76]. *Proposed by Hans Liebeck, University of Keele, England*

x and y are elements of finite order m and n respectively of a group, and $xy=yx$. What can you say about the order of xy?

Solution by the proposer. Suppose that

$$(1)\qquad \begin{aligned} m &= p_1^{\alpha_1}\cdots p_r^{\alpha_r}\cdots p_k^{\alpha_k}\\ n &= p_1^{\beta_1}\cdots p_r^{\beta_r}\cdots p_k^{\beta_k}\end{aligned}$$

where $p_1, \cdots, p_k$ are the distinct primes occurring in m or n (or both), and the primes are so arranged that $\alpha_i \neq \beta_i$ when $i=1, \cdots, r$, but $\alpha_i=\beta_i$ for all $i>r$. Define

$$\mu_i = \max\{\alpha_i, \beta_i\}, \qquad i = 1, \cdots, k.$$

We shall show that the order t of xy is restricted by the divisibility condition

$$(2)\qquad p_1^{\mu_1}\cdots p_r^{\mu_r} \mid t \mid p_1^{\mu_1}\cdots p_r^{\mu_r}\cdots p_k^{\mu_k},$$

where, as usual, "$a|b$" stands for "a divides b."

Proof. The upper bound $l=p_1^{\mu_1}\cdots p_k^{\mu_k}$ for t is simply the lowest common multiple of m and n. Clearly $(xy)^l=1$, and so by a well-known result, the order t of xy divides l.

We establish the lower divisibility condition as follows. Since $(xy)^t=1$, it follows that $x^t=y^{-t}$. Now x^t has order $m/(t, m)$, where (t, m) denotes the highest common factor of t and m, and y^{-t} has order $n/(t, n)$. Hence $m(t, n)=n(t, m)$ and so, from (1),

$$(3)\qquad p_1^{\alpha_1}\cdots p_r^{\alpha_r}(t, p_1^{\beta_1}\cdots p_k^{\beta_k}) = p_1^{\beta_1}\cdots p_r^{\beta_r}(t, p_1^{\alpha_1}\cdots p_k^{\alpha_k}).$$

For any i, $1\leqq i\leqq r$, we have $\alpha_i\neq\beta_i$. Suppose that $\alpha_i>\beta_i$. Then from (3),

$$(4)\qquad \cdots p_i^{\alpha_i-\beta_i}\cdots(t, \cdots\ p_i^{\beta_i}\cdots) = \cdots(t, \cdots\ p_i^{\alpha_i}\cdots).$$

Now suppose that the highest power of p_i dividing t is p_i^ν. We claim that $\nu\geqq\alpha_i$' For if $\nu<\alpha_i$, then the power of p_i appearing on the right hand side of (4) is p_i^ν, whereas on the left hand side appears $p_i^{\alpha_i}$ if $\nu\geqq\beta_i$ or $p_i^{\nu+(\alpha_i-\beta_i)}$ if $\nu<\beta_i$. In either case, (4) is contradicted.

Similarly, the assumption $\alpha_i<\beta_i$ leads to the condition $\nu\geqq\beta_i$. Hence $\nu \geqq \max\{\alpha_i, \beta_i\}=\mu_i$, and so $p_i^{\mu_i}| t$. This completes the proof of (2).

Finally, the following two examples show that our bounds on t are best possible:

1. In the direct product of two cyclic groups generated by x and y of orders m and n respectively, xy has order l.

2. Consider the direct product of three cyclic groups generated by a, b, c of orders $p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $p_1^{\beta_1}\cdots p_r^{\beta_r}$, $p_{r+1}^{\alpha_{r+1}}\cdots p_k^{\alpha_k}$ respectively. Put $x=ac$, $y=bc^{-1}$. Then the orders of x, y and xy are respectively m, n and $p_1^{\mu_1}\cdots p_r^{\mu_r}$.

Also solved by D. C. B. Marsh, and partially solved by R. B. Eggleton (Australia), E. K. Hayashi, J. V. Michalowicz, and Simeon Reich (Israel). There were a large number of very incomplete solutions.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (with double spacing) and should be mailed before June 30, 1969. Contributors (in North America) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5622 [1968, 910]. **Correction.** *Proposed by D. S. Lawrence, Brooklyn Polytechnic Institute, New York*

Let X be a collection of sets of real numbers, each of which is well-ordered by magnitude. Suppose also that X is *well-ordered* by inclusion. Show that X is denumerable.

5644. *Proposed by Jerrold Siegel, Purdue University*

Prove that there does not exist a continuous function f of two real variables, $f: R\times R\to R$, with the property that for any continuous $g: R\to R$, there exists a real t such that $g(x)=f(t, x)$ for all x.

Does such f exist if R is replaced by $[0, 1]$?

5645. *Proposed by Henry Guggenheimer, Polytechnic Institute of Brooklyn, N. Y.*

Given $y'=f(y)+p(x)$, with f Lipschitzian, p continuous and periodic of period T, assume sgn $f(y)$ sgn $y<0$ for $|y|>a$ and $|f(y)|>\max|p(x)|$ for $|y|>a$.

Prove there exists a periodic solution $y(x)$ of period T.

5646. *Proposed by Henry Guggenheimer, Polytechnic Institute of Brooklyn, N. Y.*

If

$$\int_{-\pi/4}^{\pi/4} f(\theta)d\theta = \int_{-\pi/4}^{\pi/4} f(\theta)\left[1+\frac{1}{\cos 2\theta}\right]^{1/2} d\theta = \int_{-\pi/4}^{\pi/4} f(\theta)\left[\frac{1}{\cos 2\theta}-1\right]^{1/2} d\theta = 0,$$

then $f(\theta)$ changes sign at least three times in $(-\pi/4, \pi/4)$, and five times if, in addition

$$\int_{-\pi/4}^{\pi/4} f(\theta)\begin{Bmatrix}\sin 2\theta\\ \cos 2\theta\end{Bmatrix} d\theta = 0.$$

5647. *Proposed by Pascual Florente, Universidad Nacional de Ingenieria, Lima, Peru*

Let A be a Euclidean domain, and let $A[x]$ be the polynomial ring in one variable over A. Prove that for every ideal $\mathfrak{a}\subset A[x]$ there is an ideal $\mathfrak{b}\subset A[x]$ such that $\mathfrak{b}$ admits a system of generators of at most 2 elements and $\sqrt{\mathfrak{b}}=\sqrt{\mathfrak{a}}$. (Here $\sqrt{\mathfrak{b}}$ denotes the radical of $\mathfrak{b}$.)

5648. *Proposed by John Shafer, University of California at Davis*

When is the set of all squares of elements in a finite group a subgroup? When is it a proper subgroup?

5649. *Proposed by Erwin Just, Bronx Community College, New York*

Let G be a group with order $p^n m$ in which p is a prime, $m<2p$ and $n>1$. Prove that G contains a normal subgroup whose order is either p^n or p^{n-1}.

5650. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur*

Let S be a set of points of the real x, y plane, such that the distance between any two of the points is rational. How big can S be when S lies (1) on a hyperbola, (2) on a parabola, (3) on a proper ellipse, and (4) when S is of the form $(x_1,+d)$, $(x_1,-d)$, $(x_2,+d)$, $(x_2,-d)$, $\cdots$?

SOLUTIONS OF ADVANCED PROBLEMS

$$\int_0^\infty \frac{dx}{A+Bx^n+Cx^{2n}}, \qquad A, B, C \geqq 0$$

5533 [1967, 1144]. *Proposed by D. S. Mitrinović, University of Belgrade Yugoslavia*

Let n be a natural number and a, b, c real numbers. Determine

$$I = \int_0^\infty \frac{1}{a^2+2b^2x^n+c^2x^{2n}}\,dx.$$

Solution by J. F. Georgatos, Stanford University. We take, without loss, a, b, $c \geqq 0$. Consider the integral

$$N(n, \phi) = \int_0^\infty \frac{1}{1+2x^n\cos\phi+x^{2n}}\,dx.$$

We will show that I is equal to

(i) $$\frac{1}{a^2}\sqrt[n]{\frac{a}{c}}\cdot N\left(n, \cos^{-1}\frac{b^2}{ac}\right) \qquad \text{if } b^2<ac,$$

(ii) $$\frac{1}{a^2}\sqrt[n]{\frac{a}{c}}\cdot \lim_{\phi\to 0} N(n, \phi) \qquad \text{if } b^2=ac,$$

(iii) $$\frac{1}{a^2}\sqrt[n]{\frac{a}{c}}\cdot K(p, q)\cdot N\left(\frac{n}{2}, \frac{\pi}{2}\right) \qquad \text{if } b^2>ac,$$

where

$$K(p, q) = \frac{p^{(1-n)/n}-q^{(1-n)/n}}{q-p}; \qquad \begin{aligned} p &= (b^2+\sqrt{b^4-a^2c^2})/ac, \\ q &= (b^2-\sqrt{b^4-a^2c^2})/ac. \end{aligned}$$

From the calculus of residues we find

$$(1) \qquad N(n, \phi) = \frac{\pi/n}{\sin(\pi/n)} \frac{\sin\left(\frac{n-1}{n}\phi\right)}{\sin \phi}.$$

The substitution $x=\sqrt[n]{a/c}\cdot y$ reduces I to the form

$$(2) \qquad I = \frac{1}{a^2}\sqrt[n]{\frac{a}{c}}\int_0^\infty \frac{dy}{1+2(b^2/ac)y^n+y^{2n}}.$$

If $b^2<ac$ we may take $\cos\phi=b^2/ac$ whence (i) is proved.

$b^2/ac=1$ is equivalent to $\phi=0$, and we note that $\lim_{\phi\to 0} N(n,\phi)$ exists. In fact from (1) the limit is

$$\frac{\pi/n}{\sin(\pi/n)}\left(\frac{n-1}{n}\right),$$

which establishes (ii).

Suppose $b^2>ac$ and take p and q as given above. We refer to (2) and write

$$\begin{aligned}\int_0^\infty \frac{dy}{1+2(b^2/ac)y^n+y^{2n}} &= \int_0^\infty \frac{dy}{(y^n+p)(y^n+q)} \\ &= \frac{1}{q-p}\left[\int_0^\infty \frac{dy}{y^n+p} - \int_0^\infty \frac{dy}{y^n+q}\right] \\ &= \frac{p^{(1-n)/n}-q^{(1-n)/n}}{q-p}\int_0^\infty \frac{dt}{t^n+1},\end{aligned}$$

since we have $p\neq q$ and $p, q>0$. Now $\int_0^\infty (t^n+1)^{-1}dt=N(n/2, \pi/2)$ completes the proof of (iii).

Also solved by Chang Chao-Ching (Taiwan), Robert Desko, D. Ž. Djoković, L. F. Epstein, S. Fempl (Yugoslavia), M. L. Glasser & V. E. Wood, Emil Grosswald, J. E. Mann, Jr., K. R. Penrose, G. W. Petrie, and Jonathan Ryshpan.

Shafer and Glasser-Wood use Formula 7, Section 6.2 in volume 1 of Erdelyi, *Table of Integral Transforms*, which also yields

$$\int_0^\infty \frac{x^m dx}{\alpha+\beta x^n+\gamma x^{2n}}.$$

Complete Sets of Orthogonal Latin Squares

5550 [1968, 83]. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Let $\mathfrak{A}$ be a complete set of $n-1$ orthogonal Latin squares of order n based on the symbols $1, 2, \cdots, n$. Denote by λ_{ij} the cell in the ith row and jth column of an $n\times n$ Latin square. Prove that if $i\neq p$ and $j\neq q$ then the cells λ_{ij} and λ_{pq} are occupied by the same symbol in exactly one of the Latin squares belonging to $\mathfrak{A}$.

Solution by A. G. Aldridge and S. J. Pierce, University of California, Santa Barbara. By relabeling the elements of the Latin squares we may assume that the ith row has the same ordering in each square, and that the (i, j) entry in each square is 1. Now the (p, q) entry cannot be equal to the (i, q) entry in any square, but the (p, q) entries must all be different in order to preserve orthogonality. Hence the (p, q) entry must be 1 in exactly one of the squares.

Also solved by T. L. Bartlow, Michael Goodman, Lionel Humblot (France), Stephen Tice, J. H. van Lint (Netherlands), and the proposer.

Boundary Conditions for Measure Preserving Transformations

5557 [1968, 84]. *Proposed by P. R. Chernoff, University of California, Berkeley*

Let E and F be measurable subsets of the unit interval having equal Lebesgue measures. Show that there is an essentially one-to-one measure preserving transformation T of the interval such that, up to sets of measure zero, $T(E)=F$.

Solution by the proposer. It suffices to construct an essentially one-to-one measure preserving transformation $T: E \rightarrow [0, \mu(E)]$ (and then extend this with a similar transformation on the complementary set). We define T for $x \in E$ by $T(x)=\int_0^x f(x)dx$, where f is the characteristic function of E. Let K be the union of the intervals of constancy of the weakly increasing function T. T is one-to-one on $E-K$ and $\mu(E-K)=\mu(E)$. Further, T is measure preserving because $T'(x)=1$ for almost all x in E by the Lebesgue differentiation theorem.

Also solved by D. A. Hejhal.

Irreducible Polynomial

5561 [1968, 197]. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let $f(x)$ be an irreducible (normal) polynomial over a field R of characteristic zero and $\psi(x) \in R[x]$. Prove or disprove: if $f(\psi(x))=f(x)h(x)$, then $h(x)$ is irreducible (normal).

Solution by D. A. Hejhal, University of Chicago. It does not follow that $h(x)$ is necessarily irreducible over R. Let $\omega=-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, let Q be the rationals and let $R=Q(\omega)$. A basis for R over Q is readily checked to be $\{1, \omega\}$ and we also have $R=\{a+bi\sqrt{3} \mid a, b \in Q\}$.

Now set $f(x)=x^2+1$ and $\psi(x)=x^3$. $f(\psi(x))=x^6+1$, but $x^6+1=(x^2+1)\cdot(x^4-x^2+1)$. Let $h(x)=x^4-x^2+1$. However, $x^4-x^2+1=(x^2+\omega)(x^2+\omega^2)$. Thus, $h(x)$ is reducible over R.

Related Sequences

5562 [1968, 187]. *Proposed by Lawrence Kuipers, Delft, Netherlands*

Two sequences $P(m, n)$ and $Q(m, n)$ are defined as follows (m, n are integers).

$P(m, 0)=1$ for $m\geqq 0$, $P(0, n)=0$ for $n\geqq 1$, $P(m, n)=0$ for $m, n<0$. $P(m, n) = \sum_{j=0}^{n} P(m-1, j)$ for $m\geqq 1$.

$Q(m, n)=P(m-1, n)+P(m-1, n-1)+P(m-1, n-2)$ for $m\geqq 1$. Express $Q(m, n)$ in terms of m and n for $m\geqq 1$.

Solution by E. W. Trost, Technikum Winterthur, Switzerland. We have $P(1, n)=1=\binom{n}{0}$, $P(2, n)=n+1=\binom{n+1}{1}$, $n\geqq 0$. Using the familiar formula

$$(1)\qquad \binom{i}{i}+\binom{i+1}{i}+\cdots+\binom{i+k}{i}=\binom{i+k+1}{i+1},$$

we get by complete induction

$$(2)\qquad P(m, n) = \binom{m-1+n}{m-1}.$$

For a slight extension of the problem we put

$$Q^*(q, n, s) = \sum_{j=0}^{s} P(q, n-j), \qquad s \leqq n.$$

We obtain for $q\geqq 0$,

$$Q^*(q, n, s) = \binom{q+n}{q} - \binom{q+n-s-1}{q}.$$

The desired special case follows from $Q(m, n)=Q^*(m-1, n, 2)$ and is

$$Q(m, n) = \binom{m+n-1}{m-1} - \binom{m+n-4}{m-1}.$$

Also solved by Einar Andresen (Norway), Bernard August, R. G. Buschman, L. Carlitz, C. A. Church, Jr., Ted Cullen, D. A. Hejhal, Robert Heller, Eric Langford, Douglas Lind, Bohuslav Míšek (Czechoslovakia), Michael Skalsky, David Zeitlin, and the proposer.

Refinements of the Cauchy Inequality

5563 [1968, 198]. *Proposed by D. E. Daykin and C. J. Eliezer, University of Malaya, Kuala Lumpur*

For which positive functions $f(x, y)$, $g(x, y)$ do we have

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leqq \sum_{i=1}^{n} f(a_i, b_i) \sum_{i=1}^{n} g(a_i, b_i) \leqq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,$$

for all real a_i, b_i?

Solution by L. Carlitz, Duke University. There will be no loss in taking a_i, $b_i>0$. For $n=1$, the inequalities reduce to

$$(ab)^2 \leqq f(a, b)g(a, b) \leqq a^2b^2,$$

so that one necessary condition becomes

$$f(a, b)g(a, b) = a^2b^2. \tag{1}$$

For $n=2$ we have

$$\begin{aligned}(a_1b_1 + a_2b_2)^2 &\leqq [f(a_1, b_1) + f(a_2, b_2)][g(a_1, b_1) + g(a_2, b_2)]\\ &\leqq (a_1^2 + a_2^2)(b_1^2 + b_2^2).\end{aligned}$$

This may be written

$$\begin{aligned}2a_1b_1a_2b_2 &\leqq f(a_1, b_1)g(a_2, b_2) + f(a_2, b_2)g(a_1, b_1)\\ &\leqq a_1^2b_2^2 + a_2^2b_1^2\end{aligned}$$

and using (1) to eliminate g, we obtain

$$2 \leqq \frac{f(a_1, b_1)}{f(a_2, b_2)} \frac{a_2b_2}{a_1b_1} + \frac{f(a_2, b_2)}{f(a_1, b_1)} \frac{a_1b_1}{a_2b_2} \leqq \frac{a_1b_2}{a_2b_1} + \frac{a_2b_1}{a_1b_2}. \tag{2}$$

In (2) replace a_1, b_1 by a, b and a_2, b_2 by λa, λb, respectively. Then (2) becomes

$$2 \leqq \frac{f(a, b)}{f(\lambda a, \lambda b)} \lambda^2 + \frac{f(\lambda a, \lambda b)}{f(a, b)} \lambda^{-2} \leqq 2,$$

so that

$$f(\lambda a, \lambda b) = \lambda^2 f(a, b). \tag{3}$$

If we set $f(a) = f(a, 1)$ we may then write

$$f(a, b) = b^2 f(a/b), \tag{4}$$

and we may put (2) in the equivalent form

$$2 \leqq \frac{f(a)/a}{f(b)/b} + \frac{f(b)/b}{f(a)/a} \leqq \frac{a}{b} + \frac{b}{a}. \tag{5}$$

If $a \geqq b$ it follows from (5) that

$$\frac{f(a)b}{f(b)a} \leqq \frac{a}{b}, \qquad \frac{f(b)a}{f(a)b} \leqq \frac{a}{b},$$

that is

$$f(b) \leqq f(a), \qquad \frac{f(a)}{a^2} \leqq \frac{f(b)}{b^2}, \qquad (a \geqq b). \tag{6}$$

We shall now show that if $f(a)$ satisfies (6) and $f(a, b)$ and $g(a, b)$ are defined through (4) and (1), then the stated inequalities hold. An equivalent formulation now is

$$2\sum a_ib_ia_jb_j \leqq \sum \left[f(a_i, b_i)g(a_j, b_j) + f(a_j, b_j)g(a_i, b_i)\right]$$
$$\leqq \sum (a_i^2b_j^2 + a_j^2b_i^2),$$

(where the summations are taken over all i, j, satisfying $1 \leqq i < j \leqq n$) which follows immediately from an application of (2).

Also solved by Bruce Berndt.

Note. The proposers state that examples for f, g were observed in the literature, e.g., 1) $f(x, y) = x^2+y^2$, $g(x, y) = x^2y^2/(x^2+y^2)$, due to E. A. Milne, may be found in Hardy, Littlewood, Polya, *Inequalities;* 2) $f(x, y) = x^{1+\alpha}y^{1-\alpha}$, $g(x, y) = x^{1-\alpha}y^{1+\alpha}$, given by D. K. Callebaut, *Generalizations of the Cauchy-Schwarz Inequality*, Journal for Mathematical Analysis and Applications, (1965) 491, ff. These produce an intermediate value in the Cauchy inequality proceeding from left to right as α increases from 0 to 1.

Rings of Ordered Pairs

5564 [1968, 198]. *Proposed by R. B. Killgrove, California State College, Los Angeles*

Consider the set S of ordered pairs of reals. Define $\oplus$, $\otimes$ on elements of S as follows ($+$ is the usual addition for reals):

$$(a, b) \oplus (c, d) = (a + c, b + d),$$
$$(a, b) \otimes (c, d) = (f(a, b, c, d), g(a, b, c, d)).$$

Let f, g be continuous and the system $\mathcal{S}(S, \oplus, \otimes)$ be a, possibly nonassociative, ring with identity (1, 0); i.e., under $\oplus$, S forms an abelian group and both distributive laws hold and a specified element is identity for $\otimes$. Certainly the complex numbers form such an $\mathcal{S}$. Characterize all other systems $\mathcal{S}$.

Solution by G. A. Heuer, Concordia College. We may define $(0, 1)\otimes(0, 1)$ arbitrarily, but having done so, all products are determined. For, let $(0, 1)^2 = (r, s)$. From distributivity and the fact that (1, 0) is an identity we see that $(a, 0)\otimes(c, d) = (ac, ad)$ for all rational a, and by continuity the same is true for all real a. Similarly, $(a, b)\otimes(c, 0) = (ac, bc)$, and $(0, b)\otimes(0, d) = (bdr, bds)$. Thus $(a, b) \otimes (c, d) = (a, 0) \otimes (c, d) + (0, b) \otimes (c, 0) + (0, b) \otimes (0, d) = (ac+bdr, ad+bc+bds)$. Conversely, it is immediate that $\otimes$ defined in this way satisfies the identity and distributivity requirements.

It follows that for any r and s the system obtained is an associative commutative ring. Indeed, it is isomorphic to the ring $R[x]/(x^2-sx-r)$, where R is the real number field, under the mapping $(a, b)\to[a+bx]$. Thus, in case $s^2+4r<0$, we have a field isomorphic to the complex numbers; in all other cases there are zero divisors.

Also solved by P. R. Chernoff, D. Ž. Djoković, M. G. Greening (Australia), D. A. Hejhal, and the proposer.

Chernoff notes that there are exactly three isomorphic classes of systems $\mathcal{S}$. If $\Delta > 0(\Delta = s^2+4r)$,

$\mathcal{S}$ contains no nonzero nilpotent elements and is generated by an element $\zeta \notin \{(a, 0)\}$ such that $\zeta^2 = 1$. If $\Delta = 0$, $\mathcal{S}$ is generated by an element $\eta \notin \{(a, 0)\}$, $\eta^2 = 0$.

A Sequence of Polynomials Converging to $f(x)$ in C

5565 [1968, 198]. *Proposed by D. A. Hejhal, University of Chicago.*

Let $f(x)$ be real and continuous over $[0, 1]$ with $f(0) = 0$. Then there exists a sequence of real polynomials $\{A_n(x)\}$, $A_n(x) = \sum_{k=0}^{\infty} a_{k,n}x^k$ (of course, for each n, $a_{k,n} = 0$ eventually), such that $A_n(x) \to f(x)$ uniformly over $[0, 1]$ as $n \to \infty$, but $\lim_{n\to\infty} a_{k,n} = 0$, $k = 0, 1, \cdots$.

Solution by R. J. Driscoll, Loyola University. For each natural number n, choose $c_n \in (0, 1)$ such that $|f(x)| < 1/n$ for $x \in [0, c_n]$. Let g be the continuous function on $[0, 1]$ which is constant on $[0, c_n]$ and for which $g(x) = x^{-n}f(x)$ for $x \in [c_n, 1]$. Choose a polynomial P_n such that $|P_n(x) - g(x)| \leq 1/n$ for $x \in [0, 1]$; then with $A_n(x) = x^n P_n(x)$ we find $|A_n(x) - f(x)| \leq 3/n$ for $x \in [0, 1]$.

Also solved by I. N. Baker (England), P. R. Chernoff, R. A. Christiansen, M. A. Ettrick, D. A. Herrero, Douglas Lind, O. P. Lossers (Netherlands), M. D. Mavinkurve (India), Dick Woodward, and the proposer.

Other solutions obtain the result as an immediate consequence of the Stone-Weierstrass theorem or of the theorem of Szasz-Müntz which imply the closure, for all k, of $\{x^n\}$, $n = k$, $k+1, \cdots$ in $C[0, 1]$.

A Pair of Nonhomeomorphic Spaces

5566, [1968, 198]. *Proposed by Forrest Dristy, Clarkson College of Technology*

Let X be the closed interval $[0, 1]$ with the topology consisting of the empty set $\varnothing$, X, and all subsets of X of the form $[0, a)$ where $0 < a \leq 1$. Let Y be the space $\{0, 1\}$ with the topology $\{\varnothing, \{0\}, Y\}$, and let Z be the cartesian product of denumerably many copies of Y. Prove or disprove that X and Z are homeomorphic.

Solution by Linda Wells, Wayne State University. The spaces X and Z are not homeomorphic. Let $p_i: Z \to Y$ be the ith coordinate projection. Then $p_1^{-1}(\{0\})$ and $p_2^{-1}(\{0\})$ are neighborhoods in Z such that neither one contains the other. But, given any two neighborhoods in X, one will contain the other.

Also solved by Yu-hua Ai, Einar Andresen (Norway), P. R. Chernoff, R. A. Christiansen, M. A. Ettrick, William Fox, M. A. Grajek, D. A. Hejhal, H. D. Keesing, Eric Langford, Dan Marcus, M. D. Mavinkurve (India), P. R. Meyer, Kenneth Miller, J. C. Morgan, Jr., P. A. Pittas, Daniel Putnam, David Ryeburn, William Smythe, L. E. Ward, Jr., W. J. Woan, and the proposer.

On Primes of the Form $ix^j + k$

5568 [1968, 199]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York City*

Let the domain of the function $f_{ijk}(x) = ix^j + k$ consist of all integers ≥ 2. Denote the range of $f_{ijk}(x)$ by R_{ijk}. If p and m are arbitrary positive integers,

prove that there exists an infinite number of primes which are not contained in the set R, where $R=\bigcup_{i=1}^{p}\bigcup_{k=1}^{m}\bigcup_{j=2}^{\infty} R_{ijk}$.

Solution by Bruce Berndt, University of Illinois. We have

$$\sum_{i=1}^{p}\sum_{k=1}^{m}\sum_{j=2}^{\infty}\sum_{x=2}^{\infty}(ix^j+k)^{-1} \leqq mp\sum_{j=2}^{\infty}\sum_{x=2}^{\infty}x^{-j} = mp\sum_{x=2}^{\infty}\frac{x^{-2}}{1-x^{-1}} \leqq 2mp\zeta(2).$$

Since the sum of the reciprocals of the primes diverges, the set R must omit infinitely many of them.

Also solved by Neal Felsinger, and by the proposers.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, Carleton College

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

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C *Modern Elementary Differential Equations.* By Richard Bellman. Addison-Wesley, Reading, Mass., 1968. xii+196 pp. $8.95.

A glance at this book will tempt many to adopt it as a text at the sophomore level for engineers and scientists. It departs from the traditional approach of a compilation of special techniques and introduces the reader to modern applications and those techniques most useful for scientists: linear differential equations, power series solutions, perturbation techniques, numerical solutions, and successive approximations.

This book was used by the reviewer in a one-semester course for sophomore engineers and scientists who had completed three semesters of calculus. The students were unable to grasp the point of many sections. Many important ideas were hidden in the problems; for example, the solution of the general first order linear equation and the method of finding a particular solution by variation of parameters. Neither the index nor the section headings would allow this book to be used for reference. The problems were quite interesting, but too involved for the students. It was necessary to supply extra problems at every

stage. The solution of partial differential equations by separation of variables was omitted entirely.

There are many unusual and good features to this book. Applications from diverse fields serve as good motivation for the reader. There is a short but valuable section on the direct use of the differential equation to obtain information about the solution. The behavior of the solutions of the second order linear equation is examined to gain an understanding of the related physical phenomena for electrical circuits. The chapter on power series is quite well done on the whole. Also, the inclusion of the chapter on numerical solutions has long been needed.

This book could be used as supplementary material or to introduce those with limited knowledge to modern techniques, but I do not recommend its use as an introductory text.

JUDITH M. ELKINS, Rutgers, The State University

L'enseignment de la Géométrie. By Gustave Choquet. Hermann, Paris, 1964. 178 pp. 33 F.

C *Algèbre linéaire et géométrie élémentaire.* By Jean Dieudonné. Hermann, Paris, 1964. 233 pp. 36 F.

These two books are both addressed to the mathematics teachers of the secondary schools and teacher training colleges in France. They are of at least equal interest and potential value to all those people everywhere who are concerned about the secondary school and undergraduate mathematics program. Both books are eloquent pleas for shifting the emphasis in school geometry to vector properties and algebraic methods. Neither book is in a form suitable for general use as a textbook in secondary schools but each provides for the teacher an admirably lucid and detailed exposition of the mathematical content of a new course in geometry for secondary school students. Both represent a sharp break from the traditional school geometry program and both are also significantly different from other new mathematics programs familiar to this reviewer.

While the aims and the point of view are very much alike the two books are nevertheless fundamentally different; they are two quite distinct solutions of the same problem. Choquet has assumed that an axiomatization of the euclidean plane and 3-space is needed which is suitable to serve as the framework on which to build something to replace the traditional geometry program which begins in the second year of the French secondary school (roughly equivalent to 7th grade in American schools). Accordingly his axioms, based on incidence, order, parallelism, perpendicularity and distance, have been chosen to be readily interpretable in terms of physical space and to be intuitively obvious. He has also succeeded however in making the choice in such a way that vector properties are introduced very early and methods of modern algebra are utilized almost from the beginning.

Dieudonné on the other hand argues that a formal axiomatic structure is

neither necessary nor desirable for introducing young students to mathematical proof. He would have the student first study geometry as a sort of physics of space in which the role of propositions is to exhibit relations among the properties of space of the form: if some properties are assumed then some other property is logically deducible from them. In this study emphasis can and should be placed on translations, symmetries, rotations and similitudes as mappings of the entire space onto itself. With a background of this sort the student would be able to understand and appreciate the need for an axiomatization of geometry. Moreover the axioms for a real vector space with a scalar product would then be interpretable for him as familiar experimentally verifiable properties of physical space. This is by far the most appropriate axiomatic system to use for introducing students to the contemporary conception of an abstract mathematical system. It is simple and it is in the main stream of contemporary mathematical thought and applications.

Accordingly Dieudonné's course is intended for the last two or three years of the French secondary school, roughly equivalent to the last year of an ordinary American high school and the first two years of the usual American college. It is intended to be given concurrently with a beginning course in calculus. The course is a development of elementary euclidean geometry of the plane and 3-space from the axioms for the real numbers and those for a real vector space with a scalar product.

The reviewer has twice taught a course based on the Dieudonné book, once to a class of very bright 16 year old high school students and once to an ordinary class of undergraduate prospective secondary school mathematics teachers at Purdue. This experience has reinforced the reviewer's highly favorable opinion of the book which, it probably ought to be noted, contrasts sharply with the views expressed by Professor Freudenthal in his review in this MONTHLY, 74 (1967) 744–8.

Taken together these two books present very persuasive arguments for using the vector space approach to geometry as a guide in efforts to improve the school mathematics program. They deserve the serious study of everyone interested in the problem.

G. N. WOLLAN, Purdue University

C *Advanced Calculus, An Introduction to Applied Mathematics, Vol.* 2. By Arthur E. Danese. Allyn and Bacon, Boston, 1965. 373 pp. $8.25. (Telegraphic Review, Feb. 1967)

This book, together with Volume One of the same title, contains most of the standard topics usually found in applied advanced calculus texts. The first part (Part V of the two volumes) is a thorough introduction to variational calculus and contains an interesting characterization of Sturm-Liouville systems as variational problems. Part VI deals with various integral transforms, the Dirac delta function via generalized functions and the general theory of operators. The inclusion of generalized functions and operators is unusual in a

junior-senior text, but they are treated at the appropriate level. Part VII contains a discussion of Laplace's equation, the heat equation and the wave equation.

Every chapter has a wealth of problems of various degrees of difficulty. This is an outstanding feature of both volumes. The reviewer used both volumes for a year course in applied analysis and obtained satisfactory results by covering a selection of topics.

ROBERT M. BULLOCK, Miami University

A Background (Natural, Synthetic and Algebraic) to Geometry. By T. G. Room. Cambridge Univ. Press, New York, 1968. viii+342 pp. $12.50. (Telegraphic Review, Aug. 1968)

The book's main theme is euclidean geometry, but in the course of this theme's development there appears substantial discussion of finite geometries and of fields. The inclusion of these latter topics is accomplished in a natural fashion, and the overall impression one obtains is of an orderly, interesting and rather novel introduction to college-level geometry. Yet, for the simple reason that the book is difficult to read—and unneccessarily so—it seems to this reviewer to be a less than satisfactory work. The problem arises from the generally sophisticated tone, the occasional lack of clarity in discussion, and the extensive use of an elaborate and unconventional symbolic language. It is especially this last feature which severely limits the book as a reference and significantly limits it as an undergraduate text.

G. P. JOHNSON, Oakland University

Elementary Topology. By Michael C. Gemignani. Addison-Wesley, Reading, Mass., 1967. xi+258 pp. $9.75. (Telegraphic Review, Jan. 1968)

The author presents ideas in elementary topology effectively by using either a geometric or an analytic approach whenever the concept is primarily geometric or analytic in nature. The book is written for the undergraduate or beginning graduate student who has had as little as three semesters of calculus, though a course in real analysis is recommended for deeper appreciation.

As stated in the preface the author has tried to motivate the concepts introduced by referring to their historical origin. In this respect the text differs from a number of others which have been published recently in the same field. Another topic, not usually discussed in an elementary topology text, is that of homotopy theory, which the author introduces in his last chapter. Chapters 1 through 10 cover the following concepts: metric and topological spaces, topologies, continuity, convergence, compactness and connectedness, and metrizability. A number of selected exercises and references are contained in every chapter. An appendix on infinite products and an index of symbols concludes the text.

DAGMAR HENNEY, George Washington University

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)–18(second graduate year). A boldface star (★) marks a notable book of general interest.

Algebra

Lattice Theory. By Thomas Donnellan. Pergamon, New York, 1968. xii+283 pp. $6.00 (cloth) $4.50 (paper). An elementary account designed to give students of mathematics and applied fields "easy access to a vast abstract theory." Included are chapters on modular and distributive lattices. T (13–15), TT, S, P, L.

The Elementary Theory of Numbers, Polynomials, and Rational Functions. By W. Eames (Lakehead Univ., Ontario, Canada). American Elsevier, New York, 1967. viii+143 pp. $6.50. The integers from the Dedekind-Peano axioms, a few pages of number theory, rational, real, and complex numbers, some algebraic structures, polynomials (as sequences of ring elements), and rational functions. T (14).

Introduction to Homological Algebra. By Sze-Tsen Hu (Univ. of California, L. A.). Holden-Day, San Francisco, Calif., 1968. ix+203 pp. $10.50. This is the eleventh book by Professor Hu that has appeared in the last nine years. It is addressed to the nonspecialist. T (17), P, L.

Ergodic Properties of Algebraic Fields. By Yu. V. Linnik. Translated by M. S. Keane. Springer-Verlag, New York, 1968. ix+192 pp. $11.00. *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Band 45. P.

Elements of Linear Algebra and Matrix Theory. By John T. Moore (Univ. of Florida). McGraw-Hill, New York, 1968. xii+370 pp. $8.95. For a one-term course. Chapters are on finite-dimensional vector spaces, linear transformations and matrices, determinants and systems of linear equations, innerproduct spaces, bilinear and quadratic forms, similarity and normal operators. T (14).

Linear Algebra. By Walter Nef (Univ. of Berne, Switzerland). Translated from the German by J. C. Ault (Univ. of Leicester). McGraw-Hill, New York, 1968. x+304 pp. $12.00. The treatment is confined to real and complex vector spaces, and modern applications are treated including linear programming, Chebyshev approximation, game theory. There are chapters on Linear functionals, Forms of the second degree, Euclidean and unitary vector spaces, Eigenvalues and eigenvectors of endomorphisms of a vector space, and Invariant subspaces and canonical forms of matrices. Based on lectures given to second year classes containing both mathematical specialists and students in the natural and social sciences. T (14).

An Introduction to the Theory of Groups. By George W. Polites (Illinois Wesleyan Univ.). International Textbook Co., Scranton, Penn., 1968. viii+80 pp. $1.75 (paper). Written for group or individual study by honors students, it might also be used for a special-topics course or mathematics seminar, or even for a short advanced undergraduate course. It begins with the definition of a group and ends with Galois theory. T, S.

An Introduction to Algebraic Structures. By Azriel Rosenfeld (Univ. of Maryland). Holden-Day, San Francisco, 1968. xi+285 pp. $12.50. Chapters are Sets, functions, and numbers; Ordered sets, lattices; Quotient sets, product sets, and cardinal numbers; Groupoids, semigroups and groups; Subgroups, factor groups, products of groups; Finiteness conditions on groups; Abelian groups; Rings; Vector spaces;

Finiteness conditions on rings. Though logically self contained, the book presumes "some degree of prior exposure to algebraic structures." The author states two central themes: first, the separation and development in a broad context of the universal-algebra parts of group theory: second, the unity achievable by developing the theories in relation to finiteness conditions. T (15–16), P, L.

Abelian l-Adic Representations and Elliptic Curves. By Jean-Pierre Serre (Collège de France). McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute. Benjamin, New York, 1968. xvi+45 pp. $8.50 (cloth) $3.95 (paper). Presents the "idèletheoretic approach to abelian representations." P.

Tables of Indices and Primitive Roots. By A. E. Western and J. C. P. Miller. Royal Society Mathematical Tables, Vol. 9. Cambridge Univ. Press, New York, 1968. liv+384 pp. $18.50. This revision and extension of previous tables, based on manuscripts left by the late Lt.-Col. Allan Cunningham, gives indices, residue-indices, and primitive roots over an "enormously" extended range of the modulus. P, L.

Matrices with Applications. By Hugh G. Campbell (Virginia Polytechnic Inst.). Appleton Century-Crofts, New York, 1968. xi+184 pp. $2.95 (paper). This paperback at a paperback price (!) gives an elementary treatment of matrix operations, systems of linear equations, linear transformations, and characteristic values. It should be useful as a supplement in the elementary calculus sequence and in courses in many fields outside of mathematics. S, TT.

Analysis

Boundary Value Problems for Second Order Elliptic Equations. By A. V. Bitsadze (Institut Matematiki, Novosibirsk, USSR). Translation editor M. J. Laird (King's College, London). Wiley, New York, 1968. 211 pp. $11.00. The original (1966) grew out of lectures on non-Fredholm elliptic boundary value problems. P.

Hilbert Spaces of Entire Functions. By Louis de Branges (Purdue Univ.). Prentice-Hall, Englewood Cliffs, N. J., 1968. ix+326 pp. $11.00. Written on a Sloan Foundation Fellowship (1963–1966), this book presents a theory of eigenfunction expansions corresponding to the physical problem of determining nuclear forces from scattering data. It begins with an exposition of the classical theory of entire functions and includes recent and previously unpublished results, an algebraic foundation for a general invariant subspace theory, and the elements of an axiomatic theory of special functions. T (18), P, L.

Chebyshev Polynomials in Numerical Analysis. By L. Fox and I. B. Parker. Oxford Univ. Press, New York, 1968. ix+205 pp. $6.75. This Oxford Mathematical Handbook collects the basic information about the theory that has developed on the polynomials discovered about a century ago by Chebyshev, promoted as a tool of numerical analysis by Lanczos about thirty years ago, and widely used today in numerical computations. Chapters are Approximation, Minimax and least squares theories, Fourier and Chebyshev series, Discrete least square approximation, Practical properties of Chebyshev polynomials and series, Chebyshev approximations for functions defined explicitly, Ordinary differential equations, and Problems in linear algebra. T (15–16), P, L.

Orthogonal Expansions and their Continuous Analogues. Proceedings of the Conference held at Southern Illinois University, Edwardsville, April 27–29, 1967. Edited by Deborah Tepper Haimo. Southern Illinois Univ. Press, Carbondale, and Fefer and Simons, London, 1968. xx+307 pp. $7.50. Twenty-two of the thirty papers presented at the Conference plus an outline of the history of orthogonal polynomials by G. Szegö and a biographical note on Mary Weiss by A. Zygmund. P, L.

Generalized Functions. By D. S. Jones (Queen's College, Dundee). McGraw-Hill, New York, 1966. xii+482 pp. $10.50. An elementary introduction based on a course for undergraduates that originally used Lighthill *An Introduction to Fourier Analysis and Generalized Functions.* Begins with a review of basic results of analysis and includes many applications. T (16–17), P.

An Introduction to Harmonic Analysis. By Yitzhak Katznelson (Hebrew University of Jerusalem). Wiley, New York, 1968. xiii+264 pp. $12.95. "Harmonic analysis is the study of objects (functions, measures, etc.) defined on topological groups. The group structure enters into the study by allowing the consideration of the translates of the object under study, that is, by placing the object in a translation-invariant space. The study consists of two steps. First: finding the "elementary components" of the object, that is, objects of the same or similar class which exhibit the simplest behavior under translation and which "belong" to the object under study (harmonic or spectral analysis); and second: finding a way in which the object can be construed as a combination of its elementary components (harmonic or spectral synthesis)." T (17–18), P, L.

Numerical Methods for Two-Point Boundary-Value Problems. By Herbert B. Keller (California Inst. of Tech.). Blaisdell, Waltham, Mass., 1968. viii+184 pp. $7.50. The techniques of initial value, finite-difference, and integral-equation methods are applied to nonlinear problems. T (16–17), P.

Foundations of Global Nonlinear Analysis. By Richard S. Palais (Brandeis Univ.). Benjamin, New York, 1968. vii+131 pp. $9.50 (cloth) $3.95 (paper). T (18) P.

Regular Matrix Transformation. By Gordon M. Petersen (Canterbury Univ., New Zealand). McGraw-Hill, New York, 1966. viii+142 pp. $8.00. In spite of the title, this book is primarily concerned with infinite sequences. P.

Ordinary Differential Equations and Stability Theory: An Introduction. By David A. Sânchez (Univ. of Calif., L. A.). Freeman, San Francisco, 1968. vi+164 pp. $3.95 (paper). Used as a supplement to existing courses in differential equations this paperback introduces modern concepts and methods related to stability theory. S (16–17).

Introduction to the Methods of Real Analysis. By Maurice Sion (Univ. of British Columbia). Holt, Rinehart and Winston, New York, 1968. x+134 pp. $8.95. Based on lectures given to seniors and first year graduate students "who are trying to pass from a sound but elementary and naive view . . . to the more sophisticated points of view of modern point set topology and measure theory," the book emphasizes participation in developing ideas and methods rather than the learning of theorems and works in a context that is more or less elementary and still recognizable by the student, rather than older techniques in an abstract setting. Part one on topological concepts deals with set theory, spaces of functions, point set topology, and continuous functions. Part two, on measure theory, covers measures on abstract spaces, Lebesgue-Stieltjes measures, integration, differentiation, and Riesz representation. The Caratheodory approach to measure is used. T (16–17), P, L.

W-Algebras.* By J. T. Schwartz (Courant Inst. of Math. Sci.). Gordon & Breach, New York, 1967. 256 pp. $13.50 (cloth) $7.95 (paper). This is one of the series *Notes on Mathematics and its Applications* edited by Schwartz and Maurice Levy. The idea of getting out lecture notes quickly without "the labor required" to bring them "up to the level of perfection which authors and public demand of formally published books" is a good one in the times of rapid development, but it is hard to see the difference between this form of publication and a "formally published book." The printing,

binding, and writing style are not unusual, and the price is high. The only unusual feature seems to be the absence of an index. It would be too bad if this idea were used in practice to publish yet another series differing from others only in having a built-in excuse for poor work on the part of the author or publisher. P.

A Hypercomplex Function-Theory Associated with Laplace's Equation. By H. H. Snyder. VEB Deutscher Verlag der Wissenschaften, Berlin, 1968. 98 pp. 24. Mk. The term "hypercomplex" refers to a linear associative algebra over a field. P.

Tables of Summable Series and Integrals Involving Bessel Functions. By Albert D. Wheelon (Hughes Aircraft Co., Culver City, Calif.). Holden-Day, San Francisco, Calif., 1968. 125 pp. $8.50. P, L.

Applications

Selected Economic Models and Their Analysis. By A. R. Bergstrom (Univ. of Auckland). American Elsevier, New York, 1967. x+131 pp. $7.50. P.

Perturbation Methods in Applied Mathematics. By Julian D. Cole (Calif. Inst. of Technology). Blaisdell, Waltham, Mass., 1968. 260. pp. $9.50. The author's purpose is to survey perturbation methods, especially those related to differential equations, in order to show the general features. The point of view is applied, and underlying ideas rather than rigor are stressed. T (16–17), P.

The Theory of Rotating Fluids. By H. P. Greenspan (M.I.T.). Cambridge Univ. Press' 1968. xii+327 pp. $15.00. This is a "unified and comprehensive account" intended "to provide a basic foundation for the support and promotion of research." S, P.

Mathematical Methods in Operations Research. By Ronald L. Gue (Southern Methodist Univ.) and Michael E. Thomas (Univ. of Florida). Macmillan, New York, 1968. xi+385 pp. $12.95. Classical optimization techniques, linear programming, nonlinear programming, dynamic programming, the maximum principle, theory of queues, decisions and games, graphs and networks, and appendices on matrix algebra and z-transforms. For students with solid mathematical background. T (15–17), P, L.

Stochastic Optimization and Control. Proceedings of an Advanced Seminar Conducted by the Mathematics Research Center and the United States Army at the University of Wisconsin, Madison, October 2–4, 1967. Edited by Herman F. Karreman. Wiley, New York, 1968. xii+217 pp. $7.95. P.

Ordinary Differential Equations. By E. R. Lapwood (Cambridge Univ.). Pergamon, New York, 1968. xi+207 pp. $9.00. Volume 1 of Topic 1 of the *International Encyclopedia of Physical Chemistry and Chemical Physics* which is to consist of about 100 volumes divided into about 20 topics, each a fairly independent treatment for the graduate and research worker in chemistry. This volume is elementary and traditional. S, P.

The Mathematical Principles of Quantum Mechanics. By Derek F. Lawden (Univ. of Canterbury, New Zealand). Methuen, London, 1967. Distributed by Barnes and Noble, in U.S.A. xiv+280 pp. $8.00. Designed for the person with "special qualifications in applied mathematics and natural philosophy" rather than for the physicist. P.

Introduction to Demography. Revised edition. By Mortimer Spiegelman. Harvard Univ. Press, Cambridge, Mass., 1968. xix+514 pp. $15.00. The first edition was published by the Society of Actuaries and the book remains of interest to people of this profession. S, P.

An Introduction to Optimal Control Theory. By Aaron Strauss (Univ. of Maryland). Springer-Verlag, New York, 1968. iv+153 pp. $3.50 (paper). This is number 3 of *Lecture Notes in Operations Research and Mathematical Economics,* edited by M. Beckmann and H. P. Kunzi. These notes are published photographically from typed manuscripts supplied by the author and are intended to avoid publication delays by producing semifinished work. This instance is intended for beginners and is not a survey of current research. S.

Computers, etc.

The Numerical Solution of Equations. By A. Balfour and A. J. McTernan (both of Heriot-Watt Univ., Edinburgh). Heinemann, London, 1967. v+85 pp. $1.80. A quick treatment of methods most suitable for computer use. S.

Introduction to an Algorithmic Language (Basic). National Council of Teachers of Mathematics, Washington, D. C. 1968. v+49 pp. $1.40 (paper). A very elementary manual with sample problems. P, TT.

An Introduction to Computer Programming with an Emphasis on Fortran IV. By Adolph C. Nydegger (College of St. Teresa). Addison-Wesley, Reading, Mass., 1968. ix+269 pp. $7.50. Developed from a service course at the College of St. Teresa, Winona, Minnesota, and tested on high school students, teachers, engineers, and accountants. T (13), S.

Tables of the Incomplete Beta-Function. Originally prepared under the direction of, and edited by, Karl Pearson. 2nd ed. with a new Introduction by E. S. Pearson and N. L. Johnson. Cambridge Univ. Press, New York, 1968. xxxii+505 pp. $15.50. A magnificent volume. But how long will it be before such volumes become of only archival interest? Problems of cost and difficulties of getting quick access to computers remain unsolved, but it is already true that computers can recalculate values more quickly than people can look in tables. L (?).

Elementary Numerical Methods. By R. E. Scraton (Northampton College of Advanced Tech., London). Heinemann, London, 1965. vii+80 pp. $1.25 (paper). A quick introduction for engineer or scientist. S.

The Computer Programmer's Dictionary and Handbook. By Donald D. Spencer (Abacus Computer Corp.). Blaisdell, Waltham, Mass., 1968. xi+244 pp. $3.75 (paper). A dictionary of 58 pages followed by 180 pages of appendices, including elementary mathematical information, numerical tables and much other information (e.g., a list of computers having FORTRAN compilers, a display of punch card codes, and a short bibliography). S, P, L.

Basic Digital Computer Concepts. By D. Whitworth. Heinemann, London, 1967. v+161 pp. $3.00. A popular simple introduction with a glossary including the atrocious word "zeroise". S.

Education

Computer-Assisted Instruction Guide. Entelek Inc., Newburyport, Mass., 1968. v+150 pp. $10.00. CAI is instruction in which the computer performs a teaching function. This guide enables the user to find existing programs on a given subject, by a given organization, written in a given computer language, or compatible with a given CAI system. P, L.

★*Educational Studies in Mathematics.* Edited by H. Freudenthal (Univ. of Utrecht). Published by D. Reidel Publ. Co., P.O. Box 17, Dortrecht-Holland. Price per vol. of four issues $22.50. Volume 1, No. 1/2, May 1968, consists of proceedings of the

Colloquium: How to teach mathematics so as to be useful, Utrecht, August 21–25, 1967. The editorial board is world-wide and distinguished. The contents are worthy of attention by every teacher of mathematics at whatever level. S, P, L.

★*Aspects of Undergraduate Training in the Mathematical Sciences.* By John Jewett and Clarence B. Lindquist. Conference Board of the Mathematical Sciences, 2100 Pennsylvania Avenue, Washington D. C. 20037. xvi+164 pp. $1.75 (paper). This is the first volume of the report of the CBMS Survey Committee chaired by Gail S. Young. It describes an extensive survey of undergraduate mathematical education and presents the results in two parts: four year institutions and two year colleges. The volume contains much valuable information on curriculum, enrollments, students, and faculty. Volume 2 will deal with graduate education and related problems of research and government support. Volume 3 will be devoted to the role of mathematical scientists in industry and government and questions of mathematical manpower. P, L.

Computer Facilities for Mathematics Instruction. National Council of Teachers of Mathematics, Washington, D. C. v+47 pp. 90¢. Prepared by the Computer-Oriented Mathematics Committee of the NCTM, this pamphlet describes the educational use of computers as devices for solving mathematical problems and for assisting the teacher in classroom and individual instruction. Some of the ideas may be of use at the college level. P, TT.

The Continuing Revolution in Mathematics. Edited by Warren C. Seyfert (Reprinted from the *Bulletin of the National Association of Secondary-School Principals* No. 327, April 1968), N.C.T.M. Washington, D. C., 1968. vi+166 pp. $2.00 (paper). Sixteen papers, including "Evolution in College Mathematics" by Bruce E. Meserve and "The Interface of Science and Mathematics" by Andrew Gleason. P, TT.

General

Mathematical Handbook for Scientists and Engineers. Definitions, Theorems, and Formulas for Reference and Review. Second, enlarged and revised edition. By Granino A. Korn (Univ. of Arizona) and Therese M. Korn. McGraw-Hill, New York, 1968. xvii+1130 pp. $25.00. A mass of well indexed and organized information followed by tables of integrals and numerical tables. Should be in mathematical libraries. P, L.

Enzyklopädie der Elementarmathematik. Edited by P. S. Alexandroff, A. I. Markuschewitsch and A. J. Chintschin. Vol. 1. Arithmetic, Vol. 2. Algebra. VEB Deutscher Verlag der Wissenschaften, Berlin, 1967. Vol. 1. xi+403 pp. 26.70 Mk. Vol. 2. ix+405 pp. 27.30 Mk. This distinguished Soviet encyclopedia of mathematics consists of seven volumes, the last five being on analysis, geometry (two volumes), various questions, and methodology and history. Though the title contains the word "elementary," the approach is highly cultured and mathematically mature. This entire encyclopedia should be translated into English. Meanwhile it should be in any serious mathematical library. L.

American Mathematical Society Translations. Series 2. Volume 70. "31 Lectures Delivered (8 in abstract) at the International Congress of Mathematicians in Moscow, 1966." A.M.S. Providence, R. I., 1968. v+256 pp. $13.60. P, L.

Geometry and Topology

An Introduction to Finite Projective Planes. By A. Adrian Albert (Univ. of Chicago), and Reuben Sandler (Univ. of Illinois). Holt, Rinehart and Winston, New

York, N. Y., 1968. viii+98 pp. \$5.50. A self contained treatment, including the necessary algebraic concepts, of the theory as it has developed over the past twenty years. T (13–15), S, P, L.

Elements of Modern Topology. By Ronald Brown (Univ. of Hull, England). McGraw-Hill, New York, 1968. xvi+351 pp. \$10.95. An introduction to point-set topology from a geometric point of view. It presupposes a basic course in analysis and set theory as well as familiarity with groups and vector spaces. The concepts of categories, functors, and groupoids play an important role. Contemporary material is included. T (16–17).

Finite Geometries. By P. Dembowski. *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Volume 44. Springer Verlag, New York, 1968. x+375 pp. \$17.00. The book appears to meet its stated purpose of providing "a reasonably complete account" of "an area of finite mathematics characterized by an interplay of combinatorial, geometric, and algebraic ideas." Topics include finite projective and affine geometries, statistical theory of designs, inversive planes, Hjelmslev planes and generalized polygons. There is an impressive bibliography of nearly 1200 titles, which suggests the intense activity in the field. S, P, L.

Mehrfarbenprobleme. By E. B. Dynkin and W. A. Uspenski. VEB Deutscher Verlag der Wissenschaften, Berlin, 1968. vii+65 pp. 5.10 Mk. (paper). This popular lecture on map coloring problems dates from the original Russian of 1952 and ought to be available in English. S, P, L.

Algebraic Topology. By Wolfgang Franz (Univ. of Frankfurt on Main). Translated from the German by Leo F. Boron, with the collaboration of Samuel D. Shore, James J. Andrews, Harry F. Joiner II, Robert C. Moore, Kyoshi Iseki. Frederick Ungar, New York, 1968. v+170 pp. \$6.50. This is the second volume on topology (of a series that appeared in the Göschen Sammlung) and presupposes the first volume or a basic knowledge of point-set topology, algebra and the theory of simplicial complexes. It is designed for the beginner rather than the specialist. Its four parts are entitled Simplicial complexes, Chain complexes and their applications, Cell complexes, Invariance and development of the theory. Geometric aspects are emphasized. T (17), P.

Geometry and Chronometry in Philosophical Perspective. By Adolf Grünbaum (Univ. of Pittsburgh). Univ. of Minnesota Press, Minneapolis, 1968. viii+378 pp. \$10.50 (cloth), \$3.45 (paper). The book begins with a reprint of a paper that first appeared in 1962 in Volume III of the Minnesota Studies in the Philosophy of Science edited by H. Feigl and G. Maxwell. There follow an expansion, commentary and a reply to a critique by Hilary Putnam. The author is currently president of the Philosophy of Science Association. S, P, L.

Regulares Parkettierungsproblem. By Heinrich Heesch. Westdeutscher Verlag, Koln und Opladen, 1968. 96 pp. \$3.50 (paper). A report of the session of a conference on tessalations of the plane by congruent figures, with a main paper of 74 pages, 14 pages of figures, a bibliography, summaries in English and French, and a report of comments and discussion. P.

Introduction to Topology. 2nd edition. Bert Mendelson (Smith College). Allyn and Bacon, Boston, Mass., 1968. vi+202 pp. \$8.50. Sets, metric spaces, topological spaces, connectedness, compactness. Intended for a one semester course. Changes from the first edition are minor but include an introductory section on categories and an explicit definition of the fundamental group. T (15).

Topics on Tournaments. John W. Moon (Univ. of Alberta). Holt, Rinehart and Winston, New York, 1968. viii+104 pp. $6.00. Tournament theory is a part of graph theory, which may be considered a topic in combinatorial topology, combinatorics or algebra. Like so many other topics of interest today it does not really fit at all into the traditional subject matter division of mathematics. This is a collection of various results scattered through the literature. P, L.

History

★*A History of Mathematics.* By Carl B. Boyer (Brooklyn College). Wiley, New York, 1968. xv+717 pp. $10.95. The first effort in many years to write a comprehensive history from earliest times to the present in a textbook form suitable for advanced undergraduates. It represents many years of work by a historian already well known for his many research papers and for his books on the history of calculus and analytic geometry. T (15–16), P, S, L.

★*Mathematics and Logic, Retrospect and Prospects.* By Mark Kac and Stanislaw M. Ulam. Praeger, New York, 1968. ix+170 pp. $5.95. "What is Mathematics? How was it created and who were and are the people creating and practising it? Can one describe its development and its role in the history of scientific thinking and can one predict its future? This book is an attempt to provide a few glimpses into the nature of such questions and the scope and the depth of the subject." These first sentences of the introduction and the reputation of the authors suggest a book of general and unusual interest. S, P, L.

Probability and Statistics

Introduction to Probability and Statistics. Fourth edition. Henry L. Alder and Edward B. Roessler (both of Univ. of California, Davis). Freeman, San Francisco, Calif., 1968. viii+333 pp. $7.00. For a "noncalculus" course with applications to natural and social sciences, agriculture, medicine, and business. In addition to minor improvements and the expansion of some chapters, the main thrust of this revision is an increased emphasis on probability. T (13).

Graphical Rational Patterns. A New Approach to Graphical Presentation of Statistics. By Roberto Bachi (Hebrew Univ., Jerusalem). Israel Univ. Press, Jerusalem. Distributed by Daniel Davey, Hartford, Conn. xvii+243 pp. $9.50. The title is descriptive. P.

Elements of Applied Probability Theory. By Petr Beckmann (Univ. of Colorado). Harcourt, Brace & World, New York, 1968. vi+189 pp. $4.50 (paper). Part of the Harbrace Series in Electrical Engineering, this paperback is an extract from the author's *Probability in Communication Engineering* (1967), omitting those parts of interest only to communication engineers. It begins with elementary probability theory and ends with Markov chains and queuing theory. T (14).

The Structure of Inference. By D. A. S. Fraser (Univ. of Toronto). Wiley, New York, 1968. x+344 pp. $16.00. A textbook on mathematical statistics presenting "a new and integrated theory of statistical inference" which "gives exact answers where traditional and contemporary methods have failed." T (15).

Topics in Regression Analysis. By Arthur S. Goldberger (Univ. of Wisconsin). Macmillan, New York, 1968. x+144 pp. $6.95. Based on lectures at the Center of Planning and Economic Research, Athens, Greece, 1965, this book is addressed to researchers already familiar with the subject. P.

Concepts of Probability. By William Guenther (Univ. of Wyoming). McGraw-Hill, New York, 1968. xi+384 pp. $8.50. Precalculus treatment with some statistical application and a chapter on Markov chains. T (13).

A Nonparametric Introduction to Statistics. By Charles H. Kraft and Constance Van Eeden (Univ. de Montreal). Macmillan, New York, 1968. x+342 pp. $9.95. By using nonparametric statistics, the authors are able to introduce modern ideas without becoming involved in complicated mathematics. The first part can be used as a one term introduction. The second describes particular tests and may be used for a second course. The third part contains extensive tables. An unusual book in a field overcrowded with books of little novelty. T (13–15).

Sampling Theory. By Des Raj (Sampling Expert, United Nations Program of Technical Assistance). McGraw-Hill, New York, 1968. 302 pp. $11.50. Intended as "a full and up to date account of sampling theory" for the "mathematically trained reader . . . more than fifty percent is new." P.

The Theory of Random Clumping. By S. A. Roach (London Sch. of Hygiene and Tropical Medicine). Methuen, London, 1968. Distributed by Barnes and Noble in U.S.A. vii+94 pp. $4.00. "When objects are scattered at random how many are hidden behind others and how many clumps of two or more will be formed? This class of problems occurs in such diverse fields as air pollution, astronomy, particle physics and bacterial counting." A systematic treatment of this relatively new and incomplete field.

NOTABLE PAPERS

Enquiry and International Colloquium on the Reform and Coordination of the Teaching of Mathematics and Physics, Lausanne, January, 1967. Special number of *Dialectica*, 21 (1–4), 1–404 (1967).

BOOKS BY PHOTOCOPY

With the cost per page of books steadily increasing and the cost of photocopy decreasing, it is now frequently the case that it is cheaper to buy a book reproduced by photocopy than by printing. This fact strongly suggests the need for revision of price policy on the part of some publishers, even though their books are protected by copyright from commercial photocopy. On the other hand, for books that are out of print, commercial photocopying is often the best way of getting a personal or library copy. At least two companies offer this service. University Microfilms (a subsidiary of Xerox Company, 300 N. Zeeb Road, Ann Arbor, Michigan 48103) offers paperbound xerographic copies at 4¢ a page and microfilms at one and one third cents a page. Duopage (Microphoto Division, Bell and Howell Company, 1700 Shaw Avenue, Cleveland, Ohio 44112) charged 5¢ a page (more for large pages) with the advantage of using both sides of the page. Both companies print catalogues but will supply copies of anything that they are able to photograph. An examination of the catalogues shows that caution is necessary. Those who need expensive rare books will find photocopying a bargain, but often a book can be purchased in the used book market for much less than the photocopy price. It is therefore advisable to check with scientific used book dealers such as Scientific Library Service (31 East 10th Street, New York 10003), Stechert-Hafner (same address) and Zeitlin and ver Brugge (815 North LaCienega Blvd., Los Angeles, California 90096).

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor W. A. Hallam, West Virginia Wesleyan College, represented the Association at the inauguration of President J. G. Harlow of West Virginia University on September 14, 1968.

University of Arizona: Dr. J. L. Brenner, Stanford Research Institute, has been appointed Professor; Associate Professor Berthold Schweizer, University of Massachusetts, has been appointed Professor; Associate Professors M. S. Cheema and D. E. Myers have been promoted to Professors.

Central Washington State College: Dr. R. Y. Dean, Battelle-Northwest, Richland, has been appointed Professor and Chairman of the Mathematics Department; Dr. Frederick Lister, Southern Oregon College, has been appointed Associate Professor; Dr. W. F. Cutlip, Michigan State University, has been appointed Assistant Professor.

Concordia College: Dr. J. M. Clark, Iowa State University, has been appointed Assistant Professor; Dr. Charles Heuer has been promoted to Associate Professor; Mr. Gerald Rowell has returned from a study leave at the University of Michigan and has been promoted to Assistant Professor.

Franklin and Marshall College: Mr. R. A. Howland, University of Massachusetts, has been appointed Assistant Professor; Assistant Professor W. F. Tyndall has been promoted to Associatê Professor; Associate Professors P. E. Bedient and Bernard Jacobson have been promoted to Professors.

University of Hawaii: Professor P. R. Halmos, University of Michigan, has been appointed Professor and Chairman of the Mathematics Department; Professor Leo Moser, University of Alberta, has been appointed Visiting Professor (1968–69); Professors L. W. Cohen, University of Maryland, and A. L. Shields, University of Michigan, have been appointed Visiting Professors (second semester 1968–69); Professor Marvin Marcus, University of California, Santa Barbara, has been appointed Visiting Colleague (first semester 1968–69); Messrs. E. A. Bertram, UCLA, M. M. Fraser, Albion College, and P. J. O'Hara, University of Miami, have been appointed Assistant Professors.

Merrimack College: Associate Professor R. E. Ozimkoski has been promoted to Professor; Assistant Professor J. W. Royal has been promoted to Associate Professor.

Dr. L. A. Aroian of TRW Systems has accepted a position at Union College as Professor of Industrial Administration.

Miss Carole Bauer has been appointed Chairman of the Mathematics Department of Triton College.

Mr. Joseph Bodenrader, Lowell Technical Institute, has been appointed Assistant Professor at State University College at Plattsburgh.

Assistant Professor Leon Cote, State University College at Fredonia, has been appointed Assistant Professor at State University College at Geneseo.

Dr. E. M. Hughes, Chadron State College, has been appointed Dean of Administration.

Assistant Professor Paul Kroll, Paterson State College, has been promoted to Associate Professor.

Mr. Michael Menn, Boston College, has been promoted to Assistant Professor.

Assistant Professor C. R. Nicolaysen, U. S. Naval Academy, Annapolis, has been appointed Director of the Computer Center, Coe College.

Associate Professor S. C. Saxena, Northern Illinois University, has been appointed Associate Professor at the University of Akron.

Associate Professor R. J. Troyer, University of North Carolina, has been appointed Associate Professor at Lake Forest College.

TOPOLOGY CONFERENCE PROCEEDINGS

The proceedings of the Point Set Topology Conference held at Arizona State University in March 1967 have been published under the title "Topology Conference, Arizona State University, 1967." Free copies for individual mathematicians are available from Topology Conference, Mathematics Department, Arizona State University, Tempe, Arizona 85281. Orders should include zip code. Library copies are available at $4.50 per copy including handling and postage from The Bookstore, Arizona State University, Tempe, Arizona 85281.

TWELFTH BIENNIAL INTERNATIONAL SEMINAR OF THE CANADIAN MATHEMATICAL CONGRESS

The 12th Biennial International Seminar of the Canadian Mathematical Congress will be held at the University of British Columbia in Vancouver, Canada, from August 11 through 27, 1969. This Seminar will be followed by the annual meeting of the Canadian Mathematical Congress at the University of Victoria in Victoria, Canada, August 28, 29, 30. Members of the Program Committee are S. A. Jennings, A. Joffe, J. G. McGregor, R. Pyke (Chairman) and D. A. Sprott, and of the Arrangements Committee are R. Harrop, S. A. Jennings (Chairman), E. Kennedy, B. N. Moyls and J. J. McNamee.

The main theme of the Seminar will be "Time Series and Stochastic Processes" although other areas of mathematics will be represented. Traditionally the Canadian Biennial Seminars embrace two series of lectures: 1) a series of research lectures given at the post-doctoral level by four internationally recognized research mathematicians and 2) a series of instructional lectures at the pre-doctoral level. Seminar participants, both graduate students and post-doctoral fellows, will be chosen primarily from Europe, the United States and Canada.

Further details will be announced at a later date.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The forty-fifth annual meeting of the Louisiana-Mississippi section of the MAA was held in the Broadwater Beach Hotel, Biloxi, Mississippi, February 16–17, 1968. Members from Mississippi State College for Women were hosts, with Professor D. A. King, Chairman. There were 250 persons registered, including 122 members of the Association.

The following officers were elected: Patrick Ford, McNeese State College, Chairman; W. E. Koss, Louisiana Polytechnic Institute, Louisiana Vice-Chairman; Porter Webster, University of Southern Mississippi, Mississippi Vice-Chairman; L. Virginia Carlton, Centenary College, Secretary-Treasurer.

Dr. M. E. Rose, Head of the Office of Computing Activities of the National Science Foundation, was the banquet speaker Friday evening. He spoke on the subject, "Computing Activities as Related to the Field of Mathematics." At the Saturday morning session Dr. R. D. Anderson, Boyd Professor of Mathematics at Louisiana State University, spoke on "Qualifications for a College Faculty in Mathematics."

The following papers were presented:

1. *Reducibility and stability in topological spaces*, by Bernard Madison, Louisiana State University.
2. *Regular matrix summability methods constructed from regular matrices and sequences of functions*, by Ed Kelly, Jr., Louisiana Polytechnic Institute.
3. *On compact divisible semigroups*, by J. A. Hildebrandt, Louisiana State University.
4. *On the symmetries of links*, by W. C. Whitten, Jr., University of Southwestern Louisiana.
5. *Bounds on the permanent of a diagonally dominant matrix*, by J. J. Johnson, University of Mississippi.
6. *Some properties of extended topologies*, by C. D. Tabor, Louisiana State University, Shreveport.
7. *Addition of a zero-network to the closed loop of an n-th order system*, by C. J. Monier, Francis T. Nicholls State College.
8. *A note on the cohomology cross product operation*, by George Butler, Louisiana Polytechnic Institute.
9. *The inverse of the Vandermonde matrix*, by B. E. Mitchell, Louisiana State University.
10. *Inversion, hypo-inversion, epi-inversion*, by Elsie C. Ozley, Louisiana Polytechnic Institute.
11. *Rational minimal surfaces*, by F. H. Eng, Nicholls State College.
12. *An evaluation of the integral from* 0 *to* $x^{-1/2}$ *of* exp $(-x\ \mathrm{Sinh}^2 t)dt$ *through terms of order* $x^{-5/2}$, by J. L. Tilley, Mississippi State University.
13. *Variable step-size integration procedures*, by E. P. Burton, Louisiana Polytechnic Institute.
14. *Concerning Padé approximants*, by J. L. Avard, Northeast Louisiana State College.
15. *A simple proof of convergence for Cimmino's method*, by J. D. Gilbert, Louisiana Polytechnic Institute.
16. *Serre duality and the Riemann-Roch theorem*, by Margaret M. LaSalle, University of Southwestern Louisiana.
17. *The periodicity of the last p digits in m^n for some integers m*, by H. J. de St. Germain and G. E. Steen, Southeastern Louisiana College.
18. *Concerning some periodic continued fractions*, by Robert Heller, Mississippi State University.
19. *The role of axiomatics and problem solving in mathematics*, a panel discussion by R. L. Pendleton, Louisiana State University, Chairman; T. K. Maddox, Southeastern Louisiana State College; R. A. Stokes, University of Mississippi; and Gail Young, Tulane University.
20. *A new type of magic cube*, by Charles Greene, Centenary College.
21. *Tidal forces on earth and moon*, by Wallace Herbert, Louisiana Polytechnic Institute.
22. *Results from a survey study on mathematics topics in a college course for prospective elementary school teachers*, by F. A. Pigno, McNeese State College.

L. VIRGINIA CARLTON, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

Fifty-Third Annual Meeting, Miami, Florida, January 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Bukhannon, April 26, 1969.

FLORIDA, Florida Atlantic University, Boca Raton, March 21–22, 1969.

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA, Purdue University, Indianapolis, May 10, 1969.

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS, Wichita State University, Wichita, March 29, 1969.

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Courant Institute of New York University, March 15, 1969.

MICHIGAN, University of Michigan, Ann Arbor, March 29, 1969.

MINNESOTA, College of St. Catherine, St. Paul, April 26, 1969.

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of Santa Clara, Santa Clara, February 8, 1969.

OHIO, Ohio State University, Columbus, April 25–26, 1969.

OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 21–22, 1969.

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, May 9–10, 1969.

SOUTHEASTERN, Winthrop College, Rock Hill, South Carolina, March 28–29, 1969.

SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.

SOUTHWESTERN, Northern Arizona University, Flagstaff, April 11–12, 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE, University of Western Ontario, London, Ont., Canada, May 1969.

WISCONSIN, Oshkosh, Wisconsin, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, University of Oregon, Eugene, Oregon, August 26–29, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Statler-Hilton Hotel, Washington, D. C., May 7–9, 1969.

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Minneapolis, April 23–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, Colorado, June 17–20, 1969.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.

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VOLUME 76 NUMBER 2

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FEBRUARY 1969

NOTICE TO AUTHORS

Articles should be typewritten on good quality paper, triple spaced with wide margins. Submit the original (and a duplicate if convenient) keeping a complete copy for your protection against loss. Please follow the format used in current issues of the Monthly.

The manuscript must make the author's intent clear to the printer. A paper which is accepted for publication by the editor may not be acceptable to the printer and will be returned to the author for retyping. Please use the American Mathematical Society *Manual for Authors of Mathematical Papers* as your guide and please conform to correct English usage.

EDITORIAL CORRESPONDENCE: to Harley Flanders, American Mathematical Monthly, Purdue University, Lafayette, IN 47907; ADVERTISING CORRESPONDENCE: to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. Willcox, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D. C. 20036.

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NOTICE

Good material for Classroom Notes is urgently needed. The publication lag is minimal for this department.

Current backlog: Main articles, 11 months; Mathematical Notes, 10 months; Research Problems, 4 months; Classroom Notes, 4 months. *Editor.*

THE ELUSIVE FIXED POINT PROPERTY

R. H. BING,* University of Wisconsin

A set X has the *fixed point property* if each map $f: X \to X$ leaves some point fixed—that is, there is a point $x \in X$ such that $f(x) = x$.

Satisfactory necessary and sufficient conditions have not been found for determining whether or not a set has the fixed point property. The sufficient conditions that have been found are too restrictive to be necessary. On the other hand, many examples have been shown to have the fixed point property—sometimes with the method of proof tailored to the example.

The paper is intended primarily as an expository article to bring together some of the interesting results about fixed points. The only new result is Theorem 15. It is hoped that a graduate student with a beginning course in geometric topology will be able to dig through most of the material. Although such specialized concepts as Betti number, Lefschetz number, Euler characteristic, indecomposable continuum, pseudo arc, contractible, degree of a map, Green's Theorem, Sperner's Lemma, absolute neighborhood retract, cyclic element, nerve of a covering, snake-like continuum, tree-like continuum, orientation preserving homeomorphism, Čech homology, etc. are mentioned when this is natural, the treatment is such that the reader can pass on and understand much of what follows, waiting until later to learn about the implications of these concepts from relevant sources if he so desires. Some theorems are included for historical information or the advanced worker, but it is hoped that many readers can provide or follow proofs of Theorems 2, 3, 6, 12–15; 20–22.

Continua discussed in the paper are compact and metrizable. All polyhedra considered are finite and connected.

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* Professor Bing is best known for his research in geometric topology. Since his graduate study and post-doctoral instructorship at the Univ. of Texas, he has been at the Univ. of Wisconsin, where he now is a Rudolph E. Langer Professor. He has been a leader in many professional organizations: AAAS (Vice President and Chairman, Section A), CBMS (Chairman), AMS (Council Member, Vice President), MAA (Section Chairman, Hedrick lecturer, President), NRC (Chairman, Div. Math. Sciences). Prof. Bing is a member of the National Academy of Sciences and currently serves on the National Science Board. *Editor.*

1. Sets with the fixed point property. Perhaps the best-known result about fixed points is the following:

THEOREM 1. *Each n-cell has the fixed point property.*

The result is frequently called the Brouwer Fixed Point Theorem although the work of Brouwer [7] was probably preceded by that of Bohl [4]. However, even before Brouwer's paper [7] appeared, reference had been made to the Brouwer Fixed Point Theorem. (See Hadamard's reference on page 472 of [23].) In proving the theorem, Bohl considered differentiable maps and used Green's Theorem to show that equivalent integrals did not match if the n-cell had a fixed point free map into itself. On the other hand, Brouwer proved the theorem by showing that homotopic maps of an $(n-1)$-sphere onto itself had the same degree; hence, there is no retraction of an n-cell onto its boundary; hence, each map of an n-cell into itself is fixed point free. Still another proof (using combinatorial techniques) was supplied by Knaster, Kuratowski, and Mazurkiewicz [18]. They used Sperner's Lemma in showing that there is no retraction of an n-cell onto its boundary. Hirsch gives a beautiful elementary proof of this result in [14].

A subset Y of a set X is a *retract* of X if there is a map $f:X\rightarrow Y$ such that $f|Y=$identity—that is, f leaves each point of Y fixed. The map f is called a *retraction* of X onto Y. The metric set Y is called an *absolute neighborhood retract* if for each metric space Z containing Y, there is an open set U in Z with $Y\subset U$ and a retraction of U onto Y.

A convenient way of showing that a set Y has the fixed point property is to use the following result.

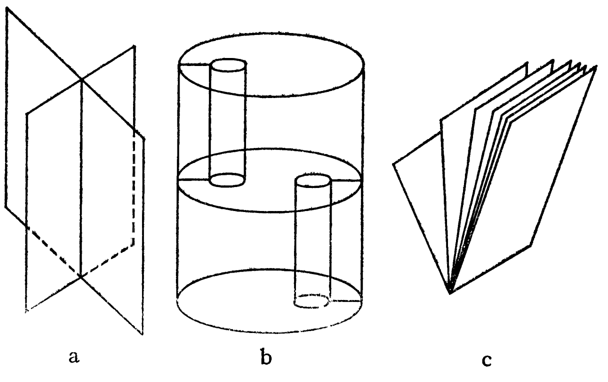

FIG. 1

THEOREM 2. *If Y is a retract of X and X has the fixed point property, then so does Y.*

For example, Theorem 2 in conjunction with Theorem 1 shows that such objects as represented by Figures 1a, 1b have the fixed point property. Figure 1a

represents four pages of a book while Figure 1b represents a house-with-two-rooms [9].

Figure 1c represents infinitely many 2-cells converging to a 2-cell such that any two of the 2-cells share a common edge. Although the object is not a retract of a 3-cell (cube), for each $\epsilon > 0$, there is a retraction of the object onto the union of a finite number of the cells that does not move any point more than ϵ. Hence the object of Figure 1c can be shown to have the fixed point property by using Theorems 1, 2, 3.

THEOREM 3. *If f is a fixed point free map of a compact set X into itself, there is a positive number ϵ such that for each $x \in X$, $\rho(x, f(x)) > \epsilon$, where ρ denotes the distance function. The number ϵ is a function of f but not of x.*

Lefschetz was able to generalize Theorem 1 by defining for each map $f: X \to X$ an integer $L(f)$. This number has since been called the Lefschetz number of f. We shall not define it here but mention that if for each $i > 0$, the ith Betti number of continuum X is 0 (as it is for a cell), then for each $f: X \to X$, $L(f) = 1$.

THEOREM 4. (Lefschetz [20]). *A sufficient condition that an absolute neighborhood retract X have the fixed point property is that for each map $f: X \to X$, $L(f) \neq 0$.*

Theorem 4 can be improved for the case in which X is a compact 1-connected manifold M. Fadell [10] and Fuller [11] have shown that M has the fixed point property if and only if for each map $f: M \to M$, $L(f) \neq 0$.

Any continuous image of an interval [0, 1] is called a *Peano continuum* or *continuous curve*. A *nondegenerate cyclic element* of a continuous curve C is the union of a simple closed curve J in C and all simple closed curves that intersect C in two or more points. Ayres provided us with the following information about continuous curves.

THEOREM 5. (Ayres [1]). *A necessary and sufficient condition that a continuous curve have the fixed point property is that each of its non-degenerate cyclic elements does.*

If X and Y are sets, the *wedge* of X and Y (denoted by $X \vee Y$) is the set obtained by joining X and Y at a point $x \in X$ and $y \in Y$—that is, x is identified with y. The reader should be able to prove the following variation of Theorem 5. A deeper variation is found in [16].

THEOREM 6. *If each of X and Y has the fixed point property, so does $X \vee Y$.*

Suppose P is a triangulated polyhedron of dimension n and $\alpha_m (m = 0, 1, \cdots, n)$ is the number of m simplexes in the triangulation of P. Then $\chi(P) = \alpha_0 - \alpha_1 + \alpha_2 - \cdots + (-1)^n \alpha_n = \sum_0^n (-1)^i \alpha_i$ has been called the Euler characteristic of P. It follows from the Euler-Poincaré formula (see pages 241–243 of [15])that $\chi(P)$ is also equal to $\sum_0^n (-1)^i p_i$ where p_i is the ith Betti number of P. Lopez has described a polyhedron with the following properties.

THEOREM 7. (Lopez [21]). *There is a connected polyhedron X with the fixed point property such that $\chi(X)$ is a positive even integer.*

We mention this result since it leads to some interesting anomalies which we mention in Section 2. We do not describe the polyhedron used by Lopez since it is of dimension 8.

QUESTION 1. *Is there a two dimensional polyhedron with the fixed point property which has even Euler characteristic?*

If $G=\{U_1, U_2, \cdots, U_k\}$ is an open covering of a space X, there is associated with the covering a geometric complex $N(G)$ with vertices $v_1, v_2, \cdots, v_k$ such that $\langle v_{i_1}, v_{i_2}, \cdots, v_{i_j}\rangle$ is a simplex of $N(G)$ if and only if $U_{i_1}\cap U_{i_2}\cap \cdots \cap U_{i_j}\neq 0$. The complex $N(G)$ is called the *nerve* of G. If the mesh of G is small, $N(G)$ is a close approximation to X in the sense that there is a map of X into $N(G)$ with small inverses.

A *snake-like* continuum is a compact metric continuum X such that for each $\epsilon>0$, there is an ϵ-covering (one of mesh less than ϵ) G of X such that $N(G)$ is an arc. A *tree-like* continuum has coverings of arbitrarily small meshes such that the nerves of these coverings are trees (contain no simple closed curves). Hamilton has proved the following:

THEOREM 8. (Hamilton [13]). *Each snake-like continuum has the fixed point property.*

We shall not give the proof here but mention that it is a variation of the *dead-end* type proof given in connection with the proof of Theorem 13.

One of the most interesting unsolved problems in geometric topology is the following:

QUESTION 2. *Does each tree-like continuum have the fixed point property?*

Each bounded plane continuum which does not separate the plane is the intersection of a decreasing sequence of topological disks (sometimes called merely disks for short). Question 2 seems related to the following which has been called the most interesting outstanding problem in plane topology.

QUESTION 3. *Does the intersection of each decreasing sequence of disks have the fixed point property?*

We mention several related partial results.

A continuum is called *indecomposable* if it is not the sum of two proper subcontinua. Hence, an arc is decomposable but a pseudo arc is not. (See [22] or [2] for a description of a pseudo arc.)

THEOREM 9. (Hamilton [12]). *If T is a tree-like continuum which contains no indecomposable continuum, then each homeomorphism of T into itself leaves some point fixed.*

THEOREM 10. (Hamilton [12]). *If X is a bounded plane continuum which*

does not separate the plane and whose boundary in the plane contains no indecomposable continuum, then each homeomorphism of X into itself leaves some point fixed.

THEOREM 11. (Cartwright and Littlewood [8]). *If h is an orientation preserving homeomorphism of the plane onto itself and $X=h(X)$ is a bounded plane continuum which does not separate the plane, then there is a point $x\in X$ such that $h(x)=x$.*

References [5, 16, 24, 25, 27] give other sets of conditions that imply that compact metric continua have the fixed point property. Rather than consider other sets of conditions here, we turn to three instructive examples.

THEOREM 12. *A disk with a spiral about its boundary has the fixed point property.*

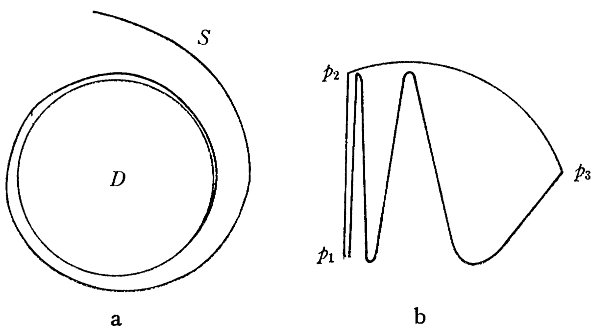

FIG. 2

This object is represented in Figure 2a as a disk D and a spiral S about the boundary of D. If $f:(D\cup S)\to(D\cup S)$, either $f(D)\subset D$ or $f(D)\subset S$ since D is arcwise connected, arcwise connectedness is preserved by maps, and there is no arc in $D\cup S$ from D to S. If $f(D)\subset D$, Theorem 1 implies that f leaves a point fixed. If on the other hand, $f(D)\subset S$, a consideration of f at any point of the boundary of D shows that f takes some point of S into S. Since S is arcwise connected, $f(D\cup S)\subset S$. Since $D\cup S$ is compact, $f(D\cup S)$ is either a point or an arc. If it is a point, the point is left fixed by f. If it is an arc, Theorem 1 implies that some point of the arc is left fixed.

THEOREM 13. *The* sin $1/x$ *circle has the fixed point property.*

This object is represented in Figure 2b as the union of the closure of the graph of $y=\sin 1/x$ $(0<x\leqq 1/\pi)$ and an arc from $p_2=(0,1)$ to $p_3=(1/\pi,0)$. The point $(0,-1)$ is labeled p_1. Suppose f is a map of the sin $1/x$ circle into itself.

We use the *dead-end* method to prove that f leaves some point fixed. Consider a dog chasing a rabbit, where a variable point x represents the dog, $f(x)$

the rabbit, and the direction along the arc in the sin $1/x$ circle from x to $f(x)$ the direction of the chase. Let the dog start at p_1. After he moves slightly beyond p_1 toward p_2 in his chase of his image, we note that the rabbit is in front of the dog in the sense that x is between p_1 and $f(x)$ on the arc $[p_1, f(x)]$ of the sin $1/x$ circle. It follows by continuity that even as x moves past p_3 and then back along the sin $1/x$ graph to near $[p_1, p_2]$, $f(x)$ stays in front of x. Each point q_0 of $[p_1, p_2]$ is the limit of a sequence of points $q_1, q_2, \cdots$, of the sin $1/x$ graph. Since $f(q_i)$ is ahead of q_i, then $f(q_1), f(q_2), \cdots$, approaches $[p_1, p_2]$. Hence $f(q_0) \in [p_1, p_2]$. Since $f[p_1, p_2] \subset [p_1, p_2]$, Theorem 1 implies that f leaves some point of $[p_1, p_2]$ fixed.

The following question suggested by Young in [27] seems related to Theorem 13. The continuum X' used in the proof of Theorem 15 of the next section shows the reason for inserting the planar assumption.

QUESTION 4. *Does each arcwise connected compact planar continuum containing no simple closed curve have the fixed point property?*

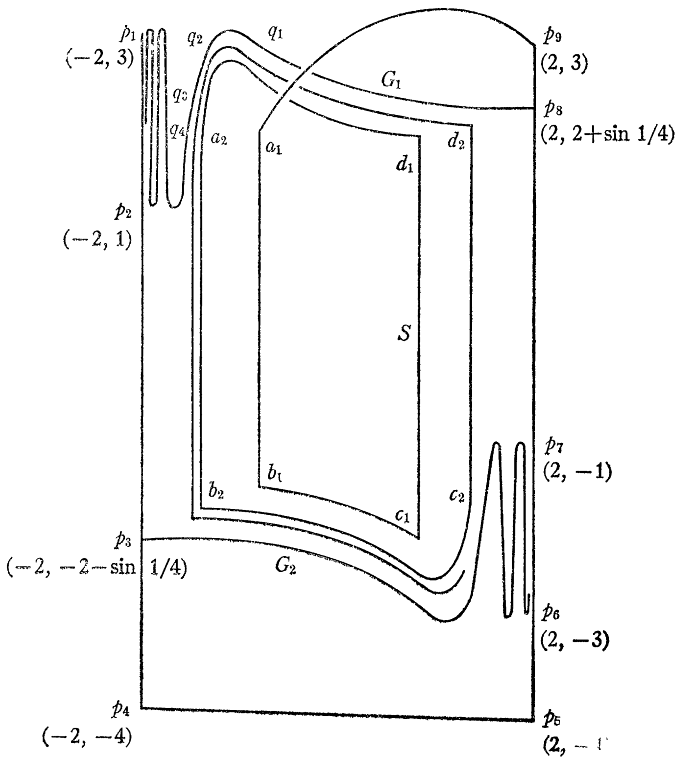

FIG. 3

We consider one more example which plays an important role in the following two sections.

THEOREM 14. *The object represented by Figure 3 has the fixed point property.*

This object (which we call X) is the union of three segments $[p_1, p_4]$, $[p_4, p_5]$, $[p_5, p_9]$, two $\sin 1/x$ curves G_1, G_2, and a spiral S. We describe them more precisely before showing that X has the fixed point property.

The points $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9$ have coordinates $(-2, 3)$, $(-2, 1)$, $(-2, -2-\sin 1/4)$, $(-2, -4)$, $(2, -4)$, $(2, -3)$, $(2, -1)$ $(2, 2+\sin 1/4)$, $(2, 3)$ as shown in Figure 3.

The $\sin 1/x$ curve G_1 has equation $y=2+\sin 1/(x+2)(-2<x\leqq 2)$ while G_2 is the reflection of G_1 through the origin $(0, 0)$.

To describe S we select a sequence of points $q_1, q_2, \cdots$ on G_1 such that the length of the arc $[q_i, p_8]$ of G_1 is $2+\sum_1^i 1/j$. Since a harmonic sequence is involved, the points $q_1, q_2, \cdots$ approach the segment $[p_1, p_2]$. On the vertical line through q_i, let a_i and b_i be the points $1/i$ below G_1 and $1/(i+1)$ above G_2 respectively. Let d_i be the reflection of b_i through the origin and c_i the point of the vertical line through d_i that is $1/(i+1)$ units above G_2.

The spiral S starts at p_9, runs to a_1 along a semicircle in a plane normal to the plane containing G_1, follows a straight path to b_1, goes from b_1 to c_1 along an arc parallel to G_2, goes straight to d_1, goes to a_2 along an arc parallel to $G_1, \cdots$. In general, the vertical segments $[a_i, b_i]$ and $[c_i, d_i]$ are in S as are arcs from b_i to c_i parallel to G_2 and arcs from d_i to a_{i+1} parallel to G_1.

To prove that X has the fixed point property, we assume $f:X\rightarrow X$ is fixed point free. The same dead-end method used in the proof of Theorem 13 can be used to show that f leaves some point fixed.

Start the dog at p_3. If $f(p_3)\in[p_1, p_3]$, the dog chases the rabbit to a dead-end along $[p_3, p_1]$ and f leaves some point of $[p_3, p_1]$ fixed. If $f(p_3)\in G_2$, move x along G_1 toward $f(x)$. It is noted that if $x\in G_2-\{p_3\}$, $f(x)$ lies ahead of x in the sense that x lies on the arc in G_2 from p_3 to $f(x)$. An argument like that used in the proof of Theorem 13 shows that f leaves some point of $[p_6, p_7]$ fixed.

If $f(p_3)\notin[p_1, p_3]\cup G_2$, one finds that after the dog leaves p_3 in his chase of the rabbit, x lies between p_3 and $f(x)$ on the arc in X from p_3 to $f(x)$. The chase continues until the dog arrives at the fork p_8. The rabbit is now on either G_1 or $S\cup[p_8, p_9]$. If it is on G_1, we have a situation similar to that in which $f(p_3)\in G_2$ so we suppose $f(p_8)\in S\cup[p_8, p_9]$.

As x moves past p_8, p_9, and into S, it is found that $f(x)$ remains ahead of x in the sense that x lies between p_9 and $f(x)$ on the arc in S from p_9 to $f(x)$. Since the tail end of S converges to $[p_1, p_3]\cup G_2\cup[p_6, p_8]\cup G_1$, if $x_1, x_2, \cdots$ is a sequence of points of S converging to p_8, then $f(x_1), f(x_2), \cdots$ converges to a point of $[p_1,p_3]\cup G_2\cup[p_6, p_8]\cup G_1$. This violates the assumption that $f(p_8)\in S\cup[p_8, p_9]$.

2. Fixed point anomalies. We find that in many ways fixed points do not behave in an expected fashion.

One might think that Theorem 6 would generalize to say that the union along an arc of two sets with the fixed point property would also have the fixed point property. We show that this is not true.

THEOREM 15. *There is a 1-dimensional continuum X with the fixed point property and a disk D such that $D\cap X$ is an arc but $D\cup X$ does not have the fixed point property.*

The continuum X is shown in Figure 3 and D is the rectangular disk with vertices p_5, p_9, p_{10}, p_{11} where $p_{10}=(3, 3)$ and $p_{11}=(3, -4)$. Note that $X\cup D$ retracts onto the continuum

$$X' = (X \cup \mathrm{Bd}\, D) - (p_8, p_9).$$

This continuum X' resembles the continuum which Young [27] showed failed to have the fixed point property. We point out a fixed point free map $f: X'\to X'$.

Restricted to $[p_1, p_3]\cup G_2\cup [p_6, p_8]\cup G_1$, f is a reflection through the origin. Also f takes homeomorphically $[p_3, p_5]$ to $[p_8, p_{11}]$; $[p_6, p_5]$ to $[p_1, p_{11}]$; $[p_5, p_{11}]$ to $[p_{11}, a_1]$; $[p_{11}, a_1]$ to $[a_1, c_1]$, and each $[a_i, a_{i+1}]$ to $[c_i, c_{i+1}]$. The only other precaution to take to see that f is continuous is to be sure that as a point moves far out on S, its image under f is very near its reflection through the origin.

QUESTION 5. *Does $X\times [0, 1]$ have the fixed point property?*

It may be noted that f is a homeomorphism on much of X. This suggests the following question suggested by Young [27].

QUESTION 6. *If C is an arcwise connected continuum which contains no simple closed curve, does each homeomorphism of C into itself leave some point fixed?*

The continuum X of Theorem 15 did not lie in the plane because of the semicircle from p_9 to a_1.

QUESTION 7. *If C is a plane continuum with the fixed point property and D is a disk that intersects C in an arc, must $C\cup D$ have the fixed point property?*

There may be some who feel that the X of Theorem 15 is a pathological example and such unusual things could not happen if the X were required to be polyhedral. This is not an accurate appraisal since Lopez [21] has constructed two continua X, Y with the fixed point property such that $X\vee Y$ has Euler characteristic $=\chi(X\cup Y)$ zero. We shall not describe this example since it is of dimension 17. However, disks can be pasted over the separating points of $X\vee Y$ so as to remove the separating points and it follows from the following result due to Wecken that the union of $X\vee Y$ and the disks does not have the fixed point property.

THEOREM 16. (Wecken [26]). *If K is a connected polyhedron without local separating points and $\chi(K)=0$, then K admits a fixed point free map.*

THEOREM 17. (Lopez [21]). *There is a polyhedron P with the fixed point property and a disk D such that $D\cap P$ is an arc but $D\cup P$ does not have the fixed point property.*

Since the dimension of Lopez's example is 17, it is natural to seek something simpler.

QUESTION 8. *What is the lowest dimension for such a polyhedron P as promised by Theorem* 17?

We shall note in Theorems 18, 23 that cartesian products need not retain the fixed point properties of their factors. However the following is open.

QUESTION 9. *If a compact* 1-*dimension continuum has the fixed point property, does its cartesian product with a segment?*

QUESTION 10. *If a bounded plane continuum has the fixed point property, does its cartesian product with an arc?*

If one takes a polyhedron with a separating point and multiplies the polyhedron with a segment, the cartesian product will have the same Euler characteristic as the polyhedron but will have lost the property of having a separating point. Hence the example of Lopez in conjunction with the result of Wecken shows the following:

THEOREM 18. (Lopez [21]). *There is a polyhedron with the fixed point property such that its cartesian product with a closed* 1-*simplex does not have the fixed point property.*

QUESTION 11. *If P and Q are polyhedra without local separating points but with the fixed point property, must $P \times Q$ have the fixed point property?*

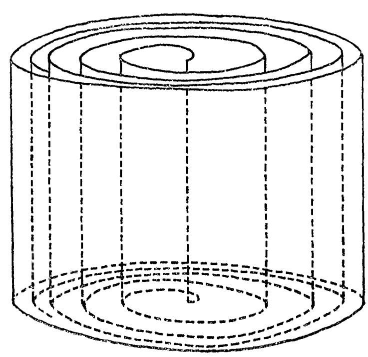

FIG. 4

Polyhedra with trivial ith Betti numbers ($i>0$) have the fixed point property as a result of Theorem 4. Hence compact contractible polyhedra have the fixed point property. In particular, a cone over a polyhedron has the fixed point property. This does not hold in the nonpolyhedral case.

Kinoshita described the example illustrated in Figure 4. It consists of a cylindrical can with a bottom but no top plus the part of the solid cylinder

which lies above a spiral on the base converging to the edge of the base. It resembles a can with a spring inside and has been called a can-with-a-roll-of-toilet-paper.

THEOREM 19. (Kinoshita [17]). *The can-with-a-roll-of-toilet-paper has the following properties: it is contractible; it does not have the fixed point property; the cone over it does not have the fixed point property.*

The above properties are treated in [17]. The can-with-a-roll-of-toilet-paper does not answer the following.

QUESTION 12. *Does each homeomorphism of a contractible continuum onto itself leave some point fixed?*

Borsuk [6] gave an example of a 3-dimension continuous curve such that there is a fixed point free homeomorphism of the continuous curve onto itself even though in Čech homology, all p cycles ($p>0$) bound. Bing [3] gave a 2-dimensional example on which the following theorem is based.

THEOREM 20. (Bing [3]). *There is a continuous curve of dimension two which is the intersection of a decreasing sequence of topological cubes and which admits a fixed point free homeomorphism onto itself.*

Consider two cones in Euclidean 3-space each with bases on the cylinder $x^2+y^2=4$ such that the cones have bases in the planes $z=1/2$ and $z=-1/2$ and vertices at $(-2, 0, 1/2)$ and $(2, 0, 1/2)$. Let X denote the image of the union of these two cones if each horizontal plane is rotated about the z axis through an angle of $\tan \pi z$. We do not draw X to scale in Figure 5 but that figure illustrates how the vertex of each cone spirals around the base of the other. For those who want an equation, we suggest the following:

$$\{[x-(2z+1)\cos\tan\pi z]^2+[y-(2z+1)\sin\tan\pi z]^2-(1-2z)^2\}$$
$$\{[x-(2z-1)\cos\tan\pi z]^2+[y-(2z-1)\sin\tan\pi z]^2-(1+2z)^2\}=0.$$

For each θ between 0 and 2π, there is a fixed point free homeomorphism h_θ of X onto itself. Each of the bases of X is rotated through an angle of size θ by h_θ. On other horizontal cross sections, h_θ is a result of a rotation of size θ, about the z-axis, a lift, and a stretch. The lift carries the line of centers of the rotated cross section into the line of centers of the higher cross section. Those wanting an equation can use $([\arctan(\theta+\tan\pi z)]/\pi)-z$ for the size of the lift. Keeping the line of centers fixed, one of the two circles in the rotated and lifted cross section is expanded and the other contracted to make the moved cross section agree with the cross section in the plane to which the lift was made.

It is natural to expect that the cone over a compact continuum would have the fixed point property. That this is true for polyhedra follows from Theorem 4. However, the following theorem due to Knill shows that the result does not hold in general. (Theorem 19 also shows this.)

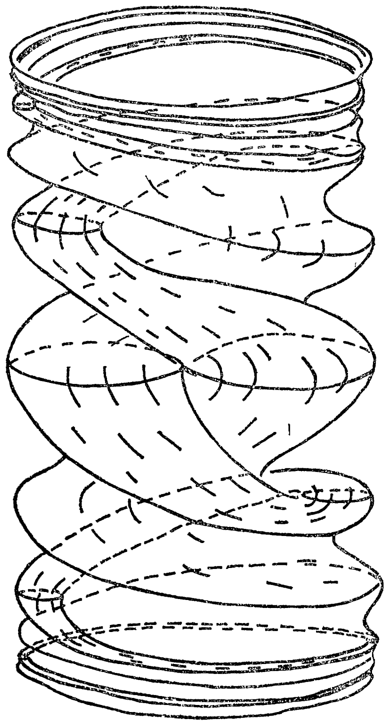

FIG. 5

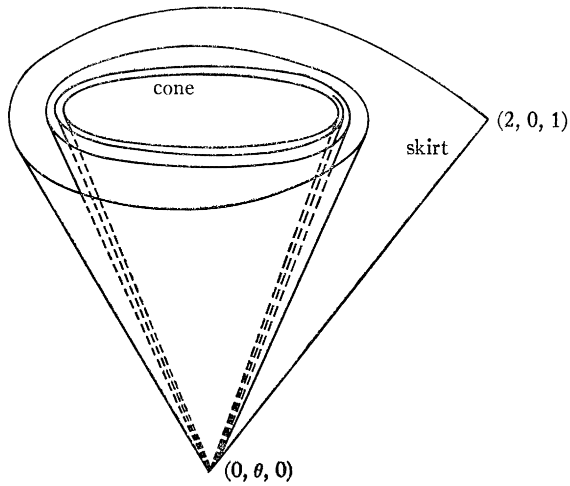

FIG. 6

THEOREM 21. (Knill [19]). *The cone over a circle with a spiral as shown in Figure* 6 *does not have the fixed point property.*

Suppose that in cylindrical coordinates (r, θ, z) the circle is defined by $[z=1,\ r=1]$, the spiral by $[z=1,\ r=1+1/(1+\theta),\ \theta\geqq 0]$, and the vertex by $(0, \theta, 0)$. We call the cone over the spiral, the *skirt* of the cone.

The fixed point free map f of the cone-with-a-skirt onto itself that we shall define sends the skirt to the skirt and the cone to a union of the cone and a segment from its vertex.

If $0\leqq z_0\leqq 1/3$, the horizontal cross section of the cone at level z_0 goes to the point that divides the segment from $(2, 0, 1)$ to $(0, \theta, 0)$ in the same ratio that z_0 divides 0 to 1/3.

Suppose f_z is the function that sends 0 to 1, 1/3 to 0, 2/3 to 1, 1 to 0 and is linear on each of $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$, while $f_\theta(\theta, z) = \theta+z-2/3$. For each point (r, θ, z) of the cone with $1/3\leqq z\leqq 1$, the fixed point free map is defined by $f(r, \theta, z) = (f_z(z), f_\theta(\theta, z), f_z(z))$. Note that $f_z(z) = z$ only at levels $z=1/4$, 1/2, 3/4. At levels 1/2 and 3/4 there is rotation and the 1/4 level is not sent onto the cone.

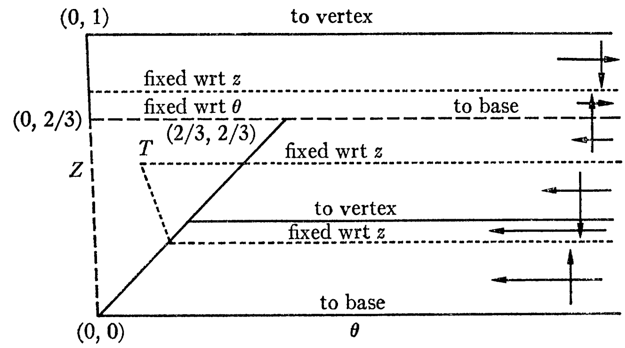

FIG. 7

In order to describe f on the skirt, we unroll it. The unrolled skirt shown in Figure 7 is not to scale since horizontal rays in the half strip represent horizontal cross sections of the skirt, vertical segments in the half strip represent segments from the vertex of the skirt to the base, and the top edge of the half strip represents the vertex of the cone. Arrows on the right of the figure indicate motion in both θ and z directions.

We use rectangular coordinates to describe f on the skirt. We ignore r since $r=z(1+1/(1+\theta))$. The θ value of $f(\theta, z)$, for a point (θ, z) of the strip, is given by 0 if $0 \leqq z \leqq 1/3$ and by max $(0, \theta+z-2/3)$ for $1/3<z\leqq 1$. There is lack of continuity on the horizontal ray to the right from (1/3, 1/3) but this does not matter since this ray will be sent into the vertex of the cone. The θ values of all points of the skirt are changed except those on the vertical segment from (0, 0) to (0, 2/3) and the horizontal ray to the right from (0, 2/3). This invariant set with respect to θ is shown with a dashed line in Figure 7.

We now describe the z values of images of points of the skirt. Except on the triangular disk T with vertices (0, 0), (0, 2/3), (2/3, 2/3), these z values are given by $f_z(z)$ where f_z was the function used to give the z values of images of points of the cone. For each point (θ, z) of T, we define the z value of its image to be 3 max $[|z-1/3|, |\theta-1/3|]$.

Points of the skirt whose z values are invariant under f are shown as dotted. Since the dotted set misses the dashed one, each point of the skirt is moved.

It is interesting to note that the cone over the cone-with-a-skirt does not have the fixed point property either. This result follows from the theorem of Kinoshita [15] which states that if a contractible set does not have the fixed point property, then the cone over it does not have the fixed point property either.

One might be misled into thinking that the reason the cone-with-a skirt did not have the fixed point property was that its base did not. This is disproved by the following result.

THEOREM 22. *If $D \cup S$ is a disk with a spiral about its boundary, shown in Figure* 2a, *the cone over $D\cup S$ does not have the fixed point property.*

There is a retraction of this cone onto a cone-with-a-skirt, so it follows from Theorem 21 and 2 that the cone over $D \cup S$ does not have the fixed point property.

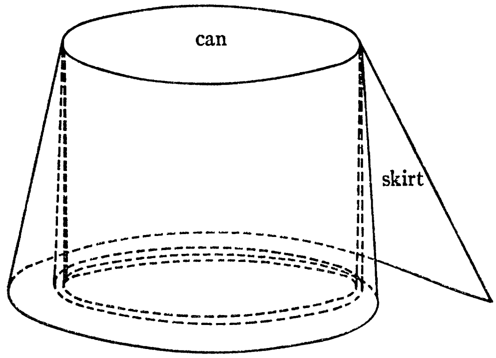

FIG. 8

Let B be a can-with-a-skirt shown in Figure 8. The can is described in cylindrical coordinates as $\{(r, \theta, z) \mid [z=0, r \leqq 1] \text{ or } [r=1, 0 \leqq z \leqq 1]\}$ while the skirt is $\{(r, \theta, z) \mid 0 \leqq z \leqq 1, \theta \geqq 0, r = 1 + (1-z)/(1+\theta)\}$. Knill has shown that this can-with-a-skirt has the following properties.

THEOREM 23. (Knill [19]). *If B is the can-with-a-skirt, then B has the fixed point property; $B \times [0, 1]$ does not have the fixed point property, and there is a 3-cell C (namely, the solid can) such that although each of B and C has the fixed point property and $B \cap C$ is a disk, $B \cup C$ does not have the fixed point property.*

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ON THE CENTRAL LIMIT THEOREM FOR THE PRIME DIVISOR FUNCTION

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Introduction. In 1920, Hardy and Ramanujan [5] showed that the typical integer m has approximately log log m prime divisors, in this sense:

For a set A of positive integers, let $N_n(A)$ be the number of m, $1 \leqq m \leqq n$, for which m lies in A; the density of A is defined as

$$D(A) = \lim_{n\to\infty} \frac{1}{n} N_n(A), \tag{1}$$

provided this limit exists. The statement that $D(A)=p$ means technically that the limit (1) exists and has value p; it means intuitively that, among all positive integers, a proportion p lie in A. Let $\nu(m)$ be the number of prime divisors of m (not counting multiplicity, so that $\nu(3^4\cdot 5^3)=2$, for example).

THE HARDY-RAMANUJAN THEOREM. *If g_m goes to infinity, no matter how slowly, then*

$$D\left\{m: \left|\frac{\nu(m) - \log\log m}{(\log\log m)^{\frac{1}{2}}}\right| \leqq g_m\right\} = 1. \tag{2}$$

If a set of density 1 is regarded as containing "practically all" integers, and if we take g_m to be, say, $(\log\log m)^{\frac{1}{4}}$, so that $g_m(\log\log m)^{\frac{1}{2}}$ is for large m very small in comparison with $\log\log m$, then (2) does say that "practically all" integers m have about $\log\log m$ prime divisors. Thus an integer in the neighborhood of 10^8 will usually have about $\log\log 10^8 \approx 3$ prime divisors, and an integer in the neighborhood of 10^{70} will usually have about $\log\log 10^{70} \approx 5$ prime divisors—remarkably few. (See [6] p. 358, where $\nu(m)$ is regarded as a measure of the "roundness" of m.)

In 1934, Turán [10] gave a greatly simplified proof of the Hardy-Ramanujan Theorem by an essentially probabilistic method. Further development of probabilistic ideas in number theory led Erdös and Kac [1 and 2] to conjecture and to prove in 1939 a remarkable refinement of (2):

THE ERDÖS-KAC THEOREM. *If $x \leqq y$, then*

$$D\left\{m: x \leqq \frac{\nu(m) - \log\log m}{(\log\log m)^{\frac{1}{2}}} \leqq y\right\} = \frac{1}{\sqrt{2\pi}} \int_x^y e^{-\frac{1}{2}u^2}\, du. \tag{3}$$

This sharpening of (2) (we shall show later that (3) does imply (2)) shows how $\nu(m)$ fluctuates about the central value $\log\log m$. For example, for $-x = y = .9$ the integral above is about .6, which is thus the approximate proportion of m for which the ratio in (3) lies in the interval $[-.9, +.9]$. If m is near 10^{70}, this ratio is approximately $(\nu(m)-5)/5^{\frac{1}{2}}$, which lies in $[-.9, +.9]$ if and only if $\nu(m)$ lies in $[5-.9\times 5^{\frac{1}{2}}, 5+.9\times 5^{\frac{1}{2}}] \approx [3, 7]$. Thus something like 60 percent of the integers in the vicinity of 10^{70} have between 3 and 7 prime divisors. (Since the abnormality of a finite stretch of integers can have no effect on the density (3), there underlies a computation of this kind the premise that the integers near 10^{70} are not atypical in their divisibility properties.)

Erdös and Kac in their original proof of (3) used difficult sieve methods. The probabilistically most natural approach to the result is that of Halberstam [4], which uses the method of moments (an idea first suggested by Kac [7]). The purpose of this paper is to show how, by the introduction of further probability to avoid some heavy calculations, Halberstam's proof can be made more transparent to the student of probability theory.

Preliminaries. On the space of positive integers, let P_n be the probability measure that places mass $1/n$ at each of $1, 2, \cdots, n$, so that among the first n positive integers the proportion that are contained in a given set A is just $P_n(A)$ and hence (1) is the same thing as

$$D(A) = \lim_{n\to\infty} P_n(A). \tag{4}$$

Thus the Erdös-Kac Theorem states that

$$\lim_{n\to\infty} P_n\left\{m: \frac{\nu(m) - \log\log m}{(\log\log m)^{\frac{1}{2}}} \leq x\right\} = \Phi(x), \tag{5}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\,du, \tag{6}$$

and the Hardy-Ramanujan Theorem can be similarly recast.

To see why (5) might be true, observe first that

$$\nu(m) = \sum_p \delta_p(m), \tag{7}$$

where $\delta_p(m)$ is 1 or 0 according as the prime p divides m or not. For a positive integer a, the number of multiples of a not exceeding n is $[n/a]$, the integral part of n/a, so that

$$P_n\{m: a \mid m\} = \frac{1}{n}\left[\frac{n}{a}\right], \tag{8}$$

which is nearly $1/a$ for large n. If $p_1, \cdots, p_k$ are distinct primes, then $p_i \mid m$ for all i if and only if $\Pi_i p_i \mid m$, so that the intersection of the k sets $\{m: \delta_{p_i}(m) = 1\}$ has P_n-measure $n^{-1}[n/\Pi_i p_i]$, which, for large n, is near the product $\Pi_i n^{-1}[n/p_i]$ of their individual P_n-measures. Thus, if m is chosen at random from 1 to n—according to P_n, that is—and if n is large, then the random variables $\delta_{p_1}(m), \cdots, \delta_{p_k}(m)$ are approximately independent, and there is some hope that the sum (7) will obey the central limit theorem when properly normalized. It is this sort of statistical plausibility argument that led originally to the surmise that (5) might be true.

In the proof of the Erdös-Kac Theorem, we shall use from number theory (in addition to the fundamental theorem of arithmetic) only the estimate

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1); \tag{9}$$

see [6; p. 351], for example. From probability theory, we shall use the facts embodied in the following four remarks, proofs of which can be found in Feller's book [3] and elsewhere.

REMARK 1. If a random variable D_n converges in probability to 0, which will be true in particular if $E\{|D_n|\} \to 0$, then a second random variable U_n (on the same probability space) has a given limiting distribution (say Φ as defined by (6)) if and only if $U_n + D_n$ does [3; p. 247]. If D_n converges in probability to 0 and the distribution of U_n converges to Φ, then $D_n U_n$ converges in probability to 0 (since $P\{|D_n U_n| > \epsilon\}$ is at most $P\{|D_n| > \epsilon/x\} + P\{|U_n| > x\}$, its limit superior is at most $2(1 - \Phi(x))$; let $x \to \infty$). If A_n converges in probability

to 1 and B_n to 0, then U_n has limiting distribution Φ if and only if $A_nU_n+B_n$ does (compare each with $A_nU_n=U_n+(A_n-1)U_n$).

REMARK 2. Since Φ is determined by its moments

$$\mu_r = \int_{-\infty}^{\infty} x^r\, d\Phi(x), \tag{10}$$

if distribution functions F_n satisfy $\int_{-\infty}^{\infty} x^r\, dF_n(x) \to \mu_r$ for $r=1, 2, \cdots$, then $F_n(x) \to \Phi(x)$ for each x [3; p. 262], which is the basis of the method of moments.

REMARK 3. If $F_n(x) \to \Phi(x)$ for each x, and if $\int_{-\infty}^{\infty} |x|^{r+\epsilon} dF_n(x)$ is bounded in n for some positive ϵ, then $\int_{-\infty}^{\infty} x^r\, dF_n(x) \to \mu_r$ [3; p. 245].

REMARK 4. If U_1, $U_2, \cdots$ are independent, uniformly bounded random variables with mean 0 and finite variances σ_i^2, and if $\sum \sigma_i^2$ diverges, then the distribution of $\sum_{i=1}^n U_i/(\sum_{i=1}^n \sigma_i^2)^{1/2}$ converges to Φ [3; p. 258], a special case of the central limit theorem.

From Remark 1 it follows that, if (5) holds for all x and $g_m \to \infty$, then $(\nu(m)-\log\log m)g_m^{-1}(\log\log m)^{-1/2}$, regarded as a random variable under the probability measure P_n, converges in probability to 0 as $n\to\infty$, which implies (2). Thus the Erdös-Kac Theorem contains the Hardy-Ramanujan Theorem.

Remark 1, together with the fact that log log m increases very slowly, can also be used to cast (5) in a more convenient form, namely,

$$\lim_{n\to\infty} P_n\left\{m: \frac{\nu(m) - \log\log n}{(\log\log n)^{1/2}} \leqq x\right\} = \Phi(x). \tag{11}$$

The equivalence of (5) and (11) will follow if we show that, for each positive ϵ,

$$\lim_{n\to\infty} P_n\left\{m: \left|\frac{\log\log m - \log\log n}{(\log\log n)^{1/2}}\right| > \epsilon\right\} = 0. \tag{12}$$

If $n^{1/2} < m \leqq n$ and the inequality in (12) holds, then

$$\log\log n^{1/2} < \log\log n - \epsilon(\log\log n)^{1/2},$$

which implies $\log\log n < \epsilon^{-2}\log^2 2$. Therefore, for all n exceeding some $n_0(\epsilon)$, the probability in (12) is not greater than $P_n\{m: m\leqq n^{1/2}\}$, which certainly goes to 0. Thus the Erdös-Kac Theorem is equivalent to (11), and this is the form in which we shall prove it.

Proof: first part. The heuristic ideas favoring the Erdös-Kac Theorem (see (7)) figure in its proof as well. We shall compare the behavior of the $\delta_p(m)$ with that of independent random variables X_p (on some probability space, one variable for each prime p) satisfying

$$P\{X_p = 1\} = \frac{1}{p}, \qquad P\{X_p = 0\} = 1 - \frac{1}{p}. \tag{13}$$

The point of the heuristic argument is that the P_n-measure of $\{m: \delta_{p_i}(m)=1, i=1, \cdots, k\}$ converges to $P\{X_{p_i}=1, i=1, \cdots, k\}$ if the p_i are distinct. Comparing the δ_p with the X_p indicates also where the norming constants in (11) come from: If $m \leqq n$, then no p actually contributing to the sum (7) can exceed n, so the distributions under P_n of $\nu(m)$ and of $\sum_{p\leqq n} \delta_p(m)$ coincide; the corresponding sum $\sum_{p\leqq n} X_p$ has mean $\sum_{p\leqq n} p^{-1}$ and variance $\sum_{p\leqq n} p^{-1}(1-p^{-1})$, each of order log log n by (9).

As the first step in the proof of (11), we shall show that it is unaffected if we still further restrict the range of p in (7), replacing $\nu(m)$ by

$$\nu_n(m) = \sum_{p\leqq\alpha_n} \delta_p(m), \tag{14}$$

where $\{\alpha_n\}$ is a sequence so chosen that

$$\alpha_n = o(n^\epsilon) \tag{15}$$

for each positive ϵ and

$$\sum_{\alpha_n<p\leqq n} \frac{1}{p} = o(\log\log n)^{\frac{1}{2}}. \tag{16}$$

The requirements (15) and (16) are met for example by

$$\alpha_n = n^{1/\log\log n};$$

this sequence goes to infinity slowly enough for (15), but because of (9), quickly enough for (16).

For a function f of positive integers, let

$$E_n\{f\} = \frac{1}{n}\sum_{m=1}^{n} f(m) \tag{17}$$

denote its expected value computed with respect to P_n. By (8),

$$E_n\left\{\sum_{p>\alpha_n} \delta_p\right\} = \sum_{p>\alpha_n} P_n\{m: \delta_p(m) = 1\} \leqq \sum_{\alpha_n<p\leqq n} \frac{1}{p},$$

and it follows by (16) and Remark 1 that (11) holds for all x if and only if

$$\lim_{n\to\infty} P_n\left\{m: \frac{\nu_n(m) - \log\log n}{(\log\log n)^{\frac{1}{2}}} \leqq x\right\} = \Phi(x) \tag{18}$$

does.

We shall compare (14) with the corresponding partial sum

$$S_n = \sum_{p\leqq\alpha_n} X_p \tag{19}$$

of the independent X_p introduced above. By (9) and (16), the quantities

$$c_n = \sum_{p \leqq \alpha_n} \frac{1}{p}, \qquad s_n^2 = \sum_{p \leqq \alpha_n} \frac{1}{p}\left(1 - \frac{1}{p}\right), \tag{20}$$

which are the mean and variance of S_n, are each $\log\log n + o(\log\log n)^{\frac{1}{2}}$, so that (18) is the same thing as

$$\lim_{n\to\infty} P_n\left\{m: \frac{\nu_n(m) - c_n}{s_n} \leqq x\right\} = \Phi(x). \tag{21}$$

Proof: second part. Since they depend only on Remark 1, the reductions thus far (from (3) to (5) to (11) to (18) to (21)) present no technical difficulty, although it is by no means trivial that such steps as the truncation in (14)—which figures in the original proof of Erdös and Kac as well as in that of Halberstam—do advance the solution of the problem. It remains to prove (21), which is the hard part.

Now (21) will follow by the method of moments (Remark 2) if we prove that, for $r = 1, 2, \cdots$,

$$E_n\{(\nu_n - c_n)^r / s_n^r\} \tag{22}$$

converges to μ_r (defined by (10)) as $n \to \infty$. To prove that (22) converges to μ_r, we shall first show that its difference with

$$E\{(S_n - c_n)^r / s_n^r\} \tag{23}$$

converges to 0 (it is here we make rigorous the heuristic argument) and then show that (23) itself converges to μ_r.

By the multinomial theorem and the definition (19), $E\{S_n^r\}$ is the sum

$$\sum_{u=1}^{r} \sum{}' \frac{r!}{r_1! \cdots r_u!} \sum{}'' E\{X_{p_1}^{r_1} \cdots X_{p_u}^{r_u}\}, \tag{24}$$

where $\sum'$ extends over those u-tuples $(r_1, \cdots, r_u)$ of positive integers satisfying $r_1 + \cdots + r_u = r$ and $\sum''$ extends over those u-tuples $(p_1, \cdots, p_u)$ of primes satisfying $p_1 < \cdots < p_u \leqq \alpha_n$. (We interpret a sum as 0 if its range is empty.) Since X_p assumes only the values 0 and 1, from the independence of the X_p and the fact that the p_i are distinct it follows that the summand in (24) is

$$E\{X_{p_1} \cdots X_{p_u}\} = \frac{1}{p_1 \cdots p_u}. \tag{25}$$

By the definition (14), $E_n\{\nu_n^r\}$ is just (24) with the summand replaced by $E_n\{\delta_{p_1}^{r_1} \cdots \delta_{p_u}^{r_u}\}$. Since $\delta_p(m)$ assumes only the values 0 and 1, from (8) and the fact that the p_i are distinct it follows that this summand is

$$E_n\{\delta_{p_1} \cdots \delta_{p_u}\} = \frac{1}{n}\left[\frac{n}{p_1 \cdots p_u}\right]. \tag{26}$$

But (25) and (26) differ by at most $1/n$, and hence $E\{S_n^r\}$ and $E_n\{\nu_n^r\}$ cannot differ by more than the sum (24) with the summand replaced by $1/n$. It now follows by the multinomial theorem that

$$(27) \qquad \left| E\{S_n^r\} - E_n\{\nu_n^r\}\right| \leqq \frac{1}{n}\left\{\sum_{p\leqq\alpha_n} 1\right\}^r \leqq \frac{\alpha_n^r}{n},$$

an inequality valid for $r=0, 1, \cdots$. Now

$$E\{(S_n - c_n)^r\} = \sum_{k=0}^{r}\binom{r}{k}E\{S_n^k\}(-c_n)^{r-k},$$

and $E_n\{(\nu_n-c_n)^r\}$ has the analogous expansion. Comparing the two expansions term for term and applying (27) we see that

$$\left| E\{(S_n - c_n)^r\} - E_n\{(\nu_n - c_n)^r\}\right| \leqq \sum_{k=0}^{r}\binom{r}{k}\frac{\alpha_n^k}{n}c_n^{r-k} = \frac{1}{n}(\alpha_n + c_n)^r.$$

Since $(\alpha_n+c_n)^r/n\to 0$, as follows by (15) and the inequality $c_n\leqq\alpha_n$, the difference between (22) and (23) does go to 0.

There remains only the purely probabilistic problem of showing that (23) converges to μ_r. Although this can be established by the sort of calculation used in the last century to prove the central limit theorem, here we shall deduce it *from* the central limit theorem. By Remark 4, the distribution of $(S_n-c_n)/s_n$ converges to Φ. That (23) converges to μ_r will therefore follow from Remark 3 if we show that the moments (23) are bounded in n when r is increased to the next larger even integer. We shall in fact show that

$$(28) \qquad \sup_n \left| E\{(S_n - c_n)^r/s_n^r\}\right| < \infty$$

for every r.

Put $Y_p=X_p-1/p$. By the multinomial theorem and the independence of the Y_p,

$$(29) \qquad E\{(S_n - c_n)^r\} = \sum_{u=1}^{r}\sum{}' \frac{r!}{r_1!\cdots r_u!}\sum{}'' E\{Y_{p_1}^{r_1}\}\cdots E\{Y_{p_u}^{r_u}\},$$

where $\sum'$ and $\sum''$ have the same ranges they have in (24). Since $E\{Y_p\}=0$, (29) still holds if we require in $\sum'$ that each r_i exceed 1. Since $|Y_p|\leqq 1$, $r_i\geqq 2$ implies $|E\{Y_p^{r_i}\}|\leqq E\{Y_p^2\}$, so that the inner sum in (29) has modulus at most

$$\sum{}'' E\{Y_{p_1}^2\}\cdots E\{Y_{p_u}^2\} \leqq \left[\sum_{p\leqq\alpha_n} E\{Y_p^2\}\right]^u = s_n^{2u}.$$

But if $r_1, \cdots, r_u$ add to r and each is at least 2, then $2u\leqq r$. For n large enough that $s_n\geqq 1$, (29) now implies

$$|E\{(S_n - c_n)^r\}| \leqq s_n^r \sum_{u=1}^{r} \sum' \frac{r!}{r_1! \cdots r_u!},$$

from which (28) follows.

This completes the proof of the Erdös-Kac Theorem in the form (11). We showed that replacing ν by ν_n as defined by (14) has no effect on (11), and we proved the modified (11) by the method of moments, showing that the moments of ν_n (normalized) are near those of the corresponding sum S_n defined by (19) and that the latter moments converge to the μ_r. We never did analyze the moments of ν itself, although it is not hard to go on and do so (see [4]).

It is easy to show that the Erdös-Kac and Hardy-Ramanujan theorems hold also if each prime divisor is counted according to its multiplicity: Let $\delta_p'(m)$ be the exponent of p in the prime factorization of $m = \Pi_p\, p^{\delta_p'(m)}$ and define $\nu'(m) = \sum_p \delta_p'(m)$. For $k \geqq 1$, $\delta_p'(m) - \delta_p(m) \geqq k$ if and only if $p^{k+1} | m$, an event which by (8) has P_n-measure at most $1/p^{k+1}$; hence $E_n\{\delta_p' - \delta_p\} = \sum_{k=1}^{\infty} P_n\{m: \delta_p'(m) - \delta_p(m) \geqq k\} \leqq 2/p^2$, which implies $E_n\{\nu' - \nu\} = O(1)$. It follows by Remark 1 that (11) persists if $\nu(m)$ is replaced by $\nu'(m)$, and by the same arguments as before we can successively deduce (5), (3), and (2) with $\nu'(m)$ in place of $\nu(m)$.

The arguments here suffice with little change, as in [4], for the central limit theorem for a completely additive f for which $f(p)$ is bounded and $\sum f^2(p)/p$ diverges. For an introduction to probability methods in number theory, see [8]; for a comprehensive account, see [9].

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Editorial Note. One should compare the famous Besicovitch result presented in the chapter "Lion and Man", pp. 135–36 of J. E. Littlewood, *A Mathematician's Miscellany*, Methuen (1953).

MINIMAL REGULAR EXTENSIONS OF ORIENTED GRAPHS

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The following problem has been solved by Erdös and Kelly [1]. Given a graph G with maximum degree r, determine the minimum number of vertices in an r-regular graph having G as an induced subgraph. In this note the corresponding problem is solved for oriented and directed graphs. The latter case is very similar to that solved by Erdös and Kelly, being a straightforward generalization of their work; however, the former case is significantly different and we will concentrate on it.

The distinction between the two classes of structures is that an oriented graph is an asymmetric directed graph; that is, in an oriented graph there is at most one arc joining two vertices, while in a directed graph there is at most one arc in each direction joining two vertices. Figure 1 shows examples of both kinds of relations.

oriented graph　　　　directed graph

FIG. 1

Other definitions which will be required are the following. The *in-degree* of a vertex v is the number of arcs to it; the *out-degree* is the number from it. The *degree-pair* of v is the ordered pair of its in-degree and out-degree. If the maximum in- or out-degree in a directed graph is r, the *in-deficiency* (resp., *out-deficiency*) of a vertex is the difference between r and its in-degree (resp., out-degree).

A directed graph is *regular* if the in- and out-degrees of all vertices are the same. If this number is r, the graph is often called *r-regular*. A subgraph of a directed graph G is said to be *induced* (by its set of vertices) if it contains all arcs of G joining two of its vertices.

Let G be an oriented (resp., directed) graph and let r denote the maximum of the in- and out-degrees in G. An r-regular oriented (resp., directed) graph

of which G is an induced subgraph is called a *regular extension* of G. That such a regular extension always exists will follow from our theorems on minimum order. Regular extensions of minimum order are shown in Figure 2 for the two structures in Figure 1.

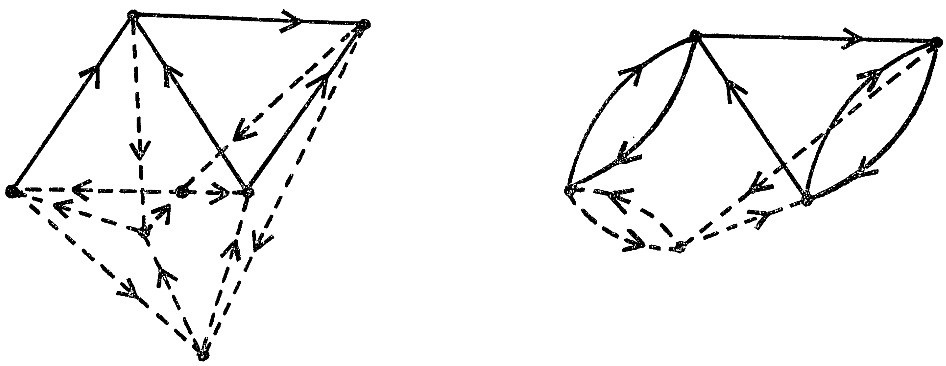

FIG. 2

Two lemmas will be useful in providing a constructive solution to the problem, and these involve sequences of pairs of nonnegative integers. Such a sequence $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ will be called *tight* if the following inequalities hold for all i, j:

$$|a_i - a_j| \leqq 1, \qquad |b_i - b_j| \leqq 1, \quad \text{and} \quad |a_i - a_j + b_i - b_j| \leqq 1.$$

In other words, two pairs in a tight sequence differ by at most 1 in each place, and if they differ in both places, the sums of their entries are equal. Tight sequences can be considered of two types, depending on whether there is a pair (a, b) with $a = \max a_i$, $b = \max b_i$. If so, the three possible kinds of pairs in the sequence are

$$(a, b), (a-1, b), \quad \text{and} \quad (a, b-1).$$

If not, the possible kinds are

$$(a-1, b-1), (a-1, b), \quad \text{and} \quad (a, b-1).$$

Of course, it is not necessary that all three kinds occur in either case, but in the former at least the first occurs and in the latter both of the last two occur.

The proof of the first lemma provides a specific way of obtaining a new tight sequence from a given one without modifying both elements of any ordered pair.

LEMMA 1. *Given a tight sequence $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ and two nonnegative integers c and d with $c+d \leqq n$, one can add 1 to c first elements and to d other second elements so that the new sequence is also tight.*

Proof. The proof consists of routine modification of the given sequence in various cases. Let $a = \max a_i$ and $b = \max b_i$.

Case I. First assume there is at least one pair (a, b) and let r, s, and t be the numbers of pairs (a, b), $(a-1, b)$, and $(a, b-1)$, respectively. We consider two subcases:

(Ia) $c>s$ and $d>t$. Add 1 to the first element of all s of the pairs $(a-1, b)$ and to $c-s$ of the pairs (a, b), and add 1 to the second element of all t of the $(a, b-1)$ and to $d-t$ of the (a, b). The result is clearly a tight sequence.

(Ib) $c \leqq s$ or $d \leqq t$. Without loss of generality we assume $d \leqq t$. Change d of the $(a, b-1)$ to (a, b). Then add 1 to the first element of c ordered pairs beginning with the $(a-1, b)$, then the $(a, b-1)$, and finally the (a, b). The resulting sequence will always be tight.

Case II. Now assume there are no pairs of the form (a, b), and let r, s, and t be the numbers of pairs $(a-1, b-1)$, $(a-1, b)$, and $(a, b-1)$, respectively. Again there are subcases to be considered:

(IIa) $c+d \leqq r$. Add 1 to the first element of c of the pairs $(a-1, b-1)$ and to the second element of d of the other $(a-1, b-1)$.

(IIb) $c>r+s$ or $d>r+t$. The two are analogous, so we assume the latter. Add 1 to the first element of c of the $(a-1, b)$ and to the second element of all of the $(a, b-1)$ and $(a-1, b-1)$ and of as many of the $(a-1, b)$ as necessary.

(IIc) Otherwise. Choose an integer k so that max $(c-s, r-d, 0) \leqq k \leqq$ min $(c, t+r-d, r)$; this is always possible. Add 1 to the first element of k of the $(a-1, b-1)$ and of $c-k$ of the $(a-1, b)$, and add 1 to the second element of the other $r-k$ of the $(a-1, b-1)$ and of $d-r+k$ of the $(a, b-1)$. In all cases the resulting sequence is clearly tight, which completes the proof.

The second lemma establishes some conditions under which a tight sequence is the sequence of degree-pairs of the vertices of an oriented graph. Its proof gives a procedure for constructing the graph.

LEMMA 2. *A tight sequence of ordered pairs of nonnegative integers* (a_1, b_1), $(a_2, b_2), \cdots, (a_m, b_m)$ *with* $\sum a_i = \sum b_i \leqq \binom{m}{2}$ *is the sequence of degree-pairs of some oriented graph.*

Proof. The proof is constructive and establishes the result by induction. The result clearly holds for the sequence $(0, 1)$, $(1, 0)$. Assume that it holds for sequences of length less than m. We give a way of modifying the given sequence of length m to obtain one of length $m-1$ belonging to an oriented graph, with the modification being such that the addition of one vertex and new arcs yields a graph with the given pairs as degree-pairs.

Let $S = \sum a_i$ and let a denote the least integer not less than S/m. There are two cases to consider depending on whether (a, a) is in the tight sequence.

If (a, a) appears, all pairs are of the types (a, a), $(a, a-1)$, or $(a-1, a)$. Delete one (a, a) from the sequence. Alternately change pairs (a, a) to $(a-1, a)$ and $(a, a-1)$, and then $(a, a-1)$ and $(a-1, a)$ to $(a-1, a-1)$ until 1 has been subtracted from a first elements and a second elements. This new sequence is clearly tight, so that by the induction hypothesis it belongs to some oriented

graph provided the sum $S-2a$ of first elements is no greater than $\binom{m-1}{2}$. But $2a \geqq 2S/m$, so that

$$S - 2a \leqq S - \frac{2S}{m} = S\left(\frac{m-2}{m}\right) \leqq \frac{m(m-1)}{2}\cdot\frac{m-2}{m} = \binom{m-1}{2}.$$

If there is no pair (a, a) in the given sequence, then each sum of elements a_i+b_i is at most $2a-1$, and hence $(2a-1)m \geqq 2S$. In this case the pairs are of the types $(a, a-1)$, $(a-1, a)$, and possibly $(a-1, a-1)$. Let s be the number of $(a, a-1)$ and remove one of these. If $a \leqq s$, change a of the pairs $(a-1, a)$ and $a-1$ of the pairs $(a, a-1)$ to $(a-1, a-1)$. If $a > s$, change all of the $(a-1, a)$ and $(a, a-1)$ to $(a-1, a-1)$ and $a-s$ of the original pairs $(a-1, a-1)$ to $(a-2, a-1)$ and $(a-1, a-2)$. In either case the resulting sequence is tight. The sum of new first elements again satisfies the desired inequality:

$$S - (2a - 1) \leqq S - \frac{2S}{m} \leqq \binom{m}{2}\left(1 - \frac{m}{2}\right) = \binom{m-1}{2}.$$

Therefore, in both cases, there is by the induction hypothesis an oriented graph with the new tight sequence as its degree-pairs. A new vertex can clearly now be added with arcs to and from the vertices corresponding to pairs which were modified so that the result is an oriented graph with the given sequence as degree-pairs.

The proofs of these lemmas provide the construction implicit in the proof of the existence of the regular extension.

THEOREM 1. *Let G be an oriented graph having maximum in- or out-degree r, sum of in-deficiences s and maximum combined deficiency t. The minimum order of a regular oriented extension of G is $m+n$, where n is the order of G and m is the least integer satisfying the conditions*

$$m \geqq t, \tag{1}$$

$$mr \geqq s, \tag{2}$$

$$\frac{m(m-1)}{2} \geqq mr - s. \tag{3}$$

In fact we prove somewhat more:

(a) In any regular oriented extension of G having order $m+n$, m satisfies the three conditions.

(b) For every m satisfying these conditions, there exists a regular oriented extension of G having order $m+n$.

Proof. We first prove statement (a). Let H be a regular oriented extension of G having order $m+n$.

(1) Clearly the number of vertices of H that are not in G can be no less than the combined deficiency of any vertex of G, so that $m \geqq t$.

(2) Since each of the m vertices which have been added to the original graph has at most r edges directed to the graph G and at most r edges directed from G, we must have $mr \geqq s$.

(3) Let F be the subgraph of H induced by the vertices not in G. The sum of the in-degrees of these vertices as vertices of F is $mr-s$. But F can have at most $m(m-1)/2$ edges, and hence at most $m(m-1)/2$ in-degrees. Thus $m(m-1)/2 \geqq mr-s$.

Statement (b) will be established by providing a construction for a regular extension of G with $m+n$ vertices, where m satisfies all three conditions. Let the vertices of G be $v_1, v_2, \cdots, v_n$, and let I be a set of m other vertices $u_1, u_2, \cdots, u_m$. Let c_i and d_i be the in- and out-deficiencies of v_i. We first make joins between G and I as follows. Join v_1 to the first c_1 vertices of I and join the next d_1 to v_1. At this stage the degree-pairs of the vertices of I form a tight sequence. Assume that the first $i-1$ vertices of G have been joined to the vertices of I so that the degree-pairs of I form a tight sequence and those $i-1$ vertices of G have degree-pairs (r, r). By condition (1), $c_i+d_i \leqq m$, so that from Lemma 1 we know that arcs can be added from v_i to c_i vertices of I and to v_i from d_i other vertices of I so that the degree-pairs of vertices in I again form a tight sequence. Therefore, by induction, joins can be made between G and I so that the result is an oriented graph in which all the vertices of G have degree-pairs (r, r) and the degree-pairs of the vertices of I form a tight sequence. Furthermore, from condition (2) we know that no in-degree or out-degree exceeds r.

Next we consider the pairs $(r-e_i, r-f_i)$, for $i=1, 2, \cdots, m$, where e_i and f_i are the number of joins with G to u_i and from u_i respectively. These form a tight sequence of length m with sum of first elements (and of second elements) being $mr-s$. By (3) this does not exceed $\binom{m}{2}$, so that Lemma 2 implies the existence of an oriented graph on I having these pairs as degree-pairs. Together with the earlier construction, this provides an r-regular oriented extension of G.

We now give three examples in Figure 3 illustrating that each of the three conditions can be the one which determines the minimum order of a regular extension.

Condition (1) will be the determining factor in choosing m if the combined deficiency of one vertex is large in comparison to the others. An example of this is given by G_1 in which $r=1$, $s=1$, and $t=2$. Conditions (2) and (3) are satisfied by $m=1$; however, condition (1) requires $m \geqq 2$, and we obtain the regular oriented graph H_1.

Condition (2) will be the important one if the sum of the in-deficiencies is large in comparison to the largest in- or out-degree. This situation occurs in G_2, where we have $r=2$, $s=5$, and $t=2$. Then conditions (1) and (3) are satisfied by $m=2$, but condition (2) demands that $m=3$, and we have the extension H_2.

Condition (3) becomes crucial when the maximum in- or out-degree of the vertices is large compared to the sum of the in-deficiencies. This is the case in G_3, where $r=2$, $s=1$, and $t=1$. Conditions (1) and (2) are seen to be satisfied by $m=1$, but (3) requires that m be at least 5. This value gives H_3 as an extension.

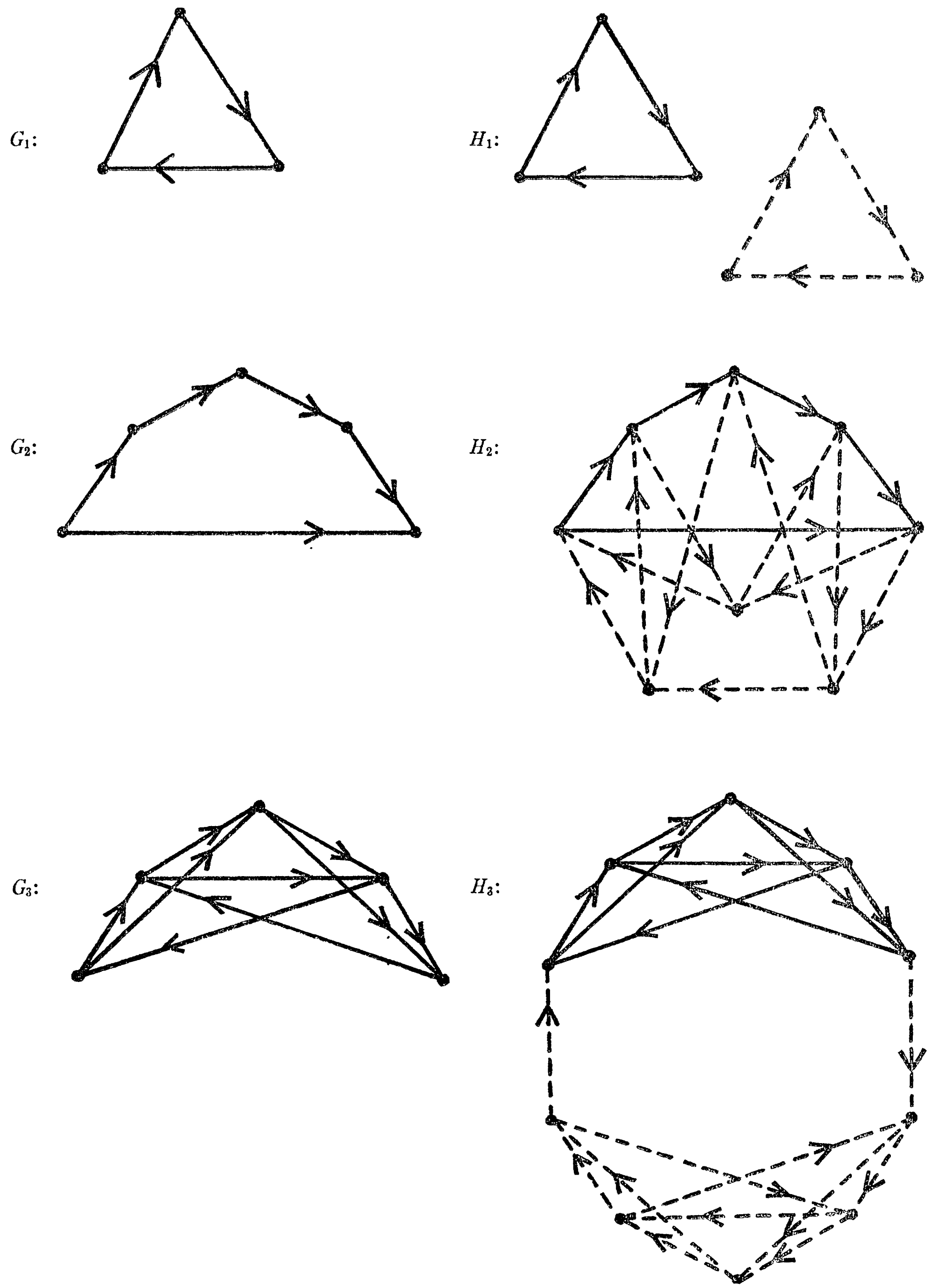

FIG. 3

The two requirements imposed on a regular extension, that the given graph be an induced subgraph and that the degree of regularity be the maximum in- or out-degree in the given graph, are important. The preceding example illustrates this. Both of the regular graphs of Figure 4 have G_3 as a subgraph and have smaller order than H_3. However, the first is 2-regular but doesn't have G_3 as an induced subgraph; the second has G_3 as an induced subgraph but is 3-regular.

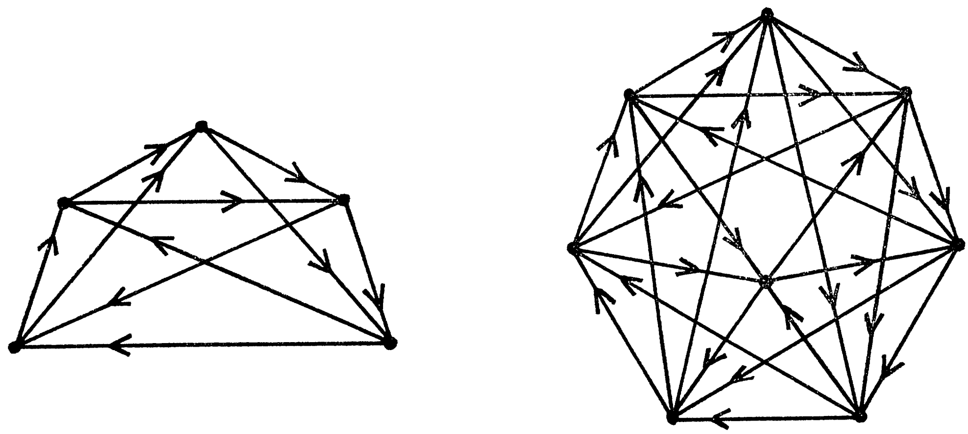

FIG. 4

Finally, we conclude with the theorem on regular extensions for directed graphs. Its proof will be omitted.

THEOREM 2. *Let G be a directed graph of order n having maximum in- or out-degree r, sum of in-deficiencies s, and maximum in- or out-deficiency q. The minimum order of a regular directed extension of G is $m+n$, where m is the least integer satisfying the conditions*

$$m \geqq q, \tag{1}$$

$$mr \geqq s, \tag{2}$$

$$m(m-1) \geqq mr - s. \tag{3}$$

Reference

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ORTHOGONAL MATRICES OVER FINITE FIELDS

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Introduction. Let $q=p^m$, p a prime, and let $GF(q)$ be the finite field of q elements. We consider square matrices of size $n\times n$ over $GF(q)$. The group of invertible matrices of this size is called the general linear group, $GL(n, q)$. The identity matrix (the unit of $GL(n, q)$)is I, and the transpose of a matrix M is denoted by M^T. The group of matrices with the property that $UU^T=I$ is called here the orthogonal group, $\mathcal{O}(n, q)$. ($\mathcal{O}(n, q)$ has a subgroup of index two consisting of these U for which det $U=+1$; this subgroup is also sometimes called the orthogonal group.) $\mathbf{x}$ denotes a row vector of n symbols of $GF(q)$.

Let A be a symmetric matrix of $GL(n, q)$. If $p\neq 2$, the quadratic form $\mathbf{x}A\mathbf{x}^T$ determines A uniquely, and $\mathcal{O}(n, q)$ is defined by the property that $U\in\mathcal{O}(n, q)$ leaves invariant the quadratic form $\mathbf{x}I\mathbf{x}^T$. (That is, $\mathbf{x}UI(\mathbf{x}U)^T=\mathbf{x}I\mathbf{x}^T\Leftrightarrow UU^T=I$.) This definition does not work for $p=2$, a fact which is generally regarded as an annoying exception to a beautiful theory. In particular, it is quite hard to find a reference which gives the order of $\mathcal{O}(n, 2^m)$.

It is possible to take a different point of view. There is an obvious way of finding the order of $\mathcal{O}(n, q)$ which turns out to be very easy for $p=2$, and much more difficult (showing in fact why one has to use quadratic forms) for $p>2$.

The method is as follows:

To each right coset $\{MU, U\in\mathcal{O}\}$ of $\mathcal{O}$ in $GL(n, q)$ is associated a unique symmetric matrix $A=MM^T$. To find the order of $\mathcal{O}$ it suffices to know the number of invertible symmetric matrices which can be factored in this way.

It was shown by A. A. Albert in 1938 [1] that a symmetric matrix in $GL(n, 2^m)$ can be factored in the form MM^T if and only if it has at least one nonzero term on the main diagonal. Thus it suffices to know the number of symmetric matrices in $GL(n, 2^m)$ and the number of these which have a main diagonal consisting entirely of zeros. Both these numbers can be calculated by very unsophisticated methods.

It is known (this is where we need quadratic forms) that an invertible symmetric matrix A over $GF(q)$, $p>2$, can be factored as $A=MM^T$ if and only if det A is a square in $GF(q)$ [2]. The total number of symmetric matrices in $GL(n, q)$ is known, so it suffices to find how many have a square or nonsquare determinant. This is trivial for odd values of n, and quite tedious for even values.

The paper is arranged as follows:

Section I contains a proof of A. A. Albert's theorem (to show that it really is easy).

In Section II we obtain the number of symmetric matrices in $GL(n, q)$ and the number of symmetric matrices with zero on the main diagonal in $GL(n, 2^m)$. This gives us the order of $\mathcal{O}(n, 2^m)$.

In Section III we discuss the parallel procedure for $p>2$.

Section I. The purpose of this section is to prove the following theorem:

THEOREM 1. *Let $A=(a_{ij})$ be an invertible symmetric $t \times t$ matrix over $GF(2^m)$. A can be factored in the form $A=MM^T$ if and only if $a_{ii} \neq 0$ for at least one i, $1 \leqq i \leqq t$.*

The proof is given in several stages.

Let $\mathbf{x}=(x_1, x_2, \cdots, x_t)$ be a vector of $GF(2^m)^t$. Then

$$\mathbf{x}A\mathbf{x}^T = \sum_{i=1}^{t} a_{ii}x_i^2 = \left(\sum_{i=1}^{t} \alpha_i x_i\right)^2,$$

where α_i is uniquely defined by the equation $\alpha_i^2 = a_{ii}$ [3].

Let $V(A)$ be the set of $\mathbf{x}$ for which $\mathbf{x}A\mathbf{x}^T=0$. $V(A)$ is the whole of $GF(2^m)^t$ if $a_{ii}=0$ all i, and otherwise is a linear subspace of dimension $(t-1)$, consisting of the $\mathbf{x}$ for which $\sum_{i=1}^{t} \alpha_i x_i = 0$.

$V(A)=V(B)$ if and only if $a_{ii}=b_{ii}$, $i=1, \cdots, t$; in particular $V(A)=V(I)$ if and only if $a_{ii}=1$, $i=1, \cdots, t$.

LEMMA 1. *If $A=MM^T$, then $a_{ii} \neq 0$ for some i.*

Proof. M is by hypothesis invertible, so the set $\mathbf{x}M$ consists of all vectors of $GF(2^m)^t$. Thus the set of $\mathbf{x}$ such that $\mathbf{x}M(\mathbf{x}M)^T=0$ ($\mathbf{x}A\mathbf{x}^T=0$) has dimension $t-1$.

This proves the necessity of the condition of Theorem 1; the sufficiency takes somewhat longer.

Let A_{ij} be the co-factor of a_{ij} in A.

LEMMA 2. *If $a_{ii} \neq 0$ for some i, then $A_{jj} \neq 0$ for some j. In words, A has at least one principal minor of rank $t-1$.*

Proof. $A^{-1}=(A_{ij})^T/(\det A)$. Suppose $A_{ii}=0$, $i=1, \cdots, t$; then $\mathbf{x}A^{-1}\mathbf{x}^T=0$ for all $\mathbf{x}$.

Take $\mathbf{x}$ such that $\mathbf{x}A\mathbf{x}^T \neq 0$. We then have the contradiction

$$0 \neq \mathbf{x}A\mathbf{x}^T = \mathbf{x}AA^{-1}A\mathbf{x}^T = (\mathbf{x}A)A^{-1}(\mathbf{x}A)^T = 0.$$

LEMMA 3. *If $a_{ii} \neq 0$ for some i, there exists an invertible matrix N such that the main diagonal of NAN^T consists entirely of 1.*

Proof. $V(A)$ is a space of dimension $t-1$. Let N be the matrix of a nonsingular linear transformation of $GF(2^m)^t$ which takes $V(I)$ into $V(A)$.

Then $\mathbf{x}NAN^T\mathbf{x}^T=0$ if and only if $\mathbf{x}I\mathbf{x}^T=0$. Thus $V(NAN^T)=V(I)$, and each element of the main diagonal of NAN^T is 1.

It now suffices to prove the sufficiency part of Theorem 1 for the case in which $A=(a_{ij})$ with $a_{ii}=1$, $i=1, \cdots, t$. The proof is by induction on t; for

$t=1$ the only choice is $A=1$, which can be factored. Moreover, by Lemma 2 we may suppose that A is of the form

$$A = \begin{bmatrix} 1 & z_2\, z_3 \cdots z_t \\ z_2 & \\ z_3 & \\ \cdot & B \\ \cdot & \\ \cdot & \\ z_t & \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{z} \\ \mathbf{z}^T & B \end{bmatrix}$$

where B is a nonsingular symmetric matrix of size $(t-1)\times(t-1)$, and $b_{ii}=1$, $i=1, \cdots, t-1$.

Since B has rank $t-1$, $\mathbf{z}$ is linearly dependent on the rows of B, say $\mathbf{z}=\mathbf{n}B$, $\mathbf{n}=(\eta_2, \eta_3, \cdots, \eta_t)$.

If $1=\sum_{i=2}^{t}\eta_i z_i$, then the first row of A is linearly dependent on the other rows, which is not the case, since A has rank t. Set $\xi^2=\sum_{i=2}^{t}\eta_i z_i=\mathbf{n}\mathbf{z}^T=\mathbf{n}B\mathbf{n}^T$; then $\xi^2\neq 1$. By induction, $B=LL^T$. Set

$$\boldsymbol{\zeta} = \mathbf{n}L, \qquad M = \begin{pmatrix} 1+\xi & \boldsymbol{\zeta} \\ \mathbf{0}^T & L \end{pmatrix},$$

where $\mathbf{0}^T$ denotes a column of zeros. Then

$$MM^T = \begin{pmatrix} 1+\xi & \boldsymbol{\zeta} \\ \mathbf{0}^T & L \end{pmatrix}\begin{pmatrix} 1+\xi & \mathbf{0} \\ \boldsymbol{\zeta}^T & L^T \end{pmatrix} = \begin{pmatrix} 1+\xi^2+\boldsymbol{\zeta}\boldsymbol{\zeta}^T & \boldsymbol{\zeta}L^T \\ L\boldsymbol{\zeta}^T & LL^T \end{pmatrix}.$$

Now

$$\begin{aligned} \boldsymbol{\zeta}\boldsymbol{\zeta}^T &= \mathbf{n}LL^T\mathbf{n}^T = \mathbf{n}B\mathbf{n}^T = \xi^2, \\ \boldsymbol{\zeta}L^T &= \mathbf{n}LL^T = \mathbf{n}B = \mathbf{z}, \\ LL^T &= B. \end{aligned}$$

Thus

$$MM^T = \begin{pmatrix} 1 & \mathbf{z} \\ \mathbf{z}^T & B \end{pmatrix} = A,$$

and the proof of Theorem 1 is complete.

Section II. Let $N(t, r)$ denote the number of symmetric matrices of size $t\times t$, rank r, with entries in a finite field $GF(q)$, $q=p^n$.

Let $N_0(t, r)$ denote the number of symmetric matrices of size $t\times t$, rank r, with entries in $GF(2^n)$, and 0 on the main diagonal.

The two theorems proved in this section are as follows:

THEOREM 2.

$$N(t, 2s) = \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1}\cdot \prod_{i=0}^{2s-1}(q^{t-i}-1), \qquad 2s \leqq t,$$

$$N(t, 2s+1) = \prod_{i=1}^{s} \frac{q^{2i}}{q^{2i}-1} \cdot \prod_{i=0}^{2s} (q^{t-i}-1), \qquad 2s+1 \leqq t.$$

THEOREM 3.

$$N_0(t, 2s) = \prod_{i=1}^{s} \frac{q^{2i-2}}{q^{2i}-1} \prod_{i=0}^{2s-1} (q^{t-i}-1), \qquad N_0(t, 2s+1) = 0.$$

The restriction to fields of characteristic 2 is essential for Theorem 3, but Theorem 2 is not so restricted. The formulae of Theorem 2, for q odd, are those obtained by Carlitz [6], Theorem 3, as a special case of a more general result.

The proof takes several steps, but the most sophisticated mathematical notion used is that the row rank of a matrix is the same as its column rank.

A symmetric matrix of size $(t+1)\times(t+1)$ may be exhibited as

$$A(t+1) = \begin{pmatrix} y_0 & y_1 \cdots y_t \\ y_1 & \\ \cdot & \\ \cdot & A(t) \\ \cdot & \\ y_t & \end{pmatrix} = \begin{pmatrix} y_0 & \mathbf{y} \\ \mathbf{y}^T & A(t) \end{pmatrix},$$

where $A(t)$ is a symmetric $t\times t$ matrix which we suppose to have rank r. The rank of $A(t+1)$ is given by the following lemma:

LEMMA 4. *From a particular $A(t)$ of rank r we obtain*

(i) $q^{t+1}-q^{r+1}$ *matrices* $A(t+1)$ *of rank* $r+2$.

(ii) $(q-1)q^r$ *matrices* $A(t+1)$ *of rank* $r+1$.

(iii) q^r *matrices* $A(t+1)$ *of rank* r.

Note that the sum of these numbers is q^{t+1}, the number of choices for $y_0, y_1, \cdots, y_t$, and that (i) *is zero if $r=t$, as it should be.*

Proof. If $\mathbf{y}$ is not linearly dependent on the rows of $A(t)$, then $A(t+1)$ has rank $r+2$; for the matrix $\binom{\mathbf{y}}{A(t)}$ has rank $r+1$, and whatever the choice of y_0, the column $(y_0, \mathbf{y})^T$ is not linearly dependent on the columns of $\binom{\mathbf{y}}{A(t)}$. There are q^t choices for $y_1, \cdots, y_t$, of which q^r are linearly dependent on the rows of $A(t)$. Hence there are $q(q^t-q^r)$ matrices $A(t+1)$ of rank $r+2$.

Suppose $\mathbf{y}$ linearly dependent on the rows of $A(t)$, say $\mathbf{y}=\mathbf{n}A(t)$, $\mathbf{n}=(\eta_1, \cdots, \eta_t)$. Then $\binom{\mathbf{y}}{A(t)}$ has rank r.

If $y_0=\mathbf{n}\mathbf{y}^T(=\sum_{i=1}^{t}\eta_i y_i)$, the column $(y_0, \mathbf{y})^T$ is linearly dependent on the columns of $\binom{\mathbf{y}}{A(t)}$ and $A(t+1)$ has rank r. There are q^r such cases.

If $y_0\neq\mathbf{n}\mathbf{y}^T$ then $A(t+1)$ has rank $r+1$. There are $(q-1)q^r$ such cases.

We have then

$$(4.1) \quad N(t+1, r) = q^r N(t, r) + (q-1)q^{r-1}N(t, r-1) + (q^{t+1}-q^{r-1})N(t, r-2).$$

It is clear that $N(t, 0)=1$, and $N(t, r)=0$ for $r>t$. We calculate, by a heroic piece of elementary algebra, that

$$N(t, 1) = q^t - 1,$$
$$N(t, 2) = (q^2/(q^2 - 1))(q^t - 1)(q^{t-1} - 1),$$
$$N(t, 3) = (q^3/(q^2 - 1))(q^t - 1)(q^{t-1} - 1)(q^{t-2} - 1).$$

We then guess that the general formula is

$$N(t, 2s) = \prod_{i=1}^{s} (q^{2i}/(q^{2i} - 1)) \cdot \prod_{i=0}^{2s-1} (q^{t-i} - 1),$$

$$N(t, 2s + 1) = \prod_{i=1}^{s} (q^{2i}/(q^{2i} - 1)) \prod_{i=0}^{2s} (q^{t-i} - 1),$$

and substitution in the recursion formula for $N(t+1, r)$ shows this to be correct.

The elementary algebra is illustrated by the calculation below;

$$\begin{aligned} N(t, 1) &= qN(t - 1, 1) + (q - 1) \\ &= q^2N(t - 2, 1) + q(q - 1) + (q - 1) \\ &= q^tN(0, 1) + (q - 1)(q^{t-1} + q^{t-2} + \cdots + q + 1) \\ &= q^t - 1. \\ N(t, 2) &= q^2N(t - 1, 2) + q(q - 1)N(t - 1, 1) + (q^t - q)N(t, 0) \\ &= q^2N(t - 1, 2) + q^2(q^{t-1} - 1). \end{aligned}$$

We could (in fact did, on the first attempt) iterate this equation as in the previous case. Hindsight shows that it is better to try the solution

$$N(t, 2) = f(q)(q^t - 1)(q^{t-1} - 1).$$

Then, from the recursion formula,

$$f(q)(q^{t-1} - 1)[(q^t - 1) - q^2(q^{t-2} - 1)] = q^2(q^{t-1} - 1)$$
$$f(q) = q^2/(q^2 - 1).$$

Since this is independent of t the guess is justified, and $N(t, 2) = (q^2/(q^2-1))(q^t-1)(q^{t-1}-1)$. Continuing in this way we obtain the formulae above.

In particular, the number of nonsingular symmetric matrices over $GF(q)$ is given by

$$N(2t, 2t) = \prod_{i=1}^{t} (q^{2t+1} - q^{2i}),$$

$$N(2t + 1, 2t + 1) = (q^{2t+1} - 1)N(2t, 2t) = \prod_{i=0}^{t} (q^{2t+1} - q^{2i}).$$

We assume now that $q = 2^m$.

A symmetric matrix of size $(t+1)\times(t+1)$ with zeros on the main diagonal may be written as

$$A(t+1) = \begin{pmatrix} 0 & y \\ y^T & A(t) \end{pmatrix}$$

where $A(t) = (a_{ij})$ has rank r and $a_{ii} = 0$, $i = 1, \cdots, t$.

LEMMA 5. *From a particular $A(t)$ we obtain*
(i) *$q^t - q^r$ matrices $A(t+1)$ of rank $r+2$.*
(ii) *q^r matrices $A(t+1)$ of rank r.*
(iii) *No matrices of rank $r+1$.*

Proof. (i) If y is independent of the rows of $A(t)$, the rank of $A(t+1)$ is $r+2$; we have now only one choice for y_0, so the number of possible cases is $q^t - q^r$.

(ii) Suppose $y = \mathbf{n}A(t)$. Then

$$\sum_{i=1}^{t} \eta_i y_i = \mathbf{n}y^T = \mathbf{n}A\mathbf{n}^T = \sum_{i=1}^{t} a_{ii} y_i^2 \text{ (characteristic 2)} = 0.$$

In this case $A(t+1)$ is always of rank r, and there are q^r choices for y.

Thus the recursion formula for N_0 is

$$N_0(t+1, r) = q^r N_0(t, r) + (q^t - q^{r-2}) N_0(t, r-2).$$

Clearly $N_0(t, 0) = 1$. Further, since

$$N_0(t+1, 1) = qN_0(t, 1) = q^2 N_0(t-1, 1) = \cdots$$

and $N_0(1, 1) = 0$, we have $N_0(t, 1) = 0$. It follows that $N_0(t, 2s+1) = 0$.

Proceeding as before by elementary algebra we find

$$N_0(t, 2) = [1/(q^2 - 1)]\,(q^t - 1)(q^{t-1} - 1),$$
$$N_0(t, 4) = [1/(q^2 - 1)] \cdot [q^2/(q^4 - 1)](q^t - 1)(q^{t-1} - 1)(q^{t-2} - 1)(q^{t-3} - 1).$$

We then guess at the general solution

$$N_0(t, 2s) = \prod_{i=1}^{s} \frac{q^{2i-2}}{q^{2i} - 1} \cdot \prod_{i=0}^{2s-1} (q^{t-i} - 1),$$

and substitution in the general formula shows this to be correct.

It is of interest to observe that for $p > 2$, $N_0(t, r)$ is the number of skew symmetric matrices $(A = -A^T)$ of size $t \times t$ and rank r over $GF(q)$. This follows immediately from the fact [4] that a skew-symmetric matrix always has even rank, so that the recursion formula for the enumerator is the same as that for N_0. This formula, with this interpretation is given in [5].

We have now accumulated the following information for the case $q = 2^m$.

$$N(2t+1, 2t+1) = \prod_{i=0}^{t} (q^{2t+1} - q^{2i}), \quad N_0(2t+1, 2t+1) = 0,$$

$$N(2t,\ 2t) = \prod_{i=1}^{t} (q^{2t+1} - q^{2i}),$$

$$N_0(2t,\ 2t) = \prod_{i=0}^{t-1} (q^{2t-1} - q^{2i}).$$

Thus $N(2t,\ 2t) = q^{2t} N_0(2t,\ 2t)$, and

$$N(2t,\ 2t) - N_0(2t,\ 2t) = (q^{2t} - 1) \prod_{i=0}^{t-1} (q^{2t-1} - q^{2i}).$$

We have then

$$\begin{aligned} |\,\mathcal{O}(2t+1)\,| &= |\,GL(2t+1,\ 2^m)\,| \,/ N(2t+1,\ 2t+1) \\ &= \prod_{i=0}^{2t} (q^{2t+1} - q^i) \Big/ \prod_{i=0}^{t} (q^{2t+1} - q^{2i}) \\ &= q^t \prod_{i=0}^{t-1} (q^{2t} - q^{2i}). \\ |\,\mathcal{O}(2t)\,| &= |\,GL(2t,\ 2^m)\,| \,/ (N - N_0) \\ &= \prod_{i=0}^{2t-1} (q^{2t} - q^i) \Big/ \left[(q^{2t} - 1) \prod_{i=0}^{t-1} (q^{2t-1} - q^{2i}) \right] \\ &= q^t \prod_{i=1}^{t-1} (q^{2t} - q^{2i}). \end{aligned}$$

We observe that $|\mathcal{O}(2t+1)| = (q^{2t}-1)|\mathcal{O}(2t)|$. The first few values for $q=2$ are tabulated below

$t = 1$	$\lvert\mathcal{O}(2)\rvert$	$= 2$
	$\lvert\mathcal{O}(3)\rvert$	$= 3!$
$t = 2$	$\lvert\mathcal{O}(4)\rvert$	$= 2\cdot 4!$
	$\lvert\mathcal{O}(5)\rvert$	$= 6\cdot 5!$
$t = 3$	$\lvert\mathcal{O}(6)\rvert$	$= 2^5\cdot 6!$
	$\lvert\mathcal{O}(7)\rvert$	$= 9\cdot 2^5\cdot 7!$
$t = 4$	$\lvert\mathcal{O}(8)\rvert$	$= 9\cdot 2^9\cdot 8!$
	$\lvert\mathcal{O}(9)\rvert$	$= 255\cdot 2^9\cdot 9!$

Section III. *The case $p>2$.* It should be noted that all of the results of this section have been obtained previously by L. Carlitz [6] as a small part of a more general problem. However it seems worthwhile to derive them again by unsophisticated methods.

Let $V_+(n,\ n)$ denote the number of symmetric matrices in $GL(n,\ q)$, $p>2$,

which can be factored in the form $A = MM^T$. The purpose of this section is to prove the following theorem.

THEOREM 4. (1) *If* $n = 2t+1$, $p > 2$, *then*

$$V_+(2t+1, 2t+1) = \tfrac{1}{2}N(2t+1, 2t+1).$$

(2a) *If* $n = 2t$, $p > 2$, *and* -1 *is a square in* $GF(q)$, *then*

$$V_+(2t, 2t) = \frac{1}{2}\,\frac{q^t+1}{q^t}\,N(2t, 2t).$$

(2b) *If* $n = 2t$, $p > 2$, *and* -1 *is not a square in* $GF(q)$, *then*

$$V_+(2t, 2t) = \frac{1}{2}\,\frac{q^t+(-1)^t}{q^t}\,N(2t, 2t).$$

To establish this we need the following result which is obtained by using the theory of quadratic forms [1], [4].

LEMMA 6. *Let A be a symmetric matrix of rank r over $GF(q)$. Then there exists an invertible matrix L such that*

$$LAL^T = \operatorname{diag}[1, 1, \cdots, 1, \delta, 0, \cdots, 0],$$

where the number of nonzero terms is r and δ is either 1 *or a nonsquare of* $GF(q)$.

It follows readily that if A has rank n, it can be factored in the form MM^T if and only if det A is a square of $GF(q)$. Suppose $n = 2t+1$. By multiplying each element of A by a primitive element of $GF(q)$, g, we obtain another symmetric matrix with determinant g^{2t+1} det A. Thus exactly half of the invertible symmetric matrices have square determinants, which proves part (1) of Theorem 4.

If $NAN^T = \operatorname{diag}[a_1, a_2, \cdots, a_r, 0, \cdots, 0]$ for invertible N, $a_i \neq 0$ all i, then A can be transformed into the form $\operatorname{diag}[1, \cdots, 1, 0, \cdots, 0]$ if and only if the product $a = \prod_{i=1}^r a_i$ is a square, (Dickson, [2], section 169). The deciding factor is the quadratic character $\psi(a)$ of this product. This remains invariant under arbitrary nonsingular linear transformations of A, and is called 'the' invariant of A.

We set

$$A(t+1) = \begin{pmatrix} y_0 & y \\ y^T & A(t) \end{pmatrix},$$

and investigate the relation between the rank and invariant of $A(t+1)$ and $A(t)$.

The transformations considered are MAM^T, where M is invertible. In particular we may permute rows and columns of A, provided the same permutation is applied to both, and we may multiply rows by a constant and add one row

to another, provided we then do the same operation on the columns.

The following matrix equations will be useful.

LEMMA 7.

$$\begin{pmatrix} 1 & 0 \\ -b/a & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & 0 \end{pmatrix}\begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -b^2/a \end{pmatrix}, \qquad (a \neq 0).$$

$$\begin{pmatrix} 1 & 1 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix} = \begin{pmatrix} 2b & 0 \\ 0 & -b/2 \end{pmatrix}.$$

LEMMA 8. *Let*

$$A(t+1) = \begin{pmatrix} y_0 & y \\ y^T & 0 \end{pmatrix}.$$

(1) *If* $A(t+1)$ *has rank* 1, *its invariant is* $\psi(y_0)$.

(2) *If* $A(t+1)$ *has rank* 2, *its invariant is* $\psi(-1)$.

Proof. (1) $A(t+1)$ has rank $1 \Leftrightarrow y=0,\ y_0 \neq 0$.

(2) $A(t+1)$ has rank 2 if and only if $y \neq 0$. In this case, by permutations and elementary operations which are symmetric on the rows and columns, we may transform $A(t+1)$ to one of the forms

$$\begin{bmatrix} a & b & 0 \\ b & 0 & \\ 0 & & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & 0 \\ b & 0 & \\ 0 & & 0 \end{bmatrix}.$$

By Lemma 7 the quadratic character of A is $\psi(-1)$.

Let

$$A(t+1) = \begin{pmatrix} y_0 & y \\ y^T & A(t) \end{pmatrix}$$

where $A(t)$ has rank $r \geqq 1$, and invariant $\psi(\delta)$.

LEMMA 9. (1) *If* $A(t+1)$ *has rank* r, *its invariant is* $\psi(\delta)$.

(2) *If* $A(t+1)$ *has rank* $r+1$, *its invariant is* $\psi(\delta)$ *for half the possible choices of* y_0, *and* $-\psi(\delta)$ *for the others.* $-\psi(\delta)$ *means that the quadratic character changes.*

(3) *If* $A(t+1)$ *has rank* $r+2$, *its invariant is* $\psi(-\delta)$.

Proof. Let L be such that $LA(t)L^T = D = \text{diag}[1, \cdots, 1, \delta, 0, \cdots, 0]$. Suppose that y is linearly dependent on the rows of A, say $y = \mathfrak{n}A$. Set

$$M = \begin{pmatrix} 1 & -\mathfrak{n} \\ 0^T & L \end{pmatrix}.$$

Then

$$\begin{aligned} MA(t+1)M^T &= \begin{pmatrix} 1 & -\mathfrak{n} \\ 0^T & L \end{pmatrix} \begin{pmatrix} y_0 & y \\ y^T & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathfrak{n}^T & L^T \end{pmatrix} \\ &= \begin{pmatrix} y_0 - \mathfrak{n}y^T & 0 \\ Ly^T & LA \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathfrak{n}^T & L^T \end{pmatrix} \\ &= \begin{pmatrix} y_0 - \mathfrak{n}y^T & 0 \\ 0^T & LAL^T \end{pmatrix} \\ &= \operatorname{diag}[y_0 - \mathfrak{n}y^T, 1, \cdots, 1, \delta, 0 \cdots 0]. \end{aligned}$$

The condition for $A(t+1)$ to have rank r is that $y_0 = \mathfrak{n}y^T$; in this case the invariant remains $\psi(\delta)$, which proves part (1) of Lemma 9.

If $A(t+1)$ has rank $r+1$, then $y_0 - \mathfrak{n}y^T = g \neq 0$. By a previous remark, the invariant remains $\psi(\delta)$ if g is a square. Since half of the nonzero elements of $GF(q)$ are squares, this proves part (2) of Lemma 9.

For part (3) of Lemma 9 we proceed in several stages. First set

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0^T & L \end{pmatrix}$$

and

$$B(t+1) = M_1 A(t+1) M_1^T = \begin{pmatrix} z_0 & z \\ z & D \end{pmatrix}.$$

We note that $z = yL^T$ is not linearly dependent on the rows of D, since by hypothesis y is not linearly dependent on the rows of A.

Write $z = (\zeta_1, \zeta_2, \cdots, \delta\zeta_r, b_{r+1}, \cdots, b_t) = \zeta D + b$, where $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_r, 0, \cdots, 0)$ and $b = (0, \cdots, 0, b_{r+1}, \cdots, b_t) \neq 0$. Take

$$M_2 = \begin{pmatrix} 1 & -\zeta - b \\ 0^T & I \end{pmatrix}, \quad \text{and} \quad C = M_2 B M_2^T = \begin{pmatrix} a & b \\ b^T & D \end{pmatrix} = \begin{bmatrix} a & 0 & c \\ 0^T & D_1 & 0^T \\ c^T & 0 & 0 \end{bmatrix}.$$

a stands for the element in the top left hand corner, which can be anything; c stands for the $(t-r)$ length vector $(b_{r+1}, \cdots, b_t)$, and $D_1 = \operatorname{diag}[1, 1, \cdots, \delta]$ of size $r \times r$. By permutations and elementary operations which are symmetrical on the rows and columns, C can be transformed to

$$\begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix}$$

where F is a 2×2 matrix of the form

$$\begin{pmatrix} a_1 & b_1 \\ b_1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix}.$$

By Lemma 7 we then have that the invariant of A is $\psi(-\delta)$.

Let $V_+(n, r)$ be the number of symmetric $n\times n$ matrices of rank r with quadratic character 1, and $V_-(n, r)$ the number with quadratic character -1. For $r\geqq 1$, $V_+(n, r)+V_-(n, r)=N(n, r)$. It is to be noted that $V_\pm(n, r)$ are the numbers $N(n, r, \pm 1)$ of [6].

The following recursion formulae are obtained directly by combining Lemmas 4, 8, 9.

(10.1) $$V_\pm(n, 1) = qV_\pm(n-1, 1) + \tfrac{1}{2}(q-1).$$

(10.2) For -1 a square
$$\begin{aligned}V_\pm(n, 2s+1) = {} & q^{2s+1}V_\pm(n-1, 2s+1) + \tfrac{1}{2}(q-1)q^{2s}N(n-1, 2s)\\ & + (q^n - q^{2s})V_\pm(n-1, 2s-1).\end{aligned}$$

(The middle term is $\frac{1}{2}(q-1)q^{2s}(V_+(n-1, 2s)+V_-(n-1, 2s))$.)

(10.3) For -1 a nonsquare
$$\begin{aligned}V_\pm(n, 2s+1) = {} & q^{2s+1}V_\pm(n-1, 2s+1) + \tfrac{1}{2}(q-1)q^{2s}N(n-1, 2s)\\ & + (q^n - q^{2s})V_\mp(n-1, 2s-1).\end{aligned}$$

(10.4) If -1 is a square in $GF(q)$,
$$\begin{aligned}V_+(n, 2) &= q^2V_+(n-1, 2) + \tfrac{1}{2}(q-1)qN(n-1, 1) + q^n - q,\\ V_-(n, 2) &= q^2V_-(n-1, 2) + \tfrac{1}{2}(q-1)qN(n-1, 1),\\ V_\pm(n, 2s) &= q^{2s}V_\pm(n-1, 2s) + \tfrac{1}{2}(q-1)q^{2s-1}N(n-1, 2s-1)\\ &\quad + (q^n - q^{2s-1})V_\pm(n-1, 2s-2).\end{aligned}$$

(10.5) If -1 is a nonsquare in $GF(q)$
$$\begin{aligned}V_+(n, 2) &= q^2V_+(n-1, 2) + \tfrac{1}{2}(q-1)qN(n-1, 1),\\ V_-(n, 2) &= q^2V_-(n-1, 2) + \tfrac{1}{2}(q-1)qN(n-1, 1) + q^n - q,\\ V_+(n, 2s) &= q^{2s}V_+(n-1, 2s) + \tfrac{1}{2}(q-1)q^{2s-1}N(n-1, 2s-1)\\ &\quad + (q^n - q^{2s-1})V_-(n-1, 2s-2),\\ V_-(n, 2s) &= q^{2s}V_-(n-1, 2s) + \tfrac{1}{2}(q-1)q^{2s-1}N(n-1, 2s-1)\\ &\quad + (q^n - q^{2s-1})V_+(n-1, 2s-2).\end{aligned}$$

Referring to the formula for $N(n, r)$ we see at once from (10.1) that
$$V_+(n, 1) = V_-(n, 1) = \tfrac{1}{2}N(n, 1),$$
and, if -1 is a square, by comparing (10.2) with the recursion formula (4.1) for $N(t, r)$,

(11.1) $$V_+(n, 2s+1) = V_-(n, 2s+1) = \tfrac{1}{2}N(n, 2s+1).$$

If -1 is a nonsquare, we still have that $V_+(n, 1)=V_-(n, 1)=\frac{1}{2}N(n, 1)$.

We are led to guess that (11.1) also can be similarly proved in this case, and it is easy to check using (10.3) and (4.1) that this is correct. [The guess is somewhat prompted by knowing that L. Carlitz has shown it to be true; see [6], (3.7).]

If -1 is a square, by the usual laborious method we obtain

$$V_+(n, 2) = \frac{1}{2} \frac{q+1}{q} N(n, 2),$$

and (making a good guess and substituting)

$$V_+(n, 2s) = \frac{1}{2} \frac{q^s+1}{q^s} N(n, 2s).$$

If -1 is a nonsquare the situation is a little more complicated. We find by algebra

$$V_+(n, 2) = \frac{1}{2} \frac{q-1}{q} N(n, 2),$$

$$V_+(n, 4) = \frac{1}{2} \frac{q^2+1}{q^2} N(n, 4),$$

$$V_+(n, 6) = \frac{1}{2} \frac{q^3-1}{q^3} N(n, 6).$$

With this evidence we are able to guess that

$$V_+(n, 2s) = \frac{1}{2} \frac{q^s+(-1)^s}{q^s} N(n, 2s),$$

and substitution in (10.5) proves this to be correct.

We now have for the order of $\mathfrak{O}(n, q)$ the known results ([5], section 172, but note that Dickson restricts the orthogonal group to matrices with determinant 1)

$$|\mathfrak{O}(2t+1, q)| = 2q^t \prod_{i=0}^{t-1} (q^{2t} - q^{2i}).$$

If -1 is a square in $GF(q)$

$$|\mathfrak{O}(2t, q)| = \frac{2}{q^t+1} \prod_{i=0}^{t-1} (q^{2t} - q^{2i}) = 2(q^t - 1) \prod_{i=1}^{t-1} (q^{2t} - q^{2i}).$$

If -1 is a nonsquare in $GF(q)$

$$|\mathfrak{O}(2t, q)| = \frac{2}{q^t+(-1)^t} \prod_{i=0}^{t-1} (q^{2t} - q^{2i}) = 2(q^t + (-1)^{t+1}) \prod_{i=1}^{t-1} (q^{2t} - q^{2i}).$$

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MATHEMATICAL NOTES

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THE PATH OF A CHARGED PARTICLE IN THE FIELD OF A MAGNETIC MONOPOLE

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The trajectory of a charged particle in the field of a magnetic monopole is considered by Lehnert [1], who derives two rather formidable simultaneous differential equations in r and z, using cylindrical coordinates. By assuming z is proportional to r (i.e., that the trajectory lies on a cone) he is able to produce a solution, but this procedure merely shows that *some* possible trajectories lie on cones. (However, one can deduce that all trajectories lie on cones from the facts that the trajectory is uniquely determined by the initial position and velocity and that a suitable cone can always be found.) The purpose of this paper is to demonstrate directly that all trajectories lie on cones, and to derive the equation of the trajectory concisely. If it is true that all trajectories lie on cones, it should be possible to prove this in a simple way, without recourse to specific coordinate systems.

The field of a magnetic monopole at the origin is given by

$$B = k'\mathbf{r}/r^3, \tag{1}$$

so the force equation $m(d\mathbf{v}/dt) = (Ze/c)\mathbf{v}\times B$ becomes, with $k = (Zek'/mc)$,

$$\frac{d\mathbf{v}}{dt} = k\frac{\mathbf{v}\times\mathbf{r}}{r^3}. \tag{2}$$

At time $t=0$, let the particle be at $\mathbf{r}_0$ with velocity $\mathbf{v}_0$. Also, let $\mathbf{u}$ be the direction of $\mathbf{r}$, so that

$$\mathbf{r} = r\mathbf{u}. \tag{3}$$

Finally, we define two constants:

$$a = r_0 \cdot v_0, \tag{4}$$

$$b = |\, r_0 \times v_0 \,| . \tag{5}$$

Then

$$\begin{aligned} b^2 &= (r_0 \times v_0)\cdot(r_0 \times v_0) = [(r_0 \times v_0) \times r_0]\cdot v_0 \\ &= [v_0 r_0^2 - r_0 r_0 \cdot v_0]\cdot v_0 = v_0^2 r_0^2 - (r_0 \cdot v_0)^2; \text{ i.e.,} \end{aligned}$$

$$b = \sqrt{(r_0^2 v_0^2 - a^2)}. \tag{6}$$

From (2), we have immediately $v\cdot(dv/dt)=0$ and $r\cdot(dv/dt)=0$. Therefore $(d/dt)v^2=(d/dt)(v\cdot v)=2v\cdot(dv/dt)=0$, so $v^2=v_0^2$ at all times:

$$v = v_0. \tag{7}$$

Furthermore, $(d/dt)(r\cdot v)=v\cdot v+r\cdot(dv/dt)=v^2+0=v_0^2$, so $r\cdot v=v_0^2 t+r_0\cdot v_0$. But then $(d/dt)r^2=(d/dt)r\cdot r=2r\cdot v=2v_0^2 t+2a$, so

$$r^2 = v_0^2 t + 2at + r_0^2, \tag{8}$$

which can be written in the form

$$r = |\, v_0 t + r_0 \,| . \tag{9}$$

We see that r is independent of k (though the direction of r is not); indeed, (7) and (9) hold whenever the right-hand side of (2) is $v\times r$ multiplied by a scalar function of r and v.

We consider next how u varies with time. Now

$$\frac{du}{dt} = \frac{d}{dt}\,\frac{r}{r} = \frac{vr - r\dot{r}}{r^2} = \frac{vr^2 - r\dot{r}r}{r^3}$$

(as usual, $\dot{r}$ means (dr/dt)), and, since $r\dot{r}=\frac{1}{2}(d/dt)r^2=\frac{1}{2}(d/dt)r\cdot r=r\cdot v$,

$$\frac{du}{dt} = \frac{vr\cdot r - rr\cdot v}{r^3} = \frac{r \times (v \times r)}{r^3}.$$

On the other hand,

$$\frac{d}{dt}(r \times v) = r \times \frac{dv}{dt} + v \times v = r \times \frac{kv \times r}{r^3} + 0 = k\frac{r \times (v \times r)}{r^3}.$$

Thus the vector

$$c \equiv ku - r \times v = ku_0 - r_0 \times v_0 \tag{10}$$

is constant in time (i.e., constant in magnitude and direction):

$$dc/dt = 0. \tag{11}$$

Since $u_0\cdot(r_0\times v_0)=0$ and $u_0=1$,

$$c^2 = k^2 + b^2. \tag{12}$$

The angle between r and c is constant:

$$\cos\phi = u\cdot(c/c) = k/c. \tag{13}$$

Therefore the particle moves on the cone with axis along c and vertex angle ϕ, given by (13).

Finally, we set up spherical coordinates, with the polar direction along c. As seen above, the polar angle ϕ of the particle remains constant in time and is equal to $\cos^{-1}(k/c)$; $r(t)$ is given by (8). To find $\theta(t)$, the azimuthal angle, we consider $|r\times v|$, which is simply r multiplied by the component of v perpendicular to r. For a particle moving on a cone with axis along the polar direction and vertex at the origin $(d\phi/dt=0)$,

$$|r \times v| = (r)(r \sin\phi(d\theta/dt)). \tag{14}$$

On the other hand, $d/dt(r\times v)=r\times(dv/dt)=-(k/r^3)r\times(r\times v)$, so $(d/dt)|r\times v|^2 =2(r\times v)\cdot(d/dt)(r\times v)=0$. Thus $|r\times v|$ is constant in time:

$$|r \times v| = b. \tag{15}$$

We combine (8), (14), and (15), and use $\sin\phi=b/c$, to get

$$d\theta/dt = c/(v_0^2t + 2at + r_0^2). \tag{16}$$

Set $\theta=0$ at $t=0$. Then, from the formula

$$\int\frac{dx}{\alpha x^2+2\beta x+\gamma} = \frac{1}{\sqrt{(\alpha\gamma-\beta^2)}}\tan^{-1}\frac{\alpha x+\beta}{\sqrt{(\alpha\gamma-\beta^2)}},$$

we finally obtain, using (6),

$$\theta = \frac{c}{b}\left[\tan^{-1}\frac{v_0t+a}{b} - \tan^{-1}\frac{a}{b}\right]. \tag{17}$$

Several comments are in order. We have tacitly assumed $b\neq 0$. If $b=0$, (12) and (13) give $\cos\phi=1$, so the particle always lies on the polar axis. Since $b=0$ implies v_0 and r_0 are parallel, we see from (9) that the motion is rectilinear, unaffected by the magnetic field: $r=r_0\pm v_0t$.

The strength of the magnetic field, represented by the parameter k, affects the particle's trajectory in the following way. The radial dependence r is independent of k, but the vertex angle ϕ goes to zero as k becomes large compared to b, and goes to $\pi/2$ as k goes to zero (the cone becomes a plane). The number of times the orbit is "wrapped around" the axis of the cone is found from (17) to be $\frac{1}{2}[1+(k/b)]^{\frac{1}{2}}$.

Let us consider "very strong" magnetic fields $(k\gg b)$. If we further assume that the velocity component parallel to B is small compared to the velocity

component perpendicular to $\boldsymbol{B}$ (i.e., $a \ll b$), we have adiabatic motion, in which the trajectory is like a spiral, with the particle describing (almost closed) circular loops while moving slowly along a $\boldsymbol{B}$ line. For such motion, the flux linked by the particle's orbit (i.e., the product of the area of the circle and B) is constant [2]. Therefore, if B varies inversely with r^2, we can deduce (at least for $k \gg b \gg a$) that the radius of the circle is proportional to r, so that the trajectory lies on a cone with vertex at the position of the monopole. In this paper we have proven that the trajectory lies on a cone no matter what k, b, and a are.

Note added in proof. Dr. Herbert H. Sauer has kindly drawn to the author's attention two related articles: V. C. A. Ferraro (Electromagnetic Theory, Athlone Press, University of London, 1956, pp. 543–544) gives a direct proof using vector calculus that the trajectory lies on a cone, and shows that the trajectory is a geodesic. J. A. van Allen (*Alfvén Invariant in the Field of a Magnetic Unipole,* Journal of Geophysical Research, 70 (1965) 1240) discusses rigorous adiabatic flux invariance.

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ON SIMILARITY AND THE DIAGONAL OF A MATRIX

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If A is a complex square matrix with trace zero, it is well known that A is similar to a matrix with main diagonal consisting of zeros (cf. [1], p. 109, Exercise 6). The purpose of this note is to record an extension of this result.

THEOREM 1. *The complex square matrix A is unitarily equivalent to a matrix with main diagonal (tr A, 0, $\cdots$, 0) if and only if tr $A \in W(A)$, the numerical range of A.*

Proof. Recall that the numerical range $W(A)$ of A is $\{(Ax, x) \mid \|x\| = 1\}$. The necessity is easy: if $x_1, \cdots, x_n$ is an orthonormal basis in which the matrix of A has main diagonal (tr A, 0, $\cdots$, 0), then tr $A = (Ax_1, x_1) \in W(A)$.

The sufficiency will be proved by induction on the size n of A. By the hypothesis there is a unit vector x with $(Ax, x) = \operatorname{tr} A$. The matrix of A in any orthonormal basis $x_1, \cdots, x_n$ with $x_1 = x$ has the form

$$\begin{pmatrix} \operatorname{tr} A & B \\ C & D \end{pmatrix},$$

where D is $(n-1)\times(n-1)$. It follows that D has trace 0. If $n=2$ we have $D=0$ and the proof is finished. If $n>2$, in order to apply the induction hypothesis to D we need to know that $0 \in W(D)$, and this can be seen as follows. If $\lambda_1, \cdots, \lambda_{n-1}$ are the eigenvalues of D (with multiplicities), then

$$\frac{1}{n-1}(\lambda_1 + \cdots + \lambda_{n-1}) = \frac{1}{n-1} \operatorname{tr} D = 0,$$

so 0 is in the convex hull of the spectrum of D. But $W(D)$ is convex [2, Problem 166] and contains the eigenvalues of D, so $0 \in W(D)$. Therefore there is a unitary matrix U such that U^*DU has main diagonal consisting of zeros. To complete the proof we observe that the matrix

$$V = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

is unitary, and that

$$V^* \begin{pmatrix} \operatorname{tr} A & B \\ C & D \end{pmatrix} V$$

has main diagonal $(\operatorname{tr} A, 0, \cdots, 0)$.

We note that in the course of the proof the following result has been established.

COROLLARY 1. *A matrix with trace zero is unitarily equivalent to a matrix with main diagonal consisting of zeros.*

Since for any $n \times n$ matrix A the matrix $A - (\operatorname{tr} A/n)I$ has trace zero, we have:

COROLLARY 2. *Any matrix is unitarily equivalent to a matrix with constant main diagonal.*

The foregoing suggests the question: which diagonals are obtainable from a given matrix under unitary equivalence? This does not seem to admit any pleasant answer, unlike the corresponding question for similarity.

THEOREM 2. *The nonscalar matrix A is similar to a matrix with main diagonal $(\lambda_1, \cdots, \lambda_n)$ if and only if $\lambda_1 + \cdots + \lambda_n = \operatorname{tr} A$.*

The proof is by induction on the size of A, with the help of the following lemma.

LEMMA. *If A is a nonscalar $n \times n$ matrix with $n \geqq 3$, and if λ is a number, then there exists an idempotent matrix P of rank one such that $PAP = \lambda P$ and $(I-P)A(I-P)$ is not a scalar multiple of $I-P$.*

Proof. Since A is not scalar, there is a vector x such that x and Ax are linearly independent. Let $x_1, x_2, \cdots, x_n$ be a basis with $x_1 = x$ and $x_2 = Ax$. If (α_{ij}) is the matrix of A in this basis, then $\alpha_{11} = 0$, $\alpha_{21} = 1$, and $\alpha_{31} = \cdots = \alpha_{n1} = 0$. Let β be a number distinct from α_{23}, and let P be the matrix with first row $(1, \lambda, \beta, 0, \cdots, 0)$ and all other rows zero. Then routine computation reveals that $P^2 = P$, $PAP = \lambda P$, and that the $(2, 3)$ entry of $(I-P)A(I-P)$ is $\beta - \alpha_{23} \neq 0$. Since the $(2, 3)$ entry of $I-P$ is 0, the proof is complete.

The second conclusion of the lemma allows it to be applied repeatedly, leaving the case $n=2$ of the theorem for consideration. Again let x be a vector such that x and Ax are linearly independent. Then x and $Ax-\lambda_1 x$ form a basis, and it is clear that in this basis the matrix of A has main diagonal $(\lambda_1, \operatorname{tr} A-\lambda_1)$, which is (λ_1, λ_2) by hypothesis.

In conclusion we note that the theorem makes sense and the proof is valid for any field.

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A COUNTABLE, CONNECTED, LOCALLY CONNECTED HAUSDORFF SPACE

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The definitions and elementary properties of the topological concepts used below can be found in [3], the number-theoretic concepts in [4].

Let $\langle a, d\rangle$ denote the infinite arithmetic progression $\{a+nd \mid n=0, 1, \cdots\}$. M. Brown [1] and S. Golomb [2] have shown that all such progressions, where a and d are relatively prime, form a basis for a connected topology D on the natural numbers $N=\{1, 2, 3, \cdots\}$.

THEOREM 1. *The topological space (N, D) is not locally connected.*

Proof. Suppose (N, D) is locally connected. Then there exist an open set U and a connected set C such that $1\in U\subseteq C\subseteq\langle 1, 2\rangle$. Choose points x and y, $x<y$, in C (C is infinite since every open set is infinite). For some m and n, $0\leqq m<n$, $x=1+2m$ and $y=1+2n$. Then

$$A=\bigcup_{i=0}^{m}\langle 1+2i, 2^{n+1}\rangle \quad\text{and}\quad B=\bigcup_{i=m+1}^{2^n-1}\langle 1+2i, 2^{n+1}\rangle$$

are disjoint open sets whose union is $\langle 1, 2\rangle$. Since $x\in A$ and $y\in B$, $A\cap C$ and $B\cap C$ separate C, a contradiction.

Now take as a subbasis for a topology D' on N all progressions $\langle a, p\rangle$, where p is a prime greater than a. We will show that (N, D') is connected, locally connected, and Hausdorff. To contrast the two topologies the referee has pointed out that a subbasis for D consists of the progressions $\langle a, p^i\rangle$, where p^i ranges through all powers of primes such that $0<a<p^i$ and $p\nmid a$. To see this, let $d=\prod_{i=1}^{m} p_i^{n_i}$ be the prime factorization of d. For each element $a+jd$ in $\langle a, d\rangle$ choose $r_{ij}\geqq n_i$ such that $p_i^{r_{ij}}>a+jd$, $i=1, 2, \cdots, m$. If we set $d_j=\prod_{i=1}^{m} p_i^{r_{ij}}$, then

$$\langle a, d\rangle=\bigcup_{j=0}^{\infty}\langle a+jd, d_j\rangle=\bigcup_{j=0}^{\infty}\left(\bigcap_{i=1}^{m}\langle a+jd, p_i^{r_{ij}}\rangle\right),$$

the last equality a result of the Chinese Remainder Theorem.

THEOREM 2. *The topology D' is connected.*

Proof. Clearly every D'-open set is D-open. Since D is connected, so is D'.

THEOREM 3. *The topology D' is Hausdorff.*

Proof. Let $a, b \in N$, with $a < b$, and let p be any prime greater than b. Then $A = \langle a, p\rangle$ and $B = \langle b, p\rangle$ are disjoint open sets which separate a and b.

THEOREM 4. *Let $A_i = \langle a_i, p_i\rangle$, $i = 1, 2, \cdots, k$, be distinct subbasis sets, and let $A = \bigcap_{i=1}^{k} \langle a_i, p_i\rangle$. If A is not empty, then the p_i are all distinct, and $A = \langle a, P\rangle$, where a is the least element common to all the A_i and $P = \prod_{i=1}^{k} p_i$.*

Proof. The set A is not empty, i.e., the system of congruences A_i is solvable, if and only if $(p_i, p_j) \mid (a_j - a_i)$ for all $i, j = 1, 2, \cdots, k$. Since $a_j \geqq a_i$ implies $p_j > a_j > a_j - a_i \geqq 0$, we must have $(p_i, p_j) = 1$ if $i \neq j$. The Chinese Remainder Theorem then gives the desired result.

Since all finite intersections of subbasis sets form a basis, we have a

COROLLARY. *If A is a (nonempty) basis set in the topology D', then A has the form $\langle a, P\rangle$, where P is square-free.*

THEOREM 5. *The topology D' is locally connected.*

Proof. We show that (N, D') has a connected basis. Let $A = \langle a, P\rangle$ be any basis set and suppose A is not connected. Then there exist open sets U and V such that $A \cap U$ and $A \cap V$ are disjoint, nonempty, and $A \subseteq U \cup V$. Let $a_1 \in A \cap U$, $a_2 \in A \cap V$. There exist basis sets $A_1 = \langle b_1, PQ\rangle$, $A_2 = \langle b_2, PR\rangle$ such that $a_1 \in A_1 \subseteq A \cap U$, $a_2 \in A_2 \subseteq A \cap V$, and PQ and PR are square-free, i.e., $(P, QR) = 1$. (Here we have used the corollary above and the fact that $\langle c, d\rangle \subseteq \langle a, b\rangle$ implies $b \mid d$.) Since P and QR are relatively prime there exist positive integers r and s such that $rQR - sP = a$. Let $a' = a + sP = rQR$ and assume without loss of generality that $a' \in A \cap U$. Then for some basis set $\langle b', PS\rangle$ we have $a' \in \langle b', PS\rangle \subseteq A \cap U$, where PS is square-free and $(S, QR) = 1$ (since any prime factor common to S and QR divides b', contradicting $(b', PS) = 1$). Therefore $(PS, PQR) = P$ and, since $(a_2 - a') \equiv 0 \pmod{P}$, there are positive integers u and v such that $uPS - vPQR = a_2 - a'$. Hence $a' + uPS = a_2 + vPQR$, and $(A \cap U) \cap (A \cap V) \neq \emptyset$, contradicting our original assumption. Therefore A is connected and (N, D') is locally connected.

A slight modification of Golomb's proof in [2] shows that D' is not T_3: The set $\{2n\}$ of even natural numbers, being the complement of $\langle 1, 2\rangle$, is closed. Any open set containing the element 1 and not meeting $\{2n\}$ must contain a basis set $\langle 1, P\rangle$, where P is even. Hence any open cover of $\{2n\}$ must contain a basis set $\langle a, Q\rangle$ containing P. Writing $P = a + rQ$, we have $(P, Q) = 1$ since $(a, Q) = 1$. But then $\langle 1, P\rangle$ and $\langle a, Q\rangle$ intersect.

Since D' is not regular it cannot be locally compact, therefore not compact. Nor is it countably compact.

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ON OPERATORS COMMUTING WITH DIFFERENTIATION

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THEOREM. *If a linear operator mapping each of the spaces $C[a, b]$ and $C^1[a, b]$ boundedly into itself commutes with differentiation, it must be a constant multiple of the identity.*

REMARK. Analogous results do not hold for linear operators applied to functions defined over infinite intervals. In this case translation, and more generally, any convolution operator

$$T(\phi) = \int_{-\infty}^{+\infty} \varphi(x - t)K(t)dt$$

clearly commute with differentiation.

Proof. Denote the operator by T and differentiation by D. Our assumption is that

$$DT(\phi) = T(D\phi) \tag{1}$$

for each function $\phi(x)$ in $C^1[a, b]$.

By repeated application of (1), we find that

$$D^mT(\phi) = T(D^m\phi), \qquad m = 1, 2, \cdots, \tag{2}$$

for each function ϕ in $C^m[a, b]$. In particular this implies that polynomials of degree m are mapped by T into polynomials of degree m or less.

We are going to show that T has the asserted form for the functions $\cos nx$ and $\sin nx$. That is, we shall prove that for some constant a,

$$T(\cos nx) = a \cos nx \quad \text{and} \quad T(\sin nx) = a \sin nx, \qquad n = 1, 2, \cdots.$$

In order to prove this we begin by observing that under T, $\cos nx$ goes onto a linear combination of $\cos nx$ and $\sin nx$. This follows by applying (2) with $m=2$ to $\phi(x) = \cos nx$, which yields

$$D^2T(\cos nx) = T(D^2 \cos nx) = -n^2T(\cos nx).$$

In other words the function $\psi = T(\cos nx)$ is a solution of the equation $D^2\psi + n^2\psi = 0$; this immediately gives us the form of $\psi = T(\cos nx)$:

$$T(\cos nx) = \alpha_n \cos nx + \beta_n \sin nx. \tag{3}$$

For the purpose of evaluating the coefficients α_n and β_n, we want to make use of Fourier series, so it will be convenient to take the interval $[a, b]$ as the interval $[-\pi, +\pi]$. This in no way restricts the generality.

Expanding the function x^2 into its Fourier series over $[-\pi, +\pi]$, we obtain

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{4}$$

where

$$a_n = \frac{4(-1)^n}{n^2}, \qquad n = 1, 2, \cdots, \tag{5}$$

and the series converges uniformly in $[-\pi, +\pi]$. Accordingly if we apply the bounded operator T to both sides of (4), we may apply it term by term on the right:

$$T(x^2) = T\left(\frac{a_0}{2}\right) + \sum_{n=1}^{\infty} a_n T(\cos nx),$$

with the resulting series again converging uniformly in $[-\pi, +\pi]$. Inserting the expression (3) for $T(\cos nx)$, this gives

$$T(x^2) - T\left(\frac{a_0}{2}\right) = \sum_{n=1}^{\infty} (a_n\alpha_n) \cos nx + (a_n\beta_n) \sin nx. \tag{6}$$

On the other hand, in view of the fact that T maps polynomials of degree m into polynomials of degree m or less, the left side of (6) must be a polynomial of degree 2 or less; so we have

$$ax^2 + bx + c = \sum_{n=1}^{\infty} (a_n\alpha_n) \cos nx + (a_n\beta_n) \sin nx,$$

where a, b, and c are constants. As the equality holds in $[-\pi, +\pi]$, the function on the left must assume the same values at $-\pi$ and $+\pi$, which is only possible if $b = 0$. Hence we obtain

$$ax^2 + c = \sum_{n=1}^{\infty} (a_n\alpha_n) \cos nx + (a_n\beta_n) \sin nx.$$

Comparing this Fourier series expansion for ax^2+c with the expansion

$$ax^2 + c = \left(\frac{aa_0}{2} + c\right) + \sum_{n=1}^{\infty} aa_n \cos nx,$$

which is obtained by multiplying (4) through by a and adding c, we find that $a_n\alpha_n = aa_n$ and $a_n\beta_n = 0$, $n = 1, 2, \cdots$. Hence, since by (5) $a_n \neq 0$, $\alpha_n = a$ and $\beta_n = 0$, $n = 1, 2, \cdots$.

Inserting the values just found for α_n and β_n into (3), we see that

$$(7) \qquad T(\cos nx) = a \cos nx, \qquad n = 1, 2, \cdots.$$

By differentiating both sides of this equation and making use of the fact that T commutes with differentiation we also obtain

$$(8) \qquad T(\sin nx) = a \sin nx, \qquad n = 1, 2, \cdots.$$

From (7) and (8) we conclude that T has the desired form for the functions $\cos nx$ and $\sin nx (n=1, 2, \cdots)$; since linear combinations of these are dense in the set of continuous functions $f(x)$ over $[-\pi, +\pi]$ satisfying the conditions

$$(9a, b) \qquad \int_{-\pi}^{+\pi} f(x)dx = 0, \qquad f(\pi) = f(-\pi),$$

it follows that T also has the desired form: $T(f(x)) = af(x)$ for these functions.

Assume now that $f(x)$ is any function in $C^2[-\pi, +\pi]$. By subtracting a suitable linear function $l(x)$ from $f(x)$, we can arrange for the difference $f(x) - l(x)$ to satisfy conditions (9 a, b). It follows that

$$T(f - l) = a(f - l).$$

Differentiating this equation we obtain $D^2T(f-l) = T(D^2(f-l)) = T(f'') = af''$. Thus for any function f in $C^2[-\pi, +\pi]$, we have $T(f'') = af''$.

Finally, since each continuous function ϕ over $[-\pi, +\pi]$ can be regarded as the second derivative f'' of some appropriate $f \in C^2[-\pi, +\pi]$, the last relation implies that $T(\phi) = a\phi$ for each ϕ in $C[-\pi, +\pi]$.

A characterization for operators which commute with D^2 can also be given. Namely, the only linear operators $T(\phi)$ mapping each of the spaces $C[-a, +a]$, $C^1[-a, +a]$ and $C^2[-a, +a]$ boundedly into themselves which commute with D^2 are those of the form

$$T(\phi) = a\phi(x) + b\phi(-x) + c \int_{-x}^{+x} [\phi(t) + \phi(-t)]dt,$$

where a, b and c are constants.

For simplicity we have stated the result for symmetric intervals $[-a, +a]$. Its proof runs along the same lines as the one above.

A NOTE ON FERMAT'S LAST THEOREM

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Fermat's Last Theorem states that there are no integers x, y, z, all nonzero, satisfying

$$(1) \qquad x^n + y^n = z^n,$$

where n is a natural number greater than two.

It is well known that we need only consider the case for n prime; x, y, and z

$$f(x) = \int_0^\infty h(xu) \int_0^\infty k(uy)f(y)dy\, du,$$

the above theorem holds in an obvious manner provided that $h_1(\tau)$ and $k_1(\tau)$ are bounded and $h_1(\tau)k_1(-\tau)=1$ where $k_1(\tau)$ is defined as above and $h_1(\tau)$ is defined similarly.

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ON EXPANSIVE HOMEOMORPHISMS

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Throughout this paper, X will be a metric space with metric d, and f will be a homeomorphism of X onto itself. If there exists $\delta>0$ such that $x, y \in X$, $x \neq y$ implies there is an integer n such that $d(f^n(x), f^n(y)) > \delta$, then f is said to be *expansive* with expansive constant δ.

It is known that if f is expansive on $X-A$, where A is a finite subset of X, then f is expansive on X ([1], Theorem 3). The purpose of this paper is to generalize the preceding result.

If $x \in X$, then the orbit of x under f is defined by $0(x) = \bigcup_{n=-\infty}^{\infty} \{f^n(x)\}$. The generalization mentioned above can now be stated.

THEOREM. *If X is compact, and if f is a homeomorphism of X onto itself which is expansive on $X - \bigcup_{i=1}^{N} 0(x_i)$, then f is expansive on X.*

Before proving this result, we need two lemmas.

LEMMA 1. *Let f be a homeomorphism of X onto itself, and let X be compact. Let $A \subseteq X$. Then f is expansive on $X-A$ if and only if f^n is expansive on $X-A$ for $n \neq 0$.*

Proof. It is known ([2], Theorem 2.2) that if X is compact, then f is expansive on X if and only if f^n is expansive for $n \neq 0$. The only reason compactness is needed is so that f will be uniformly continuous. It is clear that f is uniformly continuous on $X-A$; hence the lemma follows from the proof of Theorem 2.2 of [2].

LEMMA 2. *Let f be a homeomorphism of X onto itself, where X is compact. Let $Y \subseteq X$ with $X - Y$ at most countable and $f(Y) = Y$. Let $a \in Y$. Then if f is expansive on $Y - 0(a)$, f is expansive on Y.*

Proof. By the proof of Theorem 3 of [1], f is expansive on $(Y-0(a)) \cup \{a\}$, so let δ be an expansive constant for f on this set. Consider $x, y \in Y-0(a)$. Then there exists an integer n such that $d(f^n(x), f^n(y)) > \delta$. Next, consider $x \in Y-0(a)$,

$y \in 0(a)$. Then $y = f^k(a)$ for some integer k, so $f^{-k}(y) = a$. Also, $f^{-k}(x) \in Y - 0(a)$, since $Y - 0(a)$ is invariant under f. Hence, there exists an integer n such that $d(f^n(f^{-k}(x)), f^n(f^{-k}(y))) = d(f^{n-k}(x), f^{n-k}(y)) > \delta$. Thus, if we can show that f is expansive on $0(a)$, then f will be expansive on Y.

Suppose first that a is isolated in $0(a)$. Then there exists $\epsilon > 0$ such that $f^k(a) \neq a$ implies $d(f^k(a), a) > \epsilon$. Consider $x, y \in 0(a)$, $x \neq y$. Then $x = f^m(a)$, $y = f^n(a)$, and $f^{m-n}(a) \neq a$. Thus, $d(f^{-n}(x), f^{-n}(y)) = d(f^{m-n}(a), a) > \epsilon$, and f is expansive on $0(a)$.

Finally, suppose that a is not isolated in $0(a)$. Then a is a limit point in $0(a)$, and so is each other point in $0(a)$. ($x \in 0(a)$ implies $x = f^n(a)$, and since limit points are preserved under homeomorphisms, x is a limit point of $0(a)$.) Therefore, $\overline{0(a)}$ is a perfect set, and is therefore uncountable. Since $X - Y$ is at most countable, there exist uncountably many points x such that $x \in Y - 0(a)$, and such that x is a limit point of $0(a)$.

If δ is not an expansive constant for f on $0(a)$, then there exist $f^{m_1}(a) \neq f^{m_2}(a)$ such that $d(f^k(f^{m_1}(a)), f^k(f^{m_2}(a))) = d(f^{k+m_1}(a), f^{k+m_2}(a)) \leqq \delta$ for each k, i.e. $d(f^k(a), f^{k+m_2-m_1}(a)) \leqq \delta$ for each k, i.e., there exists $m \neq 0$ such that $d(f^k(a), f^{k+m}(a)) \leqq \delta$ for each k.

Let x be one of the uncountably many points of $Y - 0(a)$ that are limit points of $0(a)$. Choose a sequence $\{n_i\}$ such that $f^{n_i}(a) \to x$. Let n be an arbitrary integer. Then $f^{n_i+n}(a) \to f^n(x)$, and $f^{n_i+n+m}(a) \to f^{n+m}(x)$, so that since $d(f^{n_i+n}(a), f^{n_i+n+m}(a)) \leqq \delta$ for each n_i, $d(f^n(x), f^{n+m}(x)) = d(f^n(x), f^n(f^m(x))) \leqq \delta$. Since n was arbitrary, and since x and $f^m(x)$ are in $Y - 0(a)$, we have a contradiction unless $x = f^m(x)$.

Therefore, we must assume that $f^m(x) = x$ for uncountably many points $x \in Y - 0(a)$. But by hypothesis and Lemma 1, f^m is expansive on $Y - 0(a)$. Since X is compact, the uncountable collection of x's such that $f^m(x) = x$ has a limit point in X, so that for each $\epsilon > 0$, there exist $x_1 \neq x_2$ such that $x_1, x_2 \in Y - 0(a)$, $f^m(x_1) = x_1$, $f^m(x_2) = x_2$, and $d(x_1, x_2) < \epsilon$. Thus, $d((f^m)^n(x_1), (f^m)^n(x_2)) = d(x_1, x_2) < \epsilon$ for each integer n. Thus, ϵ is not an expansive constant for f^m on $Y - 0(a)$. Since ϵ was arbitrary, this is a contradiction. Thus, δ is an expansive constant for f on $0(a)$ in the case when a is not isolated in $0(a)$. Hence, f is expansive on Y.

Proof of theorem. Suppose that $N = 1$. Then f is expansive on $X - 0(x_1)$, so by Lemma 2, taking $Y = X$ and $a = x_1$, f is expansive on X.

Assume that the theorem is true for $N = k$. Suppose further that f is expansive on $X - \bigcup_{i=1}^{k+1} 0(x_i) = (X - \bigcup_{i=1}^{k} 0(x_i)) - 0(x_{k+1})$. By Lemma 2, taking $Y = X - \bigcup_{i=1}^{k} 0(x_i)$, f is expansive on Y. But by the induction hypothesis, f is then expansive on X.

If in the statement of the theorem, we replace "orbit" by "orbit closure," then the result no longer holds. To see this, take a nonexpansive orbit closure and add a fixed point.

The following example shows that the finiteness in the statement of the theorem is needed:

EXAMPLE. Consider a countable collection of concentric circles $\{C_i\}$ whose diameters go to zero. Choose on C_i a set A_i of i equally spaced points, and define f_i on C_i by cyclic permutation. If c is the center of these circles, let $X=(\bigcup_{i=1}^{\infty} A_i) \cup \{c\}$, and let f keep c fixed and be equal to f_i on A_i. Clearly f is not expansive on X.

The author wishes to thank the referee for this example as well as several other helpful suggestions.

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

CAN NINE TETRAHEDRA FORM A NEIGHBORING FAMILY?

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Two polygonal plane bodies are called *neighbors* provided that their intersection is 1-dimensional, and two polyhedral bodies in 3-space are called *neighbors* provided that their intersection is 2-dimensional. (As the term is used here, a *body* is a compact connected set which is the closure of its interior.) A family of bodies is said to be *neighboring* provided that any two of its members are neighbors. Early in the study of the four-color problem it was observed that a neighboring family of bodies in the plane has at most four members. In the late 1870's, F. Guthrie, a Scottish student of chemistry, showed there are arbitrarily large neighboring families in 3-space and asked what happens when the bodies are required to be convex. The question was repeated in 1897 by the German mathematician P. Stäckel and answered in 1905 by Tietze [11], who showed that for each n there is in 3-space a neighboring family of n convex polyhedra. (See Tietze [12] for references to Guthrie and Stäckel and for an interesting elementary discussion of neighboring families in the plane and in 3-space.) Unaware of earlier work on the problem, M. Crum repeated the question of Guthrie and Stäckel in the 1940's and Besicovitch [3] repeated Tietze's answer by a different construction in 1947. Some aspects of Besicovitch's solution were refined by Rado [10] and Eggleston [6]. Danzer, Grünbaum and Klee [5] noted that a neighboring family of n convex polyhedra in 3-space can be formed by starting from a 4-dimensional convex polyhedron with $n+1$ 3-dimensional faces, any two of which have 2-dimensional intersection, and

then projecting the first n faces into the last one in an appropriate way. (Surprisingly, such 4-dimensional convex polyhedra exist for all $n \geqq 4$. See Carathéodory [4], Gale [7, 8], and Grünbaum [9].)

In 1956 Bagemihl [1] asked what happens when the convexity condition is strengthened by requiring that all members of the family are tetrahedra; that is, he asked for the maximum cardinality N of a neighboring family of tetrahedra in 3-space. He indicated a proof that $N \leqq 17$, showed $N \geqq 8$ by exhibiting a neighboring family of eight tetrahedra, and conjectured $N=8$. If his conjecture is correct then the answer to our title question is negative. In 1965 Baston [2] showed that any neighboring family of n tetrahedra can be represented by an n-rowed matrix whose elements are all $+1$, 0, or -1, and whose minors satisfy certain conditions. Using this representation, he proved $N \leqq 9$, so it remains "only" to decide whether N is 8 or 9. However, Baston's proof fills a book of more than two hundred pages and requires the invention of several new geometric and combinatorial notions. We quote the book's last two sentences, which concern the possibility of improving his argument to show $N=8$: "Thus even though the number of nonequivalent permatrices of one dotre, one doun and seven dodos may be small, the number of permatrices to be considered would nevertheless be quite large. It therefore appears that some new idea would be required to prove the conjecture that a 9-con does not exist, for to employ the methods used so far would be a long and tedious operation."

Baston conjectured [2, p. 11] that 2^d is the maximum cardinality of a neighboring family of d-dimensional simplices in d-space (any two having $(d-1)$-dimensional intersection). However, it seems there are no published proofs of even weak bounds on the cardinality of such families.

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IS EVERY POLYGONAL REGION ILLUMINABLE FROM SOME POINT?

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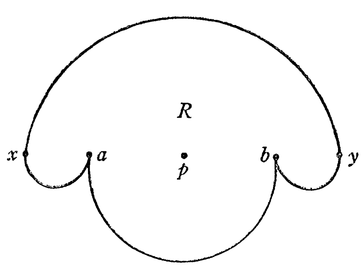
The problems and examples given here did not originate with me. I heard them from someone almost ten years ago but have been unable to trace the source.

Let R be a polygonal region in the Euclidean plane—that is, a connected open set whose boundary is the union of a finite number of line segments. For each point p of R and each direction θ, let $L(p, \theta)$ be the path followed by a light ray which issues from p in the direction θ and is reflected in the usual way when it meets the boundary of R (angle of reflection = angle of incidence). Let us agree that a light ray is absorbed if it meets a corner of the boundary; it is then the union of a finite rather than infinite sequence of line segments. The region R is said to be *illuminable* from p provided that a light source at p would illuminate the entire region—that is, $R \subset \bigcup_{\text{all } \theta} L(p, \theta)$.

The problem of the title asks whether, for every polygonal region R, there is a point p of R such that R is illuminable from p. It is also unknown whether R must be illuminable from every one of its points. The problems are open even when the boundary of R is a simple closed polygon. These problems suggest physical experiments, involving a small but powerful light source, in which the role of R's boundary is played by a system of mirrors hinged so that they can be moved to change the shape of the boundary. Of course, many difficulties would be encountered in the attempt to draw mathematical conjectures from such an apparatus. In particular, part of the region might be illuminated in the mathematical sense and yet appear dark in the experiment because the light took so long to reach it!

The above problems can be extended to plane regions R bounded by a finite number of differentiable arcs. In this case there may be a point of R from which R is not illuminable (see the figure below). However, it may be that every region is illuminable from some of its points or even from almost all of them. And it may be that every region with at most one corner is illuminable from every one of its points.



(The points a and b are corners. The semicircular arcs ab and xy are concentric at p. The region R is not illuminable from p.)

If one is given for each point x of a set X, a filter F_x of subsets containing x, and defines τ to be the collection of sets U such that U belongs to F_x for each x in U, then it is well known that (1) τ is a toplogy, and (2) for each x the filter of τ-neighborhoods is contained in F_x, and is equal to F_x provided that this compatibility property holds: If U belongs to F_x, there is a W contained in F_x, such that U belongs to F_y for each y in W.

It is easily checked that the filters just specified satisfy this property, hence are indeed the neighborhood systems in the topology they induce. It is trivial to check the neighborhood system at each point has a basis of closed sets, thus X is regular. X fails to be completely regular since, as we shall see, every real continuous function must take the same value at p_- and p_+.

In any topological space it is true that the set on which a continuous real function agrees with its value at the point x, is the intersection of the countable family of neighborhoods N_j of x ($j=1, 2, \cdots$) on which f differs from its value at x by less than $1/j$. It follows that for fixed n and k, the set of "anomalous" points of $T_{n,k}$ at which f fails to be equal to $f(p_{n,k})$ is countable. Let us denote the set of ordinates of these anomalous points by $S_{n,k}$, and let $S_n = \bigcup_k S_{n,k}$. Now pick a point p of L_{n-1} or L_{n+1} whose ordinate does not belong to the countable set S_n. Clearly

$$f(p) = \lim_{k\to\infty} f(p_{n,k}) = c_n.$$

As f takes the value c_{n+1} and c_{n-1} at all but countably many points of L_n, it must be that $c_n = c$ for all n. Then $f(p_-) = f(p_+) = c$ since f assumes the value c in every neighborhood of both points.

This work was supported by the A. E. C. and done while the author held an A.W.U. fellowship at Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico.

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A REMARK ABOUT EULER'S FUNCTION

D. L. GOLDSMITH, University of Cambridge, England

Most texts on elementary number theory prove that Euler's ϕ-function is multiplicative, and a few (see, for example, [1] page 31, problem 5) present the following generalization of the ϕ-function. Let $Q(x)$ be a polynomial with integral coefficients, and let $S(m) = \{x \mid 0 \leqq x < m,\ (Q(x), m) = 1\}$. If we define $\psi(m)$ to be the number of elements in $S(m)$, then we have

(1) $$\psi(mn) = \psi(m)\psi(n) \qquad \text{if } (m, n) = 1,$$

(2) $$\psi(p^k) = p^k - \beta_p p^{k-1},$$

where β_p is the number of integers $0 \leqq x < p$ such that $Q(x)$ is divisible by the prime p. It does not seem to be noted, however, that this generalization of Euler's function is itself just a special case of a doubly-multiplicative function. For if d is a positive divisor of m, let

$$S_d(m) = \{x \mid 0 \leqq x < m, (Q(x), m) = d\},$$

and define $\psi_d(m)$ to be the number of integers in $S_d(m)$, so that $S(m) = S_1(m)$ and $\psi(m) = \psi_1(m)$. Then we have the following

THEOREM. *If* $(m, n) = 1$, $d \mid m$, $e \mid n$, *then*

(3) $$\psi_{de}(mn) = \psi_d(m)\psi_e(n).$$

Moreover, for $k \geqq 1$, $0 \leqq l \leqq k$,

(4) $$\psi_{p^l}(p^k) = \beta_{p^l}(p^k) - \gamma_k(l)\beta_{p^{l+1}}(p^{l+1})p^{k-l-1},$$

where, for $s \mid t$, $\beta_s(t)$ *is the number of* $0 \leqq x < t$ *for which* $s \mid Q(x)$, *and* $\gamma_k(l) = 1$ *if* $0 \leqq l < k$ *and* $\gamma_k(k) = 0$.

Proof. The proof is a straightforward modification of the usual residue class proof that ϕ is multiplicative.

Since $(m, n) = 1$, there exist integers s, t such that $sn + tm = 1$. Consider the set

$$B = \{snx_i + tmy_j \mid x_i \in S_d(m), y_j \in S_e(n)\}.$$

It is easy to see that B contains $\psi_d(m)\psi_e(n)$ distinct integers, all incongruent mod mn.

Suppose now that $z \in S_{de}(mn)$, and that $z \equiv z'$ (mod m) and $z \equiv z''$ (mod n), with $0 \leqq z' < m$, $0 \leqq z'' < n$. Then

$$(Q(z'), m) = (Q(z), m) = d \quad \text{and} \quad (Q(z''), n) = (Q(z), n) = e,$$

so that $z' \in S_d(m)$ and $z'' \in S_e(n)$. Therefore $z = x_i + vm = y_j + wn$ for some x_i, y_j, v, w, hence

$$z = (sn + tm)z \equiv sn(z - vm) + tm(z - wn) \equiv snx_i + tmy_j \pmod{mn}.$$

In other words, each z in $S_{de}(mn)$ is congruent mod mn to a member of B, so $\psi_{de}(mn) \leqq \psi_d(m)\psi_e(n)$.

On the other hand,

$$Q(snx_i + tmy_j) \equiv Q(snx_i) \equiv Q(x_i) \pmod{m},$$

so $(Q(snx_i + tmy_j), m) = (Q(x_i), m) = d$. Similarly $(Q(snx_i + tmy_j), n) = e$. Therefore, since $(m, n) = 1$, we have $(Q(snx_i + tmy_j), mn) = de$. It follows that each

member of B is congruent mod mn to a member of $S_{de}(mn)$, hence $\psi_d(m)\psi_e(n) \leqq \psi_{de}(mn)$.

For the evaluation of $\psi_{p^l}(p^k)$ we will consider separately the cases when $l<k$ and $l=k$.

Suppose first that $l<k$. If $0\leqq x<p^k$, then $p^{l+1}|Q(x)$ if and only if x is of the form $x=b+tp^{l+1}$, where $0\leqq b<p^{l+1}$, $0\leqq t<p^{k-l-1}$, and $p^{l+1}|Q(b)$. Hence there are exactly $\beta_{p^{l+1}}(p^{l+1})p^{k-l-1}$ such integers. Since $(Q(x), p^k)=p^l$ if and only if $p^l|Q(x)$ but $p^{l+1}\nmid Q(x)$, we have

$$\psi_{p^l}(p^k) = \beta_{p^l}(p^k) - \beta_{p^{l+1}}(p^{l+1})p^{k-l-1} \qquad (l < k).$$

If $l=k$, then clearly $\psi_{p^k}(p^k)=\beta_{p^k}(p^k)$.

Reference

1. W. J. LeVeque, Topics in Number Theory, Addison-Wesley, Reading, Mass., 1956.

A TOPOLOGICAL CHARACTERIZATION OF THE REAL NUMBERS

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It is often desirable to include a characterization theorem in an undergraduate course in topology. This note gives a topological classification of the real numbers R, which can be undertaken in the first semester of such a course. The proof is presented as a sequence of lemmas which are not too difficult for the student to prove.

DEFINITION. Let X be a connected topological space and $x\in X$. x is called a *slice point* if $X\setminus\{x\}=A_x\cup B_x$ with A_x and B_x connected, open, nonempty, disjoint subsets of X.

THEOREM. *A nondegenerate, connected, locally connected, separable metric space in which every point is a slice point is homeomorphic to* R.

Proof: Let X have the property that every point is a slice point. Choose $p\in X$ and let $X\setminus\{p\}=A_p\cup B_p$. If $x\in A_p$, let B_x denote the component of $X\setminus\{x\}$ which contains p, and if $x\in B_p$, let A_x be the component of $X\setminus\{x\}$ containing p.

LEMMA 1. *If* $x\in A_y$ *then* $A_x\subset A_y$.

LEMMA 2. *If* $x\neq y$ *then* $x\in A_y$ *is equivalent to* $y\in B_x$.

DEFINITION. $x<y$ *if* $x\in A_y$.

LEMMA 3. "$<$" *is a simple order on* X.

LEMMA 4. *If* X *satisfies the hypothesis of the theorem, then the collection of all sets of the form* $\{z|x<z<y\}$ *is a basis for the topology on* X.

Let D be a countable dense subset of X and T the set of dyadic rationals in the open interval (0, 1).

LEMMA 5. *There is a one-to-one, onto, order preserving function* $f:D\to T$.

LEMMA 6. f *may be extended to a homeomorphism of* X *onto* (0, 1), *which is homeomorphic to* R.

COROLLARY. *If* K *is the graph of a function* $f:R\to R$, *then* $g:R\to K$ *given by* $g(x)=(x, f(x))$ *is a homeomorphism if and only if* K *is locally connected.*

It is easy to construct examples proving that each of the hypotheses is necessary to the theorem.

Historical note: The theorem, with local compactness replacing local connectedness, was proved by R. L. Moore (*Concerning simple continuous curves*, Trans. Amer. Math. Soc., 21(1920) 333–347). In the presence of the other conditions here, local compactness is equivalent to local connectedness (G. T. Whyburn, *Concerning connected and regular point sets*, Bull. Amer. Math. Soc., 33(1927) 685–689).

A NOTE ON FIXED-POINT STOCHASTIC MATRICES

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If S is a stochastic matrix, a fixed-point for S is any row probability vector t with positive components such that $tS=t$. Let us call a stochastic matrix with a unique fixed-point an f-matrix. Then it can be shown using Markov chain theory that a stochastic matrix is an f-matrix if and only if it is an ergodic matrix. An ergodic matrix is defined to be the transition matrix of a Markov chain with this property: for any two distinct states i and j, there is a positive integer k such that the probability of a transition from i to j in k steps is positive. In this note we will prove this theorem for 2×2 and 3×3 matrices without any appeal to Markov chain theory. We give a simple characterization of f-matrices from which the theorem follows immediately. It may also be mentioned that the expression for the fixed-point of f-matrices of this size is of some interest.

THEOREM 1. *Let* $S=(p_{ij})$ *be a* 2×2 *stochastic matrix. Then* S *is an* f-*matrix if and only if* $p_{12}p_{21}>0$.

Proof. S is an f-matrix if and only if $(x, 1-x)S=(x, 1-x)$, i.e., if and only if $x(p_{12}+p_{21})=p_{21}$ has a unique solution x such that $0<x<1$. This happens if and only if $p_{12}p_{21}>0$.

THEOREM 2. *Let* $S=(p_{ij})$ *be a* 3×3 *stochastic matrix, and let*

$$
\begin{aligned}
D_1 &= p_{21}p_{31}+p_{23}p_{31}+p_{21}p_{32},\\
D_2 &= p_{32}p_{12}+p_{31}p_{12}+p_{32}p_{13},\\
D_3 &= p_{13}p_{23}+p_{12}p_{23}+p_{13}p_{21}.
\end{aligned}
$$

Then S is an f-matrix if and only if $D_1D_2D_3>0$.

Proof. S is an f-matrix if and only if $(x, y, 1-x-y)S=(x, y, 1-x-y)$, i.e., if and only if the system:

$$(p_{11}-p_{31}-1)x+(p_{21}-p_{31})y=-p_{31}$$
$$(p_{12}-p_{32})x+(p_{22}-p_{32}-1)y=-p_{32}$$

has a unique solution (x, y) such that $x>0$, $y>0$, $x+y<1$. If D is the determinant of the above system, then a simple computation shows that D_1 is D when the column of x-coefficients is replaced by the column of constants. Similarly D_2 is D when the column of y-coefficients is replaced by the column of constants and $D_3=D-D_1-D_2$. Thus the system has the desired solution if and only if $D_1D_2D_3>0$.

Now let us indicate how the equivalence of ergodic and f-matrices follows from the above results. The 2×2 case: if $S=(p_{ij})$ is such that $p_{12}p_{21}>0$, then it is clearly ergodic. Conversely if at least one of p_{12}, p_{21} is zero, then this zero will persist in all positive powers of S, whence S cannot be ergodic. The 3×3 case: if $S=(p_{ij})$ is such that $D_1D_2D_3>0$, then $D_1>0$, $D_2>0$ and $D_3>0$. Now $D_1>0$ implies that at least one of p_{21}, p_{31} is positive. If both are positive, they represent the probabilities of going from states 2 and 3 to state 1 in one step. If $p_{21}=0$, then we must have $p_{31}>0$ and $p_{23}>0$. Thus the probability of going from state 3 to state 1 in one step is positive. If $S^2=(q_{ij})$, then $q_{21}=p_{23}p_{31}>0$ and this is the probability of going from state 2 to state 1 in two steps. If $p_{31}=0$, then $p_{21}>0$ and $p_{32}>0$ and again the probabilities of going from states 2 and 3 to state 1 in at most two steps are positive. The same sort of reasoning will show that $D_2>0$ implies that the probabilities of going from states 1 and 3 to state 2 in at most two steps are positive, and that $D_3>0$ implies that the probabilities of going from states 1 and 2 to state 3 in at most two steps are positive. Thus every f-matrix is ergodic. Conversely, let S be ergodic. If it were true that $D_1D_2D_3=0$, then S must be a matrix of one of the following six types:

(1) $p_{11}=1$ (2) $p_{22}=1$ (3) $p_{33}=1$

(4) $p_{21}=p_{31}=0$ (5) $p_{12}=p_{32}=0$ (6) $p_{13}=p_{23}=0$.

Such a matrix cannot be ergodic because the zeros in S will persist in all positive powers of S. Thus every ergodic matrix is an f-matrix, and the proof is complete.

Acknowledgment. We would like to thank the referee and the editor for their valuable comments.

Reference

J. G. Kemeny and L. J. Snell, Finite Markov Chains, Van Nostrand, Princeton, 1960.

PROBLEMS AND SOLUTIONS

All problems, both elementary and advanced, proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N.J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

SPECIAL NOTICE

The deadline for solutions is being gradually advanced to cut the long delay between publication of problems and their solutions. To offset slow mail, the Editor will send advance copies of new problem proposals to foreign correspondents who volunteer to distribute them in their countries. Direct inquiries to H. Flanders.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, Maine 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before June 30, 1969. Contributors (in North America) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2149. *Proposed by A. Zachariou, Oklahoma State University*

Let x and a be real numbers and let n be a nonnegative integer. Prove that

$$(x \mp a)^n(x \pm na) \leqq (x^2 + n^2a^2)^{(n+1)/2}.$$

E 2150. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be any three equilateral triangles in the plane (vertices labelled clockwise). Let the midpoints of segments C_2B_3, C_3B_1, C_1B_2 be M_1, M_2, M_3 respectively. Let the points of trisection of segments A_1M_1, A_2M_2, A_3M_3 nearer M_1, M_2, M_3, be T_1, T_2, T_3 respectively. Prove that triangle $T_1T_2T_3$ is equilateral.

E 2151. *Proposed by E. P. Starke, Plainfield, N. J.*

If a and b are consecutive integers, then $a^2+b^2+(ab)^2$ is always a perfect square. Find other integer pairs having this property. Indeed, show that corresponding to an arbitrary choice of a there are infinitely many values of b such that $a^2+b^2+(ab)^2$ is a square.

E 2152. *Proposed by H. T. Croft, Peterhouse, Cambridge, England*

(1) Does a given closed planar polygon necessarily contain an edge E and a vertex V such that the foot of the perpendicular from V to E falls within (closed) E?

(2) Does a given closed polyhedron necessarily contain a face F and a vertex V such that the foot of the perpendicular from V to F falls within (closed) F?

E 2153. *Proposed by Michael Warren, Constantine College, Middlesbrough, England*

Given n points in the Euclidean plane. Find a ruler and compass construction which will locate the point such that the maximum distance to any of the n points is minimized.

E 2154. *Proposed by Marlow Sholander, Case Western Reserve University*

Consider a double chessboard with $2n^2$ unit squares (x, y, z), $1 \leqq x \leqq n$, $1 \leqq y \leqq n$, $z=0$ or 1 (in which each $(x, y, 1)$ is superimposed upon $(x, y, 0)$). A piece called a jester can move (only) as follows:

1) From $(a, b, 0)$ to $(x, y, 1)$ where x is a or $a+1$, y is b or $b+1$ and $x+y>a+b$.
2) From $(a, b, 1)$ to $(x, y, 0)$ where x is a or $a-1$, y is b or $b-1$, and $x+y<a+b$.

Let P_1 be the set of squares on a path of jester moves from the edge $(1, y, 1)$ to the edge $(n, y, 0)$. Let P_2 be the set on a path from edge $(x, 1, 1)$ to edge $(x, n, 0)$. Prove that $P_1 \cap P_2$ is not empty.

E 2155. *Proposed by Anon, Erewhon-upon-Wabash*

Suppose $f(x)$ has a continuous $(2n)$-th derivative on $a \leqq x \leqq b$, that $|f^{(2n)}(x)| \leqq M$, and that $f^{(r)}(a) = f^{(r)}(b) = 0$ for $r=0, 1, \cdots, n-1$. Show that

$$\left| \int_a^b f(x)dx \right| \leqq \frac{(n!)^2 M}{(2n)!\,(2n+1)!}(b-a)^{2n+1}.$$

E 2156. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

Find necessary and sufficient conditions on m, n in each of the following cases ($\phi(n)$ is Euler's totient function):

(1) $\qquad m\phi(n) = n\phi(m)$. $\qquad$ (2) $\qquad n\phi(n) = m\phi(m)$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Harmonic Functions

E 2048 [1968, 76]. *Proposed by J. Barlaz, Rutgers—The State University*

(A) For what functions f can both u and $f(u)$ be harmonic? ($u=u(x, y)$, f a function of a single variable.)

(B) Prove: If u and v are harmonic and $|u+iv|\equiv$constant, then both u and v are constant. (This, of course, is well known when $u+iv$ is an analytic function of $x+iy$.)

Solution by Simeon Reich, Student, Israel Institute of Technology, Haifa. (A) We take the question to mean, what functions of a single variable $f(u)$ have the following property: for every harmonic function $u=u(x, y)$, $F(x, y) \equiv f(u(x, y))$ is also harmonic. We assume, of course, the existence and the continuity of the relevant derivatives:

$$F_x = f'(u)u_x, \qquad F_y = f'(u)u_y;$$

$$F_{xx} = f'(u)u_{xx} + (u_x)^2 f''(u), \qquad F_{yy} = f'(u)u_{yy} + (u_y)^2 f''(u);$$

$$F_{xx} + F_{yy} = f'(u)\{u_{xx} + u_{yy}\} + f''(u)\{(u_x)^2 + (u_y)^2\}.$$

Now suppose that $u_{xx}+u_{yy}=0$ and $F_{xx}+F_{yy}=0$ hold simultaneously. We get $f''(u)\{(u_x)^2+(u_y)^2\}=0$. Since there exists a harmonic function for which $(u_x)^2+(u_y)^2$ never vanishes (e.g., $u(x, y)=x+y$), we must have $f''(u)=0$ for all u, that is, $f(u)=Au+B$ where A, B are constants.

(B) $$uu_x + vv_x = 0, \qquad uu_y + vv_y = 0;$$

$$uu_{xx} + (u_x)^2 + vv_{xx} + (v_x)^2 = 0, \qquad uu_{yy} + (u_y)^2 + vv_{yy} + (v_y)^2 = 0;$$

$$u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + (u_x)^2 + (v_x)^2 + (u_y)^2 + (v_y)^2 = 0.$$

All this follows from $u^2+v^2=$constant. Since we also have $u_{xx}+u_{yy}=v_{xx}+v_{yy}=0$, we get $(u_x)^2+(v_x)^2+(u_y)^2+(v_y)^2=0$. This implies $u_x=u_y=v_x=v_y=0$, so that $u=$const., $v=$const.

Also solved by M. G. Beumer (Netherlands), Roxanne M. Byrne, J. A. Canavati, (Mexico), W. O. Egerland, R. B. Eggleton (Australia), M. A. Ettrick, M. G. Greening (Australia), Guillermo Hansen (Brazil), H. A. Heckart, Stephen Hoffman, Graham Lord, Felix Magnotta, Henry Ricardo, Judith Richman, Steve Rohde, J. S. Shipman, T. A. Straeter, H. H. Wong, P. H. Young, Lawrence Zalcman, David Zeitlin, and the proposer.

Cross-Cancellative Semigroup

E 2049 [1968, 76]. *Proposed by T. S. Frank, Le Moyne College, Syracuse, N.Y.*

Let $(S, \cdot)$ be a finite semigroup with identity in which the cross cancellation law ($a\cdot x=x\cdot b$ implies $a=b$) holds. Then $(S, \cdot)$ is an Abelian group.

Solution by M. W. Legg, New Mexico State University. Let $a, b\in S$, then $aba=aba$ and cross cancellation yields $ab=ba$. Hence S is abelian. It follows that the right cancellation law holds. A finite right cancellation semigroup with identity is a group.

Also solved by 109 other readers.

Several solvers point out that S is a cancellation semigroup by E 2007 [1967, 861]. Various

references were given to show that a cancellation semigroup is a group. See, for example, Fang, *Abstract Algebra*, Schaum's Outline Series (1963), page 70. Azriel Rosenfeld points out that " 'with identity' is unnecessary except insofar as it insures nonemptyness." Others, tacitly assuming that $S\neq\emptyset$ also note that the existence of an identity can be proved.

Maximum Value of a Set of Determinants

E 2050 [1968, 76]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let D be the maximum value of all determinants of order n whose entries are real numbers in the range $a\leqq x\leqq b$. Show that the value D is achieved by a determinant whose entries are exclusively a and b.

Solution by M. F. Neuts, Purdue University. The value of a determinant is a linear function in each entry separately. Regardless of the values of all other entries, the maximum value of the determinant will be attained when $a_{ij}=a$ or $a_{ij}=b$, depending on the sign of its cofactor. This argument applies to all entries taken separately, proving the stated result.

Also solved by Anders Bager (Denmark), Orin Chein, Red Cougar, Larry Cummings, D. Ž. Djoković, R. E. Eggleton (Australia), W. F. Fox, Liang-shin Hahn, G. A. Heuer, Peter Kornya, E. S. Langford, Dan Marcus, D. C. B. Marsh, J. B. Muskat, E. A. Parent, Simeon Reich (Israel), Eric Rosenthal, Samuel Schechter, S. Spital & J. Zelver, Bob Walcott, Gregory Wulczyn, and the proposer.

By expanding the determinant by elements and cofactors of the ith row, most solvers showed that if the element $a_{ij}\neq a$ and $\neq b$, then the value of the determinant is not decreased by replacing that element by one of a and b.

Djoković and Spital-Zelver note that this problem appears on p. 128 of the Canad. Math. Bull., 10 (1967) 605.

A Nonmultiplicative Function

E 2051 [1968, 76]. *Proposed by P. T. Bateman, University of Illinois*

If n is a positive integer, let $r_s(n)$ denote the number of solutions of the equation

$$x_1^2+x_2^2+\cdots+x_s^2=n$$

in integers $x_1, x_2, \cdots, x_s$ and let $f_s(n)=(2s)^{-1}r_s(n)$. If $s=1, 2, 4, 8$, it is known that f_s is multiplicative, that is, $f_s(mn)=f_s(m)f_s(n)$ for any pair of coprime positive integers m, n. Prove that f_s is not multiplicative for any other value of s. (Cf. J. M. Gandhi, Bull. Amer. Math. Soc., 72 (1966) 220–221.)

Solution by the proposer. We show that $f_s(2)f_s(3)\neq f_s(6)$ if $s\neq 1, 2, 4, 8$. Note that if $x_1, x_2, \cdots, x_s$ are integers with $x_1^2+x_2^2+\cdots+x_s^2=2$, then $s-2$ of the x_j are zero, while the remaining two are ± 1. Thus

$$r_s(2)=4\binom{s}{2}=2s(s-1).$$

Similarly

$$r_s(3) = 8\binom{s}{3} = \frac{4}{3}s(s-1)(s-2).$$

Again note that if $x_1, x_2, \cdots, x_s$ are integers with $x_1^2+x_2^2+\cdots+x_s^2=6$, then either $s-6$ of the x_j are zero and the remaining six are ± 1 or else $s-3$ of the x_j are zero, two of them are ± 1, and the remaining one is ± 2. Thus

$$\begin{aligned} r_s(3) &= 64\binom{s}{6} + 8s\binom{s-1}{2} \\ &= \frac{4}{45}s(s-1)(s-2)(s-3)(s-4)(s-5) + 4s(s-1)(s-2). \end{aligned}$$

Hence

$$\begin{aligned} f_s(2) &= s-1, f_s(3) = \frac{2}{3}(s-1)(s-2), \\ f_s(6) &= \frac{2}{45}(s-1)(s-2)(s-3)(s-4)(s-5) + 2(s-1)(s-2). \end{aligned}$$

Therefore by a simple calculation with polynomials

$$f_s(6) - f_s(2)f_s(3) = \frac{2}{45}s(s-1)(s-2)(s-4)(s-8),$$

which is different from zero if $s \neq 0, 1, 2, 4, 8$.

Also solved by L. Carlitz, R. B. Eggleton (Australia), M. G. Greening (Australia), and E. S. Langford.

A Matrix Property

E 2052 [1968, 76]. *Proposed by G. C. Berresford, Lawrence University*

Let A be any square matrix, and let B be a matrix formed as follows. B is identical to A except that the kth row and kth column of A are interchanged to become the kth column and kth row of B, k being arbitrary. Prove that multiplying the elements of row k by the corresponding cofactors of column k in matrix A and adding the products, gives the same result as multiplying the elements of column k by the corresponding cofactors of row k in matrix B, and adding the products.

Solution by Sidney Spital, California State College, Hayward. Let a third matrix C be identical to A (or B) except that its kth row and kth column are both the same as the kth row of A (or kth column of B). Then the two sums under consideration are Laplace expansions of det (C) by cofactors of column k and row k respectively. Hence the two are equal.

Also solved by P. L. Claypool, R. S. Eggleton (Australia), M. G. Greening (Australia), J. V. Michalowicz, Simeon Reich (Israel), and the proposer.

The Platonic Solids

E 2053 [1968, 77]. *Proposed by W. E. Buker, Pittsburgh, Pa. Public Schools*

If the five Platonic solids are inscribed in the unit sphere, the one having the greatest volume is the dodecahedron. Show this. Is this also true for surface area?

Solution by R. B. Eggleton, Avondale College, Cooranbong, N.S.W., Australia. When inscribed in a unit sphere, the Platonic solids have the following "vital statistics":

	Length of side	Volume	Surface Area
Tetrahedron	$2\sqrt{2}/\sqrt{3}=1.6329\cdots$	$0.5132\cdots$	$4.6188\cdots$
Cube	$2/\sqrt{3}=1.1547\cdots$	$1.5396\cdots$	8.0000
Octahedron	$\sqrt{2}=1.4142\cdots$	$1.3333\cdots$	$6.9282\cdots$
Dodecahedron	$(\sqrt{5}-1)/\sqrt{3}=0.7136\cdots$	$2.7851\cdots$	$10.5146\cdots$
Icosahedron	$\sqrt{(10-2\sqrt{5})/5}=1.0514\cdots$	$2.5361\cdots$	$9.5745\cdots$

Also solved by Anders Bager (Denmark), Leon Bankoff, M. G. Beumer (Netherlands), Michael Goldberg, Norman Miller, J. W. Pfaendtner, Simeon Reich (Israel), G. J. Simmons, and the proposer.

Editorial Note. The fact that of a regular dodecahedron and a regular icosahedron inscribed in the same sphere, the former has the greater volume and the greater surface area has long been known, but the unacquainted person is almost certain to guess wrong. Also, of a cube and a regular octahedron inscribed in the same sphere, the cube has the greater volume and the greater surface area. A regular dodecahedron and a regular icosahedron inscribed in the same sphere have a common inscribed sphere, as do a cube and a regular octahedron. If a regular dodecahedron and a regular icosahedron are inscribed in the same sphere, then their volumes are in the same ratio as their surface areas; the same is true of a cube and a regular octahedron. The circumcircles of the faces of a regular dodecahedron and a regular icosahedron inscribed in the same sphere are equal; the same is true of a cube and a regular octahedron. These results are interesting and contrary to general intuition.

Counting a Certain Class of Sequences

E 2054 [1968, 77]. *Proposed by L. Carlitz and R. A. Scoville, Duke University*

Find the number of sequences of positive integers $(a_1, a_2, \cdots, a_n)$ such that

$$1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n; \qquad a_i \leqq i\ (i = 1, 2, \cdots, n).$$

I. *Solution by G. A. Heuer, Concordia College.* Let $f(n)$ be the required number, and $f_k(n)$ the number of these sequences with $a_n=k$, $1\leqq k\leqq n$. Then $f_k(n+1)=\sum_{j=1}^{k} f_j(n)$, and

$$f(n) = \sum_{k=1}^{n} f_k(n) = f_n(n+1) = f_{n+1}(n+1).$$

Also, it is clear that $f_1(n)=1$ for all n, and $f_{k+1}(n+1)-f_k(n+1)=f_{k+1}(n)$. These conditions completely determine the $f_k(n)$ since f_1 is known and f_{k+1} is determined from f_k by the difference equation and initial condition

$$(*)\qquad \Delta f_{k+1}(n)=f_k(n+1),\qquad f_{k+1}(k+1)=f_k(k+1).$$

Finite integration for the first few values of k suggests the formula

$$f_k(n)=\binom{n+k-3}{n-2}-\binom{n+k-3}{n},$$

which is easily shown to satisfy the system (*). Thus

$$f(n)=f_{n+1}(n+1)=\binom{2n-1}{n-1}-\binom{2n-1}{n+1}=\frac{(2n)!}{n!(n+1)!}.$$

II. *Solution by M. G. Beumer, Technological University, Delft, Netherlands.* Denoting the number of sequences by A_n and setting $A_0=1$, we have for $n\geqq 1$:

$$(1)\qquad A_n=\sum_{k=1}^{n}A_{k-1}\cdot A_{n-k}.$$

If $f(x)=\sum_{n=0}^{\infty}A_nx^n$, then from (1) it follows that $f(x)=1+x\cdot\{f(x)\}^2$, $f(0)=1$. so that

$$(2)\qquad f(x)=(1-\sqrt{1-4x})/2x.$$

By expansion of the right hand side of (2) and comparison of coefficients we find, for $n\geqq 0$:

$$A_n=\frac{1}{n+1}\binom{2n}{n}.$$

Also solved by D. R. Anderson, Bernard August, Stephen Berman & Steven Minsker, D. M. Bloom, Rolf Bornhorst (Germany), Paul Brock, C. A. Church, Jr., J. E. Coury & D. J. Lutzer, Ted Cullen, C. D. Dixon, R. B. Eggleton (Australia), R. D. Fray, T. Fujinawa (Japan), L. M. Gaintner, M. L. Goodman, M. G. Greening (Australia), Heiko Harborth (Germany), Robert Heller C. F. Hockett, A. J. Kenzie, E. S. Langford, Renate McLaughlin, Dan Marcus, D. C. B. Marsh, Norman Miller, R. A. Mullikin, M. F. Neuts, F. D. Parker, C. B. A. Peck, V. K. Rohatgi, R. A. Savrin, G. J. Simmons, P. G. Schmitt, Jr., W. B. Smith, D. A. Spear, Michael Stolnicki, K. L. Yocom, and the proposers.

Several of the above failed to indicate the closed form of the solution. Brock notes that the problem is a special case of a problem that he and R. M. Baer discussed in their paper, *Natural sorting over spaces of binary sequences*, Berkeley Computer Center Report No. 18. Rohatgi indicates that the generalization obtained by replacing $a_i\leqq i$ by $a_i\leqq ki+r$, with $k\geqq 1$ and $0\leqq r\leqq k$ can be found in S. G. Mohanti and T. V. Narayana, *Some properties of compositions and the applications to probability and statistics I*, Biom. Zeitschrift 3 (1961) 252–258. In this case the number of sequences $B_n(k,r)$ is found to be

$$B_n(k,r)=\frac{r+1}{nk+r+1}\binom{nk+r+n}{n}.$$

Carlitz offers the following generalization. Replace $a_i \leqq i$ by $a_i \leqq ki+t$, with $k \geqq 0$ and $t \geqq -k+1$, then the number of sequences is

$$\frac{k+t}{n}\binom{(k+1)n+k+t-1}{n-1}.$$

A number of solutions indicate that the sequences could be represented by paths with unit horizontal and vertical steps. It is then pointed out that the Bertrand Ballot Theorem (Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, third ed., Wiley, New York, p. 73) can be used to count these paths.

Nature of the Roots of Real Polynomial Equations

E 2055 [1968, 188]. *Proposed by M. F. Capobianco, St. John's University*

Consider a real polynomial equation of degree n. Attention is paid to whether the roots are real and unequal, real and equal (in various combinations), or simple or multiple complex conjugates. If $n=2$ there are but three possibilities, namely, all roots real and equal, all roots real and unequal, and all roots complex. If $n=3$, there are four possibilities: three equal, two equal, three unequal, two complex. For $n=4$, there are nine possibilities. How many possibilities are there for general n?

Solution by C. F. Pinzka, University of Queensland, Australia. Denoting by m_i, $i=1, 2, \cdots, r$, the multiplicities of the conjugate pairs of imaginary roots and by n_i, $i=1, 2, \cdots, s$, the multiplicities of the real roots, we seek the number of solutions of

$$2\sum_{i=1}^{r} m_i + \sum_{i=1}^{s} n_i = n,$$

subject to the restrictions $m_1 \leqq m_2 \leqq \cdots \leqq m_r$, $n_1 \leqq n_2 \leqq \cdots \leqq n_s$. This is given by the convolution

$$\mu_n = \sum_{k=0}^{[n/2]} P(k)P(n-2k),$$

where $P(k)$ is the number of unrestricted partitions of k, with the well-known (see, for example, Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958, p. 111) generating function

$$f(t) = \sum_{k=0}^{\infty} P(k)t^k = \prod_{k=0}^{\infty}(1-t^k)^{-1}.$$

It is easily verified that the generating function for μ_n is

$$\begin{aligned} f(t)f(t^2) &= \prod_{k=0}^{\infty}(1-t^k)^{-1}\prod_{k=0}^{\infty}(1-t^{2k})^{-1} \\ &= 1+t+3t^2+4t^3+9t^4+12t^5+23t^6 \\ &\quad +31t^7+54t^8+73t^9+118t^{10}+\cdots. \end{aligned}$$

Also solved by W. D. Bouwsma, R. B. Eggleton (Australia), Michael Goldberg, Lise Hamel & Pierre Robillard, Donald Jeffords, D. C. B. Marsh, Eric Rosenthal, Steven Russ, R. W. Sielaff, and Michael Stolnicki.

A Three Circle Configuration

E 2056 [1968, 188]. *Proposed by A. W. Walker, Toronto, Canada*

Two given circles have four real common tangents passing in pairs through two points K and K' on the line of centers. A third circle C touches the given circles at points Q and R antihomologous with respect to the homothetic center K. Then the four points where the two common tangents through K' meet circle C form a quadrangle with one pair of opposite sides that are parallel to the common tangents through K and meet at a point on the line KQR. (Part of this result was given by V. Thébault, *Mathesis*, 62 (1953) 112–114.)

Editorial Note. The proposer outlines a long and tedious analytical proof. As references leading to synthetic proofs of the first part of the problem he gives (in addition to the above Thébault reference): (1) W. J. M'Clelland, *The Geometry of the Circle*, Macmillan, 1891, pp. 258–9; (2) J. Dougall, *Proc. Ed. Math. Soc.*, 33 (1915) 42. Both of these proofs use Casey's five-circle theorem. Dougall goes on to show how Feuerbach's theorem follows as a corollary. Another long analytical treatment of the first part of the problem can be found in T. C. Lewis, *Messenger of Mathematics* (2), 46 (1916) 4. Lewis uses the result to show that there are Tucker circles touching three of the four tritangent circles of a triangle. A nice solution of the second part of the problem seems still to be found, and a simpler elementary solution of the first part is desirable. As an immediate consequence of the problem it can be shown that the circle touching and surrounding the three described circles of a triangle is a Tucker circle of the triangle.

A Mean Value Type Theorem

E 2057 [1968, 188, 541]. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

(1) If $f''(x)>0$ throughout the closed interval $[a, b]$, then for each $\xi\in(a, b)$, there is a point $x_0\in[a, b]$ such that either

$$f'(\xi) = \frac{f(b) - f(x_0)}{b - x_0} \quad \text{or} \quad f'(\xi) = \frac{f(a) - f(x_0)}{a - x_0}.$$

(2) If $f''(x)>0$ in $[a, b]$ and $f'(x_0)=0$ for $x_0\in(a, b)$, then there is a point $x_1\in[a, b]$ such that either $f(a)=f(x_1)$ or $f(b)=f(x_1)$.

Solution by W. G. Dotson, Jr., North Carolina State University. (1) Define $g(x)=[f(x)-f(a)]/(x-a)$, $x\neq a$, $g(a)=f'_+(a)$; and $h(x)=[f(b)-f(x)]/(b-x)$, $x\neq b$, $h(b)=f'_-(b)$. Then $g(x)$, $h(x)$ are continuous on $[a, b]$, and $g(b)=h(a)$. Since $f'(x)$ is monotone increasing on $[a, b]$, we have $g(a)<f'(\xi)<h(b)$. If $g(a)<f'(\xi)<g(b)$, then (by continuity of $g(x)$) there exists $x_0\in(a, b)$ such that $f'(\xi)=g(x_0)$. Otherwise, we have $g(b)=h(a)\leq f'(\xi)<h(b)$, so that (by continuity of $h(x)$) there exists $x_0\in[a, b)$ such that $f'(\xi)=h(x_0)$. The case $x_0=a$ cannot be excluded, e.g., $f(x)=x^2$, $[a, b]=[0, 2]$, $\xi=1$.

(2) This follows obviously from (1) if one replaces x_0 in (2) by ξ, and replaces

x_1 in (2) by x_0. Again the case $x_0=a$ (or b) cannot be excluded, e.g., $f(x)=x^2$, $[a, b]=[-1, 1]$, $\xi=0$.

Also solved by Donald Batman, P. M. Berry, W. D. Bouwsma, Neil Cameron (Australia), F. D. Cheek II, Arnold Dunn & Chanchal Singh, P. M. Ellis, T. E. Elsner, W. F. Fox, Ray Glenn, L. S. Hahn, Robert Heller, Judith Housman, F. T. Howard, Donald Jeffords, H. E. Lahmann (Germany), E. S. Langford, R. S. Lee, P. A. Lindstrom, Beatriz Margolis (Argentina), Geoffrey Masaki, Kenneth Miller, Prof. M. Morucci's Math. 200 class, J. B. Muskat, R. A. Northcutt, T. M. Phillips, C. F. Pinzka (Australia), Francisco Ramirex (Costa Rica), V. B. Rao, Simeon Reich (Israel), Rajinder Singh, N. H. Stevens, Michael Stolnicki, Paul Sugarman, J. Treebrook, R. J. Wagner, Albert White, and the proposer. The following only found counterexamples to the original incorrect statement of the problem: R. A. Fuller, Ray Glenn, H. A. Heckart, Elaine Koppelman, J. A. Roulier, and Mark Shall.

A Divisor of $\binom{b^a}{j}$

E 2058 [1968, 189]. *Proposed by J. E. Desmond, Florida State University*

Prove b^{a-j+1} divides $\binom{b^a}{j}$ for positive integers a, $b>1$, and $j\leqq a+1$.

Solution by Donald Jeffords, Weedsport, N. Y. We first show $b^n\geqq n+1$, for $n\geqq 0$. Since $b^n\geqq 2^n$, $2^0=1$, assume for $k\geqq 0$, $2^k\geqq k+1$. Then $2^{k+1}=2(2^k)\geqq 2(k+1)$ $\geqq k+2$. Thus $b^a\geqq j$ and we have

$$\binom{b^a}{j}=\prod_{i=0}^{j-1}(b^a-i)/j!.$$

Let $i=b^rm$, $(m, b)=1$. Then $b^a-i=b^r(b^{a-r}-m)$, so that i and b^a-i, $i>0$, contain the same power of b. Thus the power of b in $\binom{b^a}{j}$ is a minus the power of b in j; however $b^{j-1}\geqq j$, $j\geqq 1$, so that $b^{a-(j-1)}=b^{a-j+1}$ divides $\binom{b^a}{j}$.

Also solved by C. Gardner, M. G. Greening (Australia), Norman Miller, and the proposer.

Inequalities for an Arbitrary Pair of Triangles

E 2059 [1968, 189]. *Proposed by A. Oppenheim, University of Reading, England*

For any two triangles prove the inequalities

$$(1)\qquad \frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\geqq 2\left(\frac{\cos A'}{a}+\frac{\cos B'}{b}+\frac{\cos C'}{c}\right),$$

$$(2)\qquad r_1+r_2+r_3\geqq 2(h_1\cos A'+h_2\cos B'+h_3\cos C'),$$

(where r_1 is the exradius corresponding to A, h_1 is the altitude from A, etc.) with equality if and only if both triangles are equilateral.

Further, deduce from (1) Barrow's inequality

$$(3)\qquad x+y+z\geqq 2(u+v+w),$$

where x, y, z are the distances of an internal point O from the vertices of a triangle XYZ and u, v, w are the distances of O from YZ, ZX, XY measured

along the internal bisectors of the angles YOZ, ZOX, XOY. (See also problem 3740 by Paul Erdös, [1937, 252] solution by D. F. Barrow.)

Solution by Leon Bankoff, Los Angeles, California. For convenience, the given inequalities may be written

$$\frac{abc}{2}\sum\frac{1}{s-a} \geqq 2(ab\cos C' + ac\cos B' + bc\cos A'), \tag{1}$$

$$(r_1+r_2+r_3)\left(\frac{abc}{2\Delta}\right) \geqq 2(ab\ \cos\ C' + ac\ \cos\ B' + bc\ \cos\ A'). \tag{2}$$

Mordell has given a simple proof of (3), the Barrow inequality, (Mathematical Gazette, October 1962, pp. 213–215) by making use of the inequality

$$a^2+b^2+c^2 \geqq 2(ab\ \cos\ C' + ac\ \cos\ B' + bc\ \cos\ A'),$$

with equality holding only when $a/\sin A' = b/\sin B' = c/\sin C'$. It is therefore sufficient to prove that $\sum a^2$ cannot exceed the left members of (1) and (2) above.

The proof depends on the relations

$$2\sum ab - \sum a^2 = 4r(4R+r), \qquad \sum a^2 \leqq 4r^2+8R^2,$$

contained in John Steinig, *Inequalities concerning the inradius and circumradius of a triangle*, Elemente der Mathematik, Nov. 1963, pp. 127–131.

Remembering that $2r \leqq R$, we have, for the proof of (1),

$$\sum a^2 \leqq 4r^2+8R^2 \leqq \frac{R}{2r}\left[4r(4R+r)\right] \equiv \frac{4R\Delta}{2}\cdot\frac{1}{4r\Delta}\left(2\sum ab - \sum a^2\right)$$

or $\quad \sum a^2 \leqq \frac{1}{2}abc\sum 1/(s-a)$, since $abc = 4R\Delta$ and

$$\frac{2\sum ab - \sum a^2}{4r^2 s} = \sum\frac{1}{s-a}\,.$$

To prove (2), we have

$$\sum a^2 \leqq 4r^2+8R^2 \leqq 2R(4R+r) \equiv \frac{abc}{2\Delta}(4R+r) \equiv \frac{abc}{2\Delta}(r_1+r_2+r_3).$$

Also solved by M. G. Greening (Australia), Simeon Reich (Israel), and the proposer.

Another Interesting Triangle Inequality

E 2060 [1968, 189]. *Proposed by F. Leuenberger, Feldmeilen, Switzerland*

Let t_a, t_b, t_c be the internal angle bisectors and m_a, m_b, m_c the medians of a triangle T, r and R its in- and circum-radius, p its semiperimeter. Prove that

$$\sum t_a^6 \leqq p^4(p^2-12rR) \leqq \sum m_a^6,$$

with equality if and only if T is equilateral.

Solution by Simeon Reich, Israel Institute of Technology, Haifa. First note that

$$p^4(p^2-12rR) = p^4\left(p^2 - 12\frac{S}{p}\cdot\frac{abc}{4S}\right) = p^4\left(p^2 - \frac{3abc}{p}\right) = p^3(p^3 - 3abc)$$
$$= p^3[(p-a)^3 + (p-b)^3 + (p-c)^3],$$

where S denotes the area of the triangle. Now

$$t_a = \frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)} \leqq \sqrt{p(p-a)}$$

with equality if and only if $b=c$. Similar expressions hold for t_b and t_c. Thus $\sum t_a^6 \leqq p^3(\sum(p-a)^3)$ with equality if and only if the triangle is equilateral. Also

$$m_a^2 = \tfrac{1}{4}(2b^2+2c^2-a^2) = \tfrac{1}{4}\{(b+c-a)(b+c+a)+b^2+c^2-2bc\}$$
$$= (p-a)(p) + \tfrac{1}{4}(b-c)^2.$$

That is, $m_a^2 \geqq p(p-a)$ with equality if and only if $b=c$. Similarly for m_b and m_c. It follows that

$$\sum m_a^6 \geqq p^3\left(\sum (p-a)^3\right)$$

with equality if and only if the triangle is equilateral.

Also solved by the proposer.

Geodesics on the Mercator Map

E 2061 [1968, 189]. *Proposed by David Singmaster, American University, Beirut, Lebanon*

For a Mercator map, it is well known that a straight line does not yield the path of shortest distance. What kind of curve does correspond to a path of shortest distance?

Solution by Judith Richman, Drexel Institute of Technology. Let units be chosen so that Cartesian coordinates on the Mercator map are z and θ, where θ is the longitude. The shortest path between two points on the surface of a sphere is along the great circle joining them. If the points have the same longitude, the minimal path is a meridian and is represented by a straight vertical line on the Mercator map. In all other cases, the equation of the great circle in terms of latitude, ϕ, and longitude is given by

$$\tan\phi = A\sin(\theta-\theta_0),$$

where the constants A and θ_0 are determined by the two points. It is a popular

misconception that the Mercator map is obtained by projection from the center of the sphere onto a cylinder tangent at the equator. If that were the case, then we would have $z=\tan\phi$, and the geodesics would be sine curves. However, Mercator maps are conformal and, ignoring the correction for the ellipticity of the earth, we should write

$$z = \log \left| \tan\phi + \sec\phi \right|,$$

so that the equations of the geodesics are

$$\sinh z = A \sin(\theta - \theta_0).$$

Reference: Bowditch, *American Practical Navigator.*

Also solved by Michael Goldberg, Norman Miller, Eric Rosenthal, Harold Simpson (England), P. D. Thomas, E. W. Trost (Switzerland), and R. H. Wilson, Jr.

Thomas considers also the more general case of the Mercator projection of an oblate ellipse of revolution. Trost finds a treatment in G. Loria, *Spezielle algebraische und transcendente Kurven*, Leipzig 1911, Vol 2, pp. 185–188.

A Certain Sequence of Composites

E 2062 [1968, 189]. *Proposed by Dale Peterson, Student, Mira Loma High School, Sacramento, Calif.*

Prove that there is a set of n composite integers in arithmetic progression, relatively prime in pairs, for any integer n.

Solution by Arne Garness, Charles Heuer, and Gerald Heuer, Concordia College. Note that if $2\leqq k\leqq N$, then $N!+k$ is composite. Given n, choose a prime $p>n$ and an integer $N\geqq p+(n-1)n!$. Then the integers $N!+p$, $N!+p+n!$, $\cdots$, $N!+p+(n-1)n!$ are composite and are in arithmetic progression. Moreover, if q were a common prime factor of two of them, q would divide their difference, $jn!$, for some $0<j<n$. Hence $q\leqq n$. But this would imply $q|N!$ and consequently $q|p$, which is impossible, so the integers are relatively prime in pairs.

Also solved by Anders Bager (Denmark), Donald Batman, R. B. Eggleton (Australia), Bengt Fornberg (Sweden), William Fox, C. Gardner, Emil Grosswald, Erwin Just, J. A. Lang, D. C. B. Marsh, J. B. Muskat, P. K. Subramanian, J. Treebrook, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before July 31, 1969. Contributors (in North America) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5651. *Proposed by Maurice Machover, St. John's University*

Consider one of the standard nonmeasurable sets, say the set E formed by

taking one representative from each one of the equivalence classes formed from the points of the circumference of a circle of diameter 1 by calling two points equivalent if the distance between them, measured along the arc, is an integer. It is easily seen that the inner measure of E is zero. What is its outer measure?

5652. *Proposed by Joseph Lehner, University of Maryland*

What is the most general function that is meromorphic in $|z| \leqq 1$ and that maps the unit circle ($|z| = 1$) into itself?

5653. *Proposed by Richard Stanley, Harvard University*

Find in each case the real-valued multiplicative number-theoretic function f which satisfies the stated condition. μ is the Möbius function, ϕ is the Euler totient function, $d(n)$ is the number of divisors of n, and $\sigma(n)$ is the sum of the divisors of n.

(A) $$\sum_{d|n} f(d) = \mu(n)f(n).$$

(B) $$|f(n)| = \sum_{d|n} f(d).$$

(C) $$\sum_{e|n} \phi(e)f(n/e) = d(n)f(n).$$

(D) $$\sum_{d|n} \frac{\mu(d)d^2f(n/d)}{\phi(d)} = \sigma(n)f(n).$$

5654. *Proposed by L. J. Lardy and J. A. Lindberg, Jr., Syracuse University*

Let A be a commutative Banach algebra with identity, and let Φ_A denote the Gelfand space of A. Call a closed subset I of Φ_A an inverting set for A if the invertible elements of A are precisely those elements whose Gelfand transform does not vanish on I. Is the intersection of two inverting sets an inverting set?

5655. *Proposed by Paul Erdös and E. C. Milner, University of Calgary*

If $<_1, <_2, <_3, \cdots$ are countably many well orderings of the set of reals, must there be a pair x, y such that $x <_n y$ for all n?

5656. *Proposed by P. A. Catlin, Carnegie-Mellon University*

Let a and n be positive integers such that $2n+1$ is prime and let the brackets denote the greatest integer function. When is $\sum_{j=1}^{n} [\sin 2\pi aj/(2n+1)]$ an even integer?

5657. *Proposed by Erwin Just, Bronx Community College, New York*

Prove that there exists a real valued function f, which is defined on $(0, \infty)$, is continuous at an infinite number of points, and has the property that for each x, $f(x) = 0$ if and only if $f(2x) \neq 0$.

SOLUTIONS OF ADVANCED PROBLEMS

Some Co-splitting Polynomials in Fields of Prime Characteristic

5560 [1968, 197]. *Proposed by G. B. Seligman, Yale University*

Let F be a field of prime characteristic p. Consider the p-polynomials with coefficients in F:

$$q(X) = X^{p^m} - a_{m-1}X^{p^{m-1}} - \cdots - a_1X^p - a_0X,$$

$$r(X) = a_0^{p^{m-1}} X^{p^m} + a_1^{p^{m-2}} X^{p^{m-1}} + \cdots + a_{m-1}X^p - X,$$

where $a_0 \neq 0$ (cf. Jacobson, *Lie Algebras*, p. 193).

(A) Show that $q(X)$ splits completely in F if and only if $r(X)$ splits completely in F. (*B*) Is it true that $q(X)$ has a nonzero root in F if and only if $r(X)$ has a nonzero root in F?

Solution by the proposer. We make the following observations: (1) If $f(X)$ is any p-polynomial, then $X \to f(X)$ is an endomorphism of F as a Z_p-vector space. (2). All roots of $q(X)$ are separable over F since $q'(X) = -a_0 \neq 0$. (3) The rule for obtaining $r(X)$ from $q(X)$ can be applied to

$$a_0^{-p^{m-1}} r(X)$$

and yields

$$q^{p^{m-2}}(a_0^{-p^{m-2}} X),$$

where q^{p^k} denotes the polynomial obtained by raising each coefficient of q to the power p^k. (We confine our attention to $m \geqq 2$, since other cases can be handled directly.) (4) If α is a root of $q(X)$, then $A\alpha^{p^k}$, $A \in F$, is a root of $q^{p^k}(A^{-1}X)$. Reversing this, each root of $q^{p^k}(A^{-1}X)$ in F determines a root of $q(X)$ in a totally inseparable extension of F which is, by (2), in F. (5) The relation between $q(X)$ and $r(X)$ can be expressed by: β is a root of $r(X)$ if and only if there is a p-polynomial $f(x) = \gamma_{m-1}X^{p^{m-1}} + \cdots + \gamma_0 X$ such that $\gamma_{m-1} = \beta$ and $[f(X)]^p - f(X)$ is divisible by $q(X)$. To prove this note that $[f(X)]^p - f(X) = \gamma_{m-1}q(X)$ if it is divisible by $q(X)$. Now compare coefficients of X, X^p, X^{p^2}, $\cdots$, $X^{p^{m-1}}$ in that order. (cf. O. Ore, *A special class of polynomials*, Trans. Amer. Math. Soc., 35 (1933) pp. 559–584, especially 570–572.)

Proof of (A). Suppose $q(X)$ splits completely in F. By (1) the roots of $q(X)$ form a Z_p-vector space S of dimension m, and each p-polynomial $f(X)$, when restricted to S, determines a Z_p-linear map: $S \to F$. Two such maps are the same only if the polynomials defining them are congruent mod $q(X)$. The set of all Z_p-linear maps: $S \to F$ has the structure of an F-vector space of dimension m. The polynomials define a subspace of the same dimension, hence the whole space. Z_p is a subfield of F, so each of the p^m linear maps: $S \to Z_p$ can be written as a map: $S \to F$, and hence as a p-polynomial $f(X)$ of degree at most p^{m-1}. If

$X \in S$, $f(X) \in Z_p$, so $q(X)$ divides $[f(X)]^p - f(X)$. If $f(X)$ is not identically zero, then it has exactly degree p^{m-1}, so the leading coefficients of the p-polynomials f: $S \to Z_p$ form a Z_p-vector space of dimension m contained in F. By (5), $r(X)$ splits completely in F.

Conversely, suppose $r(X)$ splits completely in F. (3) and (4) combined with what we have just shown imply that $q(X)$ splits completely in F.

Comments on (B). If F is finite and β, $0 \neq \beta \in F$, is a root of $r(X)$, then we form the polynomial $f(X)$ associated with β by (5). Then $q(X) = \lambda(f(X)^p - f(X))$ for some λ, $0 \neq \lambda \in F$. Any root of $f(X)$ is then a root of $q(X)$. If $f(X)$ has no nonzero root in F, then $X \to f(X)$ is one-one, hence onto. If $f(\gamma) = 1$, γ is a root of $q(X)$. So (B) is true in this case.

If F is not finite, counterexamples to (B) may be constructed. For example, in $F = Z_p(t)$ with $p \neq 2$, let $\lambda \in Z_p$, $\lambda \neq 0$, 1, and take

$$q(X) = x^{p^2} - (\lambda + t^{p-p^2})X^p + \lambda t^{1-p} X,$$

$$r(X) = -\lambda t^{p-p^2} X^{p^2} + (\lambda + t^{p-p^2})X^p - X.$$

Then t^{-1} is a root of $q(X)$. The roots of $r(X)$ are integral over $Z_p[t]$. Since this is integrally closed in F, any root of $r(X)$ in F is in $Z_p[t\ ]$. Its degree cannot be greater than 1. All linear polynomials can be shown to fail to satisfy $r(X)$.

Finite Rings

5570 [1968, 304]. *Proposed by Kwangil Koh, North Carolina State University*

Let R be a ring such that $xRy = 0$ implies $x = 0$ or $y = 0$. Let $x^\perp = \{y \mid xy = 0\}$. Let M be the set of all x such that $x^\perp \neq R$, and $x^\perp \subseteq y^\perp$ implies $y^\perp = x^\perp$ or $y^\perp = R$. Prove that if M is finite and not empty then R is finite.

Solution by Dan Marcus, York University. We assume that R is infinite and obtain a contradiction. We establish first that for each $m \in M$, $Rm \subseteq M \cup \{0\}$. Let $r \in R$; then $m^\perp \subseteq (rm)^\perp$. Either $(rm)^\perp = m^\perp$, in which case $rm \in M$; or $(rm)^\perp = R$, in which case $rmRrm = 0$ implying that $rm = 0$.

We next show that there is a $t \in R$, $t \neq 0$, such that $tM = 0$. Let $M = \{m_1, m_2, \cdots, m_n\}$. For $1 \leqq k \leqq n$, let $R_k = \{x \mid xm_i = 0,\ i = 1, 2, \cdots, k\}$. By induction it is shown that each R_k is infinite: Set $R_0 = R$. Then, assuming R_{k-1} is infinite, define ϕ: $R_{k-1} \to R$ by the formula $\phi(x) = xm_k$. It follows from the above that ϕ has finite range. Hence there is an infinite subset $S \subseteq R_{k-1}$ on which ϕ is constant. Now R_k must contain all elements of the form $S_1 - S_2$, where S_1, $S_2 \in S$, and therefore R_k is infinite.

Finally, take any $m \in M$; then $tRm \subseteq t(M \cup \{0\}) = \{0\}$, and so $tRm = 0$ with $t \neq 0$. Therefore $m = 0$, but this is not true either. The proof is complete.

Also solved by Tim Andersen, P. R. Chernoff, D. Ž. Djoković, Roger Lyndon, Surjeet Singh & Kamlesh Wasan (India), Wen-Jin Woan, and the proposer.

A slight modification of the above solution is used by Lyndon to establish $(n+1)^n$ as an upper bound for the number of elements in R.

On Compact Topological Rings

5571 [1968, 304]. *Proposed by R. Wiegandt, Mathematical Institute, Hungarian Academy of Science, Budapest*

Prove that a compact topological ring with no proper closed left ideals is a finite field or a zero-ring of prime order.

Solution by S. L. Warner, Duke University. The ring is discrete by a theorem of Kaplansky [*Topological Rings*, Am. J. Math., 69 (1947) 153–183, Theorem 9 and Corollary]. Consequently the result follows by a familiar, elementary theorem of algebra.

Note. A locally compact ring with identity and no nonzero proper closed one-sided ideals is a division ring [V. A. Andrunakievič and C. I. Arnautov, *Invertibility in topological rings* (Russian), Dokl. Akad. Nauk SSSR, 170 (1966) 755–758].

Also solved by P. R. Chernoff, D. A. Hejhal, Kwangil Koh, and the proposer.

Ideals of a Commutative Artinian Ring

5572 [1968, 304]. *Proposed by Howard Kleiman, Queensborough Community College, New York*

In an Artinian ring A which is commutative and has an identity element, let S be a multiplicatively closed subset such that $S \cap (\bigcup_{i=1}^{n} m_i) = \emptyset$, where the m_i are the maximal ideals of A. Prove that every ideal of A is a contracted ideal of $S^{-1}A$ in A, where $S^{-1}A$ is the ring of fractions of A with respect to S.

Solution by J. A. Johnson, University of California at Riverside. Since $S \cap (\bigcup_{i=1}^{n} m_i) = \emptyset$, it follows that $S \subseteq A - (\bigcup_{i=1}^{n} m_i) = \{\text{units of } A\}$. Consequently $S^{-1}A \cong A$ under the natural map. This implies $I = (I^e)^c$ for every ideal I of A, i.e., every ideal of A is the contraction of its extension to $S^{-1}A$.

Also solved by R. W. Gilmer, Jr., J. E. Humphreys, Brian Parshall, Kamlesh Wasan, and the proposer.

Wasan and Parshall refer to Zariski, *Commutative Algebra*, vol. I, Theorem 15, p. 223, from which the conclusion follows at once. See also Problem 5573, below.

Ideals of a Commutative Noetherian Ring

5573 [1968, 304]. *Proposed by Howard Kleiman, Queensborough Community College, New York*

Let R be the group of units of a Noetherian ring A which is commutative with an identity. Then every ideal of A is a contracted ideal of $R^{-1}A$, where $R^{-1}A$ is the ring of fractions of A with respect to R.

Solution by R. W. Gilmer, Jr., Florida State University. (The following includes also the result of problem 5572, above.) If R is a commutative ring with identity and if S is a multiplicative system in R such that each element of S is a unit of R, then $R = R_S$ so that each ideal of R is a contracted ideal of R_S.

$R=R_S$ follows because the elements of S, being units of R, are regular in R and $R_S=\{r/s|r\in R,\ s\in S\}=R$.

Conversely, if R is a commutative ring and if S is a multiplicative system in R such that each ideal of R is the contraction of an ideal of R_S, then R contains an identity and each element of S is a unit of R. For the proof, observe that if $s\in S$, then the extension of the ideal $A=Rs=\{rs|r\in R\}$ to R_S is R_S since $S^2\subset A$. Therefore if A is a contracted ideal of R_S, then $A=R$ and $s=es$ for some $e\in R$. If $t\in R$, then $t=rs$ for some $r\in R$ and $te=rse=rs=t$. Therefore e is an identity element for R, and the equality $R=Rs$ implies that each element s of S is a unit of R.

Also solved by J. E. Humphreys, J. A. Johnson, M. L. Laplaza (Puerto Rico), Brian Parshall, and the proposer.

The Determinant of a Partitioned Matrix

5574 [1968, 304]. *Proposed by K. Drabek and L. Beran, CVUT, Prague, Czechoslovakia*

Give a simple formula for the determinant of the matrix Q partitioned into submatrices $B_{ij}=\delta_{ij}B'+b_{ij}I$, $i,j=1,2,\cdots,n$, where δ_{ij} is the Kronecker delta, $B'=(b_{ji})$ the transpose of the matrix B, I the identity matrix, b_{ij} elements of a commutative field R.

Solution by P. J. Nikolai, Wright-Patterson Air Force Base. Let $f(\lambda)$ denote the characteristic polynomial of B, hence of B', and let $\lambda_1,\cdots,\lambda_n$ denote its zeros in a suitable splitting field of f. We view Q as the sum of the Kronecker products $B\otimes I$ and $I\otimes B'$. Then the n^2-square matrix Q has eigenvalues $\lambda_i+\lambda_j$, $i,j=1,\cdots,n$. We may thus write

$$\det Q=\prod_{i,j=1}^{n}(\lambda_i+\lambda_j)=\prod_{i=1}^{n}\det(\lambda_iI+B)=(-1)^n\prod_{i=1}^{n}\det(-\lambda_iI-B)$$
$$=(-1)^n\prod_{i=1}^{n}f(-\lambda_i)=(-1)^n\det f(-B).$$

Also solved by J. W. Bond, J. L. Brenner, G. N. de Oliveira (England), D. Ž. Djoković, J. R. Kuttler, T. L. Markham, R. C. Thompson, and the proposers.

Editorial Note. The proposers offer the formula

$$\det Q=2^n\det B\prod_{i<j}(b_{ii}+b_{jj})^2.$$

Thompson obtains the result in the form:

$$\det Q=2^n\det B\prod_{i<j}(\lambda_i+\lambda_j)^2.$$

Divisible Abelian Groups

5576 [1968, 305]. *Proposed by F. L. Sandomierski, University of Wisconsin*

Show that an Abelian group D is divisible if and only if every nonzero

homomorphic image of it is not finitely-generated.

Solution by Peter Csontos and B. M. Schreiber, University of Washington. If G is divisible, then so is every homomorphic image of G, and a finitely generated group is not divisible. If G is not divisible, then $nG \neq G$ for some positive integer n. Hence $H = G/nG$ is a nontrivial Abelian group of bounded order. It is a well known fact that such an H is a direct sum of finite cyclic groups, so it certainly has a nonzero finitely-generated homomorphic image.

Also solved by D. Ž. Djoković, Gertrude Ehrlich, Robert Gilmer, H. D. Keesing, Kwangil Koh, Z. Z. Uoiea, R. P. Ware, W. J. Woan, and the proposer.

Diagonalization of a Special Matrix

5577 [1968, 305]. *Proposed by J. L. Ercolano and F. G. Gustavson, IBM, Yorktown Heights, N. Y.*

Let $A = \{a_{ij}\}$ be an "X" matrix, i.e., a matrix with nonzero elements only on the main or alternate diagonals. Determine conditions under which A is similar to a diagonal matrix over the field of complex numbers.

Solution by R. C. Thompson, University of California, Santa Barbara. Put x_i for the element a_{ii} and y_i for $a_{i,n+1-i}$, $i = 1, 2, \cdots, n$, with all other elements 0, where, for odd n, we must have $x_{(n+1)/2} = y_{(n+1)/2}$. Make the permutation similarity of A in which the rows and columns are taken in the following order:

$$\text{for even } n: 1, n; 2, n-1; \cdots; \frac{n}{2}, \frac{n}{2}+1;$$

$$\text{for odd } n: 1, n; 2, n-1; \cdots; \frac{n-1}{2}, \frac{n+3}{2}; \frac{n+1}{2}.$$

The effect of this similarity is to convert A to the following direct sum:

$$\text{for even } n; \begin{bmatrix} x_1 & y_1 \\ y_n & x_n \end{bmatrix} \dotplus \begin{bmatrix} x_2 & y_2 \\ y_{n-1} & x_{n-1} \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} x_{n/2} & y_{n/2} \\ y_{n/2+1} & x_{n/2+1} \end{bmatrix};$$

and for odd n:

$$\begin{bmatrix} x_1 & y_1 \\ y_n & x_n \end{bmatrix} \dotplus \begin{bmatrix} x_2 & y_2 \\ y_{n-1} & x_{n-1} \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} x_{(n-1)/2} & y_{(n-1)/2} \\ y_{(n+3)/2} & x_{(n+3)/2} \end{bmatrix} \dotplus [x_{(n+1)/2}].$$

The matrix will therefore be similar to a diagonal matrix if and only if each of the 2×2 matrices

$$\begin{bmatrix} x_i & y_i \\ y_{n+1-i} & x_{n+1-i} \end{bmatrix}$$

is similar to a diagonal matrix: $i = 1, 2, \cdots, [\frac{1}{2}(n+1)]$. A sufficient condition for a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

to be similar to a diagonal matrix is $(a-d)^2+4bc\neq 0$. (If $(a-d)^2+4bc=0$, then this is possible if B is already diagonal and scalar—a case ruled out in this problem.)

Also solved by M. A. Ettrick and by the proposers.

Characteristic Values of Tridiagonal Matrices

5578 [1968, 305]. *Proposed by John H. Smith, Massachusetts Institute of Technology*

Show that a real $n\times n$ matrix with the property that $a_{ij}=0$ for $|i-j|\geqq 2$ and $a_{i,i+1}\cdot a_{i+1,i}>0$, for all i, has distinct eigenvalues.

Solution by L. W. Ehrlich and J. R. Kuttler, The Johns Hopkins University Applied Physics Laboratory. Call the matrix A. Now A is similar to the symmetric matrix DAD^{-1}, where D is the diagonal matrix with

$$d_{11} = 1, \qquad d_{i+1,i+1} = d_{ii}\sqrt{a_{i,i+1}/a_{i+1,i}}, \qquad i = 1, 2, \cdots, n-1.$$

But DAD^{-1} is a symmetric tridiagonal matrix with no zeros on the superdiagonal. Hence it has real, distinct eigenvalues by the Sturm sequence theorem. (See, e.g., J. H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965, p. 300.)

Also solved by J. L. Brenner, G. N. de Oliveira (England), M. A. Ettrick, T. L. Markham, J. C. Mettauer, J. Z. Hearon, P. V. Subbarao (India), R. C. Thompson, and the proposer.

This problem on Jacobi matrices appears in the mathematical literature, as noted by several readers. See. A. S. Householder, *The Theory of Matrices in Numerical Analysis*, p. 86, exercise 11; also a paper by S. Yamamota, Bull. Math. Statist., 1961—see Mathematical Reviews, vol. 24 A (1962), A 3171.

Mixing Homeomorphisms

5579 [1968, 305]. *Proposed by David Cohoon, Purdue University*

Let X denote an arbitrary topological space. Then a homeomorphism $f\colon X\to X$ is said to be *mixing* provided that for every pair of nonempty open sets U and V contained in X, there exists an integer $N>0$ such that $n>N$ implies $f^n(U)\cap V\neq\emptyset$. It is easy to see that any Tychonoff product X^s, with s infinite, admits a topologically mixing homeomorphism.

Define the *mixing codimension* of X to be inf $\{$cardinality of s: X^s admits a topologically mixing homeomorphism$\}$. Prove or disprove the conjecture that the mixing codimension of the unit circle, $\{\exp(i\theta)\colon 0\leqq\theta\leqq 2\pi\}$, with the topology induced by the plane E^2, is two.

Solution by P. R. Chernoff, University of California, Berkeley. The conjecture is correct. Let T denote the unit circle. We claim that a homeomorphism

$f: T \to T$ cannot be mixing. For, take V_1, V_2, V_3 to be small open arcs containing 1, ω, ω^2, respectively, ω a primitive cube root of 1. Let $U_1 = \{\exp(i\theta): -\pi/2 < \theta < \pi/2\}$, $U_2 = \{\exp(i\theta): \pi/2 < \theta < 3\pi/2\}$. If f were mixing then for all sufficiently large n, $f^{(n)}(U_i)$ would meet each V_k ($i=1, 2$; $k=1, 2, 3$). But this is clearly impossible, for $f^{(n)}(U_1)$ is a connected arc, and if it meets V_1, V_2 and V_3 it must contain one of them, preventing the disjoint set $f^{(n)}(U_2)$ from meeting it.

On the other hand T^2 does admit a mixing homeomorphism. Indeed, any ergodic automorphism of a compact abelian group is mixing (see Halmos, *Lectures on Ergodic Theory*, p. 53) and T^2 has such automorphisms. Thus, the mixing codimension of T is two.

Also solved by H. B. Keynes, and by Nicholas Passell.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

COLLABORATING EDITOR FOR FILMS: SEYMOR SCHUSTER, Carleton College

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital C in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.

Introduction to Partial Differential Equations and Boundary Value Problems. By Rene Dennemeyer. McGraw-Hill, New York, 1968. viii+376 pp. \$13.75. (Telegraphic Review, Aug. 1968)

Partial differential equations and boundary value problems are the subject matter of a vast literature, including many excellent textbooks. This book is intended to be a first introduction to the area and requires only a modest background knowledge of ordinary differential equations and vector analysis.

Chapter 1 is an introduction to the whole subject and also treats first-order equations. Chapter 2 is concerned with linear second-order equations from a broad point of view. Chapters 3, 4, and 5 treat elliptic differential equations, the wave equation, and the heat equation, respectively. Appendix 1 proves the Cauchy-Kowalewski theorem for the linear second-order partial differential equation in two independent variables with analytic coefficients. Appendix 2 provides a succinct summary of the relevant facts about Sturm-Liouville problems, Fourier series, Fourier-Legendre series, and Fourier-Bessel series. An extensive set of problems is given at the end of each chapter and answers are provided in an appendix.

On page 1 Professor Dennemeyer says: "The study of partial differential equations is a classical branch of analysis. There are many important applications of the subject in the physical sciences and engineering. Moreover it is an active field of modern mathematical research." The book provides a first-class enlargement of the first sentence. The serious mathematics student will look in vain, however, for any linkages with the theory of linear functionals and linear operators in Banach and Hilbert spaces, the theory of distributions or generalized functions, and the calculus of integral transforms. Numerical methods of solution, which are so powerful in their own right and provide also so much insight into the analytic methods, are also not mentioned. The science student will be disappointed that only the simplest classical problems of mathematical physics appear and even they are treated in such a pedestrian fashion as not to show the relevance for more complicated problems.

In short I admire the details of this book but I deplore the emphasis.

SHERMAN LOWELL, Washington State University

Studies in Modern Topology. Edited by P. J. Hilton. MAA Studies in Mathematics, Vol. 5. Mathematical Association of America. Distributed by Prentice-Hall, 1968. 212 pp. $6.00. One copy at $3.00 to members of MAA. (Telegraphic Review, Aug. 1968)

This volume in the MAA Studies in Mathematics series consists of six expository articles on topology. It is a superb and unique book which gives the reader a good orientation of the scope, directions, and great advances in the modern developments of topology.

The article "Introduction: Modern Topology" by Hilton is a comprehensive and illuminating introduction to the most active areas of topology. Several exciting achievements (associated with the names of Adams, Atiyah, Bott, Hirzebruch, Kervaire, Milnor, Papakyriekopoulos, Serre, Smale, Stallings, Thom, Zeeman, . . .) are briefly discussed.

In his paper "What is a curve?", G. T. Whyburn discusses the need for a precise definition of a curve, and applies topological consideration of the curves to the theory of functions of a complex variable.

"Some results on surfaces in 3-manifolds" by W. Haken is concerned with Heegaard-surfaces and incompressible surfaces in 3-manifolds. Here the reader is introduced to some deep geometric insights into 3-dimensional manifolds.

An account of some of the principal results of the semisimplicial homotopy theory is given by V. K. A. M. Gugenheim. "The functors of algebraic topology" by E. Dyer is a unified, comprehensive treatment of algebraic topology based on the fundamental notions of combinatorial homotopy theory.

V. Poénaru's contribution "On the geometry of differentiable manifolds" deals with embeddings, cobordism and Morse theory. The paper concludes with a proof of Smale's generalized Poincaré conjecture for differentiable manifolds of dimension $\geqq 5$.

KY FAN, University of California, Santa Barbara

Geometry and Symmetry. By Paul B. Yale. Holden-Day, San Francisco, California, 1968. xi+288 pp. $10.75. (Telegraphic Review, Aug. 1968)

This book is a landmark in undergraduate textbooks and, if not buried beneath the googol of recent publications, will certainly spur the current movement to "get geometry back into the curriculum."

The author's prefatory remark that the book is an introduction to the geometry of euclidean, affine and projective spaces will excite no one, but a serious glance at the mathematical content in its pages will pique the interest of almost everyone. The central theme is that the various geometries can best be studied through symmetries, namely those transformations which leave some subset invariant. Hence, it becomes appropriate to study euclidean symmetries (isometries), affine symmetries (dilatations, affine reflections, etc.), and projective symmetries (perspectivities, projectivities, etc.). In addition to covering the basic material, both classical and modern, the author pauses to present a beautifully lucid introduction to the crystallographic groups, presenting the details of the classification of the point and space groups.

The analysis of the crystallographic groups represents an example, *par excellence*, of applied geometry, group theory, combinatorics, and the integration of knowledge from the three. However, the relation between geometry and algebra appears throughout the entire book. Group theory—as well as the more usual linear algebra—is seriously used as an instrument for geometry, in contrast to the many textbooks that introduce the group concept and do nothing more than exhibit the fact that certain sets of transformations are, in fact, groups. However, the geometry is never submerged by the ever-present algebra.

Again, in contrast to many undergraduate textbooks, this one gives the reader the impression—and the proof—that geometry is very much alive, intimately tied up with developments in other branches of mathematics, applicable to these branches and to science, and a source of interesting open questions.

The influence of Artin's *Geometric Algebra* and Baer's *Linear Algebra and Projective Geometry* is clearly visible. In fact, the book under review makes an excellent stepping-stone to the rough-going in these treatises. In this context, the author has done a great service in giving a glossary which explains the peculiar terminology in Baer's work.

Paul Yale presents his book as his nomination for a one-semester up-to-date geometry course. While there is no doubt about its up-to-date character, it seems unlikely that undergraduates can cover the material in one semester unléss they come to the course well acquainted with the necessary linear algebra and group theory (all of which, together with preliminary combinatorics, is contained in the book). If students have the algebraic prerequisites, then it is also my nomination for a one-semester course; otherwise, for a two-semester course, which would afford the time to follow the author's suggestions for further reading and term papers. In either case, this eloquently written work can serve as the focus for a course that will be interesting in itself, and will also

provide good preparation for follow-up courses in algebraic geometry, differential geometry, algebraic topology, geometric number theory and X-ray crystallography.

SEYMOUR SCHUSTER, Carleton College

The Remarkable Sine Functions. By A. I. Markushevich. American Elsevier, New York, 1966. x+100 pp. \$6.50.

This short book begins with a geometric definition for circular, hyperbolic, and lemniscate functions which would be interesting for advanced high school students. The author then turns to the class of functions defined as inverses of the integrals

$$t = \int_0^y \frac{du}{\sqrt{1 + mu^2 + nu^4}},$$

in order to present a unified treatment of "generalized sine" functions. Much of the material in the remainder of the book requires some understanding of functions of a complex variable, but the author gives the basic definitions and theorems necessary for a good undergraduate with a calculus background to follow the presentation easily.

The language of the text is classical and makes the book valuable for independent reading for undergraduates interested in an introduction to doubly periodic functions. In general, the translation is good; however, there are a number of misprints which detract somewhat from the otherwise clear presentation.

P. E. BEDIENT, Franklin and Marshall College

An Introduction to Sequences, Series, and Improper Integrals. By O. E. Stanaitis. Holden-Day, San Francisco, 1967. vii+210 pp. \$9.50.

This well-written book, based on the author's experience teaching college sophomores and high school teachers, gives a leisurely introduction to the main classical results concerning sequences, infinite series, and improper integrals. Professor Stanaitis is competent with his subject; his name is quite familiar to those who attend the Problem sections in this MONTHLY, and he often appears on the scene when infinite series arise. His book offers the basic theory and many routine drill problems, especially in the first chapters, which include 39 pages on limits.

This reviewer would like to see Bernoulli numbers done over. There are two widely prevalent notations that should be explained. Curiously, no recurrence relations are developed and the reader is referred to Knopp. Even in Knopp and Bromwich one fails to find the elegant formulas for the Bernoulli numbers which are easily developed and which lead to short proofs of the Staudt-Clausen Theorem. Knopp, in his booklet *Infinite Sequences and Series* (Bagemihl's translation; Dover, New York, 1956) even wrote (wrongly), p. 118, that the Bernoulli

numbers "cannot be specified by means of a simple formula—except, say, by means of a determinant . . ." This . . . in spite of the fact that if we set

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x) = \frac{te^{xt}}{e^t - 1},$$

then these Bernoulli polynomials are given explicitly by

$$B_n(x) = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n,$$

with $B_n = B_n(0)$ being the Bernoulli numbers. The reviewer looks in vain in books on infinite series for any adequate introduction when such number sequences arise.

There is some entirely novel material on pp. 36–39, apparently inspired by the Putnam Competition of December 1961 and a paper by Mr. T. A. Chapman and this reviewer (this MONTHLY, 69 (1962) 651–653). The interesting situation which arises is that we know from the definition of the integral that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2} = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4},$$

but it requires more ingenuity to show that

$$\lim_{n\to\infty} \sum_{k=1}^{np} \frac{n}{n^2 + k^2} = \frac{\pi}{2},$$

for *every* positive integer $p \geqq 2$. It is refreshing to see that the book offers such novel limits (lacking from almost every other text). The reviewer wishes to note a mistake in the paper cited. A function used in deriving a limit failed to satisfy the assumed hypothesis of being nonincreasing over the interval considered. The same limit occurs in the present book as Ex. 9, p. 39, and the reviewer challenges the reader to find if the result can be validated after all by some modification. Stanaitis gives the same value for the limit in his answers.

Although this reviewer would extend the coverage of some topics (e.g., orthogonality), use variant notations, or introduce other concepts, no book on infinite series treats all that one would want or in the manner one would desire. The price is reasonable and the format pleasing in spite of some crowding as in the minuscule limits of summation on pp. 57–59.

H. W. GOULD, West Virginia University

Prediction Analysis. By John R. Wolberg. Van Nostrand, Princeton, N. J., 1967. xi+304 pp. $10.75. (Telegraphic Review, Jan. 1968)

In a useful book on the planning of experiments and the analysis of data by least square methods, the author has chosen to lay the foundations of a basic course of analysis not now available in most colleges. He presents a logical development from the theory of planned experiments to the treatment of error in

his part I, which is entitled "Theory." In part II, "Applications" he classifies his experiments according to methods of fitting the data and so lists his chapters to indicate the approach through polynomials, exponentials, sine and Gaussian functions. For his final chapter, he chooses a practical complicated experiment.

The reviewer agrees with the author that there is great "need for sophisticated methods of experimental planning and design." This book is certainly an approach to that end. There should be a good deal of concentration in the very near future upon the proper presentation of necessary background to undergraduate students, equipped with modern computers, so that they may analyze the design of experiments with a view to prediction of accuracy of results.

In introducing Chapter 4, "Prediction Analysis," Wolberg points out that it is a general method for predicting the accuracy of results that should be obtained from a proposed experiment. In this same chapter, he suggests general computer programs, with block diagrams, for prediction analysis and least squares analysis and indicates their common characteristics.

The reviewer believes this book with its problems and "computer projects" provides an interesting text for the presentation of an undergraduate course to students who have a basic background in calculus plus a working knowledge of computer programming. It is not a course for the pure mathematician, but should be a valuable challenge for the student of science and engineering.

RAYMOND E. ROTH, Rollins College

Introduction to Analytic Functions. By Wilfred Kaplan. Addison-Wesley, Reading, Mass., 1966. 1 x+212 pp. $7.95.

The text is designed to give students a working knowledge of the complex calculus; it requires no other knowledge than the real calculus as expounded in the first two years of most college curricula. The topics are standard (including Laurent series and residues, conformal mapping, analytic continuation and Riemann surfaces) except for a valuable chapter on analytic functions of several complex variables. Apparently, this is the first elementary text that incorporates this vital and vibrant topic. The emphasis throughout the text is on formal techniques. The chapter on conformal mapping contains the standard applications.

This is not a text for students with a sophisticated background. Such students will shudder at the first sentence of the book: Complex numbers are numbers of form $x+iy$, where x and y are real.

ARTHUR E. DANESE, SUNY at Buffalo

Probability: A Survey of the Mathematical Theory. By John Lamperti. Benjamin, New York, 1966. x+150 pp. $10.00 (cloth), $3.95 (paper).

After a scarcity of graduate (measure theoretic) level texts in probability theory, quite a few have been published recently. This book was not intended to be "encyclopedic" in the manner of Loève or Feller II, but rather to present material for (approximately) a one (U. S.) semester course in probability.

Attractive features of the book are sketches of proofs and proofs in special cases. These avoid many pages of technically difficult results, but still retain the essential ideas, and show why the results are (or should be) true. The "proof" of Section 18, Theorem 1, that limiting distributions of sums of doubly-indexed random variables must be infinitely divisible is especially nice. Another good device is that the problems are part of the text. Some involve working out technical details, others extensions of results.

There are a surprisingly large number of interesting "side" topics, including Gnedenko's limit theorems concerning the maximum of a sample and "convergence of type," Bernstein's proof of the Weierstrass approximation theorem, and recurrence.

The last forty-five pages deal with stochastic processes, especially Brownian motion. Some of this space might have been devoted to limit theorems or perhaps conditional expectations and martingales, but this is a matter of taste. In connection with this last statement one might want to use this book (reasonably priced in the paper back edition) in a year course along with Gnedenko and Kolmogorov *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, 1954).

For a shorter introductory course in probability or to obtain an idea of what probability theory is about, I personally prefer this book to *Probability Theory* by Krickberg (Addison-Wesley, 1964). One should also consider the lesser known book by A. Joffe *Promenades aléatoires et Mouvement Brownien* (Presses de l'Université de Montréal, 1964). The manner of presentation and the style of the Joffe book and the Lamperti book show similarities; the former covers less ground and requires somewhat less mathematical sophistication.

ROGER FISCHLER, University of Toronto

Differential Geometry. Louis Auslander (City University of New York). Harper and Row, New York, 1967. xii+271 pp. $12.50. (Telegraphic Review, Oct. 1967.)

This book departs refreshingly from "classical" introductions to differential geometry by treating the subject consistently from the group theoretic point of view, essentially in the spirit of É. Cartan and the method of "moving frames." The gist of this method is that if three orthogonal unit vectors ("frame") are attached to a point constrained to move along a surface so that two of the frame vectors are always tangent to the surface, then the direction changes of each of the frame vectors measure geometric properties of the surface (formally expressed by the "structure equations").

In the foreword the author describes his intention to "combine the modern and classical approach to the subject . . . " One consequence of this compromise is that he dulls some of the tools he spends so much time sharpening. This happens, for instance, when he stays with the classical method of using the same symbols for the coordinates of a point and the coordinate functions.

Another difficulty in this text is that many formulas are made contingent

on identifications the novice reader may find hard to make. Thus the frame bundle in Euclidean space is in one-to-one correspondence with the group of motions, but the two are hardly naturally identifiable.

The student who wants to learn differential geometry for the first time from this text faces an uphill fight, even if he has the prerequisites of one year of abstract algebra and two of calculus. There is for example no treatment of bundles, though they are central to this exposition. A precise discussion of surfaces in the large and integration theory is included. But the book somehow covers up geometric ideas—as if the author has a knack for complicating rather than simplifying. Notation is cumbersome, often unsuggestive, loaded with super- and subscripts, and sometimes not distinctive enough. Often several letters are used when one would do, and symbols when none are needed. The word "curve" is ambiguous. The number of printing errors is incredible, and there are some mathematical slips which will trouble the student. There is a nearly total lack of references, even where the author states theorems without proofs. This is a lost opportunity to bring home to the reader a taste of the vast span of differential geometry today: from the calculus of variations and topology to relativity. The book may also be criticized for its extreme lack of examples—only a handful or so. All problems and exercises are presumably incorporated into the text, but the impression remains that the author hurriedly published his lecture notes and forgot to read the galley proofs.

L. N. PATTERSON, University of East Africa

Introductory Complex Analysis. By R. A. Silverman. Prentice Hall, Englewood Cliffs, N. J., 1967. xi+372 pp. $8.75. (Telegraphic Review, June 1967.)

This is a one volume distillation of a translation by R. A. Silverman of the three volume work "The Theory of Analytic Functions" by A. I. Markushevich. The book is designed for use as an introduction to complex analysis for students with a strong background in mathematics. However, the approach is practical with strong emphasis on technique which will appear to the applied scientist.

The book begins with the first principles of complex variable and even includes some elementary set theory. The approach is geometrical rather than topological with large sections on mapping techniques, the implications of the residue theorem, and the Schwarz-Christoffel mapping. However, this is one of the few books which proves that the analytic image of a domain is a domain and which employs this approach for the maximum modulus theorem. Most theorems are done completely; the author rarely shortens the work by choosing a weakened hypothesis or by presenting only part of the proof. For example, the complete Cauchy Theorem for arbitrary closed rectifiable curves is proved. Frequently well-known theorems are presented as special cases of more general theorems. A further indication of the level of presentation is the list of those theorems which are stated but not proved, for example, the Riemann Mapping Theorem, Picard's Theorem, the Jordan Curve Theorem, and the conclusion of the Monodromy Theorem. Problems and worked examples are both numerous

and illustrative. Many of the problems are actually examples appearing in the parent three volume translation, which would allow a student to check his work in that text.

In a sense, the completeness of the work may be its only weakness. Readers who appreciate completeness might prefer more depth in the advanced sections or the inclusion of additional topics.

A. H. CAYFORD, University of British Columbia

A Brief Introduction to Probability Theory. By John P. Hoyt. International Textbook, Scranton, Pennsylvania, 1967. viii+151 pp. $3.95 (paper). (Telegraphic Review, January, 1968.)

This little book is above average among such texts presupposing elementary calculus. Many of the explanations of new ideas are well written, although there are some shortcomings such as the lack of intuitive explanation of density functions and the introduction of the Poisson Law only through the use of a limit of generating functions (and then without mention of any continuity theorem). The language is fairly precise, but there are a few lapses such as a definition of continuous random variables which allows for there to be a discrete part, and the common failing of writing only "coin tossing" to mean independent tosses of a fair coin. The material on statistical inference is less successful than that on probability theory. For example, several approaches are discussed in the chapter on estimation, but not even the simplest measures of accuracy, such as variance, are mentioned (such a measure of accuracy in an estimation problem finally occurs on page 138 in the chapter on tests). The chapter on hypothesis testing contains the misleading notion that one does not talk about power when there are composite alternatives, because no single number expresses the power. The homework exercises are a reasonable selection. The book is usable for a one-semester course, although such less mathematical texts as those of Hodges-Lehmann, Lindgren-McElrath, and Hoel are perhaps more successful.

J. KIEFER, Cornell University

TELEGRAPHIC REVIEWS

The following abbreviations indicate possible uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)—18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Applications

Macro-Economic Theory. A Mathematical Treatment. By R.G.D. Allen, Macmillan, New York, and St. Martins Press, New York, 1968. xii+420 pp. $3.50 (paper). Intended for students with training in elementary classical analysis. S, P.

Selected Applications of Nonlinear Programming. By Jerome Bracken and Garth P. McCormick (Research Analysis Corp., McLean, VA). Wiley, New York, 1968. xii+110 pp. $8.95. Primarily for those already somewhat familiar with linear pro-

gramming to develop facility in building nonlinear mathematical programming models. Applications include weapons assignment, bid evaluation, alkylation process optimization, chemical equilibrium, structural optimization, launch vehicle design and costing, parameter estimation in curve fitting. S, P.

Wave Propagation in a Random Medium. By Lev A. Chernov (Acad. of Sciences of the U.S.S.R.). Translated from the Russian by R. A. Silverman. (Reprint of the book originally published in 1960 by McGraw-Hill.) Dover, New York, 1968. viii+168 pp. $3.00 (paper). P.

Introduction to Operations Research. By A. Kaufmann (Institut Polytechnique de Grenoble, France) and R. Faure (Paris Transport Authority). Academic, New York, 1968. xi+300 pp. $14.50. In the words of the editor of the series *Mathematics in Science and Engineering*, of which this is volume 47, the authors present "eighteen charming and witty vignettes which lucidly illustrate many of the fundamental ideas and methods of operations research." The book is primarily addressed to economic decision makers, but it would also be useful in some mathematics courses for supplementary reading. S, P, L.

Readings in Mathematical Social Science. Edited by Paul F. Lazarsfeld and Neil W. Henry (both of Columbia Univ.,). MIT, Cambridge, Mass., 1966. 371 pp. $3.45 (paper). Paperback reprint. (See Telegraphic Review Jan. 1967). S, P.

Pattern Recognition. The Journal of the Pattern Recognition Society. A new journal published quarterly by Pergamon Press, New York. $40.00 per year. This new field covers the "organization of input data into identifiable classes by extraction of the significant features of the data from a background of irrelevant detail." P.

Operations Research for Management Decisions. By Samuel B. Richmond (Columbia Univ.). Ronald Press, New York, 1968. xvi+615 pp. $12.00. Presupposes only high school mathematics and includes the advanced topics required. The six parts are on optimization problems, probability theory and its applications, allocation problems, scheduling models, and decision theory. S, P.

Relativity and Cosmology. By H. P. Robertson and Thomas W. Noonan. Saunders, Philadelphia, 1968. xxxiii+456 pp. $16.50. A comprehensive treatment "from foundations to applications", based on lectures of the late Professor Robertson at California Institute of Technology, edited by his former student, Professor Noonan of the State Univ. of New York at Brockport. There is a bibliography, a list of works by Robertson, and a ten page abstract of the book itself. T (17), S, P, L.

Wave Propagation in a Turbulent Medium. By V. I. Tatarski (Acad. of Sciences of the U.S.S.R.) Translated from the Russian by R. A. Silverman. (Reprint of the translation originally published in 1961 by McGraw-Hill.) Dover, New York, 1967. xiv+285 pp. $3.25 (paper). P.

Calculus

Problems and Solutions in Ordinary Differential Equations. By Fred Brauer and John A. Nohel (both of the Univ. of Wisconsin). Benjamin, New York, 1968. x+267 pp. $3.95 (paper). A collection of 208 problems intended to accompany the authors' *Ordinary Differential Equations; A First Course* (1967),it could be used independently. Essentially complete solutions are given. S.

Ordinary Differential Equations. By George F. Carrier (Harvard Univ.) and Carl E. Pearson (University of Washington). Blaisdell, Waltham, Mass., 1968. x+229 pp. $8.50. The authors appear to have succeeded in avoiding both the traditional cook

book and fashionable logical formalism by presenting "a sequence of heuristic arguments, illustrative examples and exercises, which serve to guide the reader . . . through the arguments that underlie the invention, generalization, and usage of the techniques by which solutions of differential equations can be constructed." The approach may be as refreshing and illuminating for mathematicians as for users of mathematics. I hereby solicit comments from those who adopt this book. T (14–15).

A Short Course in Calculus. By Jack G. Ceder and David L. Outcalt (both of Univ. of California, Santa Barbara). Worth, New York, 1968. xi+238 pp. $7.95. For a one quarter or half year course for students of "biology, business, economics, psychology, and sociology." T.

Advanced Calculus. By Allen Devinatz (Northwestern Univ.). Holt, Rinehart and Winston, New York, 1968. vii+496 pp. $12.95. Not an elementary real variable course, but a second treatment of classical calculus ending with Stokes' theorem. T (16).

Mathematical Analysis. By T. M. Flett (Univ. of Liverpool) McGraw-Hill, New York, 1966. ix+439 pp. $12.00. This book differs from the usual American calculus texts in presupposing familiarity with the techniques of elementary calculus, the inclusion of more material on foundations, a chapter on topological spaces, and the omission of multiple integration. It is more elementary than the usual beginning "baby real variables" course but more theoretical than the traditional advanced calculus. Perhaps it might follow a quick freshman survey of calculus. T (14–15).

First-Year Calculus. By Einar Hille (Univ. of New Mexico), and Saturnino L. Salas (Univ. of Connecticut). Blaisdell, Waltham, Mass., 1968. xi+415 pp. $9.50. One-variable calculus "in an elementary but coherent manner." T.

Differential Equations: A Brief Course with Applications. By Lyman M. Kells (U.S. Naval Academy). McGraw-Hill, New York, 1968. ix+252 pp. $7.50 (cloth) $4.75 (paper). This appears to be a variation ot the sixth edition of the author's *Elementary Differential Equations* (1965, first edition 1932), differing from it by some omissions and by the addition of a chapter on the solution of differential equations by matrices. Unfortunately the publisher presents the book as though it were completely new without any mention of the author's previous books, and indeed without any biographical information at all about the author. This failure to fully inform the reader is similar in its ethical quality to the increasingly common practice by some publishers of reprinting old books as though they were contemporary without any indication, or with indication in very small print, of their date of origin. T (14).

Intermediate Mathematical Analysis. By Anthony E. Labarre, Jr. (Fresno State College). Holt, Rinehart and Winston, New York, 1968. xiii+272 pp. $8.95. To insert between a weak first calculus course and one on multivariate analysis. T (14).

A Complete Course in Calculus. By Serge Lang (Columbia Univ.). Addison-Wesley, Reading, Mass., 1968. xx+622 pp. $12.95. The author's *First Course* and *Second Course* bound in a single volume. T (13–14).

Calculus of One Variable. By Robert T. Seeley (Brandeis Univ.). Scott, Foresman, Glenview, Illinois, 1968. 532 pp. $10.50. Although chapter one is on the real numbers, foundational matters are treated in appendices, and the stress is on ideas and techniques rather than rigor. T.

Calculus for the Natural and Social Sciences. By Sherman K. Stein (Univ. of California, Davis). McGraw-Hill, New York, 1968. xiii+322 pp. $7.50. Although similar in

arrangement and spirit to the author's *Calculus in the First Three Dimensions* (see Telegraphic Review Dec. 1968) this is a different book and includes topics not in the longer one. It is designed for the short course. T.

Calculus of Vector Functions. 2nd edition. By Richard Williamson (Dartmouth College), Richard H. Crowell (Dartmouth College), Hale F. Trotter (Princeton Univ.). Prentice-Hall, Englewood Cliffs, N.J. 1968. ix+434 pp. $10.50. To the first edition of 1962, one of the first to really use linear algebra, is now added an introductory chapter on linear algebra, new material on vector field theory, Fourier series and eigenfunction expansions, as well as some simplification of the exposition. T (14).

General

Some Vistas of Modern Mathematics. Dynamic Programming, Invariant Imbedding, and the Mathematical Biosciences. By Richard Bellman. Univ. of Kentucky Press, Lexington, Kentucky, 1968. viii+141 pp. $7.00 (cloth) 3.95 (paper). Although focused on three particular topics, these essays are addressed to the mathematician and educated laymen. They attempt to assess the nature of contemporary mathematics and to suggest directions in which we are moving. S, P, L.

Introduction to Mathematics. By Hollis R. Cooley and Howard E. Wahlert (both of New York Univ.). Houghton Mifflin, Boston, Mass., 1968. xi+484 pp. $8.95. A revision of *Introduction to Mathematics* (1937, 1949) by Cooley, Gans, Kline, and Wahlert, intended to take account of new developments without what the authors consider to be undue attention to foundations. T (13).

A Survey of Mathematics: Elementary Concepts and their Historical Development. By Vivian Shaw Groza (Sacramento City College). Holt, Rinehart and Winston, New York, 1968. xvi+352 pp. $8.50. Designed for a terminal course in the general education curriculum, this book uses an historical framework to illuminate modern mathematical ideas. Topics include counting, sets, numeral systems, bases, symbolic logic, mathematical systems, probability, calculus, and non-Euclidean geometries. T (13), TT.

The Scope of Mathematics. A Fresh Look at Mathematics for the Non Specialist. By M. J. Holt and A. J. McIntosh. Oxford Univ. Press, New York, 1966. 266 pp. $2.00 (paper). "Our aim, then, is to show the unique role of mathematics as a link between the scientific method on the one hand and the creative spirit of the humanities on the other." The treatment is very elementary and somewhat historical, suitable for general education courses for the mathematically unwashed. T (13), TT.

★*Mathematics in the Modern World*. Readings from the *Scientific American* with introductions by Morris Kline. Freeman, San Francisco, 1968. 409 pp. $6.50 (paper) $10.00 (cloth). These are facsimile reprints of fifty fascinating articles on mathematics that have appeared in recent years in the *Scientific American*. There are three on the nature of mathematics, nine biographical articles (including Halmos on Bourbaki), thirteen on particular topics (including the recent article on non-Cantorian set theory by Cohen and Hersh), and nineteen on various modern topics related to application. Since few of the writers combine expertise in mathematics, history, and exposition, there are bound to be errors. For example, Gauss is credited with non-Euclidean geometry and even with its propagation, even though his priority is doubtful and it is certain that he very carefully concealed his views on the matter even after the publications of Lobatchevsky and Bolyai. But such criticisms are beside the point. These articles are superb for inspiration and enjoyment by both student and professor. S, P, L.

Mathematics for the Liberal Arts Student. By Fred Richman, Carol Walker, Robert J. Wisner (New Mexico State Univ.). Brooks/Cole, Belmont, Calif., 1967. vii+190 pp. $7.50. For a one- or two-semester terminal course, this book is informal and closely related to numbers and "real life." The heuristic rather than the logical side of mathematics is stressed. T (13), L.

History

The Secrets of Ancient Geometry—and its use. By Tons Brunés. Two volumes. Rhodos, International Science Publishers, Copenhagen, 1967. Distributed by "The Ancient Geometry" Nygaardsvej 41, Copenhagen. Vol. I, 331 pp. Vol. II, 252 pp. This book is solidly in the tradition of what DeMorgan called the "paradoxers". It is outside the stream of mathematical or historical scholarship and consists almost entirely of speculative reconstruction of geometrical knowledge and mysticism that the author claims existed in religious centers but was never communicated or recorded in any way and could only be reconstructed by his speculative methods. He is impressed by close approximation in ancient times to squaring the circle with ruler and compass, since "this is a problem that has not been solved and which probably cannot be solved." P (for collectors of curiosa).

Exposition de la théorie des Chances et des Probabilités. By Augustin Cournot. Facsimilie reprint of the first edition of 1843. Edizioni Bizzarri, Rome, n.d. viii+448 pp. 14,000 Lire. P, L.

Recherches sur les Principes Mathématiques de la Théorie des Richesses. By Augustin Cournot. Facsimile reprint of the first edition of 1838. Edizioni Bizzarri, Rome n.d. xi+198 pp. 5,500 Lire. Although an English translation has been available for many years, the original of this classic has been virtually unobtainable. This reprint will make it possible for libraries to possess an important classic. P, L.

Correspondance Mathématique et Physique de Quelques Célèbres Géomètres du XVIII-ème Siècle Précédée d'une Notice sur les Travaux de Leonard Euler, Tant imprimés qu'inédits et Publiée Sous les Auspices de l'Académie Impériale des Sciences de Saint-Pétersbourg. By P. H. Fuss. 2 volumes. St. Petersbourg, 1843. Reprinted as number 35 of the Sources of Science. Johnson Reprint, 111 5th Avenue, New York, 1968. Vol. I, cxxi+713 pp. Vol. II, xxiii+713 pp. Two vols., with portraits, plates and diagrams, $65.00 (cloth). A great classic containing many letters by Euler, Goldbach, the Bernoullis, and Nicholas Fuss. P, L.

Early Science in Oxford. By R. T. Gunther. Vol. I. Chemistry, Mathematics, Physics and Surveying. Facsimile reprint of the first edition of 1921–1923. Dawsons of Pall Mall, London, 1967. vi+407 pp. £8.8.0. There are fourteen volumes under this title all reprinted recently by Dawsons. P, L.

Greek Mathematical Philosophy. By Edward A. Maziarz (Loyola Univ., Chicago) and Thomas Greenwood (Montreal Univ., Quebec). Frederick Ungar, New York, 1968. xii+271 pp. $6.50. A "broad cultural survey of the interaction of mathematics with philosophy and their mutual development" from Thales through Euclid. S, P, L.

Le Livre du ciel et du monde. By Nicole Oresme. Edited by Albert D. Menut and Alexander J. Denomy. Translated with an introduction by Albert D. Menut. University of Wisconsin Press, Madison, Wisconsin, 1968. xiii+778 pp. $17.50. The original Middle French text and English translation on facing pages with substantial commentary, bibliography, and list of technical neologisms. P, L.

★*Evolution of Mathematical Concepts: An Elementary Study.* By Raymond L. Wilder (Univ. of Michigan). Wiley, New York, 1968. xviii+224 pp. $8.00. Not a history of

mathematics but an analysis of the nature of mathematics as "one of the most important cultural components of every modern society." The word "elementary" refers to the fact that illustrative material is chosen from the history of number and geometry so that the book is accessible to anyone without training in college mathematics. On the other hand the work is the fruit of many years of thoughtful investigation by a mathematician distinguished for his creativity in research, teaching, and professional leadership. Most "definitions" of mathematics are normative prescriptions expressing special points of view. In contrast, Wilder attempts a scientific description of the mathematical enterprise as it is and as it has developed, together with many interesting comments on related matters, such as mathematical education. The book should be issued in paperback to encourage the very wide distribution it deserves. T (General Education), TT, S (all courses!), P (everyone!), L.

Pre-Calculus

Elementary Algebra, Structure and Use. By Raymond A. Barnett (Merritt College). McGraw-Hill, New York, 1968. xii +402 pp. $7.95. From Chapter 1 on natural numbers to the last three chapters on quadratic equations, relations and functions, and the postulational method. Buff paper and two color printing. T.

Modern College Algebra and Trigonometry. By Edwin F. Beckenbach (Univ. of Calif., L.A.) and Irving Drooyan (Los Angeles Pierce College). Wadsworth, Belmont, Calif., 1968. x+436 pp. $9.50. A "modified revision" of *Integrated College Algebra and Trigonometry* (1964, 1965, 1966), designed for students with weak preparation. The word "modern" presumably refers to inclusion of the topics recommended for level O by the CUPM and to the use of two color printing. T.

College Algebra, 2nd ed. By Edwin F. Beckenbach (Univ. of Calif., L.A.), and Irving Drooyan (Los Angeles Pierce College) and William Wooton (Los Angeles Pierce College). Wadsworth, Belmont, Calif., 1968. vii+408 pp. $9.50. Changes from the first edition (1964) are pedagogical. T.

Arithmetic Concepts and Skills. By Murray Gechtman and James Hardesty (Los Angeles Pierce College). Macmillan, New York, 1968. viii+312 pp. $5.95. Written "for adults who are deficient in arithmetic," this book relates arithmetic to sets and uses contemporary terminology and two color printing. T (13 remedial).

Precalculus Mathematics. Algebra and Trigonometry. By Jacob F. Golightly (Jacksonville Univ.). Saunders, Philadelphia, 1968. xi+456 pp. $8.50. Designed for a one semester or one quarter prelude to calculus, the book includes materials on sets, relations, inequalities, induction, complex numbers, and linear algebra. T (13).

Trigonometry Review Manual. By William Hauck (Bucknell Univ.) McGraw-Hill, New York, 1968. ix+233 pp. $5.95 (cloth) $3.95 (paper). This manual, which is programmed in a modified Skinner mode with pre-tests, criterion frames (from which the student branches back to repeat frames if necessary), post-tests, and a final examination, may be useful to students who show weakness in elementary traditional trigonometry. S.

Modern Elementary Mathematics. By Ladis D. Kovach (U.S. Naval Postgraduate School). Holden-Day, San Francisco, 1968. 523 pp. $9.50. The first twelve chapters reprint *Introduction to Modern Elementary Mathematics*, which spiraled through the sets, real numbers, geometry, and number theory three times. This book continues with a fourth cycle of four chapters followed by three on problem solving, arithmetic operations, and supplementary topics. Very elementary. T (13), TT.

Analytic Geometry, 3rd ed. By Ross R. Middlemiss (Washington Univ), John L. Marks (San Jose S.C.) and James R. Smart (San Jose S.C.). McGraw-Hill, New York, 1968. xi+434 pp. $7.95. There is some effort to move in the fashionable direction of making analytic geometry more than a preparation for calculus. T.

Essentials of Mathematics, 2nd ed. By Russell V. Person (Capitol Inst. of Tech.). Wiley, New York, 1961. x+721 pp. $9.95. Traditional arithmetic, algebra, geometry, logarithms, and trigonometry with slight revision from the 1961 first edition. T (13).

College Algebra and Trigonometry. By Paul A. White (Univ. of Southern Calif.). Dickenson, Belmont, Calif., 1968. ix+451 pp. $9.50. " . . .traditional topics . . .in a modern manner". T.

An Introduction to Modern Mathematics, 2nd ed. By Elbridge P. Vance (Oberlin College). Addison-Wesley, Reading, Mass., 1968. xv+591 pp. $9.75. In the tradition of unified freshman texts, this book takes the student from "sets and numbers" through the calculus required in the Advanced Placement Examination of the College Entrance Examination Board. There are minor changes, revision and expansion of material on matrices and vectors, and more problems. T (13).

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Iowa: Associate Professor J. F. Jakobsen has been appointed Dean of the Graduate College; Assistant Professors T. M. Price and T. J. Robertson have been promoted to Associate Professors; Associate Professor Paul Waltman has been promoted to Professor.

University of Washington: Dr. Robert Heath, Arizona State University, has been appointed Visiting Lecturer; Dr. Robert Warfield, New Mexico State University, has been appointed Assistant Professor; Assistant Professor Thomas Hungerford has been promoted to Associate Professor; Associate Professor R. J. Nunke has been promoted to Professor.

Professor W. D. Peeples, Samford University, has been appointed Head of the Department of Mathematics.

Assistant Professor H. M. Stark, University of Michigan, has been appointed Associate Professor at the Massachusetts Institute of Technology.

Dr. H. E. Jordan, Lawrence, Kansas, died on January 6, 1968. He was a Charter Member of the Association.

Professor Emeritus P. J. Rulon, Harvard University, died on June 30, 1968. He was a member of the Association for thirty-three years.

Professor Emeritus G. W. Smith, University of Kansas, died on September 9, 1968. He was a Charter Member of the Association.

NOVEMBER 1968 MEETING AT ELMHURST COLLEGE FOCUSING ON THE TWO YEAR COMMUNITY COLLEGE

On Saturday, November 16, 1968, a meeting was held at Elmhurst College, Elmhurst, Illinois, focusing on the two year community college. The meeting was jointly sponsored by the Men's Mathematics Club of Chicago and the Illinois Section of the MAA.

The major goal of the conference was to discover a way for the Illinois community college mathematics instructors to find an effective voice, on a subject matter basis, in higher education policy making. The total number of teachers in attendance from Illinois two and four year colleges and universities was approximately 100 individuals.

The two year community college mathematics instructors in attendance resolved that Illinois junior college mathematics instructors should become more involved with existing professional organizations rather than form a new organization of their own. In particular, the body of two year college teachers resolved that junior college teachers in Illinois should become increasingly involved in the ISMAA and the Illinois Council of Teachers of Mathematics and should work to make these organizations more valuable to two year college teachers. Since the comprehensive community college mathematics curriculum overlaps both the usual college curricula as well as the high school curricula, it is very important to maintain social and professional contacts with both college professors and high school teachers. The group believed this would be difficult to accomplish if the junior college instructors formed their own independent organization.

Partly as a result of this conference, the ISMAA is planning a statewide articulation conference involving the two and four year higher education institutions. The junior colleges will also participate in planning the program of the approaching annual spring meeting of the ISMAA.

WAUKESHA MATHEMATICAL SOCIETY

The Waukesha Mathematical Society, Waukesha, Wisconsin, has decided to issue DELTA, an undergraduate journal of mathematics twice a year: once each semester, as of January, 1969. The annual subscription is one dollar. For further information, please contact R. S. Luthar, University of Wisconsin, Waukesha, Wisconsin.

THE UNIVERSITY OF TEXAS—SEVENTH ANNUAL SYMPOSIUM ON BIOMATHEMATICS AND COMPUTER SCIENCE IN THE LIFE SCIENCES

The University of Texas Graduate School of Biomedical Sciences at Houston, Division of Continuing Education, is pleased to announce the SEVENTH ANNUAL SYMPOSIUM ON BIOMATHEMATICS AND COMPUTER SCIENCE IN THE LIFE SCIENCES to be held in Houston, Texas, March 26–28, 1969. Topics for sessions of the symposium are: Quantitative and Theoretical Biology, Mathematical and Applied Statistics in Biomedical Research, Experimental Design, Bioengineering and Simulation, Information Management: Storage and Retrieval, Computer Applications, The Man-Machine Interface.

For further information contact: Office of the Dean, The University of Texas Graduate School of Biomedical Sciences, Division of Continuing Education, P.O. Box 20367, Houston, Texas 77025.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the MAA was held at Concordia College, Moorhead on October 26, 1968. There were 107 persons registered including 81 members. At the business meeting conducted by Chairman E. J. Camp, a new set of by-laws was adopted which changes the name of the section to North Central Section.

Professor R. A. Bing, University of Wisconsin, gave the invited address, "The Elusive Fixed Point Property."

The following papers were presented:

1. *Delta curves of constant width*, by J. D. E. Konhauser, Macalester College.

2. *Numerical integration by continued fractions—An example*, by G. A. Kemper, University of North Dakota.

3. *On the construction of topological examples*, by Lynn Steen and Arthur Seebach, St. Olaf College.

4. *Some necessary and sufficient conditions for commutativity of reflections*, by W. C. Ramaley, Carleton College.

5. *On the equivalence of cancellative extensions of cancellative semi-groups by groups*, by Charles Heuer, Concordia College.

6. *A countable, connected, locally connected Hausdorff Space*, by Allan Kirch, Macalester College.

7. *The preparation of Junior College Teachers of Mathematics*. Panel discussion; moderator: Wayne Roberts, Macalester College; panelists: Charles Blackstad, Worthington State Junior College; Warren Stenberg, University of Minnesota; Richard Twaddle, Anoka-Ramsey State Junior College.

WARREN THOMSEN, *Secretary*

NOVEMBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held on November 2, 1968 at Butler University. There were 85 persons in attendance, including 77 members of the Association.

The group was welcomed by Dr. Alexander Jones, President of Butler University. The following program was then presented at the morning session:

1. *On Hermite-Birkhoff interpolation*, by Kendall Atkinson, Indiana University.

2. *Homomorphism topologies and abelian groups*, by B. F. Hobbs, Olivet Nazarene College.

3. *Boundary behavior for quasi-conformal mappings*, by Glenn Schober, Indiana University.

4. *Let's be honest with the undergraduate—a numerical analyst's point of view of Cramer's rule, inverses of matrices, Laplace's equation, and other formulas: some useful and others not*, by Robert Lynch, Purdue University.

Following a brief business meeting in the afternoon, Professor George Minty of Indiana University delivered an invited address on "Kirszbraun's Theorem (geometry) and its relatives, and their applications in analysis."

M. J. MANSFIELD, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

Fifty-Third Annual Meeting, Miami, Florida, January 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 26, 1969.

FLORIDA, Florida Atlantic University, Boca Raton, March 21–22, 1969.

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA, Purdue University, Indianapolis, May 10, 1969.

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS, Wichita State University, Wichita, March 29, 1969.

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Courant Institute, New York University, March 15, 1969.

MICHIGAN, University of Michigan, Ann Arbor, March 29, 1969.

MINNESOTA, College of St. Catherine, St. Paul, April 26, 1969.

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY

NORTHEASTERN, Williams College, Williamstown, June 1969.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 25–26, 1969.

OKLAHOMA-ARKANSAS, Arkansas State University, Jonesboro, March 21–22, 1969.

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.

ROCKY MOUNTAIN, University of Colorado, Boulder, May 9–10, 1969.

SOUTHEASTERN, Winthrop College, Rock Hill, South Carolina, March 28–29, 1969.

SOUTHERN CALIFORNIA, California State College at Fullerton, March 15, 1969.

SOUTHWESTERN, Northern Arizona University, Flagstaff, April 11–12, 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE, University of Western Ontario, London, Ontario, Canada, May 1969.

WISCONSIN, Oshkosh, Wisconsin, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, University of Oregon, Eugene, Oregon, August 26–29, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Statler-Hilton Hotel, Washington, D. C., May 7–9, 1969.

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, University of Oregon, Eugene, Oregon, August 27, 1969.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Minneapolis, April 23–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, Colorado, June 17–20, 1969.

PI MU EPSILON, University of Oregon, Eugene, Oregon, August 26–27, 1969.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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MARCH 1969

NOTICE TO AUTHORS

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A MOTIVATED ACCOUNT OF AN ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM

NORMAN LEVINSON, Massachusetts Institute of Technology

1. Introduction. One of the most striking results of mathematics is the prime number theorem first conjectured, independently, by Gauss and Legendre prior to 1800 and proved, independently, by Hadamard and de la Vallée Poussin in 1896. Among the many great mathematicians of the 19th century who did not succeed in proving the theorem were Chebychef and Riemann, both of whom obtained important partial results. Riemann indicated that the prime number theorem was related to the behavior of the zeta function in the complex plane and found many properties of this function which has since borne his name. Riemann's ideas were exploited and augmented in the proofs of Hadamard and de la Vallée Poussin.

In 1949, P. Erdös and A. Selberg, using a formula previously proved by Selberg in an elementary way, jointly succeeded in giving several elementary proofs of the prime number theorem, [3]. While elementary, neither these proofs, nor another one of Selberg [6], are simple.

With the tremendous proliferation of mathematics, many mathematicians no longer study number theory. Therefore it seems worthwhile to give a self-contained and motivated account of an elementary proof of the prime number theorem.

The prime numbers (2, 3, 5, 7, 11, 13, $\cdots$) were known to ancient man and in Euclid there is a proof that they are infinite in number. The number of primes not exceeding x is called $\pi(x)$ and can be represented by

$$\pi(x) = \sum_{p \leq x} 1 \tag{1.1}$$

where the symbol p runs over the sequence of primes in increasing order. The simplest form of Legendre's conjecture was

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1. \tag{1.2}$$

Gauss' conjecture has turned out to be more profound and was that $\pi(x)$, for large x, is close to $\int_2^x dt/\log t$. He arrived at this by observing from a tabulation of prime numbers that the primes seemed to have an asymptotic density which at

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x was $1/\log x$. Because of this, for some purposes a better way to find the asymptotic behavior of the primes is to weight each p with $\log p$. This is done in the function

$$\theta(x) = \sum_{p \leq x} \log p. \tag{1.3}$$

Actually it turns out to be even more convenient to use not $\theta(x)$ but a closely related function $\psi(x)$ as will be seen.

The account that follows begins with the factorization of an integer into the product of powers of primes and proceeds with motivated proofs of the relevant discoveries of the 19th century in sections 2 and 3. This approach is continued in section 4 to prove Selberg's formula, and finally in section 5 where an exposition of proof of Selberg [6] is given as simplified by Wright [4], [9], and further simplified by the author [5].

The elementary proof of the prime number theorem has been extended to give elementary proofs of sharper forms of the theorem with a remainder by Breusch [2], Bombieri [1], Wirsing [8] and others. I am indebted to George B. Thomas for a critical reading of the manuscript.

2. The Chebychef identity and its inversion. Our starting point is that a positive integer can be factored into a product of powers of distinct primes. Thus a positive integer

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, \tag{2.1}$$

where the p_j, $1 \leq j \leq m$, are distinct primes and each k_j is a positive integer. Because addition is simpler than multiplication a more useful form of (2.1) is

$$\log n = k_1 \log p_1 + k_2 \log p_2 + \cdots + k_m \log p_m. \tag{2.2}$$

The utility of this formula is very much enhanced by the use of the von Mangoldt symbol $\Lambda(n)$, introduced in 1895, which is defined by

$$\Lambda(n) = \log p \text{ for } n = p^j, \tag{2.3}$$

where p is a prime number and j is a positive integer, and $\Lambda(n) = 0$ otherwise. Thus $\Lambda(n) \neq 0$ only if n is a power of a prime.

The symbol $\sum_{j|n}$ will be used to denote a sum on j where j runs through all of the positive divisors of the positive integer n. With this notation it will be shown that (2.2) can be written as

$$\log n = \sum_{j|n} \Lambda(j). \tag{2.4}$$

To prove (2.4) note that because of (2.1) and the definition of $\Lambda(j)$, the only nonzero terms that can appear on the right side of (2.4) are $\log p_1$, $\log p_2$, $\cdots$ log

p_m. Moreover $\log p_1$ appears for $j=p_1$, for $j=p_1^2, \cdots$, and for $j=p_1^{k_1}$. Thus $\log p_1$ appears exactly k_1 times; similarly $\log p_2$ appears k_2 times, etc., which shows that (2.4) is a consequence of (2.2). The formula (2.4) is an extremely powerful variant of (2.1) and incorporates the properties of prime numbers which are needed here. The transformation of (2.2) into the form (2.4) is not obvious and historically came relatively late.

The formula (2.4) can be written in the equivalent form

$$\log n = \sum_{ij=n} \Lambda(j), \tag{2.5}$$

where i and j are positive integers each of which takes on all possible values satisfying $ij=n$, so that indeed j runs through all positive divisors of n (as does i also).

The number of primes up to x, $\pi(x)$, is closely related to the sum

$$\psi(x) = \sum_{j\leqq x} \Lambda(j). \tag{2.6}$$

From the definition of $\Lambda(j)$,

$$\psi(x) = \sum_{p\leqq x} \log p + \sum_{p^2\leqq x} \log p + \sum_{p^3\leqq x} \log p + \cdots = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots$$

The function $\psi(x)$, expressed in the latter form, was already known to Chebychef, who gave a simple proof that the prime number theorem (1.2) is equivalent to

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1. \tag{2.7}$$

This proof will be given in Lemma 3.4 and (2.7) will be proved in Section 5. (Roughly speaking, $\psi(x)$ acts like $\pi(x)\log x$ for large x because $\psi(x)$ counts each $p\leqq x$ with weight $\log p$, (2.3), and because $\log p$ is close to $\log x$ for "most" of the $p\leqq x$. True, $\psi(x)$ also counts $\log p$ again for $p\leqq x^{1/2}$, for $p\leqq x^{1/3}$, etc., but these last are very sparse as will be seen later in the proof.)

To use (2.5) to get information about $\psi(x)$, (2.5) is summed on $n\leqq x$ to get $\sum_{n\leqq x}\log n = \sum_{n\leqq x}\sum_{ij=n}\Lambda(j)$, so that if one defines

$$T(x) = \sum_{n\leqq x} \log n, \tag{2.8}$$

then

$$T(x) = \sum_{ij\leqq x} \Lambda(j). \tag{2.9}$$

Because the logarithm is a smooth function, $T(x)$ can be readily appraised for large x, and this will be done in (3.4).

The double sum in (2.9) is taken over those lattice points in the positive quadrant of the (i, j) plane which lie on or below the hyperbola $ij=x$. If the double sum (2.9) is treated as a repeated sum, summing first on j

$$T(x) = \sum_{i\leq x} \sum_{j\leq x/i} \Lambda(j) = \sum_{i\leq x} \psi(x/i).$$

This identity was discovered by Chebychef (1850) and will be rewritten as

$$T(x) = \sum_{n\leq x} \psi(x/n). \tag{2.10}$$

The Chebychef identity (2.10) is really a transform relationship. It suggests that given a function $F(x)$, defined for $x>1$, one defines a related function $G(x)$ for $x>1$ by

$$G(x) = \sum_{n\leq x} F(x/n) = F(x) + F(x/2) + F(x/3) + \cdots F(x/[x]), \tag{2.11}$$

where as usual $[x]$ is the largest integer not exceeding x. $G(x)$ may be regarded as a transform of $F(x)$. G is seen to be a linear homogeneous function of F. Transform relationships are among the most powerful tools of the mathematician and this one is no exception.

Since $T(x)$ is a comparatively simple function, it is of interest to try to invert the relationship (2.10) to express $\psi(x)$ in terms of T, or in the more general notation, to try to invert (2.11) to find F in terms of G. To solve for F in terms of G, a first modest step would be to eliminate $F(x/2)$ from the right side of (2.11). This is easily done by writing (2.11) with x replaced by $x/2$ to get

$$G\left(\frac{x}{2}\right) = F\left(\frac{x}{2}\right) + F\left(\frac{x}{4}\right) + F\left(\frac{x}{6}\right) + \cdots.$$

Subtracting the above from (2.11) would eliminate $F(x/2)$. This process can be extended at once by writing (2.11) with x replaced by $x/2$, then by $x/3$, etc., to get

$$\begin{array}{llllllll}
G(x) & =F(x) & +F\left(\frac{x}{2}\right) & +F\left(\frac{x}{3}\right) & +F\left(\frac{x}{4}\right) & +F\left(\frac{x}{5}\right) & +F\left(\frac{x}{6}\right) & + \cdots \\
G\left(\frac{x}{2}\right) & = & F\left(\frac{x}{2}\right) & & +F\left(\frac{x}{4}\right) & & +F\left(\frac{x}{6}\right) & + \cdots \\
G\left(\frac{x}{3}\right) & = & & F\left(\frac{x}{3}\right) & & & +F\left(\frac{x}{6}\right) & + \cdots \\
G\left(\frac{x}{4}\right) & = & & & F\left(\frac{x}{4}\right) & & & + \cdots
\end{array} \tag{2.12}$$

$$(2.12)\quad \begin{aligned} G\left(\frac{x}{5}\right) &= \qquad\qquad\qquad\qquad\qquad F\left(\frac{x}{5}\right) \qquad\quad + \cdots \\ G\left(\frac{x}{6}\right) &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad F\left(\frac{x}{6}\right) + \cdots \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

If one uses the equations in sequence one can first eliminate $F(x/2)$, then $F(x/3)$, then $F(x/4)$, etc., from the right. For example up to $G(x/6)$ one gets

$$F(x) = G(x) - G\left(\frac{x}{2}\right) - G\left(\frac{x}{3}\right) - G\left(\frac{x}{5}\right) + G\left(\frac{x}{6}\right) + \cdots.$$

This suggests, and indeed, because of the diagonal form of the right side of (2.12), actually proves that (2.11) can be inverted by a formula of the type

$$(2.13)\qquad F(x) = \sum_{k\leq x} \mu(k) G\left(\frac{x}{k}\right),$$

where the $\mu(k)$ remain to be specified, (Möbius, 1832). To determine the $\mu(k)$ note that by (2.11) $G(x/k) = \sum_{j\leq x/k} F(x/jk)$ which in (2.13) gives

$$F(x) = \sum_{k\leq x} \mu(k) \sum_{j\leq x/k} F(x/jk) = \sum_{jk\leq x} \mu(k) F(x/jk).$$

If this double sum is summed first on the lattice points on the hyperbolas $jk=n$ and then for n, $1\leqq n\leqq x$,

$$(2.14)\qquad F(x) = \sum_{n\leq x} F(x/n) \sum_{jk=n} \mu(k).$$

The equation (2.14) becomes an identity if $\mu(1)=1$ and, replacing $jk=n$ by $k|n$, if

$$(2.15)\qquad \sum_{k|n} \mu(k) = 0, \qquad n \geqq 2.$$

Setting $n=2$, 3, 4, etc. successively determines the $\mu(k)$ uniquely. To find the $\mu(k)$ explicitly try the case $n=p$ to get $k=1$ and $k=p$ which gives $\mu(1)+\mu(p)=0$ and hence $\mu(p)=-1$. The case $n=p_1p_2$ gives

$$\mu(1) + \mu(p_1) + \mu(p_2) + \mu(p_1p_2) = 0$$

and hence $\mu(p_1p_2)=1$. Similarly it is easily found that

$$\mu(p_1p_2p_3) = -1, \mu(p^2) = 0, \mu(p^3) = 0, \cdots, \mu(p_1^2p_2) = 0.$$

This suggests that

$$\mu(n) = (-1)^m, \qquad n = p_1 p_2 \cdots p_m, \tag{2.16}$$

where $p_1, p_2, \cdots, p_m$ are all distinct primes and

$$\mu(n) = 0 \quad \text{if } p^2 \mid n, \tag{2.17}$$

where as usual p is a prime. The function $\mu(n)$ is known as the Möbius function.

It will now be proved that if $\mu(n)$ is defined as in (2.16) and (2.17) above, then (2.15) is indeed valid. We recall that the solution of (2.15) was unique. Because of (2.17) for $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$,

$$\sum_{j \mid n} \mu(j) = \sum_{j \mid p_1 p_2 \cdots p_m} \mu(j),$$

so only the right side need be treated to prove (2.15). If $m=1$, (2.15) is true since $\mu(1)=1$ and $\mu(p_1)=-1$. If $m \geqq 2$

$$\sum_{j \mid p_1 \cdots p_m} \mu(j) = \sum_{k \mid p_1 \cdots p_{m-1}} (\mu(k) + \mu(kp_m)). \tag{2.18}$$

But from (2.16) if k in (2.18) is the product of r primes

$$\mu(kp_m) = (-1)^{r+1} = -\mu(k).$$

Hence each term on the right of (2.18) is zero and so (2.15) is proved. Moreover by (2.16) and (2.17)

$$|\mu(n)| \leqq 1 \tag{2.19}$$

(which is the only use we shall make of the material which begins after (2.15) and ends with (2.19)).

Applying the Möbius inversion formula (2.13) to Chebychef's identity (2.10) gives the *inversion formula*

$$\psi(x) = \sum_{k \leqq x} \mu(k) T\left(\frac{x}{k}\right). \tag{2.20}$$

Using the definitions of ψ and T this can be written as

$$\sum_{n \leqq x} \Lambda(n) = \sum_{k \leqq x} \mu(k) \sum_{j \leqq x/k} \log j = \sum_{jk \leqq x} \mu(k) \log j = \sum_{n \leqq x} \sum_{jk=n} \mu(k) \log j$$
$$= \sum_{n \leqq x} \sum_{k \mid n} \mu(k) \log n/k.$$

Used for $x = 1, 2, 3, \cdots$ the above proves that

$$\Lambda(n) = \sum_{k \mid n} \mu(k) \log n/k, \qquad n \geqq 1, \tag{2.21}$$

which is the inversion formula for (2.4). The referee observes that one could also show (2.21) directly using (2.4) and (2.15).

3. Some elementary results. Although those results concerning prime numbers that follow were all discovered in the 19th century, some were not found until as much as 70 years after the prime number theorem was first conjectured by Legendre and Gauss.

It will be convenient to use the following well-known lemma in which, as usual, $[x]$ is the largest integer not exceeding x.

LEMMA 3.1. *Let $f(t)$ have a continuous derivative, $f'(t)$, for $t \geqq 1$. Let c_n, $n \geqq 1$, be constants and let $C(u) = \sum_{n \leqq u} c_n$. Then*

$$\sum_{n \leqq x} c_n f(n) = f(x)C(x) - \int_1^x f'(t)C(t)dt \tag{3.1}$$

and

$$\sum_{n \leqq x} f(n) = \int_1^x f(t)dt + \int_1^x (t - [t])f'(t)dt + f(1) - (x - [x])f(x). \tag{3.2}$$

Proof. $C(n) - C(n-1) = c_n$ and $C(u) = C([u])$ since $C(u)$ is a step function. Thus if $[x] = N$,

$$\begin{aligned} \sum_{n \leqq x} c_n f(n) &= \sum_{n \leqq x} (C(n) - C(n-1))f(n) \\ &= \sum_{n \leqq x-1} C(n)(f(n) - f(n+1)) + C(x)f(N) \\ &= -\sum_{n \leqq x-1} C(n) \int_n^{n+1} f'(t)dt + C(x)f(N) \\ &= -\int_1^N C(t)f'(t)dt + C(x)f(N). \end{aligned}$$

Since $C(t)$ is constant on $N \leqq t < x$,

$$\int_N^x C(t)f'(t)dt = C(x)(f(x) - f(N)).$$

Adding this to the previous equation and transposing the integral on the left side to the right proves (3.1).

In case $c_n = 1$, (3.1) becomes

$$\begin{aligned} \sum_{n \leqq x} f(n) &= [x]f(x) - \int_1^x f'(t)[t]dt \\ &= [x]f(x) - \int_1^x f'(t)t\,dt + \int_1^x f'(t)(t - [t])dt. \end{aligned}$$

Integrating the first integral on the right by parts proves (3.2).

It will be convenient to use the following notation. Suppose $f(x)$ is bounded

for finite x and that there is a constant K and a $g(x)$ such that for large x

$$|f(x)| \leqq Kg(x);$$

then this will be denoted by

$$f(x) = O(g(x)) \tag{3.3}$$

and, where convenient, $f(x)$ will be replaced by the right side above.

Applying (3.2) to $f(t) = \log t$ and using $0 \leqq t - [t] < 1$ gives

$$T(x) = x \log x - x + O(\log x), \tag{3.4}$$

which is a weak form of Stirling's formula.

LEMMA 3.2. (Chebychef 1850). *For large* x

$$\psi(x) < \tfrac{3}{2}x. \tag{3.5}$$

Proof. Using Chebychef's identity (2.10)

$$T(x) - 2T(x/2) = \psi(x) - \psi(x/2) + \psi(x/3) - \psi(x/4) + \cdots \geqq \psi(x) - \psi(x/2)$$

because $\psi(x/(2n-1)) - \psi(x/2n) \geqq 0$ since ψ is monotone nondecreasing. Using (3.4), $\psi(x) - \psi(x/2) \leqq x \log 2 + K \log x$, $x \geqq 2$, for some constant K. Applying the above with x replaced by $x/2^j$,

$$\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x}{2^{j+1}}\right) \leqq \frac{x}{2^j} \log 2 + K \log x \tag{3.6}$$

so long as $x/2^j \geqq 2$ which implies $j < \log x/\log 2$. Recalling that $\psi(t) = 0$, $t < 2$, and adding (3.6) for $0 \leqq j < \log x/\log 2$,

$$\begin{aligned} \psi(x) &\leqq x \log 2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) + \frac{\log x}{\log 2} K \log x \\ &= 2x \log 2 + K \log^2 x/\log 2. \end{aligned}$$

Since $\log 2 < .7$ this proves (3.5).

LEMMA 3.3. (Proved 1874 by Mertens in a slightly different form.)

$$\sum_{n \leqq x} \frac{\Lambda(n)}{n} = \log x + O(1). \tag{3.7}$$

Proof. In the double sum (2.9), sum first on i and then on j, (the opposite of what was done in the derivation of (2.10)), to get

$$\begin{aligned} T(x) &= \sum_{j \leqq x} \Lambda(j) \sum_{i \leqq x/j} 1 = \sum_{j \leqq x} \Lambda(j)\left[\frac{x}{j}\right] \\ &= x \sum_{j \leqq x} \frac{\Lambda(j)}{j} - \sum_{j \leqq x} \Lambda(j)\left(\frac{x}{j} - \left[\frac{x}{j}\right]\right). \end{aligned} \tag{3.8}$$

Moreover

$$0 \leqq \sum_{j \leqq x} \Lambda(j)\left(\frac{x}{j} - \left[\frac{x}{j}\right]\right) \leqq \sum_{j \leqq x} \Lambda(j) = \psi(x) = O(x) \tag{3.9}$$

by (3.5). Using this and (3.4) in (3.8) proves (3.7).

LEMMA 3.4.

$$\psi(x) = \pi(x) \log x + O\left(\frac{x \log \log x}{\log x}\right), \tag{3.10}$$

so that (2.7) *is equivalent to the prime number theorem.*

Proof. From its definition (2.6) and from (2.3),

$$\psi(x) = \sum_{p \leqq x} \log p + \sum_{p \leqq x^{1/2}} \log p + \sum_{p \leqq x^{1/3}} \log p + \cdots, \tag{3.11}$$

where the sums $p \leqq x^{1/j}$ above are not zero only if $x^{1/j} \geqq 2$ or if $j \leqq \log x/\log 2$. Hence

$$\psi(x) \leqq \sum_{p \leqq x} \log p + \frac{\log x}{\log 2} \sum_{p \leqq x^{1/2}} \log p.$$

From the definition (1.1) of $\pi(x)$ this gives

$$\psi(x) \leqq \log x\, \pi(x) + \frac{\log x}{\log 2} \pi(x^{1/2}) \log x^{1/2}.$$

Since $\pi(y) \leqq y$, the above gives

$$\psi(x) \leqq \log x\, \pi(x) + \frac{x^{1/2} \log^2 x}{2 \log 2}\,. \tag{3.12}$$

By (3.11)

$$\begin{aligned} \psi(x) &\geqq \sum_{x/\log^2 x < p \leqq x} \log p \geqq \log\left(\frac{x}{\log^2 x}\right) \sum_{x/\log^2 x < p \leqq x} 1 \\ &= \log\left(\frac{x}{\log^2 x}\right)\left(\pi(x) - \pi\left(\frac{x}{\log^2 x}\right)\right). \end{aligned}$$

Since $\pi(y) \leqq y$, this gives

$$\frac{\psi(x)}{\log x - 2 \log \log x} \geqq \pi(x) - \frac{x}{\log^2 x},$$

or

$$\pi(x) \log x \leqq \psi(x) \frac{\log x}{\log x - 2 \log \log x} + \frac{x}{\log x}$$

$$= \psi(x) + \psi(x)\frac{2 \log \log x}{\log x - 2 \log \log x} + \frac{x}{\log x}.$$

Using (3.5) and $2 \log \log x < (\log x)/4$ for large x,

$$\pi(x) \log x \leqq \psi(x) + \frac{4x \log \log x}{\log x} + \frac{x}{\log x}, \tag{3.13}$$

which with (3.12) proves the lemma.

It will be useful later to apply (3.2) to $f(t) = 1/t$.

LEMMA 3.5.

$$\sum_{n \leq x} 1/n = \log x + \gamma + O(1/x), \tag{3.14}$$

where γ is a constant (Euler's constant).

Proof. Applying (3.2) to $f(t) = 1/t$

$$\sum_{n \leq x} \frac{1}{n} = \log x - \frac{x - [x]}{x} + 1 - \int_1^x \frac{t - [t]}{t^2} dt.$$

If

$$\gamma = 1 - \int_1^\infty \frac{t - [t]}{t^2} dt, \tag{3.15}$$

then $\sum_{n \leq x} 1/n = \log x + \gamma + H$, where

$$H = \int_x^\infty \frac{t - [t]}{t^2} dt - \frac{x - [x]}{x} = O\left(\frac{1}{x}\right)$$

since $0 \leqq t - [t] < 1$, which proves (3.14).

REMARK. From (3.15), $0 < \gamma < 1$.

4. Selberg's elementary inequality. The Möbius inversion formula (2.20) which expresses ψ in terms of T will now be used in an attempt to find how $\psi(x)$ behaves for large x. The computation will be simplified if it is possible to find a simple $F(x)$, say $\tilde{F}(x)$, with a transform $\tilde{G}(x)$ which is close to $T(x)$. In that case subtract the Möbius inversion formula for $\tilde{F}$ (2.13), from that for ψ:

$$\psi(x) - \tilde{F}(x) = \sum_{k \leq x} \mu(k)\left(T\left(\frac{x}{k}\right) - \tilde{G}\left(\frac{x}{k}\right)\right); \tag{4.1}$$

if the right side could be shown to be small, then $\psi(x)$ would be close to $\tilde{F}(x)$.

If it were proved that $\psi(x)/x \to 1$, then $\psi(x)$ would be close to x for large x. This suggests one try $\tilde{F}(x) = F_0(x) = x$. Hence $G_0(x) = \sum_{n \leq x} F_0(x/n) = x \sum_{n \leq x} n^{-1}$

which by (3.14) becomes $G_0(x)=x\log x+\gamma x+O(1)$. This is not close enough to $T(x)$ as given by (3.4). As a refinement let $\tilde{F}(x)=F_1(x)=x-C$ where C is a constant. (There are many choices other than C that would work here.) Then

$$\begin{aligned} G_1(x) &= x\sum_{n\leq x}\frac{1}{n} - C\sum_{n\leq x}1 = x\log x+\gamma x+O(1)-C[x] \\ &= x\log x-(C-\gamma)x+O(1). \end{aligned}$$

Hence if $C=1+\gamma$ then by (3.4)

$$T(x)-G_1(x)=O(\log x) \tag{4.2}$$

which is comparatively small. Using (4.1) with $\tilde{F}=x-C$

$$\psi(x)-x+C=\sum_{k\leq x}\mu(k)\left(T\left(\frac{x}{k}\right)-G_1\left(\frac{x}{k}\right)\right). \tag{4.3}$$

Even if the right side of (4.2) were in the stronger form $O(1)$ (which is false), the fact that (by (2.19)) $|\mu(k)|\leqq 1$ would imply only that the right side of (4.3) is $O(x)$. Thus the inversion formula (4.3) gives, *not* the prime number theorem, but at most the much weaker result

$$\psi(x)=O(x), \tag{4.4}$$

(already proved more simply in Lemma 3.2). Actually (4.3) does give (4.4) as the following crude argument shows.

Since the logarithm grows more slowly than any positive algebraic power, $\log x=O(x^{1/2})$. Thus, for example, (4.2) implies the much weaker result

$$T(x)-G_1(x)=O(x^{1/2}). \tag{4.5}$$

Using this and $|\mu(k)|\leqq 1$, there is a constant K such that the right side of (4.3) is dominated by

$$\begin{aligned} Kx^{1/2}\sum_{k\leq x}k^{-1/2} &< Kx^{1/2}\left(1+\sum_{2\leq k\leq x}\int_{k-1}^{k}u^{-1/2}du\right) \\ &\leqq Kx^{1/2}\left(1+\int_1^x u^{-1/2}du\right)=O(x) \end{aligned} \tag{4.6}$$

which does in fact prove (4.4).

Thus the Möbius inversion of Chebychef's formula yields only the crude result (4.4), and herein lies the reason for the long delay in the discovery of an elementary proof of the prime number theorem.

Note that *the crude result* (4.5) *serves just as well as the much more refined* (4.2) in appraising the right side of (4.3). This suggests the following idea.

In the Möbius inversion formula

$$F(x)=\sum_{k\leq x}\mu(k)G\left(\frac{x}{k}\right) \tag{4.7}$$

(where for us $F=\psi-x+C$ and $G=T-G_1$), we can increase the terms in the sum on the right side somewhat since doing so will not change the crude appraisal $O(x)$, (4.6), for this side. On the other hand, a judicious increase of the terms on the right side might possibly replace $F(x)$ (and hence $\psi(x)$) on the left side by some growing function multiplied by $F(x)$, which would then make the appraisal $O(x)$ for the right side useful.

A little experimentation shows that the simplest case to compute explicitly is where the right side of (4.7) is replaced by

$$J(x) = \sum_{k\leq x} \mu(k) \log \frac{x}{k} G\left(\frac{x}{k}\right). \tag{4.8}$$

This must now be computed in terms of F. From the definition of G,

$$\begin{aligned} J(x) &= \sum_{k\leq x} \mu(k) \log \frac{x}{k} \sum_{j\leq x/k} F\left(\frac{x}{jk}\right) \\ &= \sum_{jk\leq x} \mu(k) \log \frac{x}{k} F\left(\frac{x}{jk}\right) = \sum_{n\leq x} F\left(\frac{x}{n}\right) \sum_{jk=n} \mu(k) \log \frac{x}{k} \\ &= \sum_{n\leq x} F\left(\frac{x}{n}\right) \sum_{k\mid n} \mu(k) \log \frac{x}{k} \cdot \end{aligned}$$

Using $\log x/k = \log x/n + \log n/k$

$$J(x) = \sum_{n\leq x} F\left(\frac{x}{n}\right) \log \frac{x}{n} \sum_{k\mid n} \mu(k) + \sum_{n\leq x} F\left(\frac{x}{n}\right) \sum_{k\mid n} \mu(k) \log \frac{n}{k} \cdot$$

By (2.15) and (2.21) this becomes $J(x) = F(x) \log x + \sum_{n\leq x} F(x/n)\Lambda(n)$. With (4.8) this gives

$$F(x) \log x + \sum_{n\leq x} F\left(\frac{x}{n}\right) \Lambda(n) = \sum_{k\leq x} \mu(k) \log \frac{x}{k} G\left(\frac{x}{k}\right), \tag{4.9}$$

and this is the Tatuzawa-Iseki identity [7] which leads easily to the inequality of Atle Selberg. Indeed by (4.2)

$$\log x(T(x) - G_1(x)) = O(\log^2 x) = O(x^{1/2})$$

and hence as already shown in (4.6)

$$\sum_{k\leq x} \mu(k) \log \frac{x}{k} \left(T\left(\frac{x}{k}\right) - G_1\left(\frac{x}{k}\right)\right) = O(x).$$

Thus (4.9) with $F(x) = \psi(x) - x + C$ becomes

$$(\psi(x) - x) \log x + \sum_{n\leq x} \left(\psi\left(\frac{x}{n}\right) - \frac{x}{n}\right) \Lambda(n) = O(x), \tag{4.10}$$

where use is made of (4.4) to incorporate $C\psi(x)$ together with $C \log x$ in $O(x)$. (4.10) is a form of the famous inequality of Atle Selberg [6].

Because of Lemma 3.3, (4.10) can be written as

$$\psi(x) \log x + \sum_{n \leq x} \Lambda(n)\psi(x/n) = 2x \log x + O(x). \tag{4.11}$$

With $c_n = \Lambda(n)$, (3.1) and (3.5) yield

$$\sum_{n \leq x} \Lambda(n) \log n = \psi(x) \log x - \int_1^x \frac{\psi(t)}{t} dt = \psi(x) \log x + O(x). \tag{4.12}$$

Also

$$\sum_{j \leq x} \Lambda(j)\psi\left(\frac{x}{j}\right) = \sum_{j \leq x} \Lambda(j) \sum_{k \leq x/j} \Lambda(k) = \sum_{jk \leq x} \Lambda(j)\Lambda(k). \tag{4.13}$$

Thus if

$$\Lambda_2(n) = \Lambda(n) \log n + \sum_{jk=n} \Lambda(j)\Lambda(k), \tag{4.14}$$

then (4.12) and (4.13) in (4.11) yield $\sum_{n \leq x} \Lambda_2(n) = 2x \log x + O(x)$ as an equivalent to (4.11). By (3.4) $\sum_{n \leq x} \log n = x \log x + O(x)$. Combining the above two inequalities,

$$Q(n) = \sum_{k \leq n} (\Lambda_2(k) - 2 \log k) = O(n), \qquad n \geqq 2, \text{ and } Q(1) = 0. \tag{4.15}$$

5. Proof of the prime number theorem. If $R(x) = \psi(x) - x$, $x \geqq 2$, and $R(x) = 0$, $x < 2$, then (4.10) becomes

$$R(x) \log x + \sum \Lambda(n) R(x/n) = O(x), \tag{5.1}$$

where the summation is self terminating since $R(x/n) = 0$ for $n > x/2$. The goal (2.7) takes the form

$$\lim_{x \to \infty} \frac{R(x)}{x} = 0. \tag{5.2}$$

The derivation of (5.2) from (5.1) is complicated because the weights $\Lambda(n)$ in the weighted sum in (5.1) depend on the location of the prime numbers which is just what we are trying to find. Because of this complication no easy derivation of (5.2) from (5.1) has been found.

The proof that follows uses several smoothing operations on (5.1) to get a more tractable inequality. Most of these smoothings involve a loss of information, and the objective is to smooth for tractability but not to degrade (5.1) completely.

First $R(x)$ will be replaced by the smoother

$$S(y) = \int_2^y \frac{R(x)}{x} dx, \qquad y \geqq ;2 \tag{5.3}$$

$S(y)=0$, $y<2$. Fortunately it is easy to show, as will be done later, that (5.2) is implied if we can prove

$$\lim_{y\to\infty} \frac{S(y)}{y} = 0. \tag{5.4}$$

LEMMA 5.1. *There exists a constant c such that*

$$|S(y)| \leqq cy \qquad y \geqq 2 \tag{5.5}$$

and

$$|S(y_2) - S(y_1)| \leqq c|y_1 - y_2|. \tag{5.6}$$

Moreover a consequence of (5.1) *is*

$$S(y) \log y + \sum \Lambda(j) S\left(\frac{y}{j}\right) = O(y). \tag{5.7}$$

Proof. From (3.5), $-x \leqq \psi(x) - x \leqq \frac{1}{2}x$ for large x. Hence

$$\limsup_{x\to\infty} \frac{|R(x)|}{x} \leqq 1 \tag{5.8}$$

and, since $|R(x)|$ is bounded for finite x, there must exist a constant c such that

$$|R(x)| \leqq cx, \qquad x \geqq 2. \tag{5.9}$$

By (5.3) $S'(y) = R(y)/y$ except at $y = p^j$ where $R(y)$ is discontinuous. By (5.9) then

$$|S'(y)| \leqq c, \qquad y \neq p^j. \tag{5.10}$$

Hence, first for the case where the interval $y_1 < y < y_2$ contains no p^j, (5.6) is true. However since $S(y)$ is continuous, the fact that the magnitude of a sum is less than or equal to the sum of the magnitudes, allows (5.6) to be extended for all y_1 and y_2. The condition (5.6) is known as a Lipschitz condition. The result (5.5) follows from (5.6) with $y_1 = 2$.

Since $||a| - |b|| \leqq |a - b|$, (5.6) yields

$$||S(y_2)| - |S(y_1)|| \leqq c|y_2 - y_1|. \tag{5.11}$$

To prove (5.7), divide (5.1) by x and integrate to get

$$\int_2^y \frac{R(x)}{x} \log x\, dx + \sum \Lambda(n) \int_2^y R\left(\frac{x}{n}\right) \frac{dx}{x} = O(y). \tag{5.12}$$

Integrating the first term by parts

$$\int_2^y \frac{R(x)}{x}\log x\,dx = \log y\,S(y) - \int_2^y \frac{S(x)}{x}\,dx = \log y\,S(y) + O(y)$$

by (5.5). Also if $\xi = x/n$

$$\int_2^y R\left(\frac{x}{n}\right)\frac{dx}{x} = \int_2^{y/n} \frac{R(\xi)}{\xi}\,d\xi = S\left(\frac{y}{n}\right).$$

These in (5.12) prove (5.7).

To make the weighted sum in (5.7) more tractable, the density of the set of points where $S(y/j)$ actually appears in the sum will be increased by iterating (5.7).

LEMMA 5.2. *With* $\Lambda_2(n) = \Lambda(n) + \sum_{ij=n}\Lambda(i)\Lambda(j)$ *as in* (4.14) *and* K_1 *a constant*

$$\log^2 y\,|S(y)| \leqq \sum \Lambda_2(m)\,|S(y/m)| + K_1 y\log y. \tag{5.13}$$

Proof. Replace y in (5.7) by y/k, multiply by $\Lambda(k)$, and sum for $k \leqq y$ to get

$$\sum \Lambda(k)S\left(\frac{y}{k}\right)\log\frac{y}{k} + \sum\sum \Lambda(k)\Lambda(j)S\left(\frac{y}{jk}\right) = O(y)\sum_{k\leqq y}\frac{\Lambda(k)}{k}\cdot$$

Setting $jk = m$ in the second sum and summing on m, and setting $\log y/k = \log y - \log k$ in the first sum and replacing this latter k by m,

$$\log y \sum \Lambda(k)S\left(\frac{y}{k}\right) - \sum_{m\leqq y} S\left(\frac{y}{m}\right)\left\{\Lambda(m)\log m - \sum_{jk=m}\Lambda(j)\Lambda(k)\right\} = O(y\log y)$$

where (3.7) is used to get the right side. The first sum above is now replaced by use of (5.7) to give

$$S(y)\log^2 y = -\sum S\left(\frac{y}{m}\right)\left\{\Lambda(m)\log m - \sum_{jk=m}\Lambda(j)\Lambda(k)\right\} + O(y\log y).$$

Replacing all terms in the sum on the right by their magnitude gives (5.13).

The inequality (4.15) suggests that on the average $\Lambda_2(m)$ acts like $2\log m$. A weighted sum with weights $2\log m$ is quite tractable and this suggests modifying (5.13) by replacing $\Lambda_2(m)$ by $2\log m$.

LEMMA 5.3. *There is a constant* K_2 *such that*

$$\log^2 y\,|S(y)| \leqq 2\sum |S(y/m)|\log m + K_2 y\log y. \tag{5.14}$$

Proof.

$$\sum |S(y/m)|\,\Lambda_2(m) = 2\sum_{m\leq y} |S(y/m)|\log m + J(y) \tag{5.15}$$

where, since by (4.15), $\Lambda_2(m) - 2\log m = Q(m) - Q(m-1)$,

$$\begin{aligned} J(y) &= \sum_{m \leq y} (Q(m) - Q(m-1)) \mid S(y/m) \mid \\ &= \sum_{m \leq y} Q(m) \mid S(y/m) \mid - \sum_{m \leq y} Q(m) \mid S(y/(m+1)) \mid \\ &= \sum_{2 \leq m \leq y} Q(m)(\mid S(y/m) \mid - \mid S(y/(m+1)) \mid) \end{aligned}$$

since $S(y) = 0$, $y < 2$. Using (4.15) and (5.11) there is a constant K_3 such that

$$\begin{aligned} J(y) &\leq K_3 \sum_{2 \leq m \leq y} m\left(\frac{y}{m} - \frac{y}{m+1}\right) \\ &= K_3 y \sum_{2 \leq m \leq y} \frac{1}{m+1} < K_3 y \int_1^y \frac{dv}{v} = K_3 y \log y. \end{aligned}$$

This and (5.15) now prove that (5.14) is a consequence of (5.13).

There is a further simplification in replacing the sum in (5.14) by an integral.

LEMMA 5.4. *There is a constant K_4 such that*

$$\log^2 y \mid S(y) \mid \leq 2 \int_2^y \mid S(y/u) \mid \log u \, du + K_4 y \log y. \tag{5.16}$$

Proof. Since $\log u$ is increasing

$$\log m \mid S(y/m) \mid \leq \int_m^{m+1} \log u \mid S(y/m) \mid du.$$

On the right use $\mid S(y/m) \mid \leq \mid S(y/u) \mid + \mid S(y/m) - S(y/u) \mid$ to get

$$\begin{aligned} \log m \mid S(y/m) \mid &\leq \int_m^{m+1} \log u \mid S(y/u) \mid du + J_m \\ J_m &= \int_m^{m+1} \log u \mid S(y/m) - S(y/u) \mid du. \end{aligned} \tag{5.17}$$

Using (5.6)

$$J_m \leq c\left(\frac{y}{m} - \frac{y}{m+1}\right) \int_m^{m+1} \log u \, du \leq \frac{cy \log(m+1)}{m(m+1)}.$$

Since $\log(m+1) \leq m$, the above in (5.17) gives

$$\log m \left| S\left(\frac{y}{m}\right) \right| \leq \int_m^{m+1} \log u \left| S\left(\frac{y}{u}\right) \right| dy + \frac{cy}{m+1}.$$

Using this in (5.14) now gives (5.16) with $K_4 = K_2 + c$.

The inequality (5.16) assumes a simpler form with an exponential change of variable. Replace u by $v = \log y/u$. Also let $x = \log y$. Then (5.16) becomes

$$x^2 |S(e^x)| \leqq 2\int_0^{x-\log 2} |S(e^v)| (x-v)e^{(x-v)}dv + K_4 x e^x. \tag{5.18}$$

If

$$W(x) = e^{-x}S(e^x) \tag{5.19}$$

then (5.18) becomes

$$|W(x)| \leqq \frac{2}{x^2}\int_0^x (x-v)|W(v)|\,dv + \frac{K_4}{x}\cdot \tag{5 20}$$

This inequality contains valuable information since it says in effect that $|W(x)|$ is dominated by a weighted average of $|W|$. This has as a consequence the following lemma. (Note that γ below is not Euler's constant.)

LEMMA 5.5. *Let*

$$\alpha = \limsup_{x\to\infty} |W(x)|, \qquad \gamma = \limsup_{x\to\infty} \frac{1}{x}\int_0^x |W(\xi)|\,d\xi; \tag{5.21}$$

then $\alpha \leqq 1$ *and*

$$\alpha \leqq \gamma. \tag{5.22}$$

REMARK. Recalling (5.4) and (5.19), our goal now is $\alpha = 0$.

Proof. That $\alpha \leqq 1$ follows from (5.19) and the fact that (5.8) and (5.3) imply that

$$\limsup_{y\to\infty} \frac{|S(y)|}{y} \leqq 1. \tag{5.23}$$

The key result $\gamma \geqq \alpha$ will be proved by use of (5.20) and *this is the only use that is made of* Lemmas 5.2, 5.3 and 5.4. Note that (5.20) can be written as

$$|W(x)| \leqq \frac{2}{x^2}\int_0^x u\,du\left(\frac{1}{u}\int_0^u |W(v)|\,dv\right) + \frac{K_4}{x} \tag{5.24}$$

as can be verified by inverting the order of integration. But

$$\frac{2}{x^2}\int_0^x u\,du = 1$$

and hence the integral on the right of (5.24) is simply a weighted average of $(1/u)\int_0^u |W(v)|dv = (1/u)\int_0^u e^{-v}|S(e^v)|dv \leqq c$ by (5.5). Hence for any fixed x_1 and $x > x_1$,

$$(5.25)\qquad \begin{aligned} I(x) &= \frac{2}{x^2}\int_0^x u\,du\left(\frac{1}{u}\int_0^u |W(v)|\,dv\right) \\ &\leqq \frac{2c}{x^2}\int_0^{x_1} u\,du + \frac{2}{x^2}\int_{x_1}^x u\,du\left(\frac{1}{u}\int_0^u |W(v)|\,dv\right). \end{aligned}$$

Given $\epsilon > 0$, for sufficiently large x_1,

$$\frac{1}{u}\int_0^u |W(v)|\,dv < \gamma + \epsilon \qquad u \geqq x_1$$

from the definition of γ. Hence (5.25) gives

$$I(x) \leqq \frac{cx_1^2}{x^2} + (\gamma + \epsilon)\left(1 - \frac{x_1^2}{x^2}\right).$$

Thus for large x, (5.24) yields

$$|W(x)| \leqq \gamma + \epsilon + \frac{cx_1^2}{x^2} + \frac{K_4}{x}\cdot$$

Letting $x \to \infty$, $\alpha \leqq \gamma + \epsilon$, and since this is true for all $\epsilon > 0$ it implies (5.22).

Two more facts are required about W to prove that $\alpha = 0$.

LEMMA 5.5. *If $k = 2c$ then*

$$(5.26)\qquad |W(x_2) - W(x_1)| \leqq k|x_2 - x_1|,$$

and hence

$$(5.27)\qquad \big||W(x_2)| - |W(x_1)|\big| \leqq k|x_2 - x_1|.$$

Proof. Since $W(x) = e^{-x}S(e^x)$,

$$|W'(x)| \leqq e^{-x}|S(e^x)| + |S'(e^x)| \qquad x \neq j \log p.$$

Hence by (5.5) and (5.10), $|W'(x)| \leqq 2c = k$ for $x \neq j \log p$. This leads to (5.26) just as (5.10) led to (5.6).

LEMMA 5.7. *If $W(v) \neq 0$ for $v_1 < v < v_2$, then there exists a number M such that*

$$(5.28)\qquad \int_{v_1}^{v_2} |W(v)|\,dv \leqq M, \qquad W(v) \neq 0, \qquad v_1 < v < v_2.$$

Proof. From (3.1) letting $c_n = \Lambda(n)$ and $f(n) = 1/n$, (3.7) implies

$$\int_2^x \frac{\psi(t)}{t^2}\,dt = \log x + O(1),$$

or since $R(t) = \psi(t) - t$,

$$\int_2^x \frac{R(t)}{t^2}\,dt = O(1). \tag{5.29}$$

But

$$\int_2^x \frac{S(y)}{y^2}\,dy = \int_2^x \frac{dy}{y^2}\int_2^y \frac{R(t)}{t}\,dt = \int_2^x \frac{R(t)}{t}\left(\int_t^x \frac{dy}{y^2}\right)dt$$
$$= \int_2^x \frac{R(t)}{t^2}\,dt - \frac{1}{x}\int_2^x \frac{R(t)}{t}\,dt.$$

Using (5.29) and (5.9), $\int_2^x (S(y)/y^2)dy = O(1)$, or letting $y = e^u$, $x = e^v$,

$$\int_{\log 2}^v W(u)du = O(1).$$

Writing this for $v = v_1$ and $v = v_2$ and subtracting, the resulting integral is bounded and hence there is a constant M such that

$$\left|\int_{v_1}^{v_2} W(u)du\right| \leqq M.$$

But if $W(u) \neq 0$, $v_1 < u < v_2$, this can be written as (5.28). Since M can be increased if convenient it can be assumed $Mk > 1$.

LEMMA 5.8. *A function $W(x)$ subject to the three conditions* (5.22), (5.27) *and* (5.28) *must in fact have $\alpha = 0$.*

Proof. Choose $\beta > \alpha$. Then from the definition of α there exists an x_β such that

$$|W(x)| \leqq \beta \qquad x \geqq x_\beta. \tag{5.30}$$

If $W(x) \neq 0$ for all large x it follows from (5.28) that $\gamma = 0$ and hence that $\alpha = 0$. Suppose then that $W(x)$ has arbitrarily large zeros. Let a and b be successive zeros of $W(x)$ for $x > x_\beta$.

CASE 1. $b - a \geqq 2M/\beta$. By (5.28), since $W(x) \neq 0$, $a < x < b$,

$$\int_a^b |W(x)| \leqq M \leqq \tfrac{1}{2}(b-a)\beta.$$

(Hence the average of $|W|$ on (a, b) is less than $\frac{1}{2}\beta$.)

CASE 2. $b - a \leqq 2\beta/k$. In this case it follows from (5.27) that if the graph of $|W(x)|$ rises as rapidly as possible going right from $x = a$ and left from $x = b$, it cannot lie above a triangle with altitude $k(b-a)/2 \leqq \beta$ and hence

$$\int_a^b |W(x)|\,dx \leqq \tfrac{1}{2}(b-a)\beta.$$

CASE 3. $2\beta/k < b - a < 2M/\beta$. Reasoning as in Case 2 for a distance β/k from each endpoint and using (5.30) otherwise,

$$\begin{aligned}\int_a^b |W(x)| &\leqq \frac{\beta^2}{k} + \left(b - a - \frac{2\beta}{k}\right)\beta \\ &= (b-a)\beta\left(1 - \frac{\beta}{k(b-a)}\right) \leqq (b-a)\beta\left(1 - \frac{\beta^2}{2Mk}\right) \\ &< (b-a)\beta\left(1 - \frac{\alpha^2}{2Mk}\right).\end{aligned} \tag{5.31}$$

Since $Mk>1$ and since $\alpha \leqq 1$, (5.31) is valid in Cases 1 and 2 also. If x_1 is the first zero of $W(x)$ to the right of x_β and $\bar{x}$ the largest zero to the left of y, then (5.31) and (5.28) imply that

$$\int_0^y |W(x)|\,dx \leqq \int_0^{x_1} |W(x)|\,dx + (\bar{x} - x_1)\beta\left(1 - \frac{\alpha^2}{2Mk}\right) + M.$$

Dividing by y and noting that $\bar{x} \leqq y$,

$$\frac{1}{y}\int_0^y |W(x)|\,dx \leqq \frac{1}{y}\int_0^{x_1} |W(x)|\,dx + \beta\left(1 - \frac{\alpha^2}{2Mk}\right) + \frac{M}{y}\cdot$$

Letting $y \to \infty$, $\gamma \leqq \beta(1-\alpha^2/2Mk)$, and since $\gamma \geqq \alpha$,

$$\alpha \leqq \beta\left(1 - \frac{\alpha^2}{2Mk}\right).$$

Since this holds for all $\beta > \alpha$ it must hold for $\beta = \alpha$. Hence $\alpha^3 \leqq 0$, and since $\alpha \geqq 0$, this implies $\alpha = 0$. Since $W(x) = e^{-x}S(e^x)$, this implies that $|S(y)|/y \to 0$ as $y \to \infty$. Hence if given $\epsilon > 0$, if y is large enough,

$$|S(y)| \leqq \tfrac{1}{3}\epsilon^2 y.$$

Thus $S(y(1+\epsilon)) - S(y) \leqq \frac{1}{3}\epsilon^2(y(1+\epsilon)+y) < \epsilon^2 y$, or

$$\int_y^{y(1+\epsilon)} \frac{R(u)}{u}\,du \leqq \epsilon^2 y.$$

Since $R(u) = \psi(u) - u$ and ψ is nondecreasing,

$$\frac{\psi(y)}{y(1+\epsilon)}\int_y^{y(1+\epsilon)} du - \int_y^{y(1+\epsilon)} du \leqq \epsilon^2 y.$$

Hence $\psi(y)/y \leqq (1+\epsilon)^2$. Similarly $S(y) - S(y(1-\epsilon)) \geqq -\epsilon^2 y$ for large enough y leads to $\psi(y)/y \geqq (1-\epsilon)^2$. Since ϵ is arbitrary this proves (2.7).

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AN INTRODUCTION TO HESTENES TERNARY RINGS

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Dedicated to Marian and Harry

Introduction. Spectral theory for rectangular matrices goes back to the general reciprocal of E. H. Moore [9] and has been studied in some detail by Penrose [10], M. R. Hestenes [2] and Lanczos [7]. In two papers [3, 4] Hestenes has cast this theory in the framework of a theory of a ternary operation based on the observation that if A, B and C are complex m by n matrices, then so is AB^*C. The purpose of this brief expository note is to indicate the possibility that Hestenes's idea extends to structure theory in the spirit of N. Jacobson [5, 6]. R. A. Stephenson [12] has already verified the rudiments of this extension. Nonetheless, many interesting questions remain.

We begin with a quick derivation of Moore's general reciprocal since this is the generic idea on which our algebra depends. In order to present an extension of the Chevalley-Jacobson Density Theorem (which we obtained jointly with R. A. Stephenson), we have taken the liberty of presenting Jacobson's original proof [5]. A well-tempered proof due to Tate [Artin, 1] is available. One should also mention Jacobson's proof in [6] which is based on a theorem very close to our extended version even though it involves only ordinary rings.

1. E. H. Moore's general inverse. Let A be an m by n complex matrix of rank r. Because $Ax=0$ if and only if $A^*Ax=0$, A^*A is an n by n nonnegative hermitian matrix of rank r. Let $x_1, \cdots, x_n$ be an orthonormal set of eigenvectors for

A^*A, so labeled that $A^*Ax_i=\lambda_i^2 x_i$ for $i=1, \cdots, n$ and with $\lambda_i^2>0$ for $i=1, \cdots, r$. With

$$X = [x_1, \cdots, x_r], \quad Z = [x_{r+1}, \cdots, x_n] \quad \text{and} \quad \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_r),$$

we have

$$X^*X = I_r, \quad XX^* + ZZ^* = I_n \quad \text{and} \quad A^*AX = X\Lambda^2, \quad AZ = 0, \quad AXX^* = A.$$

With $Y=AX\Lambda^{-1}$, we obtain $A=Y\Lambda X^*$, $Y^*Y=I_r$ and the "diagonalization" $Y^*AX=\Lambda$. Then $G=X\Lambda^{-1}Y^*$ is Moore's general reciprocal of A since $AGA=A$, $GAG=G$ and $AG=YY^*$, $GA=XX^*$ are self-adjoint. Further implications of these computations are contained in the papers [**2**, **7**, **9**, **10**, **11**].

2. The Chevalley-Jacobson Density Theorem for rings. Now let X and Y be abelian groups. Then $\operatorname{Hom}(X, Y)$ consists of all group morphisms $\alpha: X\rightarrow Y$, that is, all maps α satisfying $\alpha(x+x')=\alpha(x)+\alpha(x')$ for all x, x' in X. If we define $(\alpha+\beta)(x)=\alpha(x)+\beta(x)$ for x in X and α, β in $\operatorname{Hom}(X, Y)$, then $\operatorname{Hom}(X, Y)$ becomes an abelian group. Further, $\operatorname{Hom}(X, X)$ becomes a ring under the multiplication $(\alpha\beta)(x)=\alpha(\beta(x))$. A subring R of $\operatorname{Hom}(X, X)$ is *irreducible* if and only if $R\neq 0$ and the only R-subgroups of X are 0 and X itself. (A subgroup Y of X is an R-subgroup if and only if $R(Y)\subseteq Y$.) For the remainder of this section we assume that R is an irreducible subring of $\operatorname{Hom}(X, X)$.

SCHUR'S LEMMA. *The set* $D=\{\alpha$ *in* $\operatorname{Hom}(X, X)$; $\alpha a=a\alpha$ *for all* a *in* $R\}$ *is a division ring.*

Proof. Clearly D is a subring of $\operatorname{Hom}(X, X)$, 1_X is in D, and $1_X\neq 0$ because $R\neq 0$. Let $\alpha\in D$ be nonzero. Then $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ are R-subgroups of X and it follows from the irreducibility of R that $\operatorname{Ker}\alpha=0$ and that $\operatorname{Im}\alpha=X$. Hence $\alpha^{-1}\in\operatorname{Hom}(X, X)$ and consequently $\alpha^{-1}\in D$.

REMARK. If we define $dx=d(x)$ for $d\in D$ and $x\in X$, then X becomes a D-vector space and, since $a(dx)=a(d(x))=d(a(x))$, R becomes a ring of D-linear transformations of X into X.

DENSITY THEOREM. *Let* $x_1, \cdots, x_n$ *be* D*-linearly independent in* X *and let* $y_1, \cdots, y_n\in X$. *Then there is an* a *in* R *such that* $a(x_i)=y_i$ *for* $i=1, \cdots, n$.

Proof. Use induction on n. If $n=1$ and the theorem is false, then $Rx_1\neq X$ for some nonzero x_1 in X. But Rx_1 is an R-subgroup and hence $Rx_1=0$, $R(Dx_1)=0$ so that Dx_1 is a nonzero R-subgroup, $Dx_1=X$, $R(X)=0$, $R=0$, a contradiction. Now assume that the theorem holds for $n-1$.

LEMMA. *There is an element* b_n *in* R *such that* $b_n(x_i)=0$ *for* $i<n$ *and* $b_n(x_n)\neq 0$.

Proof. Suppose not. Then a in R and $a(x_i)=0$ for $i<n$ implies that $a(x_n)=0$. If $y\in X$, by induction there is an $a\in R$ such that $a(x_1)=y$ and $a(x_i)=0$ for $1<i<n$. If also $y=b(x_1)$ and $b(x_i)=0$ for $1<i<n$, then $a(x_n)=b(x_n)$. The assignment $y\mapsto a(x_n)$ provided $a(x_1)=y$ and $a(x_i)=0$ for $1<i<n$, therefore defines a

map $\psi: X \to X$. One checks easily that ψ is in $\mathrm{Hom}(X, X)$. For all b in R, $b(y) = (ba)(x_1)$ with $(ba)(x_i) = 0$ for $1 < i < n$, so that $\psi(b(y)) = (ba)(x_n) = b(\psi(y))$. Thus ψ is in D and $a(\psi x_1 - x_n) = \psi(a(x_1)) - a(x_n) = 0$ for all a in R. Using the theorem for $n = 1$ then gives $\psi(x_1) - x_n = 0$, violating the D-linear independence of $x_1, \cdots, x_n$.

One completes the proof by observing that the lemma together with the case $n = 1$ yields an a_n in R such that $a_n(x_i) = 0$ for $i \neq n$ and $a_n(x_n) = y_n$. Replacing x_n by x_k, we obtain an a_k in R such that $a_k(x_i) = 0$ for $i \neq k$ and $a_k(x_k) = y_k$. Then $a = a_1 + \cdots + a_n$ has the desired properties.

3. The density theorem for Hestenes ternary rings. In this section X and Y are still abelian groups. We assume that there is a map $*: \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, X)$. We call a subgroup R of $\mathrm{Hom}(X, Y)$ a *Hestenes ternary ring* if $ab^*c \in R$ whenever $a, b, c \in R$. The set of all finite sums $\sum_i a_i^* b_i$ with $a_i, b_i \in R$ is an ordinary ring R_1 since

$$(\sum_i a_i^* b_i)(\sum_j c_j^* d_j) = \sum_{i,j} a_i^*(b_i c_j^* d_j).$$

Likewise, the set of all finite sums $\sum_i c_i d_i^*$ forms a ring R_2. We call R *irreducible* if both R_1 and R_2 are irreducible. We assume that R is irreducible throughout the remainder of this section. By Schur's Lemma, we have two division rings D_1 and D_2, X is a D_1-vector space and Y is a D_2 vector space. These division rings are always isomorphic [R. A. Stephenson, 12]. We prove this here only under the assumption that both X and Y are finite-dimensional. With this assumption, the density theorem for rings implies that

$$1_X = \sum_i e_i^* e_i' \quad \text{and} \quad 1_Y = \sum_j f_j' f_j^*$$

for suitable elements e_i, e_i', f_j', f_j of R. We define

$$\zeta(d_1) = \sum_j f_j' d_1 f_j^* \quad \text{and} \quad \sigma(d_2) = \sum_i e_i^* d_2 e_i'.$$

Then ζ and σ are inverse isomorphisms between D_1 and D_2 as simple calculations quickly show. We may regard Y as a D_1-vector space if we define $d_1 y = \zeta(d_1) y$. Then for a in R,

$$d_1 a(x) = (\zeta(d_1)a)(x) = (\sum_j f_j' d_1 f_j^* a)(x) = a(d_1 x).$$

Thus all a in R are D_1-linear transformations of X into Y.

The density theorem (with D replaced by D_1 and $y_i \in X$ replaced by $y_i \in Y$) *holds*. If $n = 1$ and the theorem is false, then $Rx_1 \neq Y$ for some nonzero x_1 in X. But Rx_1 is an R_2-subgroup of Y. Hence $Rx_1 = 0$, $R^*R(D_1 x_1) = 0$, so that $D_1 x_1$ is a nonzero R_1-subgroup of X, $D_1 x_1 = X$, $R^*R(X) = 0$, $R_1 = 0$, a contradiction.

As for the proof of the lemma, we easily obtain $\psi \in D_2$ and then the equation

$$a(\sigma(\psi)x_1 - x_n) = \zeta(\sigma(\psi))a(x_1) - a(x_n) = \psi(a(x_1)) - a(x_n) = 0$$

for all a in R violates the D_1-linear independence of $x_1, \cdots, x_n$.

Further details concerning the structure of Hestenes ternary rings will be found in R. A. Stephenson [12].

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Addendum to "Autobiographical Notes" by G. C. Evans (this MONTHLY, 76 (1969) 10–12). Of all the papers that I have written in Mathematics and Economics, and the few in Physics, the one that pleased me most was the proof of "Kellogg's Lemma" in 1933 [1]. It was obvious that several mathematicians were trying to prove it, but I waited for Professor Kellogg to do it himself. He died suddenly up in the snows of New Hampshire.

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WHEN IS A TOTALLY SYMMETRIC LOOP A GROUP?

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In recent literature the general problem of characterizing among loops those which are groups has been studied by Zassenhaus [10], Parker [7] and Watson [9]. We consider here the class of totally symmetric loops and characterize those which are associative. The term *associative loop* is synonymous with the term group. We adopt the definitions and terminology of Bruck [2].

A *loop* is a set G which is closed under a binary operation $*$ such that $(G, *)$ satisfies:

(A) If any two of x, y, z are given as elements of G, the equation $x * y = z$ uniquely determines the third as an element of G.

(B) There exists an element 1 in G such that $a * 1 = a = 1 * a$ for every a in G.

A *totally symmetric loop* also satisfies

(C) $a * b = b * a$ and $a * (a * b) = b$ for all a and b in G.

A *Steiner triple system* is a set of n elements arranged in three element subsets, called *triples*, in such a way that every pair of distinct elements is contained in one and only one triple. That the class of totally symmetric loops is coextensive with the class of Steiner triple systems (designated STS(n)) was shown by Bruck [2] as follows. Let G be a totally symmetric loop with $n+1$ elements. We obtain from G an STS(n), which we call G^*, by deleting the identity element 1 and by taking the distinct elements a, b, c, to be members of the same triple if $a * b = c$. Conversely, if G^* is an STS we make it into a loop G by (1) defining, for distinct elements a and b, $a * b = c$ where c is the third element of the unique triple determined by a and b and (2) defining $1 * a = a * 1 = a$ and $a^2 = 1^2 = 1$ for all a in G^* so that the set $G = G^* \cup 1$ and properties (A), (B), and (C) are satisfied. From these considerations it follows that totally symmetric loops of order $n+1$ exist for all n such $n \equiv 1, 3$ (modulo 6) and for no other values of n since these are precisely the values of n for which Steiner triple systems exist (cf. [2], [6]).

Among the members of the class of totally symmetric loops we seek a characterization of those which are groups. It is known that a totally symmetric loop which is associative must have 2^k elements [10]. This is easily verified since each element is of order 2 inasmuch as $a^2 = 1$ for all a in G. Thus a totally symmetric loop which is associative is isomorphic to the unique group which is a product of k cyclic groups of order 2. We can also arrive at the result that there is one and only one totally symmetric loop of order 2^k which is associative by characterizing the Steiner triple systems which yield such loops. Moreover we will show, by appealing to recent work in Steiner systems, that for each associative totally symmetric loop which exists with order greater than 8, there also exists a nonassociative totally symmetric loop of the same order. (In our discussion we omit the Steiner triple systems which have $n=1$ or $n=3$. These are often considered to be trivial and the reader will note that they satisfy our theorem vacuously.)

THEOREM. *Let G be a totally symmetric loop and let G^* be the Steiner triple system derived from G. G is associative if and only if G^* has the property that every set of three elements not all in the same triple generates a Steiner triple system on seven symbols.*

Proof. We show first that if G is associative G^* has the desired property. Let a, b, c be three distinct elements of G such that $a * (b * c) \neq 1$ and none of a, b, c is the identity. Such a set of three elements in G is a set of three elements in G^* such that the three elements are not all in the same triple. We note that the choice of a, b, c insures that $a * b \neq c$, $b * c \neq a$ and $a * c \neq b$. Now we generate all the distinct products which can be obtained starting from the three given elements a, b, c. Let $a * b = p$. We show $b * c \neq p$. Suppose the contrary. Then $b * c = a * b = b * a$. Since $b * (b * c) = c$ by property (C), multiplying the preceding equality by b on the left implies $c = a$, which is contrary to the hypothesis that a, b, c are distinct. Let $b * c = q$ and let $a * c = r$. By the same argument which showed $p \neq q$, it follows that $r \neq p$ and $r \neq q$. Notice that since G is associative and $b * b = 1$, then $p * q = (a * b) * (b * c) = a * c = r$.

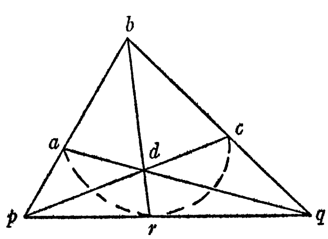
We now have six elements, namely, a, b, c, p, q, r. Consider the product $a * q$. We show that this product is a new element.

$a * q \neq a$, otherwise $q = 1$. $a * q \neq b$, otherwise $q = c$.
$a * q \neq c$, otherwise $q = b$. $a * q \neq p$, otherwise $c = 1$.
$a * q \neq q$, otherwise $a = 1$. $a * q \neq r$, otherwise $b = 1$.

Thus we have established a seventh element. Let $d = a * q$. We show that $d = p * c = b * r$ by the following sequence of equalities.

$$\begin{aligned} d &= a * q = a * (b * c) = (a * b) * c = p * c \\ &= (q * r) * c = c * (q * r) = (c * q) * r = b * r. \end{aligned}$$

We now have seven elements and have exhausted all possible products of distinct elements. When we translate our multiplication table into triples we find that we have generated an STS(7), as is shown in the following diagram which exhibits triples as collinear points. This is the well-known Fano configuration which displays the finite projective plane on 7 points.



To show the converse, let G be a totally symmetric loop whose Steiner triple system G^* has the property that every set of three elements not in the same triple generates an STS(7). We note, first, that any three elements in the same

triple of G^* obey the associative law in G since, if (abc) is a triple in G^*, then $(b*c)=a$ so that $a*(b*c)=a*a=1$ and similarly $(a*b)*c=c*c=1$. Now consider any three elements, a, b, c, not in the same triple. These generate an STS(7) by hypothesis. However, there is, up to isomorphism, one and only one STS(7) and the totally symmetric loop of order 8 obtained from STS(7) is known to be associative. (Cf. [2] and [10].) Thus any three elements not in the same triple in G^* satisfy the associative law in G. Finally, for products in G involving the element 1, associativity obviously holds. It follows that G is associative if G^* has the stated property. This completes the proof.

This theorem, which is disarmingly simple, enables us to state as corollaries several interesting conclusions which result from recent work in Steiner triple systems.

COROLLARY 1. *A totally symmetric loop G is a group if and only if its Steiner triple system G^* is isomorphic to the $k-1$ dimensional projective geometry over a Galois field of two elements, (i.e., $PG(k-1, 2)$) where the order of G is 2^k.*

Proof. G^* is of order 2^k-1 since G is a group only if the order of G is 2^k, as noted above. An STS(2^k-1) is isomorphic to $PG(k-1, 2)$ if and only if every triangle (i.e., three elements not in the same triple) generates an STS(7) (cf. [4]).

COROLLARY 2. *For each integer k such that $k \geqq 3$ there is (up to isomorphism) one and only one totally symmetric loop of order 2^k which is a group.*

Proof. This follows from the uniqueness of the $k-1$ dimensional projective geometry having three points on each line.

COROLLARY 3. *For each integer k such that $k>3$, there is at least one nonassociative totally symmetric loop of order 2^k and, in general, the number of distinct nonassociative totally symmetric loops of order 2^k tends to infinity with k.*

Proof. We note that STS(7) is unique and the loop obtained from it is associative. Therefore, this corollary does not hold for $k=3$. It is often stated without proof, that there are at least two nonisomorphic Steiner triple systems for all orders $n \equiv 1, 3 \pmod 6$. For $k=4$, there are exactly 80 Steiner triple systems of order 2^k-1. These have been counted many times. The best-known counts are those of Cole, White and Cummings [3] and Hall and Swift [5]. A recent paper by Assmus and Mattson [1] proves that for $n=2^k-1$ the number of inequivalent Steiner triple systems goes to infinity with k and, in particular, there are at least two inequivalent systems for each k such that $k \geqq 4$. This result is proved using a construction due to Vacil'ev [8]. Since by Corollary 2, the associative totally symmetric loop of order 2^k is unique, it follows, from the Assmus-Mattson result together with the coextensiveness of totally symmetric loops and Steiner triple systems noted by Bruck [2], that there exists at least one nonassociative totally symmetric loop of order 2^k for each k such that $k>4$ and, from counts of the STS(15), that there are exactly 79 nonassociative totally symmetric loops of order 2^k for $k=4$. The general Assmus-Mattson result implies

that the number of distinct nonassociative totally symmetric loops of order 2^k goes to infinity with k.

Note. This exposition is partly motivated by Problem 5567, page 312 in this issue.

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THE MEAN OF A FUNCTION $x(\nu)$ RELATIVE TO A FUNCTION $w(\xi, \nu)$

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Let $x(\nu)$ be a nonconstant continuous function on a closed interval I, which assumes each value between its minimum x_l and maximum x_u for only finitely many values of ν. If $w(\xi, \nu)$ is a positive continuous "weight function" on $[x_l, x_u]\times I$, the function

$$B(x) = \int_I \int_{x(\nu)}^{x} w(\xi, \nu)\, d\xi d\nu$$

is continuous and strictly increasing on $[x_l, x_u]$, with $B(x_l)<0<B(x_u)$. The unique number b on (x_l, x_u) for which $B(b)=0$ is then called the *mean of $x(\nu)$ relative to $w(\xi, \nu)$*.

Moreover, if $g(\xi)$ is positive, continuous, and strictly increasing on $[x_l, x_u]$, the analogous function

$$C(x) = \int_I \int_{x(\nu)}^{x} g(\xi) w(\xi, \nu)\, d\xi d\nu$$

has as *its* unique zero on (x_l, x_u) the mean c of $x(\nu)$ relative to $g(\xi)w(\xi, \nu)$, and the

intuitively obvious relation

$$b < c$$

is trivially proved.

As a simple example, one stipulates $x(\nu) > 0$, and sets $w(\xi, \nu) = \xi^{s-1}m(\nu)$, $g(\xi) = \xi^{t-s}$ where $s < t$ are arbitrary real numbers, and $m(\nu)$ is any positive continuous "density function" with $\int_I m(\nu)d\nu = 1$. One then finds that the mean b of $x(\nu)$ *relative to* $\xi^{s-1}m(\nu)$ is identical with the classical "mean M_s of order s of $x(\nu)$ *with respect to* $m(\nu)$," namely

$$M_s = \left\{\left(\int_I x^s(\nu)m(\nu)d\nu\right)^{1/s}, \quad \text{if } s \neq 0; \exp \int_I (\log x(\nu))m(\nu)d\nu, \quad \text{if } s = 0\right\}$$

and hence that $M_s < M_t$ for all $s < t$.

These "qualitative" relations were discussed in an earlier note [3] emphasizing their physical aspect. Our purpose here is to obtain upper bounds for the difference $c - b$, and consequently for $M_t - M_s$. The bounds so obtained are considerably improved for $M_t - M_1$, with integral $t > s = 1$, and, in illustration, the difference $M_3 - M_1$ is estimated for $m(\nu) \equiv 1$,

$$x(\nu) = 1 - \tfrac{1}{2}\{E_2(z(1 - \nu)) + E_2(z\nu)\}, \qquad 0 < z \text{ fixed},$$

where $E_2(z)$ is the member $n = 2$ of the sequence of functions

$$E_n(z) = \int_1^\infty u^{-n}e^{-zu}du, \qquad n = 0, 1, 2, \cdots$$

of importance in transport theory [2, 5].

Indeed, the following work was an outgrowth of an attempt to explain a remarkable agreement between the means M_1, M_2, and M_3 for this $x(\nu)$, noted by Carson Mark to whom we are indebted for the observation.

Some unfamiliar relations, which may be new, result as by-products of our very elementary methods. One may note also that all our inequalities become equalities if $x(\nu)$ is a constant function, and only then. Moreover the results generalize in an obvious way to improper integrals and/or infinite intervals I, thus yielding information on such classical means as the Γ-function, and indeed the functions $E_n(z)$ themselves. Finally, the relations below all have their analogues for the case of a discrete set of numbers $x(\nu)$, as indicated in [3], and in occasional examples below.

1. The general means $a < b < c < d$. The functions $x(\nu)$, $w(\xi,\nu)$, $g(\xi)$ being stipulated as above, we define on $[x_l, x_u]$ the four continuous increasing functions $A(x)$, $B(x)$, $C(x)$, $D(x)$ of the form

$$\int_I \int_{x(\nu)}^x W(\xi, \nu)d\xi d\nu,$$

where $W(\xi, \nu) = g^{-1}(x(\nu))g(\xi)w(\xi, \nu)$, $g^{-1}(\xi)g(\xi)w(\xi, \nu) \equiv w(\xi, \nu)$, $g(\xi)w(\xi, \nu)$, and

$g(x(\nu))w(\xi, \nu)$, respectively. Their corresponding zeros a, b, c, and d are accordingly the means of $x(\nu)$ relative to each of these four weight functions, and the linear order of the four means is trivially established in

THEOREM 1. *$B(a)$, $C(b)$, and $D(c)$ are negative; $A(b)$, $B(c)$, and $C(d)$ are positive; and $x_l<a<b<c<d<x_u$.*

Proof. Since A, B, C, and D are continuous increasing functions with the unique zeros a, b, c, and d on (x_l, x_u), consideration of a schematic figure shows that it suffices to pı ove only the first statement. For, $B(a)<0$ implies $a<b$, hence $A(b)>0$, and so on. Moreover, the method of proof is similar for B, C, and D negative as stated, and we indicate it only for D. Let L and U denote the unions of the open subintervals of I on which $x(\nu)<c$, and $x(\nu)>c$, respectively. (This convention, with respect to the appropriate second upper limit, is adhered to throughout.) Remembering that $w(\xi, \nu)$ and $g(\xi)$ are positive, the latter increasing, we have, with due regard to the direction of integration,

$$\begin{aligned}
D(c) &= \int_I \int_{x(\nu)}^{c} g(x(\nu))w(\xi, \nu)d\xi d\nu \\
&= \int_L \int_{x(\nu)}^{c} + \int_U \int_{x(\nu)}^{c} \\
&< \int_L \int_{x(\nu)}^{c} g(\xi)w(\xi, \nu)d\xi d\nu + \int_U \int_{x(\nu)}^{c} g(\xi)w(\xi, \nu)d\xi d\nu \\
&= C(c) = 0.
\end{aligned}$$

In the same fashion, one shows that $B(a)<A(a)=0$ and $C(b)<g(b)B(b)=0$.

2. **The magnitude of $c-b$.** We next obtain a way of estimating the *size* of $c-b$ for the general means in

THEOREM 2. *At the points $x=b$ and $x=c$, one has*

$$0 < -C(b) < -D(b) < (g(x_u) - g(x_l)) \min\{-B(x_l), B(x_u)\} \equiv \beta$$

$$0 < B(c) < A(c) < (g^{-1}(x_l) - g^{-1}(x_u)) \min\{-C(x_l), C(x_u)\} \equiv \gamma.$$

Proof. The simple method is the same for both statements, of which we prove only the first. With L and U defined (relative to b) as before, we have

$$\begin{aligned}
C(b) &= \int_L \int_{x(\nu)}^{b} g(\xi)w(\xi, \nu)d\xi d\nu + \int_U \int_{x(\nu)}^{b} g(\xi)w(\xi, \nu)d\xi d\nu \\
&> \int_L \int_{x(\nu)}^{b} g(x(\nu))w(\xi, \nu)d\xi d\nu + \int_U \int_{x(\nu)}^{b} g(x(\nu))w(\xi, \nu)d\xi d\nu \equiv D(b) \\
&> g(x_l) \int_L \int_{x(\nu)}^{b} w(\xi, \nu)d\xi d\nu + g(x_u) \int_U \int_{x(\nu)}^{b} w(\xi, \nu)d\xi d\nu \equiv S.
\end{aligned}$$

If we now write $g(x_l) = g(x_u) - \Delta g$, where $\Delta g = g(x_u) - g(x_l) > 0$, the last sum S may be expressed as a sum of two terms, one of which is

$$g(x_u)\int_I \int_{x(\nu)}^{b} w(\xi, \nu)d\xi d\nu = g(x_u)B(b) = 0$$

while the other is

$$-\Delta g \int_L \int_{x(\nu)}^{b} w(\xi, \nu)d\xi d\nu > -\Delta g \int_L \int_{x(\nu)}^{x_u} w(\xi, \nu)d\xi d\nu$$

$$> -\Delta g \int_I \int_{x(\nu)}^{x_u} w(\xi, \nu)d\xi d\nu = -\Delta g B(x_u).$$

In the same way, writing $g(x_u) = g(x_l) + \Delta g$ shows that $S > \Delta g B(x_l)$, and the theorem follows.

Note that other bounds on S are also obvious, and may be preferable in certain cases. Thus, for example, we may equally well write

$$\int_L \int_{x(\nu)}^{b} w(\xi, \nu)d\xi d\nu < \int_L \int_{x_l}^{b} w(\xi, \nu)d\xi d\nu < (b - x_l)(\text{length } I)(\max w(\xi, \nu))$$

and so on.

COROLLARY 1. *The means b and c of $x(\nu)$ relative to $w(\xi, \nu)$ and to $g(\xi)w(\xi, \nu)$ satisfy the inequalities*

$$0 < c - b < (g(x_u) - g(x_l)) \min\{-B(x_l), B(x_u)\}/g(b)W_1 \equiv \beta/g(b)W_1,$$
$$0 < c - b < (g^{-1}(x_l) - g^{-1}(x_u)) \min\{-C(x_l), C(x_u)\}/W_2 \equiv \gamma/W_2,$$

where $W_i = \int_I w(\xi_i, \nu)d\nu$ for some constant ξ_i on (b, c). Hence, in either relation, W_i may be replaced by $\int_I w(b, \nu)d\nu$ or by $\int_I w(c, \nu)d\nu$ in case, for each ν, $w(\xi, \nu)$ is monotone nondecreasing or nonincreasing on $[b, c]$.

Proof. Of the two similar proofs, we give only the first. By the theorem of the mean, we may write $0 = C(c) = C(b) + (c-b)C'(\xi_1)$ for some ξ_1 on (b, c). Hence by Theorem 2,

$$\beta > -C(b) = (c - b)\int_I g(\xi_1)w(\xi_1, \nu)d\nu > (c - b)g(b)W_1.$$

3. The "separable" case and convexity. The simplest example of such means results in the "separable" case, with a weight function as stipulated of the form

$$w(\xi, \nu) = w(\xi)m(\nu),$$

where $m(\nu)$ is positive continuous with $\int_I m(\nu)d\nu = 1$. The case $w(\xi) = \xi^{s-1}$ is discussed in detail in Section 4. We note here the connection with convexity.

If we define the indefinite integrals

$$W(x) = \int_{x_l}^{x} w(\xi)d\xi, \qquad G(x) = \int_{x_l}^{x} g(\xi)w(\xi)d\xi,$$

on $[x_l, x_u]$, $g(\xi)$ having the properties specified in Section 1, it is apparent that $B(x) = W(x) - \int_I W(x(\nu))m(\nu)d\nu$, $C(x) = G(x) - \int_I G(x(\nu))m(\nu)d\nu$. For the unique zeros $b < c$ of these functions on $[x_l, x_u]$ we have

$$W(b) = \int_I W(x(\nu))m(\nu)d\nu, \qquad G(c) = \int_I G(x(\nu))m(\nu)d\nu,$$

and, since $G(x)$ is increasing, we infer that $G(b) < G(c)$.

In particular, for the case $w(\xi) = 1$, one finds that $b = \int_I x(\nu)m(\nu)d\nu$ and therefore $G(b) < G(c)$ reads

$$G\left(\int_I x(\nu)m(\nu)d\nu\right) < \int_I G(x(\nu))m(\nu)d\nu.$$

The discrete version is of course

$$G\left(\sum_1^m m_\nu x_\nu\right) < \sum_1^m m_\nu G(x_\nu)$$

with $m_\nu > 0$, $\sum_1^m m_\nu = 1$, which is the standard convexity property of

$$G(x) \equiv \int_{x_l}^{x} g(\xi)d\xi.$$

One may note that our convex function $G(x)$ is *increasing*, with a continuous increasing derivative $G'(x) = g(x) > 0$.

For further ramifications, see [1], Chapter 1; [4], Section 18.43; [6], Appendix III.

4. Application to the means M_s. As an example, we suppose $x(\nu) > 0$, and take $w(\xi, \nu) = \xi^{s-1}m(\nu)$, $g(\xi) = \xi^{t-s}$, $t > s$, as specified in Section 1. By a lengthy but straightforward computation which we omit, one may verify the following explicit forms of our four functions in this case:

$$\begin{aligned}
A(x) &= \left\{t^{-1}(M_{s-t}^{s-t}x^t - M_s^s), \quad \text{if } t \neq 0; M_s^s \log(x/M_{0s}), \quad \text{if } t = 0\right\}, \\
B(x) &= \left\{s^{-1}(x^s - M_s^s), \quad \text{if } s \neq 0; \log(x/M_0), \quad \text{if } s = 0\right\}, \\
C(x) &= \left\{t^{-1}(x^t - M_t^t), \quad \text{if } t \neq 0; \log(x/M_0), \quad \text{if } t = 0\right\}, \\
D(x) &= \left\{s^{-1}(M_{t-s}^{t-s}x^s - M_t^t), \quad \text{if } s \neq 0; M_t^t \log(x/M_{0t}), \quad \text{if } s = 0\right\},
\end{aligned}$$

where M_s is the mean of order s of $x(\nu)$ with respect to the density function $m(\nu)$, as defined in Section 1, while

$$M_{0s} \equiv \exp \int_I (\log x(\nu))\left\{(x^s(\nu)/M_s^s)m(\nu)\right\}d\nu, \qquad s \lesseqgtr 0$$

is its mean of order 0 with respect to the bracketed density function.

By Theorem 1, we have for the four corresponding zeros

$$\begin{aligned} 0 < x_l < a &= \left\{ (M_s^s / M_{s-t}^{s-t})^{1/t}, \quad \text{if } t \neq 0; M_{0s}, \quad \text{if } t = 0 \right\} \\ &< b = M_s < c = M_t \\ &< d = \left\{ (M_t^t / M_{t-s}^{t-s})^{1/s}, \quad \text{if } s \neq 0; M_{0t}, \quad \text{if } s = 0 \right\} < x_u. \end{aligned}$$

This signifies that, in the three basic cases indicated,

$$\begin{array}{lll} 1. & (s \neq 0 \neq t) & (M_s^s/M_{s-t}^{s-t})^{1/t} < M_s < M_t < (M_t^t/M_{t-s}^{t-s})^{1/s} \\ 2. & (s = 0 < t) & M_{-t} < M_0 < M_t < M_{0t} \\ 3. & (s < t = 0) & M_{0s} < M_s < M_0 < M_{-s}. \end{array}$$

Among these inequalities we find nothing new, beyond the standard relation $M_s < M_t$ $(s < t)$, save for the two involving M_{0t} and M_{0s}. The former reads explicitly

$$t^{-1} \int_I x^t(\nu) m(\nu) d\nu \cdot \log \int_I x^t(\nu) m(\nu) d\nu < \int_I x^t(\nu) (\log x(\nu)) m(\nu) d\nu, \qquad t > 0$$

while the latter results from this upon replacing $x(\nu)$ by $1/x(\nu)$. A simple instance, in its discrete version, is

$$\sum_1^m (\log \nu)/\nu < \sum_1^m (1/\nu) \left\{ \log m - \log \sum_1^m (1/\nu) \right\}.$$

Turning to Theorem 2, we find that now

$$\begin{aligned} -C(b) &= \left\{ t^{-1} (M_t^t - M_s^t), \quad \text{if } t \neq 0; \log (M_0/M_s), \quad \text{if } t = 0 \right\}, \\ -D(b) &= \left\{ s^{-1} (M_t^t - M_{t-s}^{t-s} M_s^s), \quad \text{if } s \neq 0; M_t^t \log (M_{0t}/M_0), \quad \text{if } s = 0 \right\}, \end{aligned}$$

and

$$B(x) = \left\{ s^{-1} (x^s - M_s^s), \quad \text{if } s \neq 0; \log (x/M_0), \quad \text{if } s = 0 \right\},$$

so that the statement

$$0 < -C(b) < -D(b) < \beta$$

reads, in three cases referred to:

$$\begin{array}{ll} 1. & t^{-1}(M_t^t - M_s^t) < s^{-1}(M_t^t - M_{t-s}^{t-s} M_s^s) < (x_u^{t-s} - x_l^{t-s}) s^{-1} \min\{M_s^s - x_l^s, x_u^s - M_s^s\} = \beta_1, \\ 2. & t^{-1}(M_t^t - M_0^t) < M_t^t \log (M_{0t}/M_0) < (x_u^t - x_l^t) \min\{\log(M_0/x_l), \log(x_u/M_0)\} = \beta_2, \\ 3. & \log(M_0/M_s) < s^{-1} (1 - M_{-s}^{-s} M_s^s) < (x_u^{-s} - x_l^{-s}) s^{-1} \min\{M_s^s - x_l^s, x_u^s - M_s^s\} = \beta_3. \end{array}$$

Similarly, we read for $0 < B(c) < A(c) < \gamma$,

1. $s^{-1}(M_t^s - M_s^s) < t^{-1}(M_{s-t}^{s-t} M_t^t - M_s^s) < (x_l^{s-t} - x_u^{s-t}) t^{-1} \min\{M_t^t - x_l^t, x_u^t - M_t^t\} = \gamma_1,$
2. $\log(M_t/M_0) < t^{-1}(M_{-t}^{-t} M_t^t - 1) < (x_l^{-t} - x_u^{-t}) t^{-1} \min\{M_t^t - x_l^t, x_u^t - M_t^t\} = \gamma_2,$
3. $s^{-1}(M_0^s - M_s^s) < M_s^s \log(M_0/M_{0s}) < (x_l^s - x_u^s) \min\{\log(M_0/x_l), \log(x_u/M_0)\} = \gamma_3.$

We included the intermediate terms in Theorem 2 for their intrinsic interest here. The inequalities between the above extremes are seen to indicate, in one way or another, the magnitude of $M_t - M_s$ in terms of M_s and of M_t. As an illustration, the simplest (discrete) version of (2) states that

$$M_1 - M_0 < (x_u - x_l) \min\{\log(M_0/x_l), \log(x_u/M_0)\}$$

and $\log(M_1/M_0) < (x_l^{-1} - x_u^{-1}) \min\{M_1 - x_l, x_u - M_1\}$, where

$$M_0 = \left(\prod_1^m x(\nu) \right)^{1/m}, \qquad M_1 = \sum_1^m x(\nu)/m.$$

Using the β_j and γ_j above which are appropriate to the case, we find from Corollary 1 that

$$M_t - M_s < \beta_j / M_s^{t-s} W$$
$$M_t - M_s < \gamma_j / W,$$

where

$$W = \{M_s^{s-1}, \text{ if } s \geqq 1;\ M_t^{s-1}, \text{ if } s < 1\}.$$

5. The special case $M_n - M_1$. We consider here the difference $M_t - M_1$ as compared with M_1, for integral $t > 1 = s$. From the last section we see that

$$0 < -C(b) = t^{-1}(M_t^t - M_1^t) < (x_u^{t-1} - x_l^{t-1}) \min(M_1 - x_l, x_u - M_1) \equiv \beta_1$$

and $0 < M_t - M_1 < \beta_1 / M_1^{t-1}$.

In this instance however, the proof of Theorem 2 may be modified to yield a better bound $\beta_1' < \beta_1$. For we now have (with $b \equiv M_1$)

$$C(b) = \int_I \int_{x(\nu)}^b \xi^{t-1} m(\nu) d\nu = t^{-1} \int_I (b^t - x^t(\nu)) m(\nu) d\nu$$
$$= t^{-1} \int_I (b - x(\nu)) G(x(\nu)) m(\nu) d\nu,$$

where $G(x) = b^{t-1} + b^{t-2}x + \cdots + x^{t-1}$; $b = M_1$.

With conventions the same as in the proof of Theorem 2, we find

$$tC(b) = \int_L (b - x(\nu)) G(x(\nu)) m(\nu) d\nu + \int_U (b - x(\nu)) G(x(\nu)) m(\nu) d\nu$$

$$> G(x_l)\int_L (b - x(\nu))m(\nu)d\nu + G(x_u)\int_U (b - x(\nu))m(\nu)d\nu \equiv T.$$

Setting $G(x_l) = G(x_u) - \Delta G$ and $G(x_u) = G(x_l) + \Delta G$ in turn, we see first that

$$T > -\Delta G\int_L (b - x(\nu))m(\nu)d\nu > -\Delta G\int_L (x_u - x(\nu))m(\nu)d\nu$$

$$> -\Delta G\int_I (x_u - x(\nu))m(\nu)d\nu = -\Delta G(x_u - M_1).$$

Similarly, the second replacement shows that $T > \Delta G(x_l - M_1)$, and we therefore infer, for the case at hand,

THEOREM 3. *The means*

$$M_1 = \int_I x(\nu)m(\nu)d\nu \quad \textit{and} \quad M_t = \left(\int_I x^t(\nu)m(\nu)d\nu\right)^{1/t}$$

satisfy the inequalities

$$0 < -C(b) = t^{-1}(M_t^t - M_1^t) < t^{-1}\Delta G \min\{M_1 - x_l, x_u - M_1\} \equiv \beta_1'$$

and $0 < M_t = M_1 < \beta_1'/M_1^{t-1}$ *where*

$$\Delta G \equiv G(x_u) - G(x_l) = (x_u^{t-1} - x_l^{t-1}) + (x_u^{t-2} - x_l^{t-2})M_1 + \cdots + (x_u - x_l)M_1^{t-2}.$$

To see that $\beta_1' < \beta_1$ it suffices to note that

$$\Delta G < t(x_u^{t-1} - x_l^{t-1}),$$

i.e., $(x_u^{t-2} - x_l^{t-2})M_1 + \cdots + (x_u - x_l)M_1^{t-2} < (t-1)(x_u^{t-1} - x_l^{t-1})$, and this is clear, since the left side contains only $t-2$ terms, each of form

$$(x_u^{t-\nu} - x_l^{t-\nu})M_1^{\nu-1} < (x_u^{t-\nu} - x_l^{t-\nu})x_u^{\nu-1} < x_u^{t-1} - x_l^{t-1}.$$

In illustration of Theorem 3, we see that

$$M_3^3 - M_1^3 < (x_u - x_l)(x_u + x_l + M_1)\min\{M_1 - x_l, x_u - M_1\} \equiv \beta_0$$

and $M_3 - M_1 < \frac{1}{3}\beta_0/M_1^2$.

6. An example involving $E_n(z)$. As an application of the last remark, we consider the means M_1 and M_3 (with respect to $m(\nu) \equiv 1$) of the function

$$x(\nu) = 1 - f(\nu) \qquad \text{on } I = [0, 1], \quad \text{where}$$
$$f(\nu) = \tfrac{1}{2}\{E_2(z(1-\nu)) + E_2(z\nu)\}, \qquad 0 < z \text{ fixed},$$

as defined in Section 1. It follows easily from the known properties of the func-

tions $E_n(z)$ that the function $f(\nu)$, symmetric about $\nu = 1/2$, is convex since $f''(\nu) = (x^2/2)\{E_0(z(1-\nu)) + E_0(z\nu)\} > 0$ with

$$\text{minimum } m = f(1/2) = E_2(z/2) > 0$$
$$\text{maximum } M = f(0) = \tfrac{1}{2}(1 + E_2(z)) < 1$$
$$\text{and integral } F = \int_0^1 f(\nu)d\nu = (1/z)(\tfrac{1}{2} - E_3(z)).$$

Hence, for $x(\nu) = 1 - f(\nu)$, one has

$$x_l = 1 - M < M_1 = 1 - F < x_u = 1 - m.$$

Moreover, $x_u - x_l = M - m$, and the convexity of $f(\nu)$ insures $M_1 - x_l = M - F > F - m = x_u - M_1$.

Much less than convexity is required. If $h(x)$ is any continuous function on (say) $[0, 1]$ such that $h(x) < h(0) + (h(1) - h(0))x$ for all such x, then

$$H \equiv \int_0^1 h(x)dx < h(0) + \tfrac{1}{2}(h(1) - h(0))$$
$$= \tfrac{1}{2}(h(0) + h(1)), \qquad \text{i.e.,} \qquad H - h(0) < h(1) - H.$$

Hence, in these terms, we have for

$$M_1 = \int_0^1 (1 - f(\nu))d\nu \quad \text{and} \quad M_3 = \left(\int_0^1 (1 - f(\nu))^3 d\nu\right)^{1/3}$$

that

$$(M_3^3/M_1^3) - 1 < \frac{M - m}{1 - F} \cdot \frac{F - m}{1 - F} \cdot \left\{1 + \frac{1 - m}{1 - F} + \frac{1 - M}{1 - F}\right\} \equiv \beta^*$$

and $(M_3/M_1) - 1 < \frac{1}{3}\beta^*$.

For example, one finds from the tables [2] for $E_n(z)$ at $z = 1$, $E_2(1/2) = .327$, $E_2(1) = .148$, $E_3(1) = .110$, so that $m = .327$, $F = .390$, $M = .574$, $1 - M = .426$, $1 - F = .610$, $1 - m = .673$, $M - m = .247$, $F - m = .063$, and hence

$$\beta^* = .117.$$

7. The functions $\Gamma(z)$ and $E_n(z)$. Finally, we indicate without proof how these simple methods may yield interesting properties for a wider class of functions, the means M_s involved still being relative to the separable weight function $\xi^{s-1}m(\nu)$ of Section 4.

a. Let $x(\nu) = \nu$ on $I = [0, \infty)$, $w(\xi, \nu) = \xi^{s-1}m(\nu)$, $g(\xi) = \xi^{t-s}$, where $t > s > -1$, and $m(\nu) = e^{-\nu}$. Then the zero b of $B(x)$ is the mean of order s of $x(\nu)$ with respect to $e^{-\nu}$, namely

$$b = M_s = \left\{\Gamma^{1/s}(s + 1), \text{ if } s \neq 0; \exp \int_0^\infty (\log \nu)e^{-\nu}d\nu, \text{ if } s = 0\right\},$$

where $\Gamma(s+1)=\int_0^\infty \nu^s e^{-\nu}d\nu$ is the Γ-function [1] at $s+1>0$. Moreover, since $c=M_t$ is the zero of $C(x)$, we have for $s\neq 0\neq t$ (Case 1)

$$\Gamma^{1/s}(s+1) < \Gamma^{1/t}(t+1)$$

and, in particular, for integers $t>s\geqq 1$,

$$(s!)^t < (t!)^s.$$

b. The functions $E_n(z)$ defined in Section 1 may be introduced in a similar way. We take $x(\nu)=\nu^{-1}$ on $I=[1,\infty)$, $w(\xi,\nu)=\xi^{m-1}m(\nu)$, $g(\xi)=\xi^{n-m}$ for integers $n>m\geqq 0$, and $m(\nu)=e^{-z\nu}/E_0(z)$, with fixed $z>0$. We then have

$$b = M_m = \left\{(E_m(z)/E_0(z))^{1/m}, \text{ if } m\neq 0; \exp\int_1^\infty (\log(1/\nu))(e^{-z\nu}/E_0(z))d\nu, \text{ if } m=0\right\}$$

and $b<c=M_n$ asserts that $(E_m(z)/E_0(z))^{1/m}<(E_n(z)/E_0(z))^{1/n}$.

It is clear therefore how these and other such functions of classical analysis may be studied in the setting of Section 4.

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REMARKS CONCERNING THE RICCATI EQUATION

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Introduction. We consider the Riccati equation

$$\phi'(y)+\phi^2(y)+2p(y)\phi(y)+q(y)=0, \qquad \phi(a)=b. \tag{1}$$

Since it is not always possible to find a closed solution of (1), different methods have been developed for giving the solution as an infinite series and studying properties of the solution. Bellman [1] gives the solution as the minimum of a function $V(u, y)$ with respect to u, where $V(u, y)$ is the solution of a linear differential equation. Kalaba studies the problem further in [2], giving the solution

in a series known as "approximation in policy space" and proving its quadratic convergence.

Here we shall prove Bellman's results and shall give a solution of (1) different from the one given by Kalaba. Our main result is contained in Lemma 2, where the solution is given as a uniformly convergent series and the form of the convergence is discussed. Lemma 4 is a standard Sturm comparison theorem. In proving it we follow Bellman's minimization scheme, but the result can be proved by other methods.

In what follows let G be a class of functions such that if $p(y)$ and $q(y)$ are functions belonging to G and defined in an interval containing a, then in that interval a unique solution of (1) exists. If, for example, G is the class of functions continuous in the interval $[a-c, a+c]$, then, according to the Peano existence theorem, for sufficiently small c there exists a solution of (1) defined in $[a-c, a+c]$ and passing through (a, b).

The main part.

LEMMA 1. *The solution of* (1), *with* $p(y)$ *and* $q(y)$ *in* G, *is given by* $\phi(y)=\max_u V(u, y)$ for $y\leqq a$, $\phi(y)=\min_u V(u, y)$ for $y\geqq a$, where

$$V(u, y) = b \exp\left\{-2\int_a^y (u(s) + p(s))ds\right\} \tag{2}$$
$$+ \int_a^y (u^2(s) - q(s)) \exp\left\{-2\int_s^y (u(t) + p(t))dt\right\} ds,$$

and the minimizing or maximizing function $u(y)$ *is the solution* $\phi(y)$.

Proof. We begin with the observation that

$$-\phi^2 = \min_u(u^2 - 2u\phi). \tag{3}$$

This permits (1) to be written in the form

$$\phi'(y) = \min_u(u^2 - 2u\phi - 2p(y)\phi - q(y)), \qquad \phi(a) = b. \tag{4}$$

Consider now a fixed function $u(y)$ and the equation

$$V'(y) = u^2 - 2uV - 2p(y)V - q(y), \qquad V(a) = b. \tag{5}$$

This has the solution (2). From (4) and (5) we obtain $(\phi-V)'\leqq -2(u(y)+p(y))(\phi-V)$, or $((\phi-V)\exp\{-2\int_a^y(u(t)+p(t))dt\})'\leqq 0$. This implies that the function $(\phi(y)-V(u, y))\exp\{-2\int_a^y(u(t)+p(t))dt\}$ is a decreasing function of y, and since $\phi(a)=V(u, a)$, we have $\phi(y)\leqq V(u, y)$ for $y\geqq a$ and $\phi(y)\geqq V(u, y)$ for $y\leqq a$. From (3) we see that the minimizing function $u(y)$ is $\phi(y)$, so that

$$\phi(y) = \min_u V(u, y) \quad \text{for } y \geqq a, \qquad \phi(y) = \max_u V(u, y) \quad \text{for } y \leqq a. \tag{6}$$

Kalaba's paper contains a proof of this lemma on similar lines, but we have included a proof here so that this paper will be self-contained.

LEMMA 2. *Consider the equation* (1), *with* $p(y)$ *and* $q(y)$ *in* G *and* $b=0$. *Define*

$$\phi_1(y) = -\int_a^y q(x) \exp\left\{-2\int_x^y p(s)ds\right\} dx,$$

and

$$\phi_{n+1}(y) = -\int_a^y \phi_n^2(x) \exp\left\{-2\int_x^y (\phi_1(s) + \cdots + \phi_n(s) + p(s))ds\right\} dx.$$

Then the series $\sum_{i=1}^{\infty} \phi_i(y)$ *converges uniformly to the solution* $\phi(y)$ *in any compact subinterval of an interval containing* a *on which the solution* $\phi(y)$ *exists.*

Proof. Without loss of generality we may assume that $q(y) \geqq 0$ for all y, for otherwise we may put $h(y)=\phi(y)-\phi_1(y)$, and will have $h'(y)+h^2+2(\phi_1(y)+p(y))h+\phi_1^2(y)=0$, $h(a)=0$, where $\phi_1^2(y) \geqq 0$ for all y.

We will examine the case $y \geqq a$, and the case $y \leqq a$ can be examined similarly. According to (6), we have

$$\phi(y) = \min_u\left(\int_a^y (u^2(x) - q(x)) \exp\left\{-2\int_x^y (u(s) + p(s))ds\right\} dx\right).$$

Taking $u(y)=0$ for all y, we obtain

$$\phi(y) \leqq -\int_a^y q(x) \exp\left\{-2\int_x^y p(s)ds\right\} dx = \phi_1(y) \leqq 0.$$

Define $h_1(y)$ by $\phi(y)=\phi_1(y)+h_1(y)$. Then

$$h_1'(y) + h_1^2 + 2(\phi_1(y) + p(y))h_1 + \phi_1^2(y) = 0, \qquad h_1(a) = 0.$$

Generally we set

$$h_{n-1}(y) = h_n(y) + \phi_n(y), \tag{7}$$

where

$$\phi_n(y) = -\int_a^y \phi_{n-1}^2(x) \exp\left\{-2\int_x^y (\phi_1(s) + \cdots + \phi_{n-1}(s) + p(s))ds\right\} dx \leqq 0. \tag{8}$$

Then $h_n(y)$ satisfies the equation

$$h_n'(y) + h_n^2 + 2(\phi_1(y) + \cdots + \phi_n(y) + p(y))h_n(y) + \phi_n^2(y) = 0, \quad h_n(a) = 0 \tag{9}$$

Hence we have

$$\phi(y) = \phi_1(y) + \cdots + \phi_n(y) + h_n(y) \text{ and} \tag{10}$$

$$h_n(y) = \min_u\left(\int_a^y (u^2(x) - \phi_n^2(x)) \right.$$
$$\left. \cdot \exp\left\{-2\int_x^y (\phi_1(s) + \cdots + \phi_n(s) + u(s) + p(s))ds\right\} dx\right),$$

where the minimizing function $u(x)$ is $h_n(x)$. From (7), (8) and (10) we obtain that $\phi(y) \leqq h_1(y) \leqq \cdots \leqq h_n(y) \leqq \cdots \leqq 0$. Now, for a given y the sequence $h_n(y)$, $n=1, 2, \cdots$, is increasing and bounded above. Thus it converges to some limit, and we can conclude from (7) that for the given y we have $\lim_{n\to\infty} \phi_n(y) = \lim_{n\to\infty} (h_{n-1}(y) - h_n(y)) = 0$. Also, since $f_n(y) = \phi_n(y)\exp\{2\int_a^y p(s)ds\} \leqq 0$, we have

$$f_n'(y) = -2(\phi_1(y) + \cdots + \phi_n(y))f_n(y) - \phi_{n-1}^2(y) \exp\left\{2\int_a^y p(s)ds\right\} \leqq 0.$$

Thus the function $f_n(y)$ is a decreasing function of y for all n, $n=1, 2, \cdots$, and $f_n(a) = 0$. The convergence of $\phi_n(y)$ to zero as n goes to infinity implies that $\lim_{n\to\infty} f_n(y) = 0$, and, since $f_n(y)$ is a decreasing function of y, the convergence of $f_n(y)$ is uniform in any compact subinterval of an interval containing a on which the solution $\phi(y)$ of (1) exists. The uniform convergence of $f_n(y)$ in that subinterval implies in turn the uniform convergence of $\phi_n(y) = f_n(y)\exp\{-2\int_a^y p(s)ds\}$ in the same subinterval. Thus

$$\lim_{n\to\infty} \phi_n(y) = 0 \tag{11}$$

uniformly in that subinterval. We can now prove that $\lim_{n\to\infty} h_n(y) = 0$ uniformly in that subinterval. To prove this, we see from (9) that $g_n(y) = h_n(y)\exp\{2\int_a^y p(s)ds\}$ is a decreasing function of y and $g_n(a) = 0$. Also from (9) and (10) we obtain

$$h_n'(y) - h_n^2 + 2(\phi(y) + p(y))h_n + \phi_n^2(y) = 0, \qquad h_n(a) = 0. \tag{12}$$

Now if we study (12) as we did (1) in Lemma 1, we obtain a similar relation for $h_n(y)$. Indeed, for $y \geqq a$ we have

$$0 \geqq h_n(y) = \max_u\left(-\int_a^y (u^2(x) + \phi_n^2(x)) \right.$$
$$\left. \exp\left\{-2\int_x^y (u(s) - p(s) - \phi(s))ds\right\} dx\right).$$

Thus if $u(y) = 0$ for all y we obtain

$$0 \geqq g_n(y) = h_n(y) \exp\left\{2\int_a^y p(s)ds\right\}$$
$$\geqq -\exp\left\{2\int_a^y (2p(s) + \phi(s))ds\right\} \int_a^y \phi_n^2(x) \exp\left\{-2\int_a^x (p(s) + \phi(s))ds\right\} dx.$$

The functions $g_n(y)$ converge uniformly to zero as n goes to infinity; this follows from (11). Thus we have proved that $\lim_{n\to\infty} h_n(y)=0$ uniformly in any compact subinterval of an interval containing a on which the solution $\phi(y)$ of (1) exists. Relation (10) and the above result imply that $\phi(y)=\sum_{i=1}^{\infty}\phi_i(y)$, the convergence being uniform.

REMARKS. (I) We have tacitly assumed that all of the integrals considered here exist and are bounded. (II) By transforming a second order linear homogeneous differential equation into a Riccati equation, we may apply the above results to this case.

LEMMA 3. *Consider the equation*

$$\phi'(y)+\phi^2+2p(y)\phi+q(y)=0, \qquad \phi(0)=0 \tag{13}$$

with $p(y)$ *and* $q(y)$ *in G. If* (i) *for all* y, $q(y)\geqq 0$ *and* $yp(y)\leqq 0$, *then the function* $\phi(y)$ *is decreasing. Moreover, if* (ii) *for all* y, $yq'(y)\geqq 0$ *and* $p'(y)\leqq 0$, *then* $\phi(y)$ *is convex for* $y\leqq 0$, *concave for* $y\geqq 0$, *and* $y=0$ *is a point of inflection.*

Proof. If (i) holds, then according to Lemma 2, $\phi(y)\leqq 0$ for $y\geqq 0$. Now, using the condition $p(y)\leqq 0$ for $y\geqq 0$, we have from (13), $\phi'(y)\leqq 0$ for $y\geqq 0$. Thus $\phi(y)$ is a decreasing function.

If (ii) holds as well, differentiating (13) we obtain

$$\phi''(y)+2\phi\phi'+2p(y)\phi'+2p'(y)\phi+q'(y)=0.$$

Thus $\phi''(y)\leqq 0$ for $y\geqq 0$. This means that $\phi(y)$ is concave for $y\geqq 0$. In a similar way we can prove that $\phi(y)$ is convex for $y\leqq 0$.

LEMMA 4. *Consider the two equations*

$$\phi'(y)+\phi^2+2p(y)\phi+q(y)=0, \qquad \phi(0)=0,$$
$$f'(y)+f^2+2p(y)f+g(y)=0, \qquad f(0)=0,$$

with $g(y)$, $p(y)$, *and* $q(y)$ *in G. If* $q(y)\leqq g(y)$ *for all* y, *then* $\phi(y)\geqq f(y)$ *for* $y\geqq 0$ *and* $\phi(y)\leqq f(y)$ *for* $y\leqq 0$.

Proof. For $y\geqq 0$ we have

$$\begin{aligned}\phi(y)&=\min_u\left(\int_0^y (u^2(x)-q(x))\exp\left\{-2\int_x^y (u(s)+p(s))ds\right\}dx\right)\\ &\geqq \min_u\left(\int_0^y (u^2(x)-g(x))\exp\left\{-2\int_x^y (u(s)+p(s))ds\right\}dx\right)=f(y).\end{aligned}$$

For $y\leqq 0$ we have

$$\begin{aligned}\phi(y)&=\max_u\left(\int_0^y (u^2(x)-q(x))\exp\left\{-2\int_x^y (u(s)+p(s))ds\right\}dx\right)\\ &\leqq \max_u\left(\int_0^y (u^2(x)-g(x))\exp\left\{-2\int_x^y (u(s)+p(s))ds\right\}dx\right)=f(y).\end{aligned}$$

We are now in a position to state our theorem which deals with the general case.

THEOREM. *Consider the equation* (1) *with $p(y)$ and $q(y)$ in G. If*

$$\phi_1(y) = b \exp\left\{-2\int_a^y p(s)ds\right\} - \int_a^y q(x) \exp\left\{-2\int_x^y p(s)ds\right\} dx$$

and

$$\phi_{n+1}(y) = -\int_a^y \phi_n^2(x) \exp\left\{-2\int_x^y (\phi_1(s) + \cdots + \phi_n(s) + p(s))ds\right\} dx,$$

then the solution $\phi(y)$ is given by $\phi(y) = \sum_{i=1}^{\infty} \phi_i(y)$, and the convergence is uniform in any compact subinterval of an interval containing a on which the solution $\phi(y)$ exists.

Proof. It suffices to examine the case $y \geqq a$. In this case we have from (6)

$$\phi(y) = \min_u\left(b \exp\left\{-2\int_a^y (u(s) + p(s))ds\right\}\right.$$
$$\left. + \int_a^y (u^2(s) - q(s)) \exp\left\{-2\int_s^y (p(t) + u(t))dt\right\} ds\right),$$

where the minimizing function $u(y)$ is the solution $\phi(y)$. Set $u(y) = 0$ for all y. Then

$$\phi(y) \leqq \phi_1(y) = b \exp\left\{-2\int_a^y p(s)ds\right\} - \int_a^y q(s) \exp\left\{-2\int_s^y p(t)dt\right\} ds.$$

Now set $\phi(y) - \phi_1(y) = h_1(y)$. Then

$$h_1'(y) + h_1^2 + 2(\phi_1(y) + p(y))h_1 + \phi_1^2(y) = 0, \qquad h_1(a) = 0. \tag{14}$$

According to Lemma 2 the solution of (14) is given by a series which converges uniformly in y in any compact subinterval of an interval where the solution of (14) exists.

Example. Consider the equation $\phi'(y) + \phi^2 - 2By\phi + A = 0$, $\phi(0) = b$, where A and B are positive constants. Although this equation is very simple, it is a difficult task to find the solution in a closed form. Observing, however, that the conditions of Lemma 2 are satisfied, we can easily get an idea of the form of the solution. A first approximation to the solution, according to the previous theorem, is given by $\phi(y) \approx (b - A\int_0^y \exp(-Bx^2)dx)\exp(By^2)$.

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MATHEMATICAL NOTES

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TRANSCENDENTALS AND GENERATORS IN COMMUTATIVE POLYNOMIAL RINGS

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1. Introduction. Let R be a commutative ring with unit and S a *polynomial ring over R with generator x.* That is, S is a commutative unitary overring of R, $x\in S$ is transcendental over R, and every element of S can be represented (uniquely) as a polynomial in x with coefficients in R. The purpose of this article is to characterize in terms of their coefficients all the elements of S that are transcendental over R and all those transcendentals which generate S. These characterizations are given below by Corollary 2 and Theorem 3, respectively. Theorem 4, a miscellaneous result, shows that the set of transcendental elements of S is closed under multiplication by regular elements of S.

The terminology and notation used here are standard and can be found in [6, Chap. I]. But one notation should perhaps be clarified. For each y in S there is a well-defined substitution map M_y from S into S given by

$$M_y(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1y + \cdots + a_ny^n.$$

In the proofs of Theorems 1 and 3 we use such maps often without saying so explicitly. But we always indicate when one or more substitutions are forthcoming by using functional notation like $f(X)$ for an element of S and write simply $f(y)$ in place of $M_y(f(X))$.

2. Transcendentals. It is well known that zero divisors of S must be annihilated by a nonzero element of R; see [1], [2], [3], or [4]. This leads easily to a characterization of algebraic elements of S and hence to the desired description of transcendentals.

THEOREM 1. *For $y=a_0+a_1x+\cdots+a_nx^n$ in S, the following are equivalent:*
(a) *y is algebraic,*
(b) *$y-a_0$ is a zero divisor,*
(c) *$c(y-a_0)=0$ for some nonzero c in R,*
(d) *$f(y)=0$ for some $f(X)$ in S of degree* 1.

Proof. (a) implies (b). If y is algebraic, then there is $h(X)$ in S such that $h(y)=0$. The constant term in $h(y)$ is just $h(a_0)$ which must be 0. Hence $X-a_0$ divides $h(X)$. Let k be the largest integer such that $(X-a_0)^k$ divides $h(X)$. Then $h(X)=(X-a_0)^k g(X)$ for some $g(X)$ in S. So $0=h(y)=(y-a_0)^k g(y)$. Furthermore, $g(y)\neq 0$; for $g(y)=0 \Rightarrow g(a_0)=0 \Rightarrow X-a_0$ divides $g(X) \Rightarrow (X-a_0)^{k+1}$ divides $h(X)$, contradicting the choice of k. Now let m be the largest integer such that $(y-a_0)^m g(y)\neq 0$. Then $(y-a_0)(y-a_0)^m g(y)=0$ so that $y-a_0$ is a zero divisor.

The first statement in this section proves (b) implies (c). Take $f(X)=cX-ca_0$ to prove (c) implies (d). By definition of algebraic, (d) implies (a).

COROLLARY 2. *For $y=a_0+a_1x+\cdots+a_nx^n$ in S, the following are equivalent:* (a) *y is transcendental,* (b) *$y-a_0$ is regular,* (c) *0 is the only element c in R such that $ca_i=0$ for $i=1, 2, \cdots, n$.*

3. Generators. Theorem 3 stated below characterizes generators. We will prove the necessity first and then prove a lemma that will lead to an inductive proof of the sufficiency.

THEOREM 3. *An element $y=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ in S generates S if and only if a_1 is a unit of R and $a_2, a_3, \cdots, a_n$ are nilpotent.*

Proof of necessity. It is well known [5, Th. 8.1, p. 683] that the units of S are precisely those elements $c_0+c_1x+\cdots+c_mx^m$ for which c_0 is a unit of R and $c_1, c_2, \cdots, c_m$ are nilpotent. So to prove the necessity of Theorem 3 we need only show that $z=a_1+a_2x+\cdots+a_nx^{n-1}$ is a unit of S. If y generates S, then so does $y-a_0=xz$. Hence there are elements $b_0, b_1, \cdots, b_m$ in R such that $x=b_0+b_1xz+b_2x^2z^2+\cdots+b_mx^mz^m$, and b_0 must be 0. Let $w=b_1+b_2xz+\cdots+b_mx^{m-1}z^{m-1}$. Then $x=xzw$, and since x is regular, we must have $zw=1$. So z is a unit in S.

To establish the sufficiency of Theorem 3, we must show that if y has the indicated coefficients, then y is transcendental over R and every element of S can be expressed as a polynomial in y. But if a_1 is a unit in R, then y is certainly transcendental according to Corollary 2. To show that elements of S can be represented as polynomials in y, it is sufficient to show that x has such a representation. The proof of this will be by induction on n, and the following lemma is the key to the second induction step.

LEMMA. *Suppose $a\in R$ is nilpotent, $z\in S$, and $f(X)\in S$. Put $w=z+azf(z)$. Then there is $g(X)$ in S such that $z=wg(w)$.*

Proof. We can assume $af(z)\neq 0$. Let k be the largest integer such that $a^kf(z)\neq 0$. Define polynomials $g_i(X)$ for $i=0, 1, 2, \cdots$ as follows: $g_0(X)=1$ and for $i\geqq 1$, $g_i(X)=1-ag_{i-1}(X)f(Xg_{i-1}(X))$. We shall prove that $a^{k-i}z=a^{k-i}wg_i(w)$ for each $i=0, 1, \cdots, k$. Then for $i=k$, we take $g(X)=g_k(X)$ and have $z=wg(w)$ as desired.

For $i=0$, $a^kwg_0(w)=a^kw=a^kz+a^{k+1}zf(z)=a^kz$. Now suppose the result is true for $i<k$ and consider $i+1$. Then

$$a^{k-(i+1)}w = a^{k-(i+1)}z + a^{k-i}zf(z)$$

so that

$$a^{k-(i+1)}z = a^{k-(i+1)}w - a^{k-i}zf(z).$$

But by the induction assumption $a^{k-i}z = a^{k-i}wg_i(w)$. This implies that $a^{k-i}z^m = a^{k-i}w^m g_i^m(w)$ for every positive integer m. Hence, $a^{k-i}zf(z) = a^{k-i}wg_i(w)f(wg_i(w))$. So

$$\begin{aligned} a^{k-(i+1)}z &= a^{k-(i+1)}w - a^{k-i}wg_i(w)f(wg_i(w)) \\ &= a^{k-(i+1)}w(1 - ag_i(w)f(wg_i(w))) = a^{k-(i+1)}wg_{i+1}(w). \end{aligned}$$

This completes the proof.

Proof of sufficiency. Certainly y generates S if and only if $a_1^{-1}(y-a_0)$ does. So we assume $a_0=0$, $a_1=1$, and in this case we show by induction on n that there is $g(X)$ in S such that $x=yg(y)$. For $n=1$, take $g(X)=1$. Now suppose the result is true for $n-1$ and let $z=x+a_2x^2+\cdots+a_{n-1}x^{n-1}$. Then by the induction assumption there is $f(X)$ in S such that $x=zf(z)$. Thus $y=z+a_nz(z^{n-1}f^n(z))$. By the Lemma there is $h(X)$ in S such that $z=yh(y)$. Take $g(X)=h(X)f(Xh(X))$. Then $x=zf(z)=yh(y)f(yh(y))=yg(y)$. The proof is complete.

4. Miscellaneous. The following theorem shows that the product of a transcendental element of S and a regular element of S is transcendental.

THEOREM 4. *Suppose $y\in S$ is transcendental and $z\in S$. Then yz is algebraic if and only if z is a zero divisor.*

Proof. Let $y=a_0+a_1x+\cdots+a_nx^n$, $z=b_0+b_1x+\cdots+b_mx^m$, and assume yz is algebraic. Then $d(yz-a_0b_0)=0$ for some nonzero d in R. Let $m(1)$ be the largest among $0, 1, \cdots, m$ such that $db_{m(1)}\neq 0$. (If no such integer exists, then $dz=0$ and the proof is over.) Since $d(yz-a_0b_0)=0$, $db_{m(1)}a_n=0$. Let $s(1)$ be the largest among $1, 2, \cdots, n$ such that $db_{m(1)}a_{s(1)}\neq 0$. (Such an integer exists because y is transcendental.) Then $1\leqq s(1)<n$.

Repeat with $db_{m(1)}$. We have $0=db_{m(1)}(yz-a_0b_0)$. Let $m(2)$ be the largest among $0, 1, \cdots, m(1)$ such that $db_{m(1)}b_{m(2)}\neq 0$. (If no such integer exists, then $db_{m(1)}z=0$ and the proof is over.) Since $db_{m(1)}(yz-a_0b_0)=0$, $db_{m(1)}b_{m(2)}a_{s(1)}=0$ also. Let $s(2)$ be the largest among $1, 2, \cdots, s(1)$ such that $db_{m(1)}b_{m(2)}a_{s(2)}\neq 0$. If no such integer exists, then $db_{m(1)}b_{m(2)}a_i=0$ for $i=1, 2, \cdots, s(1)$ and since $db_{m(1)}a_i=0$ for $i=s(1)+1, \cdots, n$, we have $db_{m(1)}b_{m(2)}(y-a_0)=0$ so that y is algebraic, a contradiction. So $s(2)$ exists and furthermore $1\leqq s(2)<s(1)$.

This process of choosing integers $s(i)$ can not continue. So the selection of the integers $m(i)$ must also stop, say with $m(k)$. Let $p=b_{m(1)}b_{m(2)}\cdots b_{m(k)}$. Then $dp\neq 0$ but $dpb_i=0$ for $i=0, 1, \cdots, m$. Thus $dpz=0$ and z is a zero divisor.

The converse is obvious.

Added in Proof: R. Gilmer announced in Notices, Amer. Math. Soc., 13 (1966)133 results equivalent to our Corollary 2 and Theorem 3. Gilmer's results

were published in an article entitled R-Automorphisms of $R[X]$ in Proc. London Math. Soc., (3) 58(1968)328–336.

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AN IDENTITY BETWEEN PERMANENTS AND DETERMINANTS

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It appears that in general the permanent is not related in a simple manner to the determinant [2]. For special types of matrices, relationships, usually involving inequalities, between the permanent and the determinant have been found [1], [3]. We shall give a simple proof of an identity between the permanent and the determinant for semitriangular matrices.

Let $A=(a_{ij})$ be an n-square matrix. The $(n-1)$-square submatrix of A that remains, after row i and column j are removed, is denoted by A_{ij}. If $a_{ij}=0$ whenever $j>i+1$ then A is a *semitriangular matrix*. If $a_{ij}=0$ whenever $|i-j|>1$ then A is a *tridiagonal matrix*. The n-square matrix $B=(b_{ij})$ is *related* to A if

$$b_{ij} = \begin{cases} a_{ij} \text{ whenever } i \geqq j, \\ -a_{ij} \text{ whenever } i < j. \end{cases}$$

THEOREM. *If A, B are n-square semitriangular matrices with B related to A, then* per $A=\det B$.

Proof. Clearly the theorem is true for $n=2$. Suppose that A, B are m-square semitriangular matrices with B related to A and that the theorem is true for $n=m-1$. Expansion by the first rows of A and B yields

$$\begin{aligned} \text{per } A &= \sum_{j=1}^{m} a_{1j} \text{ per } A_{1j}, \\ \det B &= \sum_{j=1}^{m} (-1)^{j+1} b_{1j} \det B_{1j}. \end{aligned} \tag{1}$$

Since A, B are semitriangular and B is related to A, we have

$$\begin{aligned} a_{1j} &= (-1)^{j+1}\, b_{1j} && \text{for } j=1,2, \\ a_{1j} &= b_{1j} = 0 && \text{for } j=3,4,\cdots,m. \end{aligned} \tag{2}$$

Furthermore A_{1j}, B_{1j} are $(m-1)$-square semitriangular matrices with B_{1j} re-

lated to A_{1j} for $j=1, 2$. Hence by the inductive assumption

$$\text{per } A_{1j} = \det B_{1j} \qquad \text{for } j = 1, 2. \tag{3}$$

From (1), (2), (3) we get per $A = \det B$.

COROLLARY. *If A, B are tridiagonal matrices with B related to A, then* per $A = \det B$.

Since a tridiagonal matrix is a semitriangular matrix, this corollary is clearly true. This corollary and the extensive theory of determinants of tridiagonal matrices could be used to develop a theory of permanents of tridiagonal matrices.

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ON A CHARACTERISTIC PROPERTY OF A POLYNOMIAL AS A CONTINUOUS MAPPING FROM THE z-PLANE INTO THE w-PLANE

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1. Introduction. In [1] the following mean-value property of polynomials is presented:

Let n be a natural number different from 1. If $f(z)$ is a polynomial of degree at most $n-1$ in a complex variable z, then we have

$$\sum_{k=0}^{n-1} f(x + \omega^k y) = nf(x), \tag{1}$$

where x and y are complex variables and $\omega = \exp(2\pi i/n)$ (and thus $\omega^n = 1$).

Differentiating both sides of (1) with respect to y, we obtain

$$\sum_{k=0}^{n-1} \omega^k f'(x + \omega^k y) = 0, \tag{2}$$

where $f'(z)$ is a polynomial of degree at most $n-2$. Hence we have the following theorem.

THEOREM 1. *Let n be a natural number. If $f(z)$ is a polynomial of degree at most n in a complex variable z, then we have*

$$\sum_{k=0}^{n+1} \omega^k f(x + \omega^k y) = 0, \tag{3}$$

where x and y are complex variables and $\omega = \exp(2\pi i/(n+2))$ (and thus $\omega^{n+2} = 1$).

In this paper we shall prove the following theorem, which is a converse of Theorem 1.

THEOREM 2. *Let n be a natural number and let $f(z)$ be a complex-valued function of a complex variable z. If $f(z)$ is continuous for $|z| < +\infty$ and satisfies* (3), *then $f(z)$ is a polynomial of degree at most n.*

To this end we shall use the following lemma, which is due to T. Carleman and is proved in [2].

LEMMA. *Let n be a natural number and let P be an arbitrary regular polygon of $n+2$ sides. If $f(z)$ is a complex-valued function of a complex variable z, is continuous for $|z| < +\infty$, and satisfies $\int_P f(z)dz = 0$, then $f(z)$ is an entire function of z.*

2. Proof of Theorem 2. Let P be an arbitrary regular polygon of $n+2$ sides and let the vertices of P be $P_0, P_1, P_2, \cdots, P_k, \cdots, P_{n+1}$. Then we have

$$\int_P f(z)dz = \sum_{k=0}^{n+1} \int_{P_kP_{k+1}} f(z)dz, \tag{4}$$

where P_{n+2} denotes P_0. We represent the center of P and the vertex P_k of P by $x, x+\omega^k y$, respectively, where $k = 0, 1, 2, \cdots, n+1$ and put $\omega = \exp(2\pi i/(n+2))$. A parametric equation of the side P_kP_{k+1} is $z = x+\omega^k y + t\omega^k(\omega-1)y$, $0 \leqq t \leqq 1$. Substituting for z in the integral $\int_{P_kP_{k+1}} f(z)dz$, we obtain

$$\int_{P_kP_{k+1}} f(z)dz = \int_0^1 f(x + \omega^k y + t\omega^k(\omega - 1)y)\omega^k(\omega - 1)ydt. \tag{5}$$

From (4) and (5) we obtain

$$\int_P f(z)dz = (\omega - 1)y \int_0^1 \sum_{k=0}^{n+1} \omega^k f(x + \omega^k y + t\omega^k(\omega - 1)y)dt. \tag{6}$$

By (3), replacing x by $x+\omega^k y$ and y by $t(\omega-1)y$, we have

$$\sum_{k=0}^{n+1} \omega^k f(x + \omega^k y + t\omega^k(\omega - 1)y) = 0. \tag{7}$$

Hence, by (6) and (7), we have $\int_P f(z)dz = 0$. Now the lemma implies that $f(z)$ is an entire function of z. Differentiating both sides of (3) $n+1$ times with respect to y, using $\omega^{n+2} = 1$, and putting $y = 0$, we have, for $|z| < +\infty$, $f^{(n+1)}(z) = 0$. This is the desired result.

I wish to thank the referee for his many helpful suggestions.

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SOME SERIES FOR EULER'S CONSTANT

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In a recent paper [1] Addison, using an integral representation for the Riemann zeta function, obtained the following series for Euler's constant γ:

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \sum_{m=2^{n-1}}^{2^n-1} \frac{n}{(2m)(2m+1)(2m+2)}. \tag{1}$$

It is our purpose in this note to derive a simpler series representation of the same type directly from the standard definition of Euler's constant. Namely, we prove that

$$\gamma = 1 - \sum_{n=1}^{\infty} \sum_{m=2^{n-1}+1}^{2^n} \frac{n}{(2m-1)(2m)}. \tag{2}$$

The two series are shown to be transformable easily into one another, so that an elementary proof of (1) is thereby also furnished. Finally, we give several alternate forms of these expansions which may be interesting in themselves.

The following notation will be used:

$$s_n = \sum_{k=1}^{2^n} \frac{1}{k}, \qquad (n = 0, 1, 2, \cdots),$$

$$\sigma_n = \sum_{k=1}^{2^n} \frac{(-1)^{k+1}}{k}, \qquad (n = 0, 1, 2, \cdots).$$

Then, since

$$\sum_{k=1}^{2^n} \frac{(-1)^{k+1}}{k} = \sum_{k=1}^{2^n} \frac{1}{k} - 2\sum_{k=1}^{2^{n-1}} \frac{1}{2k},$$

we have the relation

$$\sigma_n = s_n - s_{n-1}, \qquad (n = 1, 2, \cdots). \tag{3}$$

We start from the definition

$$\gamma = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \log m \right),$$

and consider the sub-sequence in the right member corresponding to $m = 2^n$, $(n = 1, 2, \cdots)$. Then, also,

$$\gamma = \lim_{n\to\infty} (s_n - n \log 2). \tag{4}$$

In (4) we use the well-known series for log 2 and write for each integer $n \geqq 1$, $\log 2 = \sigma_n + r_n$, where

$$r_n = \sum_{k=2^n+1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

Since $0<r_n<1/2^n$, $\lim (nr_n) \to 0$ as $n \to \infty$, and (4) becomes

$$(5) \qquad \gamma = \lim_{n\to\infty} (s_n - n\sigma_n).$$

The sequence in (5) is now converted into a series in the usual way, and we have $\gamma = (s_1 - \sigma_1) + \sum_{n=1}^{\infty} [s_{n+1} - (n+1)\,\sigma_{n+1} - (s_n - n\sigma_n)]$. Using (3), this simplifies to

$$(6) \qquad \gamma = 1 - \sum_{n=1}^{\infty} n(\sigma_{n+1} - \sigma_n).$$

Equivalently, on replacing the σ_n by their values

$$(7) \qquad \gamma = 1 - \sum_{n=1}^{\infty} n\left(\frac{1}{2^n+1} - \frac{1}{2^n+2} + \cdots - \frac{1}{2^{n+1}}\right).$$

By combining consecutive pairs of terms in the parenthesis in (7), we get (2).

To show the equivalence of (1) and (2), expand the general term of (1) into partial fractions. Then

$$\begin{aligned} \gamma &= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=2^{n-1}}^{2^n-1} n\left(\frac{1}{2m} - \frac{2}{2m+1} + \frac{1}{2m+2}\right) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} n\left(\sigma_n - \sigma_{n+1} + \frac{1}{2^{n+2}}\right). \end{aligned}$$

Since $\sum_{n=1}^{\infty} n/2^{n+2} = 1/2$ (generalized geometric series), this last equation is equivalent to (6).

It follows readily that the series in (7) converges also with the parenthesis removed, in which case the resulting representation has the form

$$(8) \qquad \gamma = 1 + \sum_{t=3}^{\infty} (-1)^t \frac{[\log(t-1)/\log 2]}{t}.$$

(Here and in the sequel a bracket will denote the greatest integer function.) For choose any integer $k>3$ and set $a = [\log (k-1)/\log 2]$. Then we have

$$-\frac{a}{2^a+1} \leqq \sum_{t=2^a+1}^{k} (-1)^t \frac{[\log(t-1)/\log 2]}{t} < 0.$$

Thus as $k \to \infty$, the series in this inequality will approach zero as a limit. As this series represents the difference between the partial sums of the series in (8) and corresponding partial sums in (7), the convergence of (8) is established.

Another representation of the form (8) is obtained by modifying (6) as follows. Since $\sum_{n=1}^{\infty} n/2^{n+1} = 1$, we have from (6) that

$$\gamma = -\sum_{n=1}^{\infty} n\left(\sigma_{n+1} - \sigma_n - \frac{1}{2^{n+1}}\right).$$

As before, it may be shown that the parenthesis in this series can be removed. The resulting series representation is

$$\gamma = \sum_{t=1}^{\infty} (-1)^t \frac{[\log t/\log 2]}{t},$$

which, in a sense, is analogous to the series for log 2.

Added in Proof. The author has just discovered that this last series has been given previously by H. F. Sandham (see this MONTHLY, 56 (1949) 414, Advanced Problem 4353).

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ON IMBEDDING FIELDS IN NONTRIVIAL NEAR-FIELDS

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In [1], J. R. Clay shows that an arbitrary ring R can be imbedded in a nontrivial near-ring N. If R is a (commutative) field, it is easily verified that the near-ring N is *not* a near-field. (Recall that a near-field $N \equiv \langle N, +, \cdot\rangle$ is a set N with binary operations $+$ and $\cdot$ such that $\langle N, +\rangle$ is an abelian group, $\langle N - \{0\}, \cdot\rangle$ is a group and $(a+b)\cdot c = a\cdot c + b\cdot c$, $a, b, c \in N$.) In this note we answer affirmatively the natural question: *Can an arbitrary field be imbedded in a nontrivial near-field?*

More generally, let R be a commutative ring of cardinality $|R| \geqq 2$ without divisors of zero. Denote by F the quotient field of R and by $F[x]$ the ring of polynomials in one indeterminant over F. We form the quotient field of the integral domain $F[x]$ (i.e., the field of rational polynomials in one indeterminant over F) and denote this by $K(x) \equiv \langle K(x), +, \cdot\rangle$.

If $f = f_1/f_2$ is a nonzero element of $K(x)$, the map $d: f \to \text{degree } f_1 - \text{degree } f_2$ assigns to each nonzero element f an integer, $d(f)$, such that for $g \neq 0$ and $f \neq 0$ in $K(x)$, $d(f\cdot g) = d(f) + d(g)$. Also, the map $\theta: K(x) \to K(x)$ defined by $\theta: x \to ax + b$, $a, b \in F$, $a \neq 0$, is a field automorphism. As usual, θ^n is the automorphism of $K(x)$ defined by:

$$\theta^n = \begin{cases} \text{Identity}, & n = 0 \\ \theta \circ \theta^{n-1}, & n \geqq 1. \end{cases} \tag{1}$$

We now define a multiplication $*$ on the additive group $\langle K(x), +\rangle$ by

$$f * g = \begin{cases} 0 & g = 0 \\ \theta^{d(g)}(f)\cdot g & g \neq 0 \end{cases} \tag{2}$$

$f, g \in K(x)$, where $(x)\ \cdot$ is the multiplication in the ring $K(x)$. Zemmer [2] shows

that in general $K_\theta \equiv \langle K(x), +, *\rangle$ is a nontrivial near-field, i.e., if θ is not the identity on $K(x)$ then $K_\theta(x)$ is a near-field, which is not a field.

The function $\phi: R \to K_\theta(x)$ defined by $\phi(r) = r/1$ is a one-one function and since

$$r/1 + t/1 = (r+t)/1 \tag{3}$$

and

$$\begin{aligned} r/1 * t/1 &= \theta^{d(t/1)}(r/1)\cdot t/1 \\ &= \theta^0(r/1)\cdot t/1 = r/1\cdot t/1 = (rt)/1 \end{aligned} \tag{4}$$

for all $r, t \in R$, θ is an imbedding.

Hence we have the following:

THEOREM. *A commutative ring of cardinality greater than one without divisors of zero can be imbedded in a nontrivial near-field.*

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A NOTE ON UNIVERSAL FLOWS

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In [1], the notion of a universal transformation group is introduced. A transformation group (X, T, π) is *universal* if, given any other transformation group (Y, T, π_1) with Y homeomorphic to X, there is a homeomorphism ϕ from Y to a closed subset of X such that $\phi\pi_1{}^t = \pi^t\phi$ for all $t \in T$ (i.e., a transformation group embedding). Similar notions apply to transformation semigroups. In particular, when (X, ψ) is a discrete flow (i.e., ψ is a homeomorphism on X), then (X, ψ) is universal if given any discrete flow (Y, δ) with Y homeomorphic to X, there exists a homeomorphism ϕ: $Y \to X$ such that $\phi\delta = \psi\phi$. Two theorems of Baayen and de Groot concerning the existence of universal homeomorphisms and continuous maps on the Cantor Set are given in [1]. In this note, we generalize these results to a wider class of spaces in which the universal action can always be considered as the shift (cf. [1], p. 4).

All topological spaces are compact Hausdorff. We let Z denote the integers and Z^+ the nonnegative integers. If X and I are sets, X^I will denote the product of I copies of X.

THEOREM 1. *Suppose that X is a space which is homeomorphic to X^Z. Then there exists a universal homeomorphism on X.*

Proof. Consider the shift transformation group (X^Z, σ), i.e., $\sigma(x_i \mid i \in Z) = (x_{i+1} \mid i \in Z)$. Let θ be a homeomorphism from X^Z onto X. Then θ yields a trans-

formation group isomorphism between (X^Z, σ) and $(X, \theta\sigma\theta^{-1})$. Thus, it is sufficient to show (X^Z, σ) is a universal discrete flow.

Consider a discrete flow (X, η). Let ψ be the map from X to X^Z defined by $\psi(x) = (\eta^i(x) \mid i \in Z)$, i.e., ψ maps x to its orbit under η. It is trivial to check that ψ is one-to-one, continuous, into, and hence closed. Since $\sigma\psi(x) = \sigma(\eta^i(x) \mid i \in Z) = (\eta^{i+1}(x) \mid i \in Z) = \psi\eta(x)$, ψ is the desired embedding. Finally, if $\phi: Y \to X$ is a homeomorphism, then the discrete flow (Y, δ) is isomorphic to $(X, \phi\delta\phi^{-1})$. The result follows.

Suppose that $X = X_0^Z$. Then clearly X satisfies the hypothesis of Theorem 1. Also, X^Z is canonically homeomorphic to $X_0^{Z \times Z}$ by $((x_{(i,j)} \mid j \in Z) \mid i \in Z) \to (x_{(i,j)} \mid i, j \in Z)$. Thus, any bijection $\rho: Z \times Z \to Z$ gives a homeomorphism θ_ρ from X^Z onto X defined by $\theta_\rho(x_{(i,j)} \mid i, j \in Z) = (x_{\rho(i,j)} \mid i, j \in Z)$. Moreover, since $(X, \theta_\rho\sigma\theta_\rho^{-1})$ is isomorphic to (X^Z, σ), any two different maps ρ, ρ_1 yield isomorphic universal flows. These remarks show that the Cantor set (and, in fact, any space F^I, where F is a finite set with the discrete topology and I is infinite), any infinite dimensional cube $[0, 1]^I$, and any infinite dimensional torus C^I have universal homeomorphisms. The "infinite shift" described in [1] can be recovered by choosing an appropriate ρ (i.e., if $\rho(Z \times \{j\})$ is infinite in both directions), and [1, Theorem 4] in this setting asserts the existence of the isomorphism between (X^Z, σ) and $(X, \theta_\rho\sigma\theta_\rho^{-1})$. So it is simply the flexibility of representing the space that yields universal flows; the action itself can be considered as fixed.

Under the hypothesis of Theorem 1, one can also assert the existence of a universal semiflow (i.e., a continuous map) on X. For X is clearly homeomorphic to X^{Z^+} and any homeomorphism θ yields an isomorphism between (X^{Z^+}, σ_0), the one-sided shift, and $(X, \theta\sigma_0\theta^{-1})$. Embedding any semiflow on X by one-sided orbits into (X^{Z^+}, σ_0) then yields the desired result.

Theorem 1 immediately generalizes to a larger class of transformation groups. Let T be any discrete group and X any space. Then the *left symbolic transformation group over T to X* is the transformation group (X^T, T) with action $(x_t \mid t \in T)s = (x_{st} \mid t \in T)$, i.e., T acts on the index set T by group left multiplication.

THEOREM 2. *Suppose X is a space which is homeomorphic to X^T. Then there exists a universal action of T on X.*

Proof. Let θ be a homeomorphism from X^T onto X. Let α be the action of T on X defined by $(x, t)\alpha = \theta(\theta^{-1}(x)t)$. Since $(\theta(x), t)\alpha = \theta(\theta^{-1}\theta(x)t) = \theta(xt)$, θ yields an isomorphism between (X^T, T) and (X, T, α).

Let (X, T, μ) be a transformation group. Then the mapping ψ from X to X^T defined by $\psi(x) = ((x, t)\mu \mid t \in T)$ is the desired embedding. The result follows.

By defining the left symbolic transformation semigroup when T is a discrete semigroup, one can prove a result analogous to Theorem 2 for discrete semigroups. Thus, given a discrete group or semigroup T, all spaces F^I, $[0, 1]^I$, and C^I, where crd $I \geqq$ crd $T \geqq \aleph_0$, support a universal action of T.

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A KREIN-MILMAN THEOREM FOR PARTIALLY ORDERED SETS

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The familiar Krein-Milman Theorem for locally convex topological linear spaces T states that any compact convex subset S of T is the closed convex hull of the set of its own extreme points [5, Th. 11.5, p. 138]. (It will be recalled that p is an extreme point of S when closed line segments lying in S can contain p only as an end point.)

For partially ordered sets there is also a notion of convexity, in which intervals play the rôle of line segments. We shall give an analogue of the Krein-Milman theorem for this case.

If S is a subset of a partially ordered set P, the *convex hull* of S consists of all $x \in P$ such that $s \leqq x \leqq t$ for some $s, t \in S$. S is a *convex* subset of P if S is its own convex hull. Equivalently, S is convex if, whenever $s, t \in S$ and $s \leqq t$, S must contain the whole interval $[s, t] = \{x \in P : s \leqq x \leqq t\}$ [1, p. 7].

If S is convex, it is natural to call $c \in S$ an *extreme point* of S if for any interval $[s, t] \subseteq S$, $c \in [s, t]$ implies $c = s$ or $c = t$. More simply, c is an extreme point of S if c is a minimal or a maximal element of S.

To express the condition of compactness, it is necessary to choose a topology on P. A reasonable requirement to impose is that for each $x \in P$, the "(closed) rays" $J(x) = \{y \in P : y \geqq x\}$ and $M(x) = \{y \in P : y \leqq x\}$ be closed sets. We shall choose the weakest topology meeting this requirement, namely the Frink interval topology [3], where the rays of P constitute a subbase for the closed sets.

THEOREM 1. *Let S be a convex subset of a partially ordered set P, and let S be compact with respect to the Frink interval topology on P. Then S is the convex hull of its set of extreme points.*

Proof. It is sufficient to show that for each $x \in S$ there exist elements s minimal in S and t maximal in S, such that $s \leqq x \leqq t$. Let $x \in S$ be given. We shall show the existence of t; the existence of s is dual.

Let C be a maximal chain in $J(x) \cap S$, and let $M = \bigcap_{c \in C} J(c)$. Since each of the nested closed sets $J(c)$ has a nonempty intersection with the compact set S, the set $M \cap S$ is likewise nonempty. Let $t \in M \cap S$. Then $t \geqq x$. The fact that $c \leqq t$ for all $c \in C$ and the maximality of C together imply that t is a maximal element of C and hence a maximal element of S, as required. The proof is thus complete.

Even if P is a lattice, P may not be a Hausdorff space under the Frink interval topology, and compact convex subsets of P need not be closed. But the following theorem, when applied to convex subsets which *are* closed, shows that the set of extreme points is at least the union of two Hausdorff subspaces.

THEOREM 2. *Let L be a lattice and let S be a subset of L closed in the Frink interval topology on L. Then the set H^+ of maximal elements of S is a Hausdorff space in the relativization to H^+ of the Frink interval topology on L. The same is true for the set H^- of minimal elements of S.*

Proof. Let $p_1, p_2 \in H^+$, $p_1 \neq p_2$. We want to show that p_1 and p_2 can be separated by disjoint relatively open subsets of H^+, or, equivalently, that there exist relatively closed subsets V_1, V_2 of H^+ such that (a) $p_i \notin V_i (i=1, 2)$ and (b) $V_1 \cup V_2 = H^+$. We shall show that in fact there exist sets T_1, T_2 closed in L such that (a′) $p_i \notin T_i$ $(i=1, 2)$ and (b′) $S \subseteq T_1 \cup T_2$. Then $V_1 = T_1 \cap H^+$ and $V_2 = T_2 \cap H^+$ will satisfy our requirements.

Since p_1 and p_2 are maximal and distinct in S, $p_1 \vee p_2 \notin S$. S is closed, and so there is a basic closed set B of L which contains S but not $p_1 \vee p_2$. Each basic closed set is a finite union of subbasic closed sets, i.e., of (closed) rays. Thus $B = R_1 \cup \cdots \cup R_n$, where each R_j is a ray. For $i=1$, 2 let T_i be the union of those rays R_j for which $p_i \notin R_j$. Then T_i is closed and $p_i \notin T_i$. Thus (a′) holds. To verify (b′), suppose that for some j, R_j is contained in neither T_1 nor T_2. Then $p_1 \in R_j$ and $p_2 \in R_j$. Since R_j is a ray, $p_1 \vee p_2 \in R_j$, so that $p_1 \vee p_2 \in B$, contrary to assumption. Therefore every R_j, hence B, hence S, is contained in $T_1 \cup T_2$. Thus (b′) holds.

The proof for H^- is dual.

Let a subset of L which consists of mutually noncomparable elements be called *horizontal.*

COROLLARY. *A closed, compact, convex subset S of a lattice L with the Frink interval topology is a "sandwich" consisting of all elements of L which lie between two horizontal Hausdorff subspaces of L.*

REMARK. All other common topologies on partially ordered sets (or lattices), such as the order topology [1, p. 244], the Birkhoff interval topology [2] and the ideal topology [4], are at least as strong as the Frink interval topology whenever they are defined. Theorem 1 therefore remains true if the Frink interval topology is replaced by one of these others. (Theorem 2 does not remain true.)

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A NOTE ON DOUBLY-STOCHASTIC MATRICES

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This note proves that a necessary and sufficient condition for the limiting matrix A of a finite irreducible Markov chain with transition matrix P to have all elements equal to $1/N$, where N is the number of states, is that P be doubly-stochastic. The fact that a doubly-stochastic matrix of an irreducible Markov chain has a limiting matrix A with all elements equal to $1/N$ is well known and demonstrated in Doob ([1], p. 182), Feller ([2], p. 358), and Parzen ([4], p. 255). A check of the literature, however, shows that the stronger statement proved below has apparently been overlooked.

THEOREM. *Let P be the transition matrix for a finite, irreducible Markov chain and let $A=\{a_{ij}\}$ be its limiting matrix, defined as*

$$(1)\qquad \begin{aligned} A &= \lim_{n\to\infty} P^n \qquad \textit{for } P \textit{ regular and} \\ A &= \lim_{n\to\infty} (\lambda I - (1-\lambda)P)^n \qquad \textit{for } P \textit{ ergodic}, \quad 0<\lambda<1. \end{aligned}$$

A necessary and sufficient condition for

$$(2)\qquad a_{ij} = 1/N, \qquad i, j = 1, 2, \cdots, N,$$

is that the matrix P satisfy

$$(3)\qquad \sum_{i=1}^{N} p_{ij} = 1, \qquad j = 1, 2, \cdots, N.$$

Proof. The proof is by contradiction. Let P be a stochastic matrix with $\sum_{i=1}^{N} p_{ij} \neq 1$ for some j and assume that the limiting matrix A satisfies (2).

To be the limiting matrix, A and P must satisfy

$$(4)\qquad AP = PA = A.$$

Take any row $a_i=(a_{i1}, a_{i2}, \cdots, a_{iN})$ of A and any column $p_j=(p_{1j}, p_{2j}, \cdots, p_{Nj})$ of P for which condition (3) is not satisfied. Then $a_i p_j \neq 1/N$ which contradicts (4). Hence (3) is a necessary condition for the limiting matrix of P to be of the form of (2). To prove sufficiency, assume (2) is satisfied, then it follows immediately that

$$a_i p_j = a_{ij} = 1/N, \qquad i, j = 1, 2, \cdots, N.$$

While this result may be intuitively obvious, it is, nevertheless, rather deceptive and we feel that several research applications might result from this presentation.

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IDEALS AND POLYNOMIAL FUNCTIONS OVER MATRIX RINGS

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Lewis [3], and Niven and Warren [5] have characterized those polynomials $f(x)$ in $Z_m[x]$ such that $f(a)=0$ for each a in Z_m. Given a finite field k of $q=p^n$ elements and a positive integer u, we shall characterize those polynomials $f(x)$ in $k[x]$ such that $f_L(A)=(0)$ for each A in $M(u;k)$, where $f_L(A)$ is the left functional value of $f(x)$ at A in the sense of MacDuffee [4], and $M(u;k)$ is the algebra of all u-by-u matrices over k. In particular, we shall show that these polynomials form the principal ideal generated by

$$h(x) = \prod_{t=1}^{u} e(t)$$

where $e(t)=x^{q^t}-x$.

For each A in $M(u;k)$, the minimum polynomial of A exists and is of degree at most u. The polynomials in $k[x]$ whose left functional values at A are (0) form the principal ideal generated by the minimum polynomial of A. Also, every monic polynomial $f(x)$ in $k[x]$ of degree at most u is the minimum polynomial of some A in $M(u;k)$. If $d(x)$ is in $k[x]$ and $d_L(A)=(0)$ for each A in $M(u;k)$ then $f(x)$ divides $d(x)$ over $k[x]$ for each $f(x)$ in $k[x]$ with the degree of $f(x)$ at most u.

If r is a real number, let $[r]$ denote the greatest integer not exceeding r. Dickson [2] has shown that the product of all irreducible polynomials over k of degree m, $V(m;k)$, may be expressed as a quotient of factors of the form $e(t)$. If $g(x)$ is in $k[x]$ and $g_L(A)=(0)$ for each A in $M(u;k)$, then for each m, $1 \leqq m \leqq u$, $(V(m;k))^{[u/m]}$ divides $g(x)$ over $k[x]$. If $g(x)$ is also of minimum degree, then

$$g(x) = \prod_{m=1}^{u} (V(m;k))^{[u/m]}.$$

However, Carlitz [1] has shown that $h(x)=g(x)$.

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

WHAT IS THE EXPECTED VOLUME OF A SIMPLEX WHOSE VERTICES ARE CHOSEN AT RANDOM FROM A GIVEN CONVEX BODY?

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In Euclidean d-space a convex body B of unit volume is given and $d+1$ points $x_0, x_1, \cdots, x_d$ of B are chosen independently at random. What is the expected volume V_B of their convex hull (which is a d-simplex except in degenerate cases not affecting the expectation)? The volume of the convex hull is $1/d!$ times the absolute value of the determinant

$$D(x_0, x_1, \cdots, x_d) = \begin{vmatrix} x_0^1 & x_0^2 \cdots x_0^d & 1 \\ x_1^1 & x_1^2 \cdots x_1^d & 1 \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ x_d^1 & x_d^2 \cdots x_d^d & 1 \end{vmatrix},$$

where $x_j^1, \cdots, x_j^d$, are cartesian coordinates of the point x_j. Hence the expectation V_B is $1/d!$ times the value of the $(d+1)$-fold integral

$$\text{(I)} \qquad \int_{x_0\in B}\int_{x_1\in B}\cdots\int_{x_d\in B} |D(x_0, x_1, \cdots, x_d)|$$

or, as ratios of volumes are invariant under nonsingular affine transformations, $1/d!$ times the quotient

$$\int_{y_0\in C}\int_{y_1\in C}\cdots\int_{y_d\in C} |D(y_0, y_1, \cdots, y_d)| \,/(\text{volume of } C)$$

for any body C in d-space affinely equivalent to B. It is almost immediate that $V_{\text{unit interval}} = 1/3$. Indeed, for two points u and v chosen at random from $[0, 1]$ the expected value of the distance between them (the 1-dimensional "volume" of their convex hull) is

$$\int_0^1\int_0^1 |v-u|\,dudv = 2\int_0^1\int_0^v (v-u)dudv = 2\int_0^1 (v^2/2)dv = 1/3.$$

But even when $d=2$ the direct evaluation of (I) involves a six-fold iteration of ordinary 1-dimensional integration and it is not surprising that difficulties arise. As was remarked in 1885 by Crofton [3], "The intricacy and difficulty to be encountered in dealing with such multiple integrals and their limits is so great that little success could be expected in attacking such questions directly by this

method; and most of what has been done in the matter consists in turning the difficulty by various considerations, and arriving at the result by evading or simplifying the integration."

With the aid of a useful "evasion" of Crofton, the evaluation of V_B can be reduced to a similar problem in which only $x_1, \cdots, x_d$ are chosen at random in B while x_0 is chosen at random in B's boundary. This simplifies the calculation and has led to evaluation of V_B for various plane convex bodies B (triangle, quadrilateral, regular hexagon, ellipse) of unit area. In particular, $V_{\text{triangle}}=1/12$ and $V_{\text{ellipse}}=35/48\pi^2$. Crofton [3] proposed a proof that for plane convex bodies, V's minimum is attained for ellipses. In 1903 Czuber [5] acknowledged Crofton's proof and conjectured that the maximum is attained for triangles. The story nicely illustrates the vicissitudes of mathematical rigor, for when Deltheil [6] provided a rigorous treatment in 1926 he was able to prove Czuber's conjecture about triangles but was forced to leave Crofton's "theorem" about ellipses as a conjecture, remarking that "il est sans doute bien difficile d'établir en toute rigueur." The story also illustrates the vagaries of mathematical communication, for rigorous proofs of Crofton's "theorem" and Czuber's conjecture had been given by Blaschke 1, 2 in 1917 and 1923.

For more details concerning the material of this paragraph, see the works of Crofton [3], Czuber [4, 5], and Deltheil [6], and the valuable survey of Kendall and Moran [7] and Moran [9]. Each of these authors treats the evaluation of V_B in discussing J. J. Sylvester's problem of finding the probability P_B that four points chosen at random in B form the vertices of a convex quadrilateral; note that $P_B=1-4V_B$ when B is of unit area.

Let $v(d)$ denote the value of V_B when B is a d-simplex. Thus $v(1)=1/3$ and $v(2)=1/12$. In attempting to compare the efficiencies of two procedures for the phase analysis of multicomponent systems, I became interested in evaluating $v(d)$ for $d>2$ (see [8]). I conjectured $v(3)=1/60$ but Monte Carlo experiments conducted independently by Dr. G. Marsaglia of the Boeing Scientific Research Laboratories and Mr. J. Baker of the University of Washington show that $1/57$ is the integer-reciprocal closest to $v(3)$. We ask:

What is the expected volume of a tetrahedron whose vertices are chosen at random from a tetrahedron of unit volume?

While $v(3)$ might yield to brute force, the numbers $v(4), v(5), \cdots, v(d)$ are also of interest. For [8] we should also know the expected volume, when $x_0, x_1, \cdots, x_d$ are chosen at random from a d-simplex S of unit volume, of the smallest convex polytope which contains all the chosen points and has all of its facets parallel to those of S.

A recent communication from Prof. J. F. C. Kingman shows that when B is an $(n-1)$-dimensional spherical ball of unit volume, V_B has the value

$$\left(\frac{1}{2}\right)^{n-1} \psi_n^n/\psi_{n^2} \quad \text{for even } n, \quad \left(\frac{1}{2\pi}\right)^{n-1} \psi_{n^2}/\psi_n^n \quad \text{for odd } n,$$

where $\psi_n=n!/([(n/2)!]^2)$. In particular, the value is $9/715$ in the 3-dimensional case.

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IS THERE AN n FOR WHICH $\phi(x)=n$ HAS A UNIQUE SOLUTION?

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The ϕ of the title is Euler's function, $\phi(x)$ denoting the number of natural numbers $\leqq x$ which are relatively prime to x. The question comes from what may be, in a special sense, the longest research paper in existence. Carmichael [1] claimed to prove in 1906 that for each natural number x there exists $y\neq x$ such that $\phi(y)=\phi(x)$; that is, for no n does the equation, $\phi(x)=n$, have exactly one solution. The statement of this as an exercise in his 1914 book [2] led to discovery of an error in the 1906 proof and to publication of a correction [3] in 1922. In this, the earlier assertion was left as a conjecture and it was proved that $n>10^{37}$ if there is a unique solution of $\phi(x)=n$. In 1948 Carmichael [4] corrected another statement in his 1906 paper and in 1949 he published [5] a list of misprints in the 1948 correction. Thus it seems fair to describe the source of our title problem as a paper which is about forty-three years long!

There has been only a little progress on Carmichael's conjecture since 1922, and Erdös [8] remarks that it "seems very deep." For each n, let $S(n)$ denote the number of solutions of the equation, $\phi(x)=n$, and for each s let N_s denote the set of all n for which $S(n)=s$. Carmichael's lower bound of 10^{37} on the members (if any) of N_1 was improved by Klee [10] to 10^{400}. Wright [12] corrected some misprints and improved some of the results of [10]. Erdös [8] showed for each s that the set N_s is infinite if it is nonempty; indeed, if $n\in N_s$ then $n(p-1)\in N_s$ for infinitely many primes p.

The function S is finite-valued, for $\phi(x)\rightarrow\infty$ as $x\rightarrow\infty$. Of the many results in the literature implying that S is unbounded (Erdös [6, 7] et al.), the simplest is Schinzel's observation [11] that

$$S((p_1-1)(p_2-1)\cdots(p_k-1))>k$$

when $p_1,\cdots,p_k$ are the first k primes. Klee [9] proved that $S(2m)$ is 0, 2, or 4

when m is odd, and that the set N_0 includes infinitely many numbers of the form $2m$. Schinzel [11] proved that for each t, N_0 includes infinitely many multiples of t, and that the sets N_2 and N_3 are infinite. He also reported Sierpinski's conjecture that N_s is infinite for all $s>1$.

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

SOME EXPLICIT FORMULAS FOR THE EXPONENTIAL MATRIX e^{tA}

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In a recent paper, E. J. Putzer [1] described two methods for calculating exponential matrices of the form e^{tA}, where t is a scalar and A is any square matrix. Putzer's methods are particularly useful in practice because they are valid for all square matrices A and require no preliminary transformations of any kind. All that is needed is the factorization of the characteristic polynomial of A, that is, a knowledge of the eigenvalues of A and their multiplicities. Both methods are based on the fact that e^{tA} is a polynomial in A whose coefficients are scalar functions of t that can be determined recursively by solving a simple system of first-order linear differential equations.

A purely algebraic method for computing e^{tA} is given by the Lagrange-Sylvester interpolation formula described on pp. 101–102 of Gantmacher's *Theory of Matrices* [2]. This formula requires a knowledge of the factorization of the minimal polynomial of A and is usually more complicated than Putzer's methods.

Another algebraic method for computing e^{tA} was developed recently by R. B. Kirchner [3] who gave an explicit formula for calculating e^{tA} in terms of A and the factorization of the characteristic polynomial of A. Kirchner's method requires the inversion of a certain matrix polynomial $q(A)$, although as Kirchner points out, this inversion can sometimes be avoided.

General methods often have the disadvantage that they are not the simplest

methods to use in certain special cases. The purpose of this note is to point out that explicit formulas for the polynomial e^{tA} can be obtained very easily (a) if all the eigenvalues of A are equal, (b) if all the eigenvalues of A are distinct, or (c) if A has two distinct eigenvalues, exactly one of which has multiplicity 1. We state these formulas in the following three theorems.

THEOREM 1. *If A is an $n \times n$ matrix with all its eigenvalues equal to λ, then we have*

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} (A - \lambda I)^k. \tag{1}$$

Proof. Since the matrices λtI and $t(A-\lambda I)$ commute, we have

$$e^{tA} = e^{\lambda tI} e^{t(A-\lambda I)} = (e^{\lambda t} I) \sum_{k=0}^{\infty} \frac{t^k}{k!} (A - \lambda I)^k.$$

The Cayley-Hamilton Theorem implies that $(A-\lambda I)^k = 0$ for $k \geqq n$, so the theorem is proved.

NOTE. If $(A-\lambda I)^m = 0$ for some $m < n$, then the same proof shows that we can replace $n-1$ by $m-1$ in the upper limit of summation in (1).

Formula (1) is precisely the result obtained by applying Putzer's second method or Kirchner's explicit formula. The foregoing proof seems to be the simplest and most natural way to derive this result.

THEOREM 2. *If A is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then we have*

$$e^{tA} = \sum_{k=1}^{n} e^{t\lambda_k} L_k(A),$$

where the $L_k(A)$ are Lagrange interpolation coefficients given by

$$L_k(A) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{A - \lambda_j I}{\lambda_k - \lambda_j} \qquad \textit{for } k = 1, 2, \cdots, n.$$

Proof. Although this theorem is a special case of the Lagrange-Sylvester interpolation formula, the following alternate proof may be of interest.

Define a matrix-valued function of the scalar t by the equation

$$F(t) = \sum_{k=1}^{n} e^{t\lambda_k} L_k(A). \tag{2}$$

To prove that $F(t) = e^{tA}$ we show that F satisfies the differential equation $F'(t) = AF(t)$ and the initial conditions $F(0) = I$. From (2) we see that

$$AF(t) - F'(t) = \sum_{k=1}^{n} e^{t\lambda_k} (A - \lambda_k I) L_k(A).$$

By the Cayley-Hamilton Theorem we have $(A-\lambda_k I)L_k(A)=0$ for each k, so F satisfies the differential equation $F'(t)=AF(t)$. Since

$$F(0)=\sum_{k=1}^{n} L_k(A)=I,$$

this completes the proof.

NOTE. My colleague Professor John Todd points out that another simple proof of Theorem 2 can be based on the fact that F satisfies the functional equation $F(s+t)=F(s)F(t)$.

The next theorem treats the case when A has two distinct eigenvalues, exactly one of which has multiplicity 1.

THEOREM 3. *Let A be an $n\times n$ matrix ($n\geqq 3$) with two distinct eigenvalues λ and μ, where λ has multiplicity $n-1$ and μ has multiplicity* 1. *Then we have*

$$e^{tA}=e^{\lambda t}\sum_{k=0}^{n-2}\frac{t^k}{k!}(A-\lambda I)^k + \left\{\frac{e^{\mu t}}{(\mu-\lambda)^{n-1}}-\frac{e^{\lambda t}}{(\mu-\lambda)^{n-1}}\sum_{k=0}^{n-2}\frac{t^k}{k!}(\mu-\lambda)^k\right\}(A-\lambda I)^{n-1}.$$

Proof. As in the proof of Theorem 1 we write

$$\begin{aligned} e^{tA} &= e^{\lambda t}\sum_{k=0}^{\infty}\frac{t^k}{k!}(A-\lambda I)^k = e^{\lambda t}\sum_{k=0}^{n-2}\frac{t^k}{k!}(A-\lambda I)^k + e^{\lambda t}\sum_{k=n-1}^{\infty}\frac{t^k}{k!}(A-\lambda I)^k \\ &= e^{\lambda t}\sum_{k=0}^{n-2}\frac{t^k}{k!}(A-\lambda I)^k + e^{\lambda t}\sum_{r=0}^{\infty}\frac{t^{n-1+r}}{(n-1+r)!}(A-\lambda I)^{n-1+r}. \end{aligned}$$

Now we evaluate the series over r in closed form by using the Cayley-Hamilton Theorem. Since $A-\mu I=A-\lambda I-(\mu-\lambda)I$, we find $(A-\lambda I)^{n-1}(A-\mu I)=(A-\lambda I)^n-(\mu-\lambda)(A-\lambda I)^{n-1}$. The left member is 0 by the Cayley-Hamilton Theorem so

$$(A-\lambda I)^n=(\mu-\lambda)(A-\lambda I)^{n-1}.$$

Using this relation repeatedly we find

$$(A-\lambda I)^{n-1+r}=(\mu-\lambda)^r(A-\lambda I)^{n-1}.$$

Therefore the series over r becomes

$$\begin{aligned} \sum_{r=0}^{\infty}\frac{t^{n-1+r}}{(n-1+r)!}(\mu-\lambda)^r(A-\lambda I)^{n-1} &= \frac{1}{(\mu-\lambda)^{n-1}}\sum_{k=n-1}^{\infty}\frac{t^k}{k!}(\mu-\lambda)^k(A-\lambda I)^{n-1} \\ &= \frac{1}{(\mu-\lambda)^{n-1}}\left\{e^{t(\mu-\lambda)}-\sum_{k=0}^{n-2}\frac{t^k}{k!}(\mu-\lambda)^k\right\}(A-\lambda I)^{n-1}. \end{aligned}$$

This completes the proof of Theorem 3.

The explicit formula in Theorem 3 can also be deduced by applying Putzer's

method or by using Kirchner's formula, but the details are much more complicated.

The explicit formulas in Theorems 1, 2, and 3 cover all matrices of order $n\leqq 3$. Since the 3×3 case is often discussed in the classroom, the formulas in this case are listed below for easy reference.

1. If a 3×3 matrix A has eigenvalues $\lambda, \lambda, \lambda$, then

$$e^{tA} = e^{\lambda t}\{I + t(A - \lambda I) + \tfrac{1}{2}t^2(A - \lambda I)^2\}.$$

2. If a 3×3 matrix A has distinct eigenvalues λ, μ, ν, then

$$e^{tA} = e^{\lambda t}\frac{(A-\mu I)(A-\nu I)}{(\lambda-\mu)(\lambda-\nu)} + e^{\mu t}\frac{(A-\lambda I)(A-\nu I)}{(\mu-\lambda)(\mu-\nu)} + e^{\nu t}\frac{(A-\lambda I)(A-\mu I)}{(\nu-\lambda)(\nu-\mu)}\cdot$$

3. If a 3×3 matrix A has eigenvalues λ, λ, μ, with $\lambda\neq\mu$, then

$$e^{tA} = e^{\lambda t}\{I + t(A - \lambda I)\} + \frac{e^{\mu t} - e^{\lambda t}}{(\mu-\lambda)^2}(A - \lambda I)^2 - \frac{te^{\lambda t}}{\mu - \lambda}(A - \lambda I)^2.$$

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BASIS-FREE PROOF OF $\partial J/\partial t = J$ div V

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The well-known formula $\partial J/\partial t = J$ div V (the symbols are explained below) involving a Jacobian determinant J depending on a parameter, is proved here without the introduction of a basis. The proof usually given may be found in [1, page 131]. For information about the exponential of a matrix, which we use below, one may consult [2, page 121].

Consider a one-parameter family of transformations $F_t: R^n \to R^n$, each F_t being continuously differentiable and invertible $(0<t<\infty)$. Let the differential be denoted by $F_t'(x)$. We assume that

$$F_{t+\epsilon}(x) = F_t(x) + \epsilon V_t(x) + R(x, t, \epsilon),$$

where $\|R'(x, t, \epsilon)\| = o(\epsilon)$ (hence, *a fortiori* $|R| = o(\epsilon)$) and V_t is continuously differentiable. (Double bars denote the operator norm, and single bars denote the Euclidean norm.) We interpret F_t as a fluid motion: that is, $F_t(x_0)$ is the position at time t of a particle which began at x_0 at $t=0$. Then $V_t(x_0)$ is the velocity of this particle at time t. The Eulerian velocity field $V(x, t)$ is $V_t(x_0)$ with $x_0 = F_t^{-1}(x)$, i.e., $V(x, t) = V_t F_t^{-1}(x)$. Let $J(x_0, t) = \det F_t'(x_0)$.

THEOREM. *Under the assumptions made above,*

$$\partial J(x_0, t)/\partial t = J(x_0, t) \operatorname{div} V(x, t).$$

Proof. In the following proof $Q(\epsilon)$ stands for any linear transformation from R^n to R^n satisfying $\|Q(\epsilon)\| = o(\epsilon)$; each time Q is used it may stand for a different such transformation.

Let $G_t = F_t^{-1}$ and $x = F_t(x_0)$. Then

$$V'(x) = (V_tG_t)'(x) = V_t'(x_0)G_t'(F_t(x_0)) = V_t'(x_0)(F_t'(x_0))^{-1}$$

by the chain rule and the inverse function theorem. Then

$$\begin{aligned} F_{t+\epsilon}'(x_0) &= F_t'(x_0) + \epsilon V_t'(x_0) + R' \\ &= [I + \epsilon V_t'(x_0)(F_t'(x_0))^{-1} + Q(\epsilon)]F_t'(x_0) \\ &= [I + \epsilon V'(x) + Q(\epsilon)]F_t'(x_0) \\ &= \exp(\epsilon V'(x))F_t'(x_0)[I + Q(\epsilon)]. \end{aligned}$$

Taking the determinant of both sides, we find

$$J(x_0, t + \epsilon) = J(x_0, t) \exp(\epsilon \operatorname{trace} V'(x))(1 + o(\epsilon))$$

(using the easily verified fact that $\det(I + Q(\epsilon)) = 1 + o(\epsilon)$). Since trace $V'(x) = \operatorname{div} V$, the result follows from this estimate.

The formula just proved is frequently used to prove the rule for differentiating multi-dimensional integrals with respect to a parameter. For completeness we include this proof.

THEOREM. *Let R_0 be a bounded domain in R^n and $H: R^n \to R$ be continuously differentiable. Then*

$$\frac{d}{dt}\int_{F_tR_0} H(x)dx = \int_{F_tR_0} \operatorname{div}(HV)dx.$$

Proof.

$$\begin{aligned} \frac{d}{dt}\int_{F_tR_0} H(x)dx &= \frac{d}{dt}\int_{R_0} H(F_t(x_0))J(x_0, t)dx_0 \\ &= \int_{R_0} (J \operatorname{grad} H \cdot V + H\partial J/\partial t)dx_0 \\ &= \int_{R_0} J \operatorname{div}(HV)dx_0 \\ &= \int_{F_tR_0} \operatorname{div}(HV)dx. \end{aligned}$$

This theorem is attributed to O. Reynolds. For applications to the derivation of the partial differential equations of fluid motion, see [1].

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CONTINUOUS DEPENDENCE ON PARAMETERS

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The Picard Existence Theorem for initial value problems in differential equations is often proved using the fixed point property of contraction mappings in a complete metric space. When this approach is taken the following becomes an interesting way in which to present a theorem about continuous dependence of solutions on parameters.

We make these assumptions: (X, ρ) and (T, σ) are metric spaces, X complete; F is a map from $X \times T$ into X; for each fixed parameter t in T, $F(x, t)$ is a contraction map on X. In particular, for each t in T there is some real number $M(t)$ such that $0 \leqq M(t) < 1$ and

$$\rho(F(x, t), F(y, t)) \leqq M(t)\rho(x, y)$$

for all x and y in X. Then for each t in T there exists an element $x(t)$ of X such that $x(t)$ is a fixed point of the function $F(x, t)$ as a function of x, that is, $F(x(t), t) = x(t)$ for all t in T.

THEOREM. *If M is a continuous function of t and $F(x, t)$ is a continuous function of t for each x in X, then the fixed point $x(t)$ is a continuous function of t.*

Proof. Let s and t be elements of T. We shall try to show $x(t)$ is continuous at t. We have

$$\begin{aligned}\rho(x(s), x(t)) &= \rho(F(x(s), s), F(x(t), t)) \\ &\leqq \rho(F(x(s), s), F(x(t), s)) + \rho(F(x(t), s), F(x(t), t)) \\ &\leqq M(s)\rho(x(s), x(t)) + \rho(F(x(t), s), F(x(t), t)).\end{aligned}$$

Hence

$$\rho(x(s), x(t)) \leqq \frac{\rho(F(x(t), s), F(x(t), t))}{1 - M(s)} .$$

Because of the continuity of $M(s)$ and $F(x(t), s)$ as functions of s, the right hand side of the above inequality is a continuous function of s (with value zero at t). Hence, given $\epsilon > 0$ there exists a $\delta > 0$ such that $\sigma(s, t) < \delta$ implies $\rho(x(s), x(t)) < \epsilon$. This shows $x(t)$ is a continuous function at any t in T.

As an example, consider the initial value problem

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{1}$$

and the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))d\xi,$$

where $I=[a, b]$, f is continuous and satisfies a Lipschitz condition (with constant K) on $I\times R$. Also, $x_0\in I$, $y_0\in R$ and $y\in X=C[a, b]$. In X, let ρ be the metric generated by the sup norm. Let $Y=I\times R$ with the Euclidean metric.

For each $y\in X$ and $(x_1, y_1)\in Y$, define

$$F(y, (x_1, y_1))(x) = y_1 + \int_{x_1}^{x} f(\xi, y(\xi))d\xi.$$

Then F defines a mapping from $X\times Y$ into X. For a fixed $y\in X$, there exists an $M>0$ such that $|f(\xi, y(\xi))|\leqq M$ for all $\xi\in I$.

Let $w_1=F(y, (x_1, y_1))$, $w_0=F(y, (x_0, y_0))$. We have

$$\begin{aligned}\rho(w_1, w_0) &= \max_{x\in I}\left| y_1 + \int_{x_1}^{x} f(\xi, y(\xi))d\xi - y_0 - \int_{x_0}^{x} f(\xi, y(\xi))d\xi\right| \\ &\leqq |y_1 - y_0| + M|x_1 - x_0|.\end{aligned}$$

From this inequality it is clear that for any given y in X, $F(y, (x_0, y_0))$ is a continuous function of the parameter (x_0, y_0). By a standard argument, for $y, z\in X$, $(x_0, y_0)\in Y$,

$$\rho(F(y, (x_0, y_0)), F(z, (x_0, y_0)) \leqq K(b - a)\rho(y, z).$$

If $M(t)=K(b-a)<1$, then the hypotheses of the above theorem are satisfied and the solutions of (1) are continuous functions of the parameter (x_0, y_0).

If $K(b-a)\geqq 1$, we can divide $[a, b]$ into subintervals of length δ with $0<K\delta<1$, apply the above result on these subintervals and appropriately "splice" them together to obtain the desired result on $[a, b]$.

AN ELEMENTARY EXAMPLE OF A CONTINUOUS SINGULAR FUNCTION

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Let C denote the familiar Cantor-Lebesgue function which rises monotonely and continuously from 0 to 1 across the unit interval (we regard C as extended to the whole real line by making it constant outside the unit interval). The function C provides the simplest and best-known example of a continuous, nonconstant function which is singular, i.e., has the property that its derivative is zero a.e. Lebesgue. One of the features of this example that has greatly contributed to its popularity is the ease with which singularity is verified. Indeed, C is locally constant on the complement of a set of measure zero. This very fact, however, suggests another question. Can a continuous function be singular for

some less obvious reason? In particular, can a continuous, singular function be strictly increasing? The answer is affirmative, of course, and in text-books dealing with such matters there is an example to show that this is possible. The standard example adduced to this end, however, seems, at least to this writer, to be too erudite for the purpose it serves.

(The example alluded to is not difficult to construct. In roughest outline it goes this way. Let $0<\theta<1$, and let $f_0(x)=x$, $0\leqq x\leqq 1$. First "break" f_0 by replacing it by the function f_1 that rises linearly by the amount θ between 0 and $\frac{1}{2}$, and by the amount $(1-\theta)$ from $\frac{1}{2}$ to 1. Next construct f_2 by "breaking" in like manner each of the linear halves of f_1, and so on by induction. Finally, pass to the limit. The resulting function, call it f_θ, is clearly strictly increasing and is readily seen to be continuous. Its singularity (for $\theta\neq\frac{1}{2}$) has been verified in at least three different ways. Details may be found in [1, pp. 306–310] or in [6, pp. 48–49]. Still another treatment appears in [4]. The construction has become folkloric, though as a matter of historical fact it is properly attributed to Hellinger, in whose thesis [3] it first appears. The same idea was subsequently rediscovered by Salem [7], who contributed an interesting generalization; in this connection, see [2]. A closely related circle of ideas is treated in [5].)

The aim of this note is to point out that, if all that is required is what was stated above, viz., an example of a strictly increasing, continuous, singular function, then a considerably less delicate approach will work.

To begin with, it is an easy matter, starting with C, to construct a continuous, monotone, singular function that is constant outside a specified interval I, and that rises by a specified amount across I. Thus,

$$k\cdot C\left(\frac{x-\alpha}{\beta-\alpha}\right)$$

rises continuously from 0 to k across the interval $[\alpha,\beta]$. Let $\{I_n\}_{n=1}$ be an enumeration of all the subintervals of the unit interval that have rational end-points, and, for each n, let f_n denote a continuous, singular function that rises from 0 to $1/2^n$ across I_n and is constant outside I_n. Then, if we define

$$S=\sum_n f_n,$$

it is obvious that S is continuous and strictly increasing across $[0, 1]$. But also, since $S'=\sum_n f_n'$ (see, e.g., [6], p. 11) S is singular as well. Thus S provides an example meeting all of the specified conditions.

In conclusion it should be stated that this simple construction emphatically does not replace the example f_θ referred to above. In fact, the outstanding merit of that example lies somewhat deeper than has been suggested so far. The point is (as Hellinger shows) that f_θ is not only singular with respect to Lebesgue measure for $\theta\neq\frac{1}{2}$, but, in fact, f_θ is singular with respect to $f_{\theta'}$ for any $\theta\neq\theta'$. (I.e., df_θ and $df_{\theta'}$ are mutually singular Borel measures. Note that $f_{\frac{1}{2}}(x)=x$, so that $df_{\frac{1}{2}}$ is Lebesgue-Borel measure itself.) As a corollary of this, we have the rather strik-

ing fact that the unit interval supports an entire continuum of mutually singular, atom-free Borel measures. (For a different proof of the same fact, see [5].)

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SOME SIMPLE SINGULAR AND MIXED PROBABILITY DISTRIBUTIONS

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It is not unusual in post-calculus level courses in probability theory to give the Lebesgue decomposition of the distribution function of random variables $X_1, \cdots, X_n$:

$$F = F_d + F_{ac} + F_s,$$

where

(1) $F(x_1, \cdots, x_n) = P(X_1 \leqq x_1, \cdots, X_n \leqq x_n), \ -\infty < x_1, \cdots, x_n < \infty$,
(2) $F_d(x_1, \cdots, x_n)$ is the pure discrete component which is completely determined by the probability function $p(x_1, \cdots, x_n) = P(X_1 = x_1, \cdots, X_n = x_n)$,
(3) $F_{ac}(x_1, \cdots, x_n)$ is the pure absolutely continuous component which is completely determined by the probability density function $f(x_1, \cdots, x_n) = \partial^n/(\partial x_1, \cdots, \partial x_n) F(x_1, \cdots, x_n)$, and
(4) $F_s(x_1, \cdots, x_n)$ is what we shall call the singular component.

(In the terminology of [1], F_s is the nonatomic part of the singular component relative to Lebesgue measure.) The singular component describes the probability mass which is not concentrated at discrete points yet is concentrated on sets of n-dimensional Lebesgue measure zero. See [1, p. 140] for a more precise and complete statement of the decomposition theorem.

The vast majority of probability models require only one or the other of the first two pure components and traditionally the singular component, if mentioned at all, is quickly "swept under the carpet." The reason is that in the most studied case of a single random variable, singular distributions are difficult to construct and are uninteresting as models for physical phenomena. However, for $n \geqq 2$, singular components are both easily constructible *and* interesting from the viewpoint of certain applications. It is the purpose of this note to point out a simple class of purely singular distributions and mixed distributions with

singular components for $n=2$ which will provide the probability student with a better understanding of the above decomposition. A class of experiments from statistics leading to models composed of singular distributions will illustrate that singular distributions do occur in "real life."

For convenience in stating the examples, we shift to the conventional (X, Y), (x, y) notation for a two dimensional random variable and its values in the Euclidean plane. Let $A=\{(x, y): x<0 \text{ or } y<0\}$, $B_x=\{(x, y): 0\leqq x\leqq 1, y>1\}$, $B_y=\{(x, y): x>1, 0\leqq y\leqq 1\}$, $B_{xy}=\{(x, y): 0\leqq x, y\leqq 1\}$, $C=\{(x, y): x>1 \text{ and } y>1\}$.

EXAMPLE 1. *A class of pure singular distributions.* Let $F(x, y)=0$ on A, $\frac{1}{2}(x+y)$ on B_{xy}, $\frac{1}{2}(x+1)$ on B_x, $\frac{1}{2}(1+y)$ on B_y and 1 on C. It is easily checked that $F(x, y)$ is a distribution function (see [1]). The probability function

$$p(x,y) = F(x,y) - F(x^-,y) - F(x,y^-) + F(x^-,y^-),$$

where, as usual, "$-$" means limit from below, is zero at each point, as is the density function $f(x, y)=(\partial^2F)/(\partial x\partial y)$, where it is defined. Thus, the distribution is pure singular and the only question is—where is the probability mass? Let $L_1=\{(x, y): x=0, 0\leqq y<1\}$ and $L_2=\{(x, y): 0\leqq x\leqq 1, y=0\}$. Then $P[(X, Y)\in L_1]=F(0, 1)-F(0^-, 1)-F(0, 0^-)+F(0^-, 0^-)=1/2-0-0+0$. Similarly $P[(X, Y)\in L_2]=1/2$. Thus, all of the mass is concentrated on these two line segments. Note that the marginals $F(x, \infty)$ and $F(\infty, y)$ are of mixed discrete-absolutely continuous type having a discrete mass of $1/2$ at 0 and the remaining probability uniformly distributed on $[0, 1]$.

Clearly, *any* polynomial in two variables with positive coefficients and no constant or cross product terms can be converted into a pure singular distribution function in the same way. The probabilities on L_1 and L_2 and the marginal densities can be varied by varying the weights and degree of the polynomial.

EXAMPLE 2. *Mixed singular-absolutely continuous distributions.* The distribution of Example 1 is modified by including a cross product term: Let $F(x, y)=0$ on A, $\frac{1}{3}(x+xy+y)$ on B_{xy}, $\frac{1}{3}(1+2x)$ on B_x, $\frac{1}{3}(1+2y)$ on B_y, and 1 on C. Again $p(x, y)\equiv 0$, but now $f(x, y)=1/3$ on B_{xy} and 0 otherwise. This yields a total probability of $\iint f(x, y)dxdy=1/3$ for the absolutely continuous component and singular mass $1/3$ on each of the lines L_1 and L_2. The marginals are again mixed discrete-absolutely continuous. As before, a wide variety of mixtures and marginals can be obtained by taking polynomials of higher degree to define $F(x, y)$ and by varying the weights of the components.

EXAMPLE 3. *Mixed singular-absolutely continuous-discrete distributions.* A discrete mass can be added to either of the above examples by adding a constant term to $F(x, y)$ on A^c: Take $F(x, y)=0$ on A, $\frac{1}{4}(1+x+xy+y)$ on B_{xy}, $\frac{1}{2}(1+x)$ on B_x, $\frac{1}{2}(1+y)$ on B_y, and 1 on C. Now $p(0, 0)=1/4$, $f(x, y)=1/4$ on B_{xy}, 0 otherwise and singular masses of $1/4$ are concentrated on each of L_1 and L_2.

Singular distributions need not always lead to mixed marginal distributions. A "classical" example is the distribution with distribution function $F(x, y) = 0$ on A, $\min(x, y)$ on B_{xy}, x on B_x, y on B_y and 1 on C. It is easily checked that $p(x, y) \equiv 0$ and $f(x, y) \equiv 0$. By evaluating the probabilities of rectangles over the regions where $F(x, y)$ is not constant, it is found that these probabilities are positive only when the rectangles intersect the line segment $\{(x, y): y = x\} \cap B_{xy}$. Thus this line segment has (singular) probability 1. The marginals are clearly those of random variables uniformly distributed on $[0, 1]$; thus, are pure absolutely continuous.

Mixed distributions occur naturally in the field of sequential analysis in statistics. The following is a rather brief description of a simple type of sequential procedure. If $X_1, X_2, \cdots$ are absolutely continuous random variables which constitute the potential observations in an experiment, a sequential decision procedure is based upon a positive integer-valued random variable N called a stopping variable and a sequence of functions $T_1(x_1)$, $T_2(x_1, x_2), \cdots$, $T_n(x_1, \cdots, x_n), \cdots$. In a given "trial" of the experiment, a value of N is realized which determines the number of observations $X_1, X_2, \cdots$ to be made (i.e., when sampling is to stop) and then a "decision" is made on the basis of the value of the random variable $T_N(X_1, \cdots, X_N)$. Under rather weak conditions each $T_n(X_1, \cdots, X_n)$ is absolutely continuous and $P(N < \infty) = 1$. It follows that the joint distribution of the two dimensional random variable $(X, Y) = (N, T_N)$ is singular. This is clear because all of the probability is continuously distributed on the lines $L_n = \{(x, y): x = n\}$, $n = 1, 2, \cdots$, and $\bigcup_{n=1}^{\infty} L_n$ is a set of two dimensional Lebesgue measure 0.

In most cases of singular distributions there is no convenient function with which probabilities can be calculated as is the case for discrete and absolutely continuous distributions via the probability function and density function. In this case however, probabilities can be conveniently computed by means of a "hybrid" probability density-probability function $\mathrm{pf}(n, y) = (d/dy)P(X = n, Y \leqq y)$. Simple examples of this type can be easily constructed for classroom demonstration. Note that the X marginal is pure discrete, and the Y marginal is absolutely continuous by Beppo Levi's theorem ([2] p. 35–6).

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PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before July 31, 1969. Contributors (in North America) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2157. *Proposed by D. M. Bloom, Brooklyn College*

Prove that S_n is false for all n such that $17 \leqq n \leqq 1000$, where S_n is the statement: Every set of n consecutive integers contains an integer which is relatively prime to the others in the set.

E 2158. *Proposed by Gregory Wulczyn, Bucknell University*

For what integral values of $n>1$ will there be a finite or infinite number of solutions a, b, to the Diophantine equation

$$1\cdot a^2 + 2(a+1)^2 + 3(a+2)^2 + \cdots + n(a+n-1)^2 = b^2.$$

E 2159. *Proposed by G. M. Lee, San Mateo, California*

Prove:

$$\sum_{r=1}^{n} (-1)^{t-1} \frac{1}{r} \sum_{i=1}^{r} (-1)^{t-1} \binom{r}{i} i^n = 0,$$

where $n = 2, 3, 4, \cdots$.

E 2160. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let p_i, x_i be the distances of an interior or a boundary point P of a triangle $A_1A_2A_3$ from the vertex A_i and the side opposite to A_i, $i=1, 2, 3$, with r the inradius. Prove the inequalities

(a) $$\sum_{i=1}^{3} p_i(\tfrac{1}{2}\sin A_i) \leqq \sum_{i=1}^{3} x_i \leqq \sum_{i=1}^{3} p_i \sin(\tfrac{1}{2}A_i).$$

(b) $$p_2p_3 + p_3p_1 + p_1p_2 \geqq 8x_1x_2x_3/r.$$

E 2161. *Proposed by T. Kaucký, Slovak Academy of Sciences, Bratislava*

Let α be an arbitrary number. Evaluate the determinant

$$D(\alpha, s) = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & \binom{\alpha+1}{1} \\ 0 & 0 & \cdots & 1 & \binom{\alpha+2}{1} & \binom{\alpha+2}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \binom{\alpha+s-1}{1} & \cdots & & \binom{\alpha+s-1}{s-2} & \binom{\alpha+s-1}{s-1} \\ \binom{\alpha+s}{1} & \binom{\alpha+s}{2} & \cdots & & \binom{\alpha+s}{s-1} & \binom{\alpha+s}{s} \end{vmatrix}$$

E 2162. *Proposed by D. Rameswar Rao, Osmania University, India*

Let p_i be distinct prime numbers. Show that

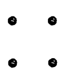
$$\frac{(p_1+1)(p_2+1)\cdots(p_n+1)}{p_1\cdot p_2\cdots p_n} \leqq 2 \leqq \frac{p_1\cdot p_2\cdots p_n}{(p_1-1)(p_2-1)\cdots(p_n-1)}$$

is a necessary condition for $A = \prod_{i=1}^{n} p_i^{k_i}$ to be perfect.

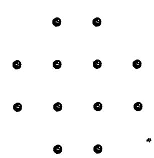
E 2163. *Proposed by E. F. Bell, Washington and Jefferson College*

Given $8n-4$ points arranged in the form of a cross, e.g.,

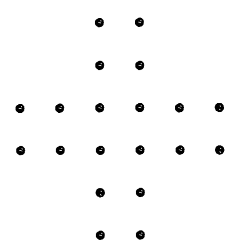
$n = 1$,



$n = 2$,



$n = 3$,



What is the largest number of squares which can be superimposed on the nth cross figure with each vertex of each square on one of the $8n-4$ points?

2164. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha*

Let $x_n = p_1 + p_2 + \cdots + p_n$, where $p_1, p_2, \cdots, p_n$ are the first n primes. Prove that between x_n and x_{n-1} there always lies a square number.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Fixed-point Property

E 2063 [1968, 189]. *Proposed by W. G. Dotson, Jr., North Carolina State University*

Prove: If S is a set of real numbers such that every continuous function from S into S has a fixed point, then S either consists of a single point or is a closed bounded interval.

Solution by K. D. Juhlin, Student, University of Illinois.

(1) S is connected: Assume S has more than one point, and suppose there exist $a, b \in S$ and $x \in R$ such that $a < x < b$, but $x \notin S$. Define $f: S \to S$ as

$$f(y) = \begin{cases} b & \text{if } y < x \\ a & \text{if } y > x. \end{cases}$$

Clearly f is continuous, but has no fixed point.

(2) S is bounded: If not, then either $f(y) = y+1$ or $f(y) = y-1$ will work (since we have the result of (1)).

(3) S is closed: If not, then either (i) $S = (a, b)$, (ii) $S = [a, b)$, or (iii) $S = (a, b]$. In case (i) or (iii) define $f(x) = \frac{1}{2}(x+a)$, or in case (ii) define $f(x) = \frac{1}{2}(x+b)$. Again it is easy to see that f is continuous but has no fixed point.

Also solved by A. G. Aldridge & J. S. Pierce, Stephen Berman & Steven Mensker, W. D. Bouwsma, Sung-sheng Chang (Taiwan), Orin Chein, F. A. Climenti, D. M. Cohen, Gertrude Ehrlich, M. A. Ettrick, William Fox, Marvin Gruber, C. V. & G. A. Heuer, R. A. Jacobson, Donald Jeffords, Erwin Just & Bertram Kabak, H. E. Lahmann (Germany), Eric Langford, B. L. Lientz, W. G. McArthur, Kenneth Miller, J. C. Morgan II, D. E. Penney, T. M. Phillips, Ira Rosenholtz, Steven Russ, B. T. Sims, P. K. Subramanian, Paul Sugarman, Linda E. Wells, and the proposer.

Editorial Note. Several solvers observe that the converse of the proposition holds. Langford proposes this generalization: Suppose S is a subset of E^n such that every $f: S \to S$ has a fixed point. Need S be compact and connected? More generally, for what topological spaces is there an affirmative answer?

A Recursion Formula

E 2065 [1968, 292]. *Proposed by R. R. Poole, University of Redlands, Cal.*

Find a solution for the recursion formula $x_n = (n-1)(x_{n-1} + x_{n-2})$, $n \geqq 4$, $x_2 = 1$, $x_3 = 2$.

Solution by Anders Bager, Hjørring, Denmark. Putting $x_k = k! y_k$ we get

$$y_k - y_{k-1} = -\frac{1}{k}(y_{k-1} - y_{k-2}), \qquad k \geqq 4. \tag{1}$$

Using (1) for $k = 4, 5, \cdots, m$ we get

$$y_m - y_{m-1} = (-1)^m/m!, \qquad m \geqq 4. \tag{2}$$

Summing (2) for $m=4, 5, \cdots, n$ we get finally

$$x_n = n! \sum_{m=2}^{n} \frac{(-1)^m}{m!},$$

which is easily seen to be true also for $n=2, 3$.

Also solved by forty-three other readers.

Editorial Note. This problem is not new. Alexander Zujus points out that it appears as "the Bernoulli-Euler problem of the misaddressed letters" in H. Dorrie, *100 Great Problems of Elementary Mathematics.* Henry Ricardo observes that it is solved as Example 2, p. 366 of Brand, *Differential and Difference Equations.* Several other references were cited.

Square Roots of the 3×3 Identity Matrix

E 2066 [1968, 293]. *Proposed by George Grossman, Board of Education, New York City*

Find the most general square root of the 3×3 identity matrix if the elements are to be (a) integers, (b) any real numbers, (c) complex numbers.

Solution by D. C. B. Marsh, Colorado School of Mines. If X is a square root of the 3×3 identity matrix, then we have $(X-I)(X+I)=0$. Since $\rho(AB)\geqq\rho(A)+\rho(B)-n$ for any two $n\times n$ matrices ($\rho(A)$ is the rank of A), we must have at least one factor with rank $\leqq 1$. If $\rho(X+I)=1$, then $X+I=(a, b, c)^T(x, y, z)$, whence

$$\begin{aligned} 0 = (X+I)(X-I) &= (a, b, c)^T(x, y, z)[(a, b, c)^T(x, y, z) - 2I] \\ &= (a, b, c)^T[(x, y, z)(a, b, c)^T - 2](x, y, z) \\ &= (a, b, c)^T[ax + by + cz - 2](x, y, z). \end{aligned}$$

Thus we need $ax+by+cz=2$. The case $\rho(X-I)=1$ is treated similarly.

Thus all square roots are given by $\pm I$, $\pm\{I-(a, b, c)^T(x, y, z)\}$ with $ax+by+cz=2$. Integral (real, complex) roots occur for a, b, c, x, y, z integral (real, complex) respectively.

Also solved by J. V. Michalowicz, C. F. Pinzka (Australia), and E. T. Treebrook.

NOTE. See also Problem E 1882 [1967, 1139] and E 1942 [1968, 409]. The latter provides text references to the more general problem of finding the mth root of an $n\times n$ matrix.

Near Simple Groups

E 2067 [1968, 293]. *Proposed by Colonel Johnson, Jr., Southern University, Baton Rouge, La.*

We say that a group is near-simple if it has exactly one proper normal subgroup. Give a characterization of the near-simple Abelian groups.

Solution by Azriel Rosenfeld, University of Maryland. Since each subgroup of an Abelian group is normal, we have to characterize the Abelian groups G which have exactly one proper subgroup. In fact, we can do this without the Abelian

assumption. Let G have exactly one proper subgroup H, and suppose that G is not cyclic: then we have $(x)=H$ for each $x\neq e$ in G, so that every element of G is in H, contradiction. Thus G is cyclic and, readily, it must have order p^2 for some prime p.

Also solved by Anders Bager (Denmark), R. W. Ball, W. S. Butler, J. P. Comiskey, J. H. C. Creighton, Thomas Elsner, César Fernández (Chile), R. M. Firestone, C. M. Geschke, M. G. Greening (Australia), D. A. Hejhal, C. V. Heuer, R. A. Jacobson, Donald Jeffords, Erwin Just, Roman Kaluzniacki, Geoffrey Kandall, J. M. Katz, Peter Kornya, Eric Langford, F. M. Markel, W. G. McArthur, D. C. B. Marsh, W. A. Thrash, Jr., E. J. Treebrook, Z. Z. Uoiea, W. W. Whitman, Mrs. Barbara Yanosko, and the proposer.

A Five-digit Number

E 2068 [1968, 293]. *Proposed by S. R. Conrad, Francis Lewis High School, Flushing, N. Y.*

The five-digit number (in decimal notation) $xy57z$ is divisible by 729. Find x, y, z.

Solution by Paul Sugarman, Student, Massachusetts Institute of Technology. The only five-digit numbers with a 7 in the tens place that are divisible by 729 are 21870, 26973, 32076, 37179, 49572, 54675, 59778, 72171, 77274, 82377, 94770, and 99873. The only one of these that is of the form $xy57z$ is 49572. Hence $x=4$, $y=9$, $z=2$.

Also solved by seventy-six other readers.

A Tower of x's

E 2069 [1968, 293]. *Proposed by R. P. Sheets, University of Chicago*

Let it be defined that $x_0=1$, $x_{n+1}=x^{x_n}$, $n=0, 1, \cdots$. For $n=1, 2, \cdots, 8$ it seems that the derivative of x_n is given by

$$\frac{d}{dx}(x_n) = \frac{1}{x}\sum_{j=0}^{n-1}(\ln x)^j \prod_{i=n-j-2}^{n-1} x^{x_i},$$

where for $j=n-1$, the index i takes only the values $0, 1, \cdots, n-1$. Prove (or disprove) the formula in general.

Solution by Kenneth Miller, Macalester College. It will be shown by induction that the formula is valid in general. For $n=1$, the right side of the formula reduces to

$$\frac{1}{x}(\ln x)^0 x^{x_0} = \frac{x}{x} = 1$$

which equals $d(x_1)/dx$. If we assume the validity of the formula for $n=m$, then for $n=m+1$,

$$\begin{aligned}\frac{dx_{m+1}}{dx} &= \frac{\partial x_{m+1}}{\partial x} + \frac{\partial x_{m+1}}{\partial x_m}\frac{dx_m}{dx}\\ &= x_m x^{x_m-1} + (\ln x)x^{x_m}\frac{1}{x}\sum_{j=0}^{m-1}(\ln x)^j \prod_{i=m-j-2}^{m-1} x^{x_i}\\ &= \frac{1}{x}x_m x^{x_m} + \frac{1}{x}\sum_{j=0}^{m-1}(\ln x)^{j+1}\prod_{i=m-j-2}^{m} x^{x_i}\\ &= \frac{1}{x}(\ln x)^0 x^{x_{m-1}}x^{x_m} + \frac{1}{x}\sum_{j=1}^{m}(\ln x)^j\prod_{i=m-j-1}^{m} x^{x_i}\\ &= \frac{1}{x}\sum_{j=0}^{m}(\ln x)^j\prod_{i=m-j-1}^{m} x^{x_i}.\end{aligned}$$

Also solved by Anders Bager (Denmark), Bruce Berndt, Arthur Bolder, W. D. Bouwsma, W. G. Brady, D. Ž. Djoković, Thomas Elsner, R. F. Emnett, R. A. Jacobson, Donald Jeffords, Roman Kaluzniacki, B. W. King, Eric Langford, Terry Mackin, Beatriz Margolis (Argentina), Norman Miller, Ž. M. Mitrović (Yugoslavia), Simeon Reich (Israel), Henry Ricardo, Chanchal Singh, Paul Sugarman, R. G. Van Meter, and David Zeitlin.

Several readers note that this problem is the same as Problem 653, Mathematics Magazine, 40 (1967) 283–4.

The Square of the Multiplication Table of GF(q)

E 2070 [1968, 293]. *Proposed by J F. Burke, University of Vermont*

Let A be the matrix formed from the elements of the multiplication table of the multiplicative group of a Galois field GF(q), where $q=p^n$, p a prime and n a positive integer. Show that for $q\geqq 4$, $A^2=[\theta_{ij}]$, where $\theta_{ij}=0$.

Solution by R. G. Van Meter, St. Lawrence University. Suppose $q\geqq 4$ and c is a generator of the (cyclic) multiplicative group of GF(q). Let A be the matrix of a multiplication table of this group. Then $A=[a_{ij}]$, where $a_{ij}=c^{\beta_i}c^{\beta_j}$ for all i, $j\in\{1,\cdots,q-1\}$ and some $\beta_1,\cdots,\beta_{q-1}$ such that $\{\beta_1,\cdots,\beta_{q-1}\}=\{1,\cdots,q-1\}$. Hence $A^2=[\theta_{ij}]$, where

$$\theta_{ij} = \sum_{k=1}^{q-1} c^{\beta_i+\beta_k}c^{\beta_k+\beta_j} = c^{\beta_i+\beta_j}\sum_{k=1}^{q-1} c^{2\beta_k} = c^{\beta_i+\beta_j}\sum_{k=1}^{q-1} c^{2k}$$

for all $i, j\in\{1,\cdots,q-1\}$. As the order of c is $q-1$ $(\geqq 3)$, $c^2\neq 1$; hence $\theta_{ij}=c^{\beta_i+\beta_j}[(c^2)^q-c^2]/(c^2-1)$ for $i, j\in\{1,\cdots,q-1\}$. As $(c^2)^q-c^2=c^2-c^2=0$ in GF(q), this proves the desired result for *each* of the $(q-1)!$ possible multiplication tables.

By examining all (three) possible tables for $q=2$ and $q=3$, we see that this result does not hold; in fact, for all appropriate (i, j), $\theta_{ij}\not\equiv 0 \pmod{p}$.

Also solved by Anders Bager (Denmark), P. A. Catlin, M. E. Chernesky, D. Ž. Djoković, N. J. Fine, M. G. Greening (Australia), Donald Jeffords, Roman Kaluzniacki, and C. G. Wagner.

Primitive mth Roots of Unity

E 2071 [1968, 293]. *Proposed by C. J. Mozzochi, University of Connecticut*

Let $P=\{e^{ix_1}, \cdots, e^{ix_n}\}$ be any partition of the unit circle $(0=x_0<x_1<\cdots<x_n=2\pi)$. Prove, or disprove: There exists a composite integer m such that in each cell of P there exists at least one primitive mth root of unity.

Solution by Eric Langford, Naval Postgraduate School, Monterey, Calif. Choose a prime p such that $4\pi/p<\min(x_{i+1}-x_i)$. By this choice of p, there are at least two pth roots of unity in each cell. Call any two such successive roots $e^{2\pi ik/p}$ and $e^{2\pi i(k+1)/p}$. Since $k/p=kp/p^2<(kp+1)/p^2<[(k+1)p]/p^2=(k+1)/p$, and $kp+1$ and p^2 are relatively prime, $e^{2\pi i(kp+1)/p^2}$ is a primitive p^2th root between the successive pth roots. $m=p^2$ is composite and satisfies the conditions.

Also solved by Anders Bager (Denmark), H. M. Edgar, J. E. Hafstrom, D. A. Hejhal, C. V. Heuer, R. A. Jacobson, Donald Jeffords, and Erwin Just.

Note. There are other suitable choices of m. Thus $m=2p$ was a favorite.

An Old Problem

E 2072 [1968, 293]. *Proposed by Dorembus Leonard, Tel-Aviv University, Israel*

Find necessary and sufficient conditions for a $k\times n$ matrix $(k<n)$ with integral elements, in order that it be a submatrix of an integral $n\times n$ matrix with determinant 1.

Solution by R. C. Thompson, University of California at Santa Barbara. Let A be the $k\times n$ matrix with integral elements. The necessary and sufficient condition is that the greatest common divisor of the $k\times k$ minors of A be one. It is an old and well-known result [B. W. Jones, *The Arithmetic Theory of Quadratic Forms*, (Carus Monographs No. 10), p. 62; C. C. MacDuffee, *The Theory of Matrices*, p. 31] that this condition is sufficient. The necessity can be seen as follows: let B be the $n\times n$ integral matrix with A as the first k rows, and det $B=1$. By a Laplace expansion of det B across its first k rows, it is easy to see that an integral linear combination of the $k\times k$ minors of A equals one.

The special case $k=1$ of this problem has already appeared twice in this MONTHLY. See E 1911 [1968, 81] and the remarks following the solution.

Also solved by A. Zujus, and by the proposer.

Editorial Note. The result is also known for matrices with coefficients in a principal ideal domain. It is best proved by using exterior algebra and the theorem that each submodule of a free module is free.

A Set of Limit Points

E 2073 [1968, 293]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, N. Y.*

Let I be the real line, and let S be the set of points each of whose distances

from the origin is the sum of any two terms of the sequence $\{1/n\}$, $n=2, 3, \cdots$. Find the set of limit points of S in I.

Solution by Dennis McMacken, Northern Arizona University. Clearly we have

$$\left\{\pm \frac{1}{n}\right\} \cup \{0\} \subset S' \qquad \text{since } \lim_{m\to\infty} \pm\left(\frac{1}{n}+\frac{1}{m}\right) = \pm \frac{1}{n}$$

and

$$\lim_{n,m\to\infty} \pm\left(\frac{1}{n}+\frac{1}{m}\right) = 0.$$

We show that no other point is a limit point of S. Obviously, no point whose distance from the origin is greater than 1 will be a limit point.

Let $0<r\leqq 1$, $r\notin\{1/n\}$ and suppose that $1/k$ is the largest element of $\{1/n\}$ less than r. [For $-1\leqq r<0$, $r\notin\{-1/n\}$ the proof goes through with the obvious changes.] The elements of S which are contained in the neighborhood $N(r, \delta)$, where

$$\delta = \min\left(\frac{r-(1/k)}{2}, \frac{1/(k-1)-r}{2}\right),$$

are a subset of the finite set $\{(1/n)+(1/m): 1<n, m\leqq T\}$, where T is the largest integer such that $((1/k)+(1/T))\in N(r, \delta)$. Since we can find a neighborhood of r which contains only a finite number of elements of S, r is not a limit point of S. Thus the set of limit points of S is $\{\pm 1/n\}\cup\{0\}$.

Also solved by Bernard Arbic, Anders Bager (Denmark), W. D. Bouwsma, D. L. Carlson, F. D. Cheek II, Thomas Elsner, R. F. Emnett, M. A. Ettrick, Horacio Feliciángeli (Paraguay), William Fore, Michael Goldberg, M. G. Greening (Australia), Marvin Gruber, D. A. Hejhal, C. V. Heuer, G. A. Heuer, R. A. Jacobson, Donald Jeffords, Roman Kaluzniacki, D. Z. Kilhefner, Mark Mandelker, J. V. Michalowicz, Thomas Mickewich, Norman Miller, Bohuslav Míšek (Czechoslovakia), T. M. Phillips, E. J. Treebrook, David Wille, and the proposers.

NOTE. Many readers observed that essentially the same problem appears in *Topology* by J. Dugundji, p. 91, no. 6, and in *Mathematical Analysis* by Apostle, p. 58, no. 3–2g.

A Slope Condition For Parabolas

E 2074 [1968, 294]. *Proposed by Erwin Just and John Furst, Bronx Community College, N. Y.*

Let m_i denote the slope of the line containing the points P_i and P_{i+1}, $(i=0, 1, \cdots, n)$ with $P_{n+1}=P_0$. If $f(x)$ is not a constant and f is defined for all real x, prove that a necessary and sufficient condition for the graph of $y=f(x)$ to be a parabola is that there exists a point P_0 on the graph of f such that for any set of n points $P_1, \cdots, P_n$ on the graph, $\sum_{i=0}^{n}(-1)^i m_i=0$.

Solution by M. G. Greening, University of New South Wales, Australia. (i) If

the graph is a parabola, then $f(x)=ax^2+bx+c$, with $-\infty<x<+\infty$ as its domain. Then

$$\sum_{i=0}^{n}(-1)^i m_i = \sum_{i=0}^{n}\{a(x_i+x_{i+1})+b\}(-1)^i = \begin{cases} 0 & \text{for } n \text{ odd} \\ 2ax_0+b & \text{for } n \text{ even.}\end{cases}$$

As $2ax_0+b=f'(x_0)$, it is sufficient to take P_0 as

$$\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right).$$

(ii) Taking P_0 as the origin and $n=2$, we derive

$$\frac{y_1}{x_1}-\frac{y_2-y_1}{x_2-x_1}+\frac{y_2}{x_2}=0.$$

Keeping P_2 constant, we find that (x_1, y_1) must satisfy

$$y=\frac{1}{x_2}x^2+\left\{\frac{x_2}{y_2}-\frac{y_2}{x_2}\right\}x,$$

the graph of a parabola defined for all real x. The given condition is therefore both necessary and sufficient.

Also solved by Anders Bager (Denmark), W. D. Bouwsma, Ted Cullen, G. A. Heuer, B. E. Litov, W. D. Markel, Thomas Mickewich, Norman Miller, E. J. Treebrook, and the proposer.

$4^x+4^y+4^z$ = a Square

E 2075 [1968, 403]. *Proposed by S. R. Conrad, Francis Lewis High School, Flushing, N. Y.*

Find all integral values of x, y, z for which $4^x+4^y+4^z$ is a perfect square.

Solution by Eleanor G. Jones, Virginia State College. If $x\leqq y\leqq z$, then $4^x+4^y+4^z$ is a perfect square implies that there exist a positive integer m and a positive odd integer t such that

$$1+4^{y-x}+4^{z-x}=(1+2^m t)^2.$$

Thus,

$$4^{y-x}(1+4^{z-y})=2^{m+1}t(1+2^{m-1}t)$$

which indicates that $m=2y-2x-1$. If follows that

$$\begin{aligned} t-1 &= 4^{y-x-1}(4^{z-2y+x+1}-t^2) \\ &= 4^{y-x-1}(2^{z-2y+x+1}+t)(2^{z-2y+x+1}-t), \end{aligned}$$

which implies $t=1$, and consequently $z=2y-x-1$. Hence the only integral solutions are $\{x, y, 2y-x-1\}$, with x, y arbitrary. Finally these do produce the square $(2^x+2^{2y-x-1})^2$.

Also solved by fifty-six other readers.

NOTE. Most of the solutions were incomplete. The most common failing being that it was assumed that $n^x+n^y+n^z$ is a perfect square only if it is of the form $a^2+2ab+b^2$. This is not true, e.g., $2^1+2^1+2^5$, $2^0+2^9+2^4$, $3^1+3^1+3^1$, $12^0+12^2+12^2$. It does happen to be true for $n=4$ as the above proof shows.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before August 31, 1969. Contributors (in North America) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5658. *Proposed by Stephen Gelbart, Princeton University*

Let E be a locally convex topological vector space, and E' its topological dual equipped with the weak topology $\sigma(E', E)$. Is it true that E nonmetrizable implies E' nonmetrizable?

5659. *Proposed by K. L. Singh, Memorial University of Newfoundland*

Prove that a continuous rectifiable curve in a uniformly convex space possesses a tangent at almost all of its points.

5660. *Proposed by W. G. Dotson, Jr., North Carolina State University*

(1) Show that if T is a continuous function from the reals to the reals and there is a number x such that the sequence $\{(x+Tx+T^2x+\cdots+T^{n-1}x)/n\}$ of Cesàro means of the Picard iterates is bounded, then T has a fixed point. (2) Find a continuous function T from the reals to the reals which has a unique fixed point p, and such that if $x\neq p$ then the sequence $\{(x+Tx+T^2x+\cdots+T^{n-1}x)/n\}$ is bounded but does not converge to p.

5661. *Proposed by G. J. Foschini, Bell Telephone Laboratories, Holmdel, N. J.*

Find all solutions in the complex plane of $z=\sum_{q=2}^{\infty}\sum_{p=1}^{q}e^{2\pi ipz/q}$.

5662. *Proposed by Irving Kaplansky, University of Chicago*

Let R be an associative ring with unit. Suppose in R whenever $ab=1$ then $ba=1$. Prove the same thing holds in $R[x]$, the polynomial ring over R.

5663. *Proposed by D. Ž. Djoković, University of Waterloo, Ontario*

Show that

$$\sum\frac{1}{p_1!p_2!\cdots p_n!}=\frac{1}{k!}\binom{n-1}{k-1},$$

where the summation is over all nonnegative integers $p_1, p_2, \cdots$, such that $p_1+2p_2+\cdots+np_n=n$, $p_1+p_2+\cdots+p_n=k$.

5664. *Proposed by Harry Pollard, Purdue University*

Suppose a particle is attracted to a fixed center O by a force proportional to the inverse cube of its distance r from O. Without determining the possible orbits, show that one of these three things must occur:

(a) the particle moves in a circle;

(b) the particle collides with O in a finite time;

(c) $r \to \infty$ as $t \to \infty$.

Conclude that the inverse cube law is a poor substitute for the inverse square law in designing a solar system.

SOLUTIONS OF ADVANCED PROBLEMS

A Generalized Mean Value Theorem

5526 [1967, 1014]. *Proposed by R. O. Davies and Allan Hayes, The University, Leicester, England.*

If P is due south of Q, then anyone travelling from P to Q on a Friday, without either crossing his own path or stopping, must at some instant be travelling due north. Discuss this assertion.

I. *Solution by Steven Bank, University of Illinois.* The solution is given in Lemma 1, p. 303, of E. W. Chamberlain, *The univalence of functions asymptotic to nonconstant logarithmic monomials*, Proc. A. M. S., Vol. 17, No. 2 (1966), pp. 302–309. This lemma states that if $z(t)=x(t)+iy(t)$ (for $t\in[0, 1]$) is a continuous curve in the plane which is 1-1 and is such that $z'(t)$ is nowhere zero on $(0, 1)$, then for some $t_0\in(0, 1)$, the oriented tangent vector to the curve at $z(t_0)$ is in the same direction as the vector from $z(0)$ to $z(1)$.

II. *Solution by the proposers.* In taking a spiral path around the globe one would cross the International Date Line, whereupon the day would change from Friday, unless one crossed exactly at midnight, which we shall regard as a foul and ignore. In leaving the South or approaching the North Pole at finite speed, one would be travelling due north, so we may assume that neither P nor Q is at a pole. After flattening, the required theorem therefore reduces to: *On every sufficiently smooth simple planar arc PQ there is a point at which the forward tangent is in the direction from P to Q.* (If the words *simple* and *forward* are dropped this is essentially Lagrange's Mean Value Theorem.)

Call a sub-arc $P'Q'$ (P' reached before Q') *admissible* if the direction from P' to Q' is the same as that from P to Q; call the time from P' to Q' its *duration.*

LEMMA. *PQ contains admissible sub-arcs of arbitrarily small durations.*

Proof. If not, a limiting argument yields an admissible sub-arc $P'Q'$ of smallest duration. Then $\gamma=P'Q'$ cannot contain any other point R' on the line $P'Q'$, otherwise one of the arcs $P'R'$, $R'Q'$ would be admissible and of smaller duration; hence γ lies entirely on one side of the line $P'Q'$. Now any sufficiently close parallel line on this side must intersect γ in points P'', Q'' near P', Q' re-

spectively, and then $P''Q''$ is an admissible sub-arc of smaller duration, a contradiction.

REMARK. It follows easily that PQ contains admissible interior sub-arcs (i.e. with neither endpoint at P or Q) of arbitrarily small durations.

Proof of the Theorem. By the Lemma and Remark, PQ contains an admissible interior sub-arc P_1Q_1 of duration less than half that of PQ. By the same (applied to P_1Q_1 instead of PQ), P_1Q_1 contains an admissible interior sub-arc P_2Q_2 of duration less than half that of P_1Q_1; and so on. At the point common to all the arcs P_nQ_n, the forward tangent (assumed to exist everywhere between P and Q) must be in the direction PQ.

NOTES. 1. The proof could be simplified if the tangent were assumed to vary continuously, because it would then no longer be necessary to construct nested sub-arcs P_nQ_n.

2. This simple example shows that the theorem fails if the word simple is deleted.

III. *Solution by Marlow Sholander, Case Western Reserve University.* One can spiral into the north pole through 180° of longitude (avoiding the International Date Line) by a path which meets circles of longitude at angles which decrease to zero, and then head south to Q.

Also solved by Mrs. A. C. Garstang, Michael Goldberg, B. W. Miller, and Oswald Wyler.

Projections of a Simple Closed Curve in 3-Space

5527 [1967, 1014]. *Proposed by Frederic Cunningham, Jr., Bryn Mawr College*

Prove that for every simple closed curve in 3-space there is a plane projection of it having at least two double points (or a multiple point of order greater than two). Give an example of such a curve for which no plane projection has more than two double points (or one triple point).

I. *Solution (to first part) by Harley Flanders, Purdue University.* Let S be the unit circle and $f:S\to E^3$ a one-one C' mapping defining the curve. We define a continuous mapping g on the torus $T=S\times S$ into the projective plane P as follows:

Identify P with the plane at infinity. If $x\neq y$, the secant $f(x)\cup f(y)$ meets P in $g(x, y)$. If $x=y$, the tangent line t_x to the curve at x meets P in $g(x, x)$.

The continuity of g follows from the continuous differentiability of f.

The mapping g cannot be one-one. Suppose it were; then $g:T\to P$ is one-one on a compact space into a Hausdorff space, hence is a homeomorphism on T into P, actually *onto* P since T and P are both two-manifolds and T is compact (invariance of domain). This is impossible since T and P have different homology groups.

Each of the various cases in which g sends two distinct points of T to the same point of P yields a direction of projection onto a plane curve with the desired singularities. Notation: distinct letters = distinct points.

(In each case the minimal possible singularity is listed.)

(i) $g(x, y) = g(z, w)$	Two double points.
(ii) $g(x, y) = g(x, z)$	Triple point.
(iii) $g(x, x) = g(y, z)$	Cusp and double point.
(iv) $g(x, x) = g(y, y)$	Two cusps.

II. *Solution* (*to second part*) *by the proposer.* An example of a curve all of whose plane projections have at most two double points is given parametrically by

$$x = \cos t, \qquad y = \sin t, \qquad z = \sin 2t + \sin 3t.$$

There are no three parallel lines each meeting the curve twice. Since all lines parallel to the z-axis meet the curve only once, we can use azimuth θ and slope relative to the xy-plane to distinguish the direction of lines. The strategy is to take any fixed θ, and for each t compute the slope u of the unique line of azimuth θ meeting the curve in two points, one of which corresponds to the parameter value t, and show that u takes any value for at most two values of t. To do this, first turn the curve around the z-axis by an angle θ, i.e. replace t by $t-\theta$ in z, then compute the slopes of chords parallel to $y=0$. We get

$$u = \frac{\sin 2(t-\theta) + \sin 3(t-\theta) + \sin 2(t+\theta) + \sin 3(t+\theta)}{2 \sin t}$$

$$= 2\cos 2\theta(\cos t) + \cos 3\theta(1 + 2\cos 2t).$$

For each θ we must consider this function on the interval $0 \leqq t \leqq \pi$. Note that the coefficients $2\cos 2\theta$ and $\cos 3\theta$ are never simultaneously zero. The derivative

$$\frac{du}{dt} = -2\sin t(\cos 2\theta + \cos 3\theta \cos t)$$

vanishes at the ends of the interval, and otherwise only at most once. Therefore u is monotone on each of two complementary subintervals, giving the desired result.

Totally Symmetric Loops

5567 [1968, 198]. *Proposed by Donna J. Seaman, Olympia College, Bremerton, Wash.*

Given a loop L_n with n elements such that for every pair a, b in L_n (a, b need not be distinct)

$$\text{(i) } a\cdot b = b\cdot a, \qquad \text{(ii) } a\cdot(a\cdot b) = b.$$

(1) For which values of n do such loops exist?

(2) Is there a nonassociative loop L_n with properties (i) and (ii) for which $n=2^k$?

Solution by Jane W. Di Paola, New York University. The loops satisfying (i) and (ii) are called *totally symmetric* and the class of such loops is coextensive with the class of Steiner triple systems (cf. R. H. Bruck, *What is a loop?* MAA Studies in Modern Algebra, New York, 1963). Such loops exist for all values of n such that $n-1\equiv 1$, 3 (modulo 6) since, as shown by Bruck, each L_n gives rise to a Steiner triple system of order $n-1$ and these are known to exist for the stated values of n. Nonassociative loops L_n with properties (i) and (ii) exist for all $n=2^k$ such that k is an integer satisfying $k>3$. This follows from the Assmus-Matson result (Journal of Combinatorial Theory, 1 (1966), 301–305) that there exist at least two inequivalent Steiner triple systems of order 2^k-1 for $k>3$. Each of these yields a loop L_n of order 2^k. For each k, one of these loops is associative and all the others are nonassociative.

This material is explained in detail in the paper, *When is a totally symmetric loop a group?* by Jane W. Di Paola, this issue of the MONTHLY, pp. 249–252.

Also solved by the proposer.

Set of Discontinuities of a Lower Semi-continuous Function

5581 [1968, 414]. *Proposed by J. E. Shirey, Purdue University*

What can be said about the points of discontinuity of a function $f:R\to R$, if every point is a local minimum?

I. *Solution by Charles Riley, Keene (N. H.) State College.* The set $D(f)$ of discontinuities of f must be nowhere dense. If not, let the closure $\overline{D(f)}$ contain the open interval I. In the following, the interval I_n may be taken with length $<1/n$. Take $p_1\in I$, and a closed neighborhood $I_1\subset I$, such that the minimum value of f is $f(p_1)$. I_1^0 (Interior of I_1) contains a point q of D. If $f(q)>f(p_1)$, take $p_2=q$. If $f(q)=f(p_1)$, then I_1^0 contains a point r such that $f(r)>f(q)$, in which case take $p_2=r$. In any case $f(p_2)>f(p_1)$. Let I_2 be a closed neighborhood of p_2, contained in I_1, and on which $f(p_2)$ is the minimum value assumed. Let p_3 be a point of I_2^0 such that $f(p_3)>f(p_2)$. Continue, to generate a nested sequence $\{I_n\}$ of closed intervals and let $\bigcap_{n=1}^{\infty} I_n=\{p\}$. For each n, $f(p)\geqq f(p_n)>f(p_{n-1})$ since $p\in I_n$. Since $\{p_n\}$ is eventually in any neighborhood of p, it is clear that f does not have a local minimum at p.

II. *Solution by Michael von Renteln, Giessen, Germany.* Let F be a nowhere-dense closed set. Then $f(x)=1$ on $R-F$ and $f(x)=0$ on F is a function with the desired characteristic for which $D(f)=f$.

However, $D(f)$ need not be closed, as is seen by taking

$$f(x) = \begin{cases} 1, & x \notin [-\tfrac{1}{2}\cdot\tfrac{1}{2}] \\ 1/n, & x \in \left(\dfrac{1}{n+1}, \dfrac{1}{n}\right) \quad \text{or} \quad x \in \left(-\dfrac{1}{n}, -\dfrac{1}{n+1}\right) \\ 0, & \text{otherwise.} \end{cases}$$

Also solved (partially) by M. D. Mavinkurve (India), J. C. Morgan II, and the proposer.

The question as to whether a preassigned F_σ nowhere dense set can be the exact set of discontinuities remains unanswered.

Eigenvectors of Transposed Matrices

5582 [1968, 414]. *Proposed by Olga Taussky, California Institute of Technology*

Let a_i be complex numbers. Let α be a root of

$$f(x) \equiv a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n = 0.$$

It can be shown easily that $g(x) \equiv f(x)/(x-\alpha) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ where $b_r = \sum_{s\geq 0} a_{r+s+1}\alpha^s$, with $a_n = 1$, $a_{n+1} = a_{n+2} = \cdots = 0$ (see e.g. E. Artin, *Theory of Algebraic Numbers*, Göttingen 1959). Let β be a root of $g(x)$. Let C be the companion matrix of $f(x)$ and C' its transpose. Interpret $\sum_{i=0}^{n-1} b_i\beta^i = 0$ as an orthogonality condition on characteristic vectors of C and C'.

Solution by David Carlson, University of Minnesota, Duluth. Suppose x and y are characteristic vectors of C and C', respectively, and suppose α and β are their characteristic values. Then x and y are orthogonal if β is a root of $g(x)$.

To see this, write $xC = \alpha x$, $yC' = \beta y$ and let $x = [x_1, \cdots, x_n]$ and $y = [y_1, \cdots, y_n]$. A straightforward calculation shows that $x = x_n[b_0, b_1, \cdots, b_{n-1}]$ and $y = y_1[1, \beta, \beta^2, \cdots, \beta^{n-1}]$. Then $xy' = x_ny_1g(\beta) = 0$.

If $\beta \neq \alpha$, the above can be extended: Let A be any $n \times n$ matrix over a field. Suppose x and y are characteristic vectors of A and A', respectively. If their corresponding characteristic values are distinct, then x and y are orthogonal. For if $xA = \alpha x$, $yA' = \beta y$ and $\alpha \neq \beta$, then $\alpha x = xA \Rightarrow \alpha xy' = xAy' = x(yA')' = x(\beta y)' = \beta xy' \Rightarrow xy' = 0$.

Also solved by Harley Flanders, R. C. Thompson, and the proposer.

The Cardinality of Discrete Subsets in a Regular Topological Space

5583 [1968, 414]. *Proposed by S. W. Williams, Lehigh University*

Let D be a dense subset and S a discrete subset of a topological space X. If $|D| = m$ and $|S| > 2^m$ for some cardinal m, then X is not regular. ($|\cdot|$ denotes cardinality; regular does not imply T_1.)

Solution by R. V. Fuller, University of North Carolina. Assume X is regular and $|D| = m$. Then for each $s \in S$ there is a closed neighborhood U_s such that $U_s \cap S = \{s\}$. Let $E_s = D \cap U_s$ for each $s \in S$. Then s is the only limit point of E_s in S. Hence $E_s \neq E_t$ for $s \neq t$. Thus $E: S \to P(D)$ (power set of D) is a 1-1 function. Therefore $|S| \leq |P(D)| = 2^m$.

Also solved by D. H. Anderson, M. A. Ettrick, Donald Hartig, D. A. Hejhal, P. O. Kirley, Dan Marcus, M. D. Mavinkurve (India), Charles Riley, P. S. Schnare, R. E. Smithson, D. P. Sumner, Linda E. Wells, Wan-jui Woan, and the proposer.

Factoring an Abelian Group

5584 [1968, 414]. *Proposed by Erwin Just, Bronx Community College, New York*

Let $H_i (i=1, 2, \cdots, m-1)$ be a set of $m-1$ subgroups of an abelian group G in which $H_j \cap H_k = \{1\}$ for each $j \neq k$. Prove that there exists a subgroup $H_m \subset G$ such that $G/\prod_{i=1}^{m} H_i$ is a torsion group and $H_i \cap H_m = \{1\}$, $1 \leqq i \leqq m-1$.

Solution by D. P. Sumner, University of Massachusetts. Let $C = \{F \subset G : F$ is a subgroup, $F \cap H_i = \{1\}$ for $i=1, \cdots, m-1\}$. $C \neq \varnothing$ since $\{1\} \in C$. Partially order C by set inclusion. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a chain in C and $F = \bigcup_{\lambda \in \Lambda} F_\lambda$. Then F is a group and if $x \in F \cap H_i$ for some i, then $x \in F_\lambda \cap H_i$ for some $\lambda \in \Lambda$, so $x = 1$. Thus $F \in C$. By Zorn's lemma, C contains a maximal element H_m. Let $x \in G$, then $H_m \subset \langle H_m, x\rangle$ (the group generated by $H_m \cup \{x\}$). So, by the maximality of H_m, either $H_m = \langle H_m, x\rangle$ or $\langle H_m, x\rangle \cap H_i \neq \{1\}$ for some i. If $H_m = \langle H_m, x\rangle$, then $x \in H_m$. If $\langle H_m, x\rangle \cap H_i \neq \{1\}$ for some i, then there exists an $h \in H_m$ and integers r and s such that $1 \neq h^r x^s \in H_i$ ($s \neq 0$ since $H_m \cap H_i = \{1\}$). In any event, for every $x \in G$, there exists $n \neq 0$ such that $x^n \in \prod_{i=1}^{m} H_i$, and so $G/\prod_{i=1}^{m} H_i$ is a torsion group.

Also solved by Prabir Bhattacharya (India), P. R. Chernoff, Allan Cooper, M. L. Laplaza (Puerto Rico), M. D. Mavinkurve (India), Charles Riley, Z. Z. Uoiea, and the proposer.

Nonanalytic Functions in C^∞

5585 [1968, 414]. *Proposed by H. D. Keesing, Northern Illinois University*

Let K be the Cantor set on the real line R (the standard one in $[0, 1]$ plus all its integer translates). Suppose $f: R \to R$ is of class C^∞, and that the restriction of f to each component of $R-K$ is real analytic. Prove or disprove that f must be real analytic on R.

Solution by Nicholas Passell, Roosevelt University, Chicago. The well-known function e^{-1/x^2} provides an example which is in C^∞ but which is not analytic at $x=0$. If (a_n, b_n) is an open component of $R-K$, we define $f(x)$ on (a_n, b_n) so that it behaves like e^{-1/x^2} at a_n and at b_n. It is then possible to have $f(x)$ in C^∞ on R, analytic on $R-K$; moreover each point of K is a singularity of $f(x)$.

Also solved by R. P. Boas, P. R. Chernoff, W. G. Dotson, Jr., Harley Flanders, G. J. Foschini, D. A. Hejhal, D. A. Herrero, M. D. Mavinkurve (India), George Piranian, Charles Riley, and the proposer.

Entire Functions

5586 [1968, 415]. *Proposed by H. D. Keesing, Northern Illinois University*

Let U be a dense open subset of E^2. Suppose $f: E^2 \to E^2$ is of class C^∞ and $f|U$ is complex analytic. Prove or disprove that f must be entire.

Solution by D. A. Herrero, University of Chicago. Under the conditions of the statement, f is an entire function. Furthermore, it is sufficient to assume that f is complex analytic in the dense open set U and $f\in C^1(\mathcal{C})$, ($\mathcal{C}$ = the complex numbers). In fact, if $f\in C^1$ then we proceed by examining the continuous function

$$k(x, y) = \left|\frac{\partial \operatorname{Re}(f)}{\partial x} - \frac{\partial \operatorname{Im}(f)}{\partial y}\right| + \left|\frac{\partial \operatorname{Im}(f)}{\partial x} + \frac{\partial \operatorname{Re}(f)}{\partial y}\right|.$$

Since $k(x, y)$ is identically zero on the dense subset U, the Cauchy-Riemann conditions are obviously fulfilled in all of C^2 and f is entire.

Also solved by P. R. Chernoff, W. G. Dotson, Jr., M. A. Ettrick, G. J. Foschini, D. A. Hejhal, M. D. Mavinkurve (India), Charles Riley, and the proposer.

ϵ-Numbers

5587 [1968, 415]. *Proposed by G. F. Schumm, University of Chicago*

Let $\beta\neq 0$ be an ordinal number of the second kind. (β is of the second kind if it has no predecessor.) Then for $\gamma\leqq\omega^\alpha$ and $\alpha\neq 0$, prove that $(\omega^\alpha+\gamma)^\beta=\alpha\beta$ if and only if $\alpha\beta$ is an ϵ-number.

Solution by A. L. Rubin, Purdue University. If $\gamma=\omega^\alpha$ and β is a limit ordinal then it follows from Theorem 9.1.8(a), p. 227 of Rubin, *Set theory for the Mathematician* (Holden-Day, San Francisco (1967)) that

$$(\omega^\alpha + \gamma)^\beta = \omega^{\alpha\beta}.$$

Thus, by the definition of an ϵ-number, $(\omega^\alpha+\gamma)^\beta=\alpha\beta$ if and only if $\alpha\beta$ is an ϵ-number.

Also solved by Allan Cooper, T. E. Gantner, E. C. Milner, C. R. Platt, Charles Riley, and the proposer.

Platt notes that by expressing γ in normal form, the proposition can be obtained replacing the given condition with $\gamma<\omega^{\alpha+1}$.

Behavior of a Seminorm

5588 [1968, 415]. *Proposed by C. B. Mehr, Ohio University, Athens*

Let X be a real linear space and f a seminorm on X. Let $x, y\in X$ and $a\in R$. Prove

$$\lim_{n\to\infty} f[(n + a)x + y] - f(nx + y) = af(x).$$

Solution by W. G. Dotson, Jr., North Carolina State University. For all positive integers $n>-a$ we have

$$\begin{aligned}&| f[(n + a)x + y] - f(nx + y) - af(x) |\\ &\quad= \left| (n + a)f\left(x + \frac{1}{n + a} y\right) - nf\left(x + \frac{1}{n} y\right) - af(x)\right|\end{aligned}$$

$$\leqq n\left|f\left(x+\frac{1}{n+a}y\right)-f\left(x+\frac{1}{n}y\right)\right|+|a|\cdot\left|f\left(x+\frac{1}{n+a}y\right)-f(x)\right|$$

$$\leqq nf\left(\frac{1}{n+a}y-\frac{1}{n}y\right)+|a|\cdot f\left(\frac{1}{n+a}y\right)$$

$$= nf\left(\frac{-a}{n(n+a)}y\right)+\frac{|a|}{n+a}f(y)=\frac{2|a|}{n+a}f(y)\to 0 \qquad \text{as } n\to\infty.$$

Also solved by R. G. Bilyeu, P. R. Chernoff, Mary R. Embry, N. L. J. Hautus, D. A. Hejhal, D. A. Herrero, T. M. Phillips, D. E. Putnam, Charles Riley, C. V. L. Smith, B. L. D. Thorp (Wales, U.K.), J. K. Washenberger, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, Carleton College

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital C in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.

Basic Real and Abstract Analysis. By John F. Randolph. Academic Press, New York, 1968. ix+515 pp. $14.00. (Telegraphic Review, Oct. 1968.)

As a coauthor of a text having virtually the same title as this volume, the reviewer opened this text with some misgivings. However, the book *is* accurately described by its title. It is written by a man whose first love seems to be classical analysis, and it contains an immense amount of it. The book is designed to follow a two-year calculus sequence. Good third-year students, under the guidance of a careful teacher, should have no trouble with the concrete parts. The chapter on measure theory may prove overly abstract for immature minds. Standard "real variable" topics are treated: construction of the real numbers; a bit about cardinal numbers; metric spaces; sequences and series; Lebesgue measure and Lebesgue integration on the line; some abstract measure theory; Fubini's Theorem; continuity; derivatives; Stieltjes integrals. The book contains an amazing collection of problems. The most avid of problem-solving students will find plenty here to keep him busy.

If the book lacks anything, it may be a sense of unity. The author rides a number of pet topics to what may be an unwarranted extent in a book designed

for university juniors. For example, ergodic theory occupies 27 pages, and many other recondite topics are taken up. The trigonometric, exponential, and logarithmic functions are treated with precision, but only in the next to the last chapter. It is almost as if the author had put in everything he could think of, instead of selecting judiciously what he believes to be the basic facts of analysis.

Of the traditional topics for advanced calculus texts, one notices the absence of the theorems of Gauss, Green, and Stokes. Among latterly fashionable topics for advanced calculus texts, one notices the absence of the Hahn-Banach Theorem, and indeed of any part of functional analysis.

With omissions and additions, *ad libitum*, a teacher will be able to use this text to make a highly effective advanced calculus course.

EDWIN HEWITT, University of Washington

A Second Introduction to Analytic Geometry. By G. Hochschild (Univ. of California, Berkeley). Holden-Day, San Francisco, 1968. 63 pp. $4.50 (paper). (Telegraphic Review, Nov. 1968.)

The usual treatment of analytic geometry is based upon synthetic Euclidean geometry and proceeds without reference to the related subjects of linear algebra, group theory, and analysis. On the other hand, courses in these more advanced topics only rarely discuss applications of these topics to geometry. The purpose of this book is to fill this "geometry gap." It should be useful for students who wish to do transitional reading at this stage of their careers, but it can hardly be considered to be a text in the ordinary sense.

The content is elementary, but the exposition is sophisticated. The first chapter is a quick summary of real numbers, sets, maps, groups, and vector spaces. The second chapter treats the plane as a two-dimensional real vector space with an inner product. Lines are defined as translates of the one-dimensional vector subspaces. Rotations and reflections are special cases of distance preserving maps. The exponential function is rigorously introduced and is used to define sine and cosine and the measure of an angle.

The third chapter treats three-space in a similar fashion. A novel feature is the introduction of quaternions and their use to define the concepts of vector product and determinant.

The lack of an index prevents the author from giving an appropriate reference to Halmos. (*Editor's Note*: See the entry Hochschild in *Finite Dimensional Vector Spaces* (1958) by Halmos.)

C. B. ALLENDOERFER, University of Washington

Foundations of Euclidean and Non-Euclidean Geometry. By Ellery B. Golos. Holt, Rinehart and Winston, New York, 1968. xiii+227 pp. $7.95. (Telegraphic Review, Nov. 1968.)

"This book is an attempt to present, at an elementary level, an approach to geometry in keeping with the spirit of Euclid, and in keeping with the modern developments in axiomatic mathematics." (First sentence of preface.) This goal

has been achieved in admirable fashion. The emphasis is on presentation suitable for the beginning high school teacher as well as for the student who wants to understand the motivation for the axiomatic treatment of euclidean geometry by geometers like Pasch and Hilbert. On the American standard textbook scene, the book brings back into proper focus the synthetic axiomatic treatment of geometrical truth, albeit on a level which excludes all post-Hilbertian developments (like Hilbert-Moore's topologization, Reidemeister-Thompson-Artin-Bachmann's group theoretic upheaval and Freudenthal's Helmholtz-Lie-theoretic revival). Compare, e.g., Chapter 1 (1.1 Definitions and Undefinitions, 1.2 Axioms, 1.3 Logic, 1.4 Sets) and later on the careful exposition of Saccheri's contributions to absolute geometry with Chapter 1 (The algebra of the real numbers) and with the passage '10.5 A historical comedy' from Edwin Moise's substantial book on *Elementary Geometry from an Advanced Standpoint.* TT, T(14–15), L.

HANS ZASSENHAUS, Ohio State University

The Theory of Gambling and Statistical Logic. By Richard A. Epstein (Hughes Aircraft Company). Academic Press, New York, 1967. xiii+492 pp. $10.00. (Telegraphic Review, Feb. 1968.)

This book is a lot of fun and makes most rewarding reading for the neophyte in the subject of games of chance, skill, or varying combinations thereof. Its title may be justified by two breezy introductory chapters, called "mathematical preliminaries" and "fundamental principles of a theory of gambling". They are neither very mathematical, nor fundamental for the remainder of the book. In fact any reader with only very modest background in probability, statistics and game theory can and should jump right into the subject of his choice, whether it be "coins, wheels, and oddments", "blackjack", or the more intellectual "games of pure skill and competitive computers". There he will find detailed authoritative information. For example the optimal strategy for the game of Nim and its generalizations, or the reason why the games of Hex and Bridg-it cannot end in a draw and therefore in theory always permit the first player to win. Concerning less frivolous pastimes such as horse racing and the stockmarket the information tends to be of a more speculative character. The author excels in providing historical insight into subjects ranging from the antics of passionate dice players in ancient Rome to machine programs developed by the modern aero-space industry to play games of more recent origin.

F. SPITZER, Cornell University

Analysis, Vol. 2. By Einar Hille. Blaisdell, Waltham, Mass. 1966. xii+672 pp. $11.75. (Telegraphic Review, Feb. 1967.)

This book is recommended as an advanced calculus text for mathematics majors and many physical science majors. It may be also be profitably used by college teachers of mathematics to strengthen their mathematical background,

or as a first year text for honors sections. Volume 2 is virtually self-contained and could be used without Volume 1.

The book steers a reasonable path between abstract theory and concrete applications meaningful for the student. The topics treated are those usually met in the classical advanced calculus texts: Riemann-Stieltjes and Laplace-Stieltjes integrals, Linear Algebra, Solid Analytical Geometry, Functions of Several Variables, Implicit Functions, Multiple Integrals and Ordinary Differential Equations. Each topic is motivated in a readable manner and interspersed with interesting historical comments. Much space is devoted to carefully worked out problems illustrating the theory or emphasizing the importance of given hypotheses. The proofs are clear, rigorous, and not obscured by a morass of detail. There is a wealth of graded problems for each section followed by numerous miscellaneous problems for each chapter.

The approach of this book spans the gap usually found between undergraduate and graduate analysis. Techniques and concepts of modern analysis, especially functional analysis, are used in a natural and integrated manner. For example, the modulus of continuity is used in obtaining the properties of continuous functions of several variables. Complete metric spaces are first introduced in the discussion of vector-valued functions and later the fixed point theorem for contraction maps on complete metric spaces is used to obtain implicit function theorems and local solutions of first order differential equations. Multiple integrals are interpreted as measures and the reader encounters concrete examples and elementary techniques of measure theory.

This book is written on a level appropriate for undergraduates and in a manner designed to develop their mathematical maturity. It seems to be the advanced calculus text which so many have been looking for, since it treats the classical topics while avoiding the usual dryness and it uses modern analysis without becoming an elementary real-variable text. Furthermore, it provided the desired background for the author's graduate text *Analytic Function Theory*.

JUDITH MOLINAR ELKINS, Rutgers—The State University

TELEGRAPHIC REVIEWS

The following abbreviations indicate possible uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)–18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Algebra

Theory and Problems of Group Theory. By B. Baumslag and B. Chandler (both of New York Univ.). Schaum's Outline Series, McGraw-Hill, New York, 1968. 279 pp. $3.95 (paper). Not a mere outline but a text in the Schaum format. Chapters are sets, mappings and binary operations, groupoids, groups and subgroups, isomorphism theorems, finite groups, abelian groups, permutational representations, free groups and presentations, appendices on number theory, and a guide to the literature. Does not presuppose previous work in modern algebra. T (15–17), S.

A University Algebra. 2nd ed. By D. E. Littlewood (University College of North Wales, Bangor). Heinemann, London, 1965. viii+324 pp. $6.00. This reprint of the 1958

edition contains minor alterations and additions, a virtually new chapter on "the laws of algebra" and more details on the theory of ideals. The book is intended to "include . . . all the algebra that reasonably would be required for an honors degree course in mathematics." The level of abstraction is about one or two decades below that which has become customary in recent U.S. algebra textbooks. T (15–16), L.

Introduction to Combinatorial Mathematics. By C. L. Liu (Elec. Eng. Dept., MIT). McGraw-Hill, New York, 1968. x+393 pp. $13.50. A very wide coverage (including necessary elementary mathematics, enumerative analysis, theory of graphs, optimization techniques, transport networks, matching theory, linear programming, dynamic programming and block designs) from an applied and elementary point of view, not presupposing any modern algebra. Suitable for lower division mathematics courses or for courses at a more advanced level for students in computer science, engineering, or operations research. T (14–17), S.

The Theory of Groups. By Ian D. MacDonald (Univ. of Queensland, Australia). Oxford, New York, 1968. vi+254 pp. $7.20 (cloth) $4.50 (paper). Presupposes no previous training in algebra. Concentrates exclusively on groups, excluding other mathematical topics and applications. Begins with simple examples, gives equal emphasis to finite and infinite groups, includes normal structure, Sylow Theorems, generators and relations, nilpotent groups, and soluble groups. Nothing on group representation. T (15–16).

Lessons on Rings, Modules and Multiplicities. By D. G. Northcott (Univ. of Sheffield). Cambridge Univ. Press, New York, 1968. xiv+444 pp. $14.50. The purpose is to give a "virtually self-contained introduction" to special topics in algebra and to provide a bridge between graduate and undergraduate study. Topics include prime ideals and primary submodules, rings and modules of functions, Noetherian rings and modules, Hilbert rings and zeros, multiplicity theory, the Koszul complex, and filtered rings and modules. T (17), S, P.

Analysis

The Elements of Complex Analysis. By J. Duncan (Univ. of Aberdeen, Scotland). Wiley, New York, 1968. ix+313 pp. $11.50 (cloth) $5.75 (paper). Intended as a first course "for students who take their mathematics seriously" and who have had one year of analysis beyond elementary calculus, this book begins with a chapter on metric space preliminaries and includes chapters on Cauchy's theory for starlike domains, local analysis, global analysis, conformal mapping, and analytic continuation. There are suggestions for further study and a short bibliography. T (16), S.

Generalized Functions. Volume 2. Spaces of Fundamental and Generalized Functions. By I. M. Gel'fand and G. E. Shilov (both of the Academy of Sciences, Moscow, USSR). Translated by Morris D. Friedman, Amiel Feinstein, and Christian P. Peltzer. Academic Press, New York, 1968. x+261 pp. $12.50. The other volumes in this five volume treatise have already been published beginning in 1964. P, L.

Introduction to Analysis. By Bernard Kripke (Univ. of California, Berkeley). Freeman, San Francisco, 1968. vii+274 pp. $8.50. Intended for the transition from calculus to modern abstract analysis, this book deals with the real number system, vector spaces, metric and normed spaces, complex numbers, compactness, connectedness, and mathematical applications of these basic ideas. The author calls his exposition "frankly opinionated," but a glance at the book suggests that this is his way of referring to his lively style, sense of humor, and frankness in telling the student what is going on. T (14–15).

Repertorium der Theorie der Differentialgleichungen. By F. G. Tricomi. Springer-Verlag, New York, 1968. viii+167 pp. $7.00. L.

Special Functions and the Theory of Group Representations. By N. Ja. Vilenkin. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, 1968. x+613 pp. $28.00. The author uses the theory of group representation to "comprehend the theory of the most important classes of special functions from the single point of view" and thereby to "establish some kind of order" in a theory that has appeared as a "chaotic collection of formulas." There is a bibliography of 303 titles. P.

Applications

Dynamic Programming of Economic Decisions. By Martin J. Beckmann. *Econometrics and Operations Research. Vol. IX.* Springer-Verlag, New York, 1968. xii+143 pp. $7.00. The subject is described as the "analysis of multistage decision in the sequential mode." Intended as an introductory monograph rather than a complete treatise, this book surveys the field and includes references for further reading. T, S, P.

Classical Network Theory. By V. Belevitch (Univ. of Louvain). Holden-Day, San Francisco, 1968. x+440 pp. $19.50. A "complete fundamental theoretical treatise on analysis and synthesis of lumped linear time-invariant passive networks in the complex frequency domain." Presupposes knowledge of matrix algebra and analytic functions. T (17), S, P, L.

Mathematische Methoden der Zuverlassigkeitstheorie I. By B. W. Gnedenko, J. K. Beljajew and A. D. Solowjew. Translated from the Russian by Peter Franken. Akademie-Verlag, Berlin, 1968. vii+222 pp. $5.73. Reliability theory preceded by the necessary probability and statistics. This volume contains four out of seven chapters. P.

Mathematical Structures of Language. By Zellig Harris (Univ. of Pennsylvania). Interscience, New York, 1968. ix+230 pp. $11.95. This is not a general survey of mathematical linguistics but a report on the author's work toward constructing an abstract system which characterizes natural language "as a system of sets of arbitrary objects, the sets being closed with respect to particular operations, with certain mappings of these sets into themselves or into or onto related sets. The operations and mappings have interpretations which yield the meanings of the utterances . . . " P.

Linear Programming. By Bela Kreko (Marx Karoly Univ. of Economics, Budapest). Translated by J. H. L. Ahrens and Carolyn M. Safe. American Elsevier, New York, 1968. xii+355 pp. $14.50. A translation of a revision of a German book first published in 1962, this is a fairly comprehensive treatment designed primarily for economists. S, P.

Introduction to Physical Statistics. By Robert Bruce Lindsay. (Unabridged and corrected republication of a book first published in 1941 by Wiley.) Dover, New York, 1968. xix+306 pp. $2.75 (paper). At first sight, it may seem strange to reprint a physics textbook almost three decades old, but this one is of interest because it gives a thoughtful explanation of the nature of deterministic and probabilistic physical theory as well as a survey of the latter in many areas: thermodynamics, Maxwell-Boltzmann statistics, kinetic theory of gases, classical statistical mechanics, Darwin and Fowler statistical mechanics, quantum statistics, specific heats, electrical and thermal properties of metals, and emission of electrons from surfaces. Incidentally the title points to an unfortunate usage that is standard among physicists. The word "statistics" as used by them does not refer either to descriptive or inferential statistical theory as understood by mathematicians, but rather to the use of probabilistic concepts in physical laws and theories. It would be more appropriate to say "prob-

abilistic mechanics" or "stochastic mechanics" rather than "statistical mechanics." S, P.

Folk Song Style and Culture. By Alan Lomax. American Association for the Advancement of Science. Washington, D.C. xix+363 pp. $16.75. This is a first venture into "cantometrics" and "choreometrics," the mathematical analysis of song and dance as cultural phenomena. There are charts, profiles, graphs, and correlation matrices. S, P, L.

Fundamental Theory of Servomechanisms. By LeRoy A. MacColl. (Reprint of the first edition published by Van Nostrand in 1945.) Dover, New York, 1968. xviii+130 pp. $2.00 (paper). In his preface to the Dover edition, R. W. Hamming describes the book as "irreplaceable as an elementary introduction to the field" that is closely related to many branches of modern mathematics. S, P.

Set Theory and Linguistics. By Alejandro Ortiz and Ernesto Zierer (both of Universidad Nacional de Trujillo, Peru). Mouton, The Hague, 1968. 61 pp. 10 Dutch Guilders (paper). Quite elementary set algebra with simple applications to linguistics and some cautions against applying mathematics indiscriminantly. S, P.

Great Ideas of Operations Research. By Jagjit Singh. Dover, New York, 1968. vii+228 pp. $2.25 (paper). A popularization by a successful practitioner of this art. S, P.

Computers, etc.

Computing Methods for Scientists and Engineers. By L. Fox and D. F. Mayers (Oxford Univ. Comp. Lab). Oxford, New York, 1968. xii+255 pp. $7.20. This book is neither on numerical analysis as a mathematical discipline nor on the use of computers. It is rather a link between the two and intended to meet the need for books "at a medium level, which use simple mathematics, which do not make a fetish of mathematical rigor, and which try to communicate to the reader some of the 'numerical sense' that good computing men exhibit." Topics include error analysis and floating-point arithmetic, recurrence relations, polynomials, matrices, polynomial approximation, Chebyshev approximation, interpolation and differentiation, numerical integration, and ordinary differential equations. T (15–16), P.

Logic Machines, Diagrams and Boolean Algebra. By Martin Gardner. (A corrected reprint of the book first published in 1958 by McGraw-Hill under the title *Logic Machines and Diagrams.*) Dover, New York, 1968. xi+157 pp. $2.00 (paper). There is a brief preface referring to more recent developments. The book contains much interesting historical, bibliographic and descriptive material. S, P, L.

Mathematical Theory of Switching Circuits and Automata. By Sze-Tsen Hu. Univ. of California Press, Berkeley, 1968. xiv+261 pp. $9.00. The author undertakes to establish a new branch of mathematics by abstracting from "hardware" consideration and extraneous mathematical techniques. The book is designed for mathematics courses on computing machines and engineering courses on switching. Chapters are switching functions, minimization methods, decomposition algorithms, and sequential machines and automata. T (14), S, P, L.

Computer Dictionary and Handbook. Charles J. Sippl. Howard W. Sams, Indianapolis, 1968. 766 pp. $16.95. After a 349 page dictionary on computer terminology, including both software and hardware, there follow 400 pages containing 26 appendices describing hardware, software, various programming languages, mathematical and statistical vocabulary, etc., etc. High on quantity, low on quality. "Linear equations are those graphed as a straight line" (p. 521). "A quantity (A) is said to be a function of another quantity (B) when no change can be made in B without producing a corresponding change in A and vice versa" (p. 516).

Time-Sharing Computer Systems. By M. V. Wilkes (Univ. of Cambridge). American Elsevier, New York, 1968. 102 pp. $4.95. Time-Sharing is described as "a development that is revolutionizing our idea of what a computer system should be." P, L.

Geometry and Topology

Projective Plane Geometry. By John W. Blattner (San Fernando Valley State Coll.). Holden-Day, San Francisco, 1968. xi+297 pp. $10.75. The aim of this book is to achieve relevance to the modern mathematics curriculum, especially to modern abstract algebra. T (15).

Outline of General Topology. By R. Engleking (Polish Acad. of Sci.). Translated from the Polish by K. Sieklucki. North-Holland Publishing Co., Amsterdam; PWN-Polish Scientific Publishers; Wiley, New York, 1968. 388 pp. $17.50. The Polish edition was published in 1966. Chapters include operations on topological spaces, compact spaces, metric and metrizable spaces, paracompact spaces, connected spaces, dimension of topological spaces, uniform spaces, proximity spaces. Features include easy exercises after each section, harder problems at the end of the chapters, historical and bibliographic notes at the end of each chapter, a list of special symbols and a fourteen page bibliography. T (16–17), P, L.

Differential Geometry and the Calculus of Variations. By R. Hermann (Univ of Calif., Santa Cruz). Academic, New York, 1968. x+440 pp. $18.50. Directed to those who are not specialists in differential geometry but who have a good background in advanced calculus, and possibly some knowledge of differential forms, Lie groups and vector fields, it brings together the classical and modern "global" approaches. The first half is an exposition of the geometric side of the "classical one-independent variable calculus of variations and the Hamilton-Jacobi theory." The second is devoted to Riemannian geometry. S, P, L.

Semisimpliziale algebraische Topologie. By Klaus Lamotke. *Die Grundlehren der mathematischen Wissenschaften, Band 147.* Springer-Verlag, New York, 1968. vii+285 pp. $12.00. The last five chapters are on homology of semisimplicial sets, spectral sequence of a fiber ring, homotopy groups, Eilenberg-MacLane sets, and cohomology operations. P.

Elementary College Geometry. By David A. Ledbetter (Pasadena City College). McGraw-Hill, New York, 1968. xvii+253 pp. $7.95. SMSG high-school plane geometry for college students, using incidence, betweenness and ruler axioms. T (13).

Geometry for Teachers. By G. Y. Rainich (Univ. of Michigan) and S. M. Dowdy (Ball State Univ.). Wiley, New York, 1968. ix+228 pp. $7.95. Based on lectures given at National Science Foundation Academic Year Institutes at the University of Michigan and at the University of Notre Dame, this book is not on methods but is intended to provide the deeper knowledge that should underlie good teaching. Topics include intuitive and axiomatic vector geometry, projective geometry, inversive geometry, hyperbolic geometry, axiom systems, groups, numbers in geometries. TT.

Elements of the Theory of Algebraic Curves. By A. Seidenberg (Univ. of Calif., Berkeley). Addison-Wesley, Reading, Mass., 1968. viii+216 pp. $11.00. By abandoning inessential parts of ring and field theory, the author presents material formerly given at the graduate level in a form suitable for undergraduates with a course in modern algebra behind them. The level of difficulty is about that of a first differential geometry course. Some headings toward the end are Noetherian conditions, linear series and rational mappings, and infinitely near points. T (16–17), L.

History

Passages from the Life of a Philosopher. By Charles Babbage. London, 1864. Reprinted by Dawsons of Pall Mall, London, 1968. xii+496 pp. £6.6.0. The autobiography of the inventor of the Analytical Engine, of which a woodcut is reproduced as the frontispiece. The dedication is to Victor Emmanuel II of Italy in recognition of the fact that his father, King Charles Albert, provided the opportunity at a meeting in Italy "for the first public and official acknowledgement" of Babbage's invention. P, L.

A History of Conic Sections and Quadric Surfaces. By Julian Lowell Coolidge (Reprint of a work first published by Oxford in 1945). Dover, New York, 1968. xi+214 pp. $2.75 (paper). A welcome publishing event is the reprinting of this excellent book that had become quite difficult to obtain. The story begins with the Greeks and ends with recent developments. An excellent course on analytic geometry could be developed around this little book by simply having the class carry through the mathematical developments that are considered. T, S, P, L.

American Science in the Age of Jackson. By George H. Daniels. Columbia Univ. Press, New York, 1968. viii+282 pp. $7.95. Mathematics gets even less discussion than it deserves in this book. Benjamin Peirce is named without any real discussion of his contributions, Robert Adrain is consistently misnamed "Adrian," and Nathaniel Bowditch is not even mentioned. P.

Newton, His Friend and His Niece. By the late Augustus DeMorgan. Edited by his wife, and by his pupil Arthur Cowper Ranyard. London, 1885. Reprinted with a new introduction by E. A. Osborne. Dawsons of Pall Mall, London, 1968. vi+161 pp. £4. 0. This is DeMorgan's defense of Newton against the story that, for his political advancement, he prostituted his niece Catherine Barton to his patron, the dissolute Charles Montagu. P, L.

Cybernetics. Key Papers. Edited by C. R. Evans and A. D. J. Robertson. Butterworths, London, 1968. viii+289 pp. $8.75 (paper). A collection of sixteen items, beginning with a paper on the Analytical Engine, continuing with recent work, and closing with a brief comment by Lady A. A. Lovelace, a contemporary of Babbage. An interesting anthology that might even be used as a challenging introductory textbook. T, S, P, L.

The Mathematical Principles of Natural Philosophy. By Sir Isaac Newton. Translated into English by Andrew Motte. 1729. Reprinted in facsimile with an introduction by I. Bernard Cohen, in two volumes. Dawsons of Pall Mall, London, 1968. Vol. I. xvii+320 pp. Vol. II. 393+71 pp. £15 (for both). In the introduction, Cohen, who is editing the forthcoming critical edition of the Principia, says "Motte's translation is generally an exact rendition of the original, for which reason it is greatly to be preferred to the 'modernization' made by Florian Cajori and published in 1934." Indeed the Cajori edition is a poor one, and it is unfortunate that it is widely distributed in paperback. As soon as possible, it should be allowed to go out of print and be replaced by a cheap facsimile edition or an accurate modernization. P, L.

★*Hermann Weyl. Gesammelte Abhandlungen.* Edited by K. Chandrasekharan. Four volumes. Springer-Verlag, New York, 1968. xviii+2830 pp. $42.00. This includes practically all of Weyl's scientific papers, but not his books or lecture notes. Volume 4 contains a reprint of the biographical article by C. Chevalley and A. Weil that originally appeared in *l'Enseignement Mathématique* in 1957. Volume 1 begins with a good portrait and three quotations from manuscripts. The breadth of Weyl's contributions testifies that the universality is still possible and is even characteristic of the greatest contributors. L (!).

Logic and Foundations

Foundations of Mathematics. By William S. Hatcher (Univ. of Toledo). Saunders, Philadelphia, 1968. xiii+327 pp. $12.75. Chapter headings are first-order logic, the origin of foundational studies, Frege's system and the paradoxes, the theory of types, Zermelo-Fraenkel set theory, Hilbert's program and Gödel's incompleteness theorems, the foundational systems of W. V. Quine, and categorical algebra. Bibliography and glossary. T (16–17).

Real Numbers. A Development of the Real Numbers in an Axiomatic Set Theory. By G. L. Isaacs (State Univ. of New York). McGraw-Hill, New York, 1968. viii+112 pp. $8.50. In the tradition of Landau, this book begins with axioms for abstract set theory, constructs the real numbers and shows that they are a model of the categorical axioms for an infinite, totally ordered complete field. The book concludes with chapters on the axioms of completeness and the elementary functions of analysis. T (14–15).

Allgemeine Mengenlehre. I. Ein Fundament der Mathematik. By Dieter Klaua (Karl-Marx-University, Leipzig). 2nd revised edition. Akademie-Verlag, Berlin, 1968. ix+379 pp. DM38 (cloth). A solidly packed volume covering axiom systems, class algebra, finite sets, ordering, cardinals, ordinals, and order types. The second volume will deal with the arithmetic of ordinals and cardinals and with some applications of set theory in mathematics. P.

Philosophy of Science. A Formal Approach. By Henry E. Kyburg, Jr. (Univ. of Rochester). Macmillan, New York, 1968. xii+332 pp. $7.95. This book is of special interest to mathematicians because it is written by a mathematician, is based on the concept of a formal system, and considers many questions of interest to mathematicians. Indeed it might serve as a text for a course in the philosophy of mathematics. T, S, P, L.

Constructive Real Numbers and Function Spaces. By N. A. Šanin. Translations of Mathematical Monographs, Vol. 21. American Mathematical Society, 1968. iv+325 pp. $17.60. There are interesting general remarks in the long introduction and in the appendix entitled "On Criticism of Classical Mathematics." P.

First Order Logic. By R. M. Smullyan (City Univ. of New York). *Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 43.* Springer Verlag, New York, 1968. xii+158 pp. $9.00. A treatise that is logically self-contained but presupposes, in practice, considerable sophistication and some previous knowledge of mathematical logic. The approach is via analytic tableaux. T, S, P.

Probability and Statistics

Convergence of Probability Measures. By Patrick Billingsley (Univ. of Chicago). Wiley, New York, 1968. xii+253 pp. $12.50. The theory of weak convergence of probability measures on metric spaces, together with applications sufficient to show their power and utility. There is no statement of prerequisites or possible uses, but the book appears to be suitable for advanced courses presupposing considerable knowledge of analysis. T (17).

Distribution-Free Statistical Tests. James V. Bradley (Antioch College). Prentice-Hall, Englewood Cliffs, N.J. 1968. xii+388 pp. $11.50. Intended as a reference work and "a provocative and enlightening dissertation," this book organizes tests according to mathematical derivation or rationale under the following headings: randomization applied to ranks, normal scores, binomial distribution, hypergeometric distribution, multivariate hypergeometric distribution, multinomial distribution, runs having con-

stant probability under H_0, runs up and down, and miscellaneous tests. There are 66 pages of tables. S, P.

Fundamentals of Probability Theory and Mathematical Statistics. By V. E. Gmurman. English edited by I. I. Berenblut. Iliffe Books, Ltd., London. American Elsevier, New York, 1968. 249 pp. $9.75. A mediocre translation of an inadequate elementary text in a field already well covered in the English literature.

Information Theory and Statistics. (Revised and augmented edition of a book first published in 1959 by Wiley.) By Solomon Kullback (George Washington Univ.). Dover, New York, 1968. xvii+395 pp. $3.00 (paper). This is not a treatise on information theory but rather on "the study of logarithmic measures of information and their application to the testing of statistical hypotheses." It presupposes some previous work in probability and statistical theory. There are a substantial bibliography and a glossary. T (16), S, P.

Handbook of Nonparametric Statistics. By John E. Walsh (System Development Corporation). Three volumes. I. Investigation of Randomness, Moments, Percentiles, and Distributions. II. Results for Two and Several Sample Problems, Symmetry, and Extremes. III. Analysis of Variance. Van Nostrand, Princeton, N.J., 1962, 1965, 1968. I. xxvi+549 pp. $16.00. II. xxvi+686 pp. $18.50. III. xxvi+747 pp. $18.00. The author's original plan was to cover comprehensively "the published material that appeared in the nonparametric statistics field up to 1958." The project expanded from an expected 2000 hours over three years, to an actual 10,000 over twelve years. Moreover the third and last volume does not cover regression analysis, discriminant analysis or multivariate analysis as the author had hoped. The volumes contain an extraordinary amount of well indexed information, including bibliographies. They are testimony to the overwhelming volume of published material in any field of mathematics and the tremendous difficulty involved in summarizing existing information. P, L.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at BUFFALO

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D.C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of British Columbia: Dr. L. G. Roberts, Harvard University, has been appointed Assistant Professor; Associate Professor R. A. Restrepo has been promoted to Professor.

Carleton University: Associate Professor J. D. Dixon, University of New South Wales, has been appointed Associate Professor; Assistant Professor K. S. Williams has been promoted to Associate Professor.

University of Houston: Dr. Henry Decell, NASA, Manned Spacecraft Center, Houston, has been appointed Associate Professor; Dr. J. A. Johnson, University of California, Riverside, has been appointed Assistant Professor; Associate Professor Arnold Vobach, University of Georgia, has been appointed Associate Professor.

Oberlin College: Dr. R. J. Hemstead, California Institute of Technology, and Mr. D. C. Kelly, Talladega College, have been appointed Assistant Professors.

Wisconsin State University—Superior: Assistant Professor Margaret Marchand, Lakehead University, has been appointed Professor; Professor Gloria Olive, Anderson College, has been appointed Professor; Mr. Donald Weyers has been promoted to Assistant Professor.

University of Wisconsin: Associate Professors R. A. Askey and C. C. Conley have been promoted to Professors; Assistant Professors R. A. Brualdi and Michael Voichick have been promoted to Associate Professors; Assistant Professor Charles MacCluer, Michigan State University, has been appointed Visiting Lecturer.

Assistant Professor George Beers, Middle Tennessee State University, has been promoted to Associate Professor.

Mr. Harold Carda, South Dakota School of Mines and Technology, has been promoted to Assistant Professor.

Dr. M. N. Manougian, University of Texas at Austin, has been appointed Assistant Professor at the University of South Florida, Tampa.

Dr. M. Warten, IBM Scientific Center, Palo Alto, has been appointed Associate Professor at California State Polytechnic College, San Luis Obispo.

Mr. W. D. Lambert, former chief of the section of gravity and astronomy at the U.S. Coast and Geodetic Survey, died on October 27, 1968. He was a Charter Member of the Association.

Professor Emeritus Helen G. Russell, Wellesley College, died on October 24, 1968. She was a member of the Association for thirty-two years.

Professor Emeritus Charles T. Salkind, Brooklyn Polytechnic Institute and Pace College, died on November 10, 1968. He was a member of the Association for twenty-five years. He had served for many years as Chairman of the Committee on High School Contests.

ADVANCED PLACEMENT CONFERENCE IN MATHEMATICS AT TEXAS A&M UNIVERSITY

The Advanced Placement Conference in Mathematics will be held at Texas A&M University, June 26, 27, and 28, 1969. The featured speakers will be Mr. Julius Hlavaty, President of the National Council of Teachers of Mathematics; Mr. Paul Kelley, Director of Testing Center, University of Texas at Austin; Professor Daniel Finkbeiner, Kenyon College; Mr. John Bunnell, Asst. Director of Admissions, Stanford University; and Professor William T. Guy, University of Texas at Austin. Much of the program will be devoted to the use of the new AB and BC syllabi for advanced placement and to beginning an Advanced Placement Program. There will also be panel discussions relating to many phases of the Advanced Placement Program. Inquiries for information about the conference should be addressed to the Department of Mathematics, Texas A&M University, College Station, TX 77840.

150th ANNIVERSARY CELEBRATION OF INDIANA UNIVERSITY IN 1970

The Mathematics Department of Indiana University is currently making plans to hold a special year in Functional Analysis and its applications during the 1969–1970 academic year in honor of the 150th Anniversary celebration of Indiana University in 1970.

Several visitors in the field of functional analysis or its applications will be invited

to spend either the whole academic year or a shorter period in Bloomington. Further information can be obtained by writing to George Springer, Chairman, Department of Mathematics, Swain Hall East, Indiana University, Bloomington, Indiana 47401.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE NEW JERSEY SECTION

The thirteenth annual meeting of the New Jersey Section of the MAA was held at Rutgers, The State University, on November 2, 1968. Mr. F. A. Brooks, Jr., of Mutual Benefit Life presided at the morning session and Professor Bernard Greenspan, senior member of the Executive Council, presided at the afternoon meeting. Seventy individuals attended the meeting, sixty of whom were members of the Association.

Professor Bernard Greenspan, Drew University, was elected Chairman of the section at the afternoon business meeting. John Reckzeh, Jersey City State College, was elected Secretary-Treasurer; Professor Myron White, Stevens Institute, was elected Associate Secretary-Treasurer and Professor Theodore Faraklas was elected Member-at-Large of the Executive Committee. Francis Varrichio, Secretary-Treasurer, and F. A. Brooks, Jr., of the High School Contest Committee, presented reports.

The morning program was as follows:

1. *Mathematics of physical quantities*, by Hassler Whitney, Institute of Advanced Study (by invitation).
2. *Function spaces: a history*, by Michael Bernkopf, Pace College (by invitation).

The afternoon program was as follows:

1. *Some mathematical models of a random function*, by Lawrence Shepp, Bell Telephone Laboratories (introduced by the Secretary).

JOHN RECKZEH, *Secretary-Treasurer*

NOVEMBER MEETING OF THE UPPER NEW YORK STATE SECTION

The Fall Meeting of the Upper New York State Section of the MAA was held at Rensselaer Polytechnic Institute on November 9, 1968. Professor F. R. Olson, Chairman of the Section, presided at the sessions at which the following papers were presented:

1. *Application of recurrence relations to prove algebraic identities*, by E. T. Frankel, Schenectady, New York.

2. *Integer approximation of rationals*, by A. Thuswaldner, Northern Electric Company Ltd., Ottawa, Canada.

3. *Multicoherence in graphs*, by D. V. Vittum and C. J. Houghton, SUNY at Binghamton.

4. *Things like the Euler characteristic in graphs*, by D. E. Scheim, undergraduate, University of Rochester.

5. *Preliminary reports from CUPM panels on "Mathematics in Two Year Colleges" and "Qualifications for Two Year College Faculties in Mathematics,"* by M. W. Pownall, Colgate University and R. D. Larsson, Mohawk Valley Community College.

6. *Report on the Upstate New York MAA Contest Section—British Mathematical Olympiad*, by N. D. Turner, SUNY at Albany.

7. *Uniform convergence on classes of subsets*, by R. C. Shiflett, Wells College.

8. *On a new class of dynamical systems*, by U. D'Ambrosio, SUNY at Buffalo.

9. *A note on nonlinear models in two dimensional cavitating flow*, by K. M. Agrawal and A. C. Smith, University of Windsor.

10. *Rotational gas flows with straight and circular streamlines*, by Om Parkash Chandna and A. C. Smith, University of Windsor.

P. SCHAEFER, *Secretary-Treasurer*

INSTITUTIONAL MEMBERS OF THE ASSOCIATION

At its meeting at Stillwater on August 30, 1961, the Board of Governors took action to provide for institutional membership in the Association, and since that time has elected such members at each of its meetings. The following is a complete list of all present corporate and academic members of the Association:

Corporate Members

American Mathematical Society, Providence, Rhode Island
Boeing Scientific Research Laboratories, Seattle, Washington
International Business Machines Corp., Yorktown Heights, New York

Academic Members

Adelphi University, Garden City, New York
University of Alabama, University, Alabama
University of Alaska, College, Alaska
University of Alberta, Edmonton, Alberta, Canada
Albertus Magnus College, New Haven, Connecticut
Allen University, Columbia, South Carolina
Amherst College, Amherst, Massachusetts
Andrews University, Berrien Springs, Michigan
Arizona State University, Tempe, Arizona
University of Arizona, Tucson, Arizona
Arkansas Polytechnic College, Russellville, Arkansas
Arlington State College, Arlington, Texas
Auburn University, Auburn, Alabama
Bethel College, North Newton, Kansas
Boston College, Chestnut Hill, Massachusetts
Bowling Green State University, Bowling Green, Ohio
University of Bridgeport, Bridgeport, Connecticut
Brigham Young University, Provo, Utah
University of British Columbia, Vancouver, B. C., Canada
Brock University, St. Catharines, Ontario, Canada
Bronx Community College, Bronx, New York
Brooklyn College, Brooklyn, New York
Polytechnic Institute of Brooklyn, Brooklyn, New York
Brown University, Providence, Rhode Island
Bucknell University, Lewisburg, Pennsylvania
California State College at Long Beach, Long Beach, California
California State College, Los Angeles, California
California State College at San Bernardino, San Bernardino, California.
University of California, Berkeley, California
University of California, Davis, California
University of California, Irvine, California
University of California, Los Angeles, California
University of California, Riverside, California
University of California, Santa Barbara, California
Canisius College, Buffalo, New York
Carnegie-Mellon University, Pittsburgh, Pennsylvania

Case Western Reserve University, Cleveland, Ohio
Catholic University of America, Washington, D. C.
Central Michigan University, Mt. Pleasant, Michigan
Central Missouri State College, Warrensburg, Missouri
Chamberlayne Junior College, Boston, Massachusetts
Chatham College, Pittsburgh, Pennsylvania
University of Cincinnati, Cincinnati, Ohio
Clarkson College of Technology, Potsdam, New York
Colgate University, Hamilton, New York
College Jean-de-Brebeuf, Montreal, Quebec, Canada
College of Petroleum & Minerals, Dhahran, Saudi Arabia
Colorado State University, Ft. Collins, Colorado
University of Colorado, Boulder, Colorado
Concordia College, Moorhead, Minnesota
Connecticut College for Women, New London, Connecticut
Cornell University, Ithaca, New York
Cumberland College, Williamsburg, Kentucky
Dartmouth College, Hanover, New Hampshire
University of Dayton, Dayton, Ohio
Denison University, Granville, Ohio
University of Denver, Denver, Colorado
University of Detroit, Detroit, Michigan
Drexel Institute of Technology, Philadelphia, Pennsylvania
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HENRY L. ALDER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

Fifty-Third Annual Meeting, Miami, Florida, January 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, May 3, 1969.

FLORIDA

ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.

INDIANA, Purdue University, Indianapolis, May 10, 1969.

IOWA, University of Northern Iowa, Cedar Falls, April 18, 1969.

KANSAS

KENTUCKY, Morehead State University, Morehead, Spring 1969.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, St. Mary's College of Maryland, St. Mary's City, April 26, 1969.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, College of St. Catherine, St. Paul, April 26, 1969.

MISSOURI, St. Louis University, St. Louis, April 26, 1969.

NEBRASKA, Nebraska Center for Continuing Education, Lincoln, April 25–26, 1969.

NEW JERSEY, Drew University, Madison, May 3, 1969.

NORTHEASTERN, Williams College, Williamstown, June 28, 1969.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 25–26, 1969.

OKLAHOMA-ARKANSAS

PACIFIC NORTHWEST, University of Oregon, Eugene, August 1969.

PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.

ROCKY MOUNTAIN, University of Colorado, Boulder, May 9–10, 1969.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, Northern Arizona University, Flagstaff, April 11–12, 1969.

TEXAS, Texarkana College, Texarkana, April 18–19, 1969.

UPPER NEW YORK STATE, University of Western Ontario, London, Ontario, Canada, May 1969.

WISCONSIN, Oshkosh, Wisconsin, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, University of Oregon, Eugene, Oregon, August 26–29, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Statler-Hilton Hotel, Washington, D. C., May 7–9, 1969.

ASSOCIATION FOR SYMBOLIC LOGIC

CERTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, University of Oregon, Eugene, Oregon, August 27, 1969.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Minneapolis, April 23–26, 1969.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, Colorado, June 17–20, 1969.

PI MU EPSILON, University of Oregon, Eugene, Oregon, August 26–27, 1969.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.

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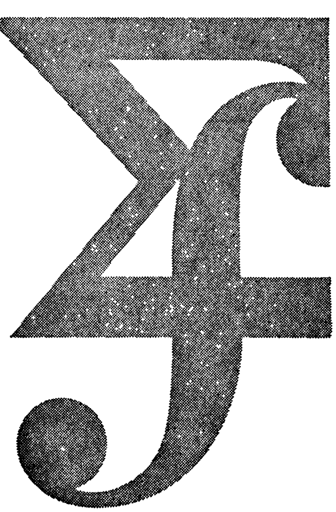

CALCULUS
Lipman Bers, Columbia University
This modern text for the basic course treats elementary calculus as the art of setting up and solving differential equations. The text presupposes no special preparation beyond the ability to perform ordinary algebraic operations and rudimentary knowledge of geometry. Geometric intuition, motivation and applications are stressed throughout. *Solutions Manual.* March 1969 / 720 pages / $12.50 (tent.)

CALCULUS OF THE ELEMENTARY FUNCTIONS
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PRECALCULUS ELEMENTARY FUNCTIONS AND RELATIONS
Donald R. Horner, Eastern Washington State College
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MODERN INTERMEDIATE ALGEBRA
Vivian Shaw Groza and Susanne Shelley, both of Sacramento City College
Beginning with a review of elementary algebra, this modern presentation uses set concepts consistently. The manner of development is axiomatic, with all important concepts logically justified by valid proofs. The text contains many illustrative examples and exercises. *Solutions Manual.* April 1969 / 352 pages / $8.50 (tent.)

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APRIL 1969

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GRAMMARS AND LANGUAGES

E. SPANIER, University of California at Berkeley

1. Introduction. An automaton can be used to describe a set of words in a free semigroup. A grammar is also a means of describing a set or "language." Thus, a connection between automata and grammars is that both can be used to define sets. Before attempting to make precise the concepts of grammar and language, we consider natural language from an intuitive point of view by way of motivation for the definition given in Section 2.

A language may be thought of as a set of sentences, each sentence being a string of words in the language. The language is completely known, therefore, when its sentences are specified. The number of words in the language is finite, but there are potentially infinitely many sentences, inasmuch as one sentence can appear as a proper part of another sentence.

A grammar may be viewed as a finite set of rules for generating the sentences of the language. The grammatical rules of interest to us will generate the strings of words which are well formed sentences and which may or may not have meaning: that is, our grammars will deal with syntax but not semantics.

The grammatical rules we shall consider will generate the sentences by building them up from component pieces. For example, it is possible to construct a sentence in English by juxtaposing a subject with a predicate. If we know how to construct subjects and predicates, we know how to generate some sentences. Symbolically this means of constructing a sentence is represented by the "rewriting rule"

$$(\text{sentence}) \to (\text{subject})(\text{predicate}).$$

(The parentheses are used to indicate that the whole expression enclosed therein is to be treated as a single unit. In linguistics these units are called syntactic variables.)

Symbols will also be used to represent the various phrases and parts of speech that can occur in a sentence, and rewriting rules will specify how to build complicated components from simpler ones. Finally, there will be rewriting rules involving words of the language such as (noun)$\to$man, (adjective)$\to$good, etc. (Parentheses have not been used to enclose a word when it is the word itself that is meant rather than its use as part of a syntactic variable.)

As an illustration note that the sentence

The car hit a wall

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can be constructed by using the following set of rewriting rules

$$\begin{gathered}
(\text{sentence}) \to (\text{subject})(\text{predicate}) \\
(\text{subject}) \to (\text{article})(\text{noun}) \\
(\text{predicate}) \to (\text{verb})(\text{direct object}) \\
(\text{direct object}) \to (\text{article})(\text{noun}) \\
(\text{article}) \to \textit{the}, \ (\text{article}) \to a \\
(\text{noun}) \to \textit{car}, \ (\text{noun}) \to \textit{wall} \\
(\text{verb}) \to \textit{hit}.
\end{gathered}$$

The sentence is then generated by starting with the symbol "(sentence)" and applying one rewriting rule at a time to some syntactic variable until a string of words in the language is obtained. For the sentence above one such generation is given by

$$\begin{aligned}
(\text{sentence}) &\to (\text{subject})(\text{predicate}) \\
&\Rightarrow (\text{subject})(\text{verb})(\text{direct object}) \\
&\Rightarrow (\text{subject}) \text{ hit } (\text{direct object}) \\
&\Rightarrow (\text{article})(\text{noun}) \text{ hit } (\text{direct object}) \\
&\Rightarrow \text{The } (\text{noun}) \text{ hit } (\text{direct object}) \\
&\Rightarrow \text{The car hit } (\text{direct object}) \\
&\Rightarrow \text{The car hit } (\text{article})(\text{noun}) \\
&\Rightarrow \text{The car hit a } (\text{noun}) \\
&\Rightarrow \text{The car hit a wall.}
\end{aligned}$$

Other generations of the same sentence using the same rewriting rules are possible. For example, there are generations in which the following string occurs

(subject)(verb) a wall.

The same set of rewriting rules also generates the following three other sentences:

A car hit a wall
A car hit the wall
The car hit the wall

as well as the corresponding sentences where "car" and "wall" have been interchanged or one has been replaced by the other such as

The wall hit a car
The car hit a car.

All of these sentences are to be regarded as well formed even though some of them may not be meaningful.

The mathematical model abstracted from this will use single symbols for the syntactic variables of the grammar and other single "terminal" symbols for the words of the language. A sentence will be a string or word in the free semigroup over the terminal symbols. Our primary interest is in the mathematical properties of the corresponding sets of strings.

For a detailed treatment as well as other results we suggest the book by Ginsburg [6] as a general reference.

2. Phrase structure grammars. Let Σ be a finite set and let Σ^* be the free semigroup generated by Σ. Σ^* consists of all finite sequences $a_1, a_2, \cdots, a_m$, $m \geqq 0$ with $a_i \in \Sigma$ including the empty sequence ϵ for $m = 0$. Such a sequence is called a *word* or *string* over Σ, is usually written $a_1 a_2 \cdots a_m$ without commas, and its *length* is m.

A *phrase structure grammar* G is a 4-tuple (V, Σ, P, σ), where:

(1) V is a finite set (called the *vocabulary*).

(2) Σ is a subset of V (called the *terminal symbols*).

(3) P is a finite set of ordered pairs $u \rightarrow v$ with u a nonempty word of $(V-\Sigma)^*$ and v a word of V^* (called the *productions* or *rewriting rules*). (We use $u \rightarrow v$ instead of (u, v) to denote an ordered pair with first component u and second component v which is a production.)

(4) σ is an element of $V-\Sigma$ (*called the start variable*).

Elements of $V-\Sigma$ are called the *variables* of the grammar. In the linguistic applications, Σ is the set of words of the language, $V-\Sigma$ is the set of syntactic variables, P is the set of rewriting rules, and σ stands for "sentence." The term "phrase structure" is used because a generation in the grammar is intended to provide a representation of the constituent structure of the phrase generated.

We now show how such a grammar is used. Given words w, w' in V^* we define $w \Rightarrow w'$ if there exist words $x, y, u, v, \in V^*$ such that $w = xuy$, $w' = xvy$ and $u \rightarrow v$ is in P. We define $w \overset{*}{\Rightarrow} w'$ if there is a sequence $w_0, w_1, \cdots, w_n$, $n \geqq 0$, of words in V^* such that $w = w_0$, $w_n = w'$, and $w_{i-1} \Rightarrow w_i$ for $1 \leqq i \leqq n$. Such a sequence of words w_i is called a *derivation* of w' from w. Note that $\overset{*}{\Rightarrow}$ is just the transitive reflexive extension of the relation $\Rightarrow$.

The *language generated by* G, denoted by $L(G)$, is defined by

$$L(G) = \{w \in \Sigma^* \mid \sigma \overset{*}{\Rightarrow} w\}.$$

Thus, the language generated by G is the set of words in the terminal symbols that can be derived from σ.

A set $L \subset \Sigma^*$ is called a *phrase structure language* if there exists a phrase structure grammar G such that $L = L(G)$. It is known [5] that the phrase structure languages are identical with the recursively enumerable sets. That is, they can be characterized in terms of acceptance by Turing machines. The recursively enumerable sets have been extensively studied, and there seems to be no particular advantage in thinking of them as phrase structure languages. It is more promising to consider restrictions on the grammars leading to other classes of languages.

A phrase structure grammar G is said to be *context-sensitive* (or *context-dependent*) if each production has the form $u \rightarrow v$, where the length of u is less than or equal to the length of v (i.e., no production is length decreasing). These grammars generate the class of *context-sensitive languages*. Note that no context-

sensitive language contains the empty word ϵ. The term "context sensitive" is due to the fact that the family of context-sensitive languages is known to be the family generated by phrase structure grammars in which each production has the form $u_1\xi u_2 \rightarrow u_1 v u_2$ with $u_1, u_2 \in (V-\Sigma)^*$, $\xi \in V-\Sigma$, and $v \in V^* - \{\epsilon\}$. This production can be thought of as the rewriting rule $\xi \rightarrow v$ in the "context" u_1, u_2.

EXAMPLES:

(1) The set $L = \{ww \mid w \in \Sigma^* - \{\epsilon\}\}$ is a context-sensitive language.

(2) The set $L = \{a^n b^n c^n \mid n \geqq 1\} \subset \{a, b, c\}^*$ is a context-sensitive language.

The context-sensitive languages are known to be recursive sets so they form a proper subclass of the class of phrase-structure languages. They are characterized in terms of acceptance by linear bounded automata [**11**, **12**], which are Turing machines in which the tape used during computation is bounded by the length of the input word.

A *context-free grammar* G is a phrase structure grammar in which each production has the form $\xi \rightarrow v$ with $\xi \in V-\Sigma$ and $v \in V^*$. The corresponding class of languages is the class of context-free languages.

EXAMPLES:

(1) Any finite subset of Σ^* is a context-free language. In fact, the finite set $\{w_i\}_{i=1,\cdots,n}$ is generated by the context-free grammar $(\Sigma \cup \{\sigma\}, \Sigma, P, \sigma)$ where $P = \{\sigma \rightarrow w_i\}_{i=1,\cdots,n}$.

(2) Σ^* is a context-free language generated by the context-free grammar $(\Sigma \cup \{\sigma\}, \Sigma, P, \sigma)$ where $P = \{\sigma \rightarrow \sigma a \mid a \in \Sigma\} \cup \{\sigma \rightarrow \epsilon\}$.

(3) The set $\{a^n b^n \mid n \geqq 0\} \subset \{a, b\}^*$ is a context-free language generated by the context-free grammar $(\{a, b, \sigma\}, \{a, b\}, P, \sigma)$ where $P = \{\sigma \rightarrow a\sigma b, \sigma \rightarrow \epsilon\}$.

(4) The set $\{ww^R \mid w \in \Sigma^*\}$, where w^R is the *reverse* of w (i.e., if $w = a_1 \cdots a_k$ with $a_i \in \Sigma$, then $w^R = a_k \cdots a_1$), is a context-free language generated by the grammar $(\Sigma \cup \{\sigma\}, \Sigma, P, \sigma)$ where $P = \{\sigma \rightarrow a\sigma a \mid a \in \Sigma\} \cup \{\sigma \rightarrow \epsilon\}$.

Every context-free language is a recursive set, and if L is a context-free language, then $L - \{\epsilon\}$ is a context-free, context-sensitive language [**1**]. However, not every context-sensitive language is context free (e.g., $\{a^n b^n c^n \mid n \geqq 1\}$ is not context free). Thus, the context-free languages not containing ϵ form a proper subfamily of the context-sensitive languages. The context-free languages are characterized in terms of acceptance by pushdown store automata [**4**, **16**], which are finite state devices with a pushdown store memory.

The three families of languages described above were introduced by Chomsky [**2**, **3**] in his attempt to find models for natural language. The family of context-free languages has also proved important in the theory of programming languages since it is known [**9**] that they are equivalent to the languages obtained from "Backus normal form." For this reason they have also been called ALGOL-like languages.

Another important family of languages is the family of *regular languages*. These languages are generated by *right linear grammars*, which are context-free grammars in which each production has the form $\xi \rightarrow w$ or $\xi \rightarrow w\xi'$ with $\xi, \xi' \in V-\Sigma$

and $w \in \Sigma^*$. The regular languages are characterized in terms of acceptance by finite state automata, and by the Kleene Theorem [14], they form the smallest family of subsets of Σ^* containing all finite sets and closed with respect to the operations of union, product (i.e., juxtaposition), and star (i.e., semigroup closure). They have been much studied in other connections and form a proper subclass of the family of context-free languages ($\{a^nb^n \mid n \geqq 1\}$ is context free but not regular).

3. Properties of languages. The families of recursively enumerable sets and regular languages are closed under Boolean operations, homomorphism, and various other operations. We consider all the families with respect to these operations, however our main interest will be the context-sensitive languages and the context-free languages.

(1) Each family is closed under union.

In fact, if L_1 and L_2 are phrase-structure languages, we can find grammars $G_1 = (V_1, \Sigma, P_1, \sigma_1)$ and $G_2 = (V_2, \Sigma, P_2, \sigma_2)$ such that $V_1 - \Sigma$ and $V_2 - \Sigma$ are disjoint and $L_1 = L(G_1)$, $L_2 = L(G_2)$. Let σ be an element disjoint from $\Sigma_1 \cup \Sigma_2$ and define a phrase structure grammar $G = (V_1 \cup V_2 \cup \{\sigma\}, \Sigma, P, \sigma)$ with $P = P_1 \cup P_2 \cup \{\sigma \to \sigma_1, \sigma \to \sigma_2\}$. Then

$$L(G) = L(G_1) \cup L(G_2) = L_1 \cup L_2.$$

Notice that if G_1, G_2 are both context-sensitive, context free, or right linear, the same is true of G.

(2) Each family is closed under product.

If $G_1 = (V_1, \Sigma, P_1, \sigma_1)$ and $G_2 = (V_2, \Sigma, P_2, \sigma_2)$ are such that $V_1 - \Sigma$ and $V_2 - \Sigma$ are disjoint, let σ be an element not in $\Sigma_1 \cup \Sigma_2$ and define

$$G = (V_1 \cup V_2 \cup \{\sigma\}, \Sigma, P_1 \cup P_2 \cup \{\sigma \to \sigma_1\sigma_2\}, \sigma).$$

Then $L(G) = L_1L_2$.

(3) Each family is closed under plus (for $A \subset \Sigma^*$, $A^+ = \bigcup_{i \geqq 1} A^i$).

Given $G = (V, \Sigma, P, \sigma)$, let σ' be an element not in V and define $G' = (V \cup \{\sigma'\}, \Sigma, P', \sigma')$ where $P' = P \cup \{\sigma' \to \sigma'\sigma, \sigma' \to \sigma\}$. Then $L(G') = [L(G)]^+$.

(4) The intersection of two context-free languages need not be a context-free language.

The set $L_1 = \{a^ib^ic^j \mid i, j \geqq 1\}$ is the product of the context-free language $\{a^ib^i \mid i \geqq 1\}$ with the context-free language $\{c\}^*$ so is a context-free language. Similarly $L_2 = \{a^ib^jc^j \mid i, j \geqq 1\}$ is also a context-free language. However, $L_1 \cap L_2 = \{a^nb^nc^n \mid n \geqq 1\}$ is not a context-free language.

In fact, it is recursively unsolvable whether the intersection of two context-free languages is a context-free language. It is also unsolvable of context-free languages L_1, L_2 whether $L_1 \subset L_2$ and whether $L_1 = L_2$.

(5) The family of context-sensitive languages is closed under intersection.

This is a consequence of the characterization of context-sensitive languages

as the sets accepted by linear bounded automata.

It follows that the intersection of two context-free languages is a context-sensitive language.

(6) The complement of a context-free language need not be a context-free language.

This is a consequence of (4).

It is an open question whether the complement of a context-sensitive language is a context-sensitive language. The answer would be affirmative if every context-sensitive language were accepted by some deterministic linear bounded automaton, but this is another open question. However, every context-free language is accepted by some deterministic linear bounded automaton, so the complement of a context-free language is a context-sensitive language.

(7) Each family is closed under regular intersection (i.e., intersection with a regular language).

This is an important result proved in [**1**] for the context free case.

(8) Each family is closed under nonerasing homomorphism and arbitrary inverse homomorphisms.

All the families except the context-sensitive one are closed under arbitrary homomorphism. It is not hard to show that any phrase-structure language is the homomorphic image of some context-sensitive language. This implies that the family of context-sensitive languages with the empty word added freely is not closed under arbitrary homomorphism.

(9) All the families are closed under substitution.

By substitution we mean the replacement of each symbol a in Σ by a word in a language L_a in each word of some language $L\subset\Sigma^*$. If L and all the languages L_a belong to one of the four families, then the set of all such possible substitutions is a language of the same family.

The property of "ambiguity" is particularly important for context-free grammars. A grammar is said to be *ambiguous* if there are two or more "essentially different" derivations of some word in the language generated by the grammar. A language is said to be *inherently ambiguous* if every grammar generating it is ambiguous and is said to be *unambiguous* otherwise. It is not hard to show that every regular language is unambiguous.

(10) There exist inherently ambiguous context-free languages.

For example, $\{a^ib^jc^id^k \mid i, j, k > 1\} \cup \{a^ib^jc^kd^j \mid i, j, k \geqq 1\}$ is an inherently ambiguous context-free language [**13**]. It is recursively unsolvable whether a given context-free language is unambiguous or inherently ambiguous.

In view of the undecidability of many important questions for the family of context-free languages, there is interest in finding subfamilies of context-free languages for which these questions are decidable. One such subfamily is the family of "bounded" context-free languages (a set $X\subset\Sigma^*$ is called *bounded* if there exist words $w_1, \cdots, w_r \in \Sigma^*$ such that $X \subset w_1^* \cdots w_r^*$). This family turns

out to have a much more satisfactory theory [10] than the family of all context-free languages. In particular, it is recursively solvable whether a given context-free language is bounded, and for two context-free languages at least one of which is bounded, the inclusion relation is decidable.

In the general area of the structure of languages little is known. A typical question here might be the decomposition of an arbitrary context-free language into simpler ones. A closely related question concerns the algebraic properties of context-free languages (for example, as in [15]).

Another general question is the study of invariants of languages. Here we ask for quantities associated to a grammar that depend only on the corresponding language. Ideally the quantities should be easily calculated and should determine the language or determine it up to simple modification in some suitable sense.

4. Abstract families of languages. The general closure properties of the four families of languages discussed so far, as well as other families of languages, have been abstracted by Ginsburg and Greibach [7] in the concept of an "abstract family of languages," which we now discuss. Given an infinite set Σ, an *abstract family of languages* $\mathcal{L}$ over Σ is a family $\mathcal{L} = \{L\}$ of subsets $L \subset \Sigma^*$ such that:

(1) For each $L \in \mathcal{L}$ there is some finite subset $\Sigma_L \subset \Sigma$ such that $L \subset \Sigma_L^*$.

(2) There is some $L \in \mathcal{L}$ with $L \neq \varnothing$.

(3) $\mathcal{L}$ is closed under union, product, plus, regular intersection, nonerasing homomorphism, and arbitrary inverse homomorphism.

The four families discussed in Section 3 are abstract families of languages. So also are many families described by means of acceptors such as stack automata [8].

Any abstract family of languages contains all regular sets not containing ϵ and is closed under nonerasing generalized sequential machine mappings. Many of the properties that hold for the four families of phrase structure languages considered earlier also hold for any abstract family of languages. The unifying concept of abstract family of languages has served to emphasize other families of languages and is currently being studied in various connections.

There is also a definition of an "abstract family of acceptors" which serves as an abstract description of a family of automata. A main result asserts that abstract families of languages that are closed under arbitrary homomorphisms are identical with the families defined by acceptance by means of an abstract family of acceptors.

It is an open problem to formalize the concept of an "abstract family of grammars" in such a way that the corresponding families of languages coincide with the abstract families of languages.

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ON THE DISTRIBUTION OF FIRST SIGNIFICANT FIGURES

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1. Introduction. According to F. Benford, in what appears to be the earliest paper on this subject [2], someone once noticed that library books of logarithm tables used by students were dirtier towards the beginning than towards the end. While such a phenomenon is to be expected in bad novels, it seemed curious that students of chemistry and engineering should have been more interested in the logarithms of numbers beginning (say) with the digit 1 than with 2, and so on. This curiosity suggested that the first digits of the numbers dealt with by these students did not occur with equal frequency, but that the earlier digits appeared more often than the later.

Benford made many counts, using numerical tables of various sorts. Typical examples of such tables might be:

(a) The street addresses of the first thousand persons listed in *Who's Who*;

(b) The areas, in square miles, of a thousand rivers in the U.S.A.; and

(c) The freezing points, in degrees centigrade, of a thousand chemical compounds.

In his collection of such "random" cases, Benford found that the fraction of all entries whose first digit is $\leqq p$ is approximately $\log_{10}(p+1)$, for $p=1, 2, \cdots, 9$. The law does not hold, of course, in "systematic" tables, such as the square roots of the first thousand integers.

Benford's explanation of this law, partly mathematical but necessarily partly philosophical, did not end the matter; [9] and [6] list some of the papers which followed. The recent paper of B. J. Flehinger [6] shares with Benford's paper a summability attitude towards the problem. Flehinger regards the universe of all possible entries in lists such as (a), (b), and (c) (let us call these numbers "constants") as represented by N, the class of all positive integers, since the position of the decimal point is irrelevant to the problem of first digits. If $E_p \subset N$ is the set of all integers whose first digit (in decimal expansion) is $\leqq p$, the "probability" that a given integer is in E_p should be measured by some summability scheme on the sequence of 0's and 1's given by the characteristic function of E_p:

$$
\begin{aligned}
f(n) &= 1 \qquad \text{if } n \in E_p; \\
&= 0 \qquad \text{if } n \notin E_p.
\end{aligned}
$$

Flehinger shows that the $(C, 1)$ method applied to f produces a new sequence $f^{(2)}$, where $f^{(2)}(n) = (1/n)\sum_{k=1}^{n} f(k)$, which is also oscillatory; thus f is not summable $(C, 1)$. But iteration of the procedure, defining $f^{(j)}(n) = (1/n)\sum_{k=1}^{n} f^{(j-1)}(k)$ yields progressively more narrowly oscillatory sequences, whose later entries cluster around $\log_{10}(p+1)$. To be precise, $\lim_{j\to\infty} \limsup_{n\to\infty} f^{(j)}(n) = \lim_{j\to\infty} \liminf_{n\to\infty} f^{(j)}(n) = \log_{10}(p+1)$.

Of course, there is nothing surprising in the notion that *some* method of summability, indeed even a Toeplitz method, should produce the logarithm law, though there is something natural about $(C, 1)$ and its relatives as a measure of frequency. Still, Flehinger's method has no obvious justification as a description of the universe of constants.

In the paper of Pinkham [9], and more recently in unpublished work of Bumby and Ellentuck [3], another philosophical hypothesis of a sort is given, from which a mathematical development drives one to the logarithm law.

Consider example (b) above. If in the list of rivers and their areas, "acres" were substituted for "square miles," each numerical entry would be multiplied by 640. Whatever statistical regularity the original table possessed in common with other tables of its type should still be apparent in the new table, and indeed in every table obtained from it by a scale change of whatever size, since there is no end to the variety of units of area which could be employed. Pinkham therefore invoked a scale-invariance principle for the measures, or probabilities, of the sets he considered, in the following statistical model:

R^+ (the positive real numbers) is regarded as the potential universe of physical constants, and a cumulative distribution function $F:R^+\to[0, 1]$ is assumed to exist, with properties

(1) $F(0)=0$, $F(\infty)=1$.

(2) F is continuous and monotone increasing. (Continuity because one hesitates to assign a positive probability to a single number.)

Then F induces a countably additive probability measure ϕ on R^+. If $[a, b)$ is any half-open interval, say, then $\phi([a, b))=F(b)-F(a)$, and is interpreted as the probability that a number lies in $[a, b)$.

The set of all positive real numbers whose first digit in decimal expansion is $\leqq p$ is $D_p=\bigcup_{n=-\infty}^{\infty}[10^n, (p+1)10^n)$. The measure of this set, $\phi(D_p)$, is of course $\sum_{n=-\infty}^{\infty}-\{F(10^n)-F((p+1)10^n)\}$, and it is this number which it is hoped will turn out to be $\log_{10}(p+1)$ when a suitable invariance property is demanded of F.

If $k>0$, $kD_p=\bigcup_{n=-\infty}^{\infty}[k\cdot 10^n, k\cdot(p+1)10^n)$. If a physical constant lies in D_p in one system of units, it will lie in kD_p when the system of units has been rescaled by the factor k. Pinkham therefore asked F to be of such a nature that

(3) $\phi(D_p)=\phi(kD_p)$ for $p=1, 2, \cdots, 9$ and for all $k\in R^+$. With properties (1)–(3), then, Pinkham proved that $\phi(D_p)=\log_{10}(p+1)$.

Pinkham's hypothesis of scale-invariance is reasonable, given that there *is* a distribution function F describing the occurrence of members of R^+ in all the possible physical tables in all possible worlds. But such a function cannot in fact exist. For, if the first-digit behavior of F is invariant, why are not all the other statistical features of the distribution also invariant? What is so special about first digits? In other words, if $A\subset R^+$ is *any* measurable subset, it seems that $\Pr\{x\in A\}=\Pr\{x\in kA\}$ should be demanded, for any $k\in R^+$, and not only when A is D_p. But if F satisfies (1) and (2) and this requirement, we would have

$$F(1) = \Pr\{x\in(0, 1]\} = \Pr\{x\in(0, k]\} = F(k)$$

for all k, i.e., F is constant, denying (1) and (2).

The same objection can be put another way. Pinkham's model gives the existence of some $T>0$ such that $\Pr\{x\leqq T\}=1/2$, i.e., half of all physical constants are less than T. There may be such a T for a given sample (e.g., the Chemical Rubber tables [7]), but if one contemplates the possibility of an endless number of such books employing various systems of units, T takes on a mystical significance. I cannot bring myself to believe in such a number, yet this is what we must believe when we admit that scale-invariance is not to be demanded for sets like $\{x\leqq T\}$.

What I shall present is an "unorthodox" probability model related to Pinkham's. What is unorthodox about it is that it employs a finitely, but not countably, additive measure. There is no reason *a priori* why all things subjectively regarded as "probabilities" should be countably additive, though there are of course many mathematical conveniences. Dropping the requirement of countable additivity in the present problem, however, permits a good number of

other mathematical conveniences, and—more important, to me anyway— a closer model of reality.

2. Banach measures. Let $(R, +)$ be the additive group of all real numbers. It is well known that there exists a large class $\mathfrak{I}$ of real-valued set-functions, defined for all subsets of R, which are translation-invariant, finitely-additive, positive and normalized, i.e., for any $\theta \in \mathfrak{I}$,

(1) $\theta(A+s)=\theta(A)\ \forall s\in R, A\subset R$;

(2) $\theta(A\cup B)=\theta(A)+\theta(B)$ if $A\cap B=\emptyset$;

(3) $\theta(A)\geqq 0\ \forall A\subset R$;

(4) $\theta(R)=1$.

Each such θ is obtained from a "Banach limit," i.e., a functional θ' on the space of all bounded real valued functions on R, which has corresponding properties [**1**; p. 33], by setting $\theta(A)$ equal to the value of θ' on the characteristic function of A. We shall call $\mathfrak{I}$ the set of *Banach measures* on R.

For certain subsets $A\subset R$, if $\theta_1\neq\theta_2$, it may be that $\theta_1(A)\neq\theta_2(A)$. We call A *$\mathfrak{I}$-measurable* if $\{\theta(A):\theta\in\mathfrak{I}\}$ is a single number, its *Banach measure*. The following important set is Banach measurable:

$$K_a = \bigcup_{n=-\infty}^{\infty} [n, n + a),$$

where $[n, n+a)$ is the half-open interval from n to $n+a$, and $0<a\leqq 1$. The Banach measure of K_a is a, as may easily be deduced from properties (1)–(4).

3. Remark. For any $x\in R$, the interval $(-\infty, x)$ is *not* Banach measurable. Indeed, there exist Banach measures θ_1 and θ_2 such that $\theta_1((-\infty, x))=1$ and $\theta_2((-\infty, x))=0$, as may be proved by applying Theorem 1 of [**10**] to the characteristic function of the interval $(-\infty, x)$.

4. Scaled measures on R^+. The mapping $\log: R^+\to R$, where log means $\log_{10}$, maps the multiplicative group $(R^+, \cdot)$ isomorphically onto the additive group $(R, +)$. Hence there exists, by transportation, a set $\mathfrak{S}$ of positive, finitely-additive, normalized, *scale-invariant* set functions defined for all subsets of R^+, i.e., if $\sigma\in\mathfrak{S}$,

(1′) $\sigma(As)=\sigma(A)\forall s\in R^+, A\subset R^+$;

(2′) $\sigma(A\cup B)=\sigma(A)+\sigma(B)$ if $A\cap B=\emptyset$;

(3′) $\sigma(A)\geqq 0\ \forall A\subset R^+$;

(4′) $\sigma(R^+)=1$.

Indeed if $\theta\in\mathfrak{I}$, its image $\sigma\in\mathfrak{S}$ may be defined by $\sigma(A)=\theta(\log(A))$. This correspondence is clearly a 1:1 mapping of $\mathfrak{I}$ onto $\mathfrak{S}$, where $\mathfrak{I}$ is the set of *all* Banach measures on R, and $\mathfrak{S}$ is the set of *all* scaled measures on R^+ (i.e., set functions satisfying (1′)–(4′)). (1′) is a consequence of (1) and the arithmetic properties of log, and (2′)–(4′) only require the 1:1 onto property of log. Thus, in particular, scaled measures exist on R^+, and since log is an isomorphism the entire process could have been reversed: *every* scaled measure on R^+ comes from a Banach measure on R. A subset $A\subset R^+$ will be called scale-measurable if $\{\sigma(A):\sigma\in\mathfrak{S}\}$

reduces to a single point. Clearly A is scale-measurable iff $\log(A)$ is Banach measurable.

5. The first-digit problem. Let $D_p \subset R^+$ be the set of all real positive numbers whose first digit in decimal notation is $\leqq p$. Thus $D_p = \bigcup_{-\infty}^{\infty} [10^n, (p+1)10^n)$. We look for a measure of D_p as a subset of R^+ which should satisfy (2′), (3′) and (4′) of course, if it is to mean anything like "frequency of occurrence," and which must satisfy (1′) if, as a measure of the frequency of occurrence in tables of physical constants (say), it is to be independent of the units employed. Thus we wish to compute $\sigma(D_p)$ for some $\sigma \in \mathcal{S}$. But if $\sigma \in \mathcal{S}$, $\sigma(D_p) = \theta(\log D_p)$, where θ is the Banach mean corresponding to σ. Now $\log D_p = \bigcup_{-\infty}^{\infty} [n, n + \log(p+1))$, so that $\theta(\log D_p) = \sigma(D_p) = \log(p+1)$, no matter which $\theta \in \mathfrak{I}$ is employed, because $\log D_p = K_{\log(p+1)}$ in the notation of Section 2. Thus every scaled measure on R^+ gives measure $\log(p+1)$ to the set D_p.

6. Remark. The interval $(0, T] \subset R^+$, which in Pinkham's model had probability $\frac{1}{2}$, straining one's credulity, is in the scaled measure model merely not scale-measurable. It is the image of $(-\infty, \log_{10} T] \subset R$, and this set, by Remark 4, has no Banach measure.

Any finite interval $[x, y) \subset R^+$ is the image of $[\log_{10} x, \log_{10} y) \subset R$, which has Banach measure 0. This seems philosophically defensible. If all units are equally valid in the construction of physical tables, then almost all entries ought to be near 0 and ∞.

7. Affinely invariant measures. The set $\mathcal{S}$ of scale invariant measures on R^+, described by (1′)–(4′) of Section 4 above, contains a nonempty proper subset $\mathcal{A}$ satisfying the additional invariance property: For any $\delta \in \mathcal{A}$,

$$\delta(A + s) = \delta(A) \ \forall s \in R^+, \qquad A \subset R^+. \tag{5'}$$

I shall omit the proof that there actually exist such measures; it is a fairly technical exercise in the theory of *amenable semigroups* [**4**], and depends on the fact that the collection of "positive affine mappings" $x \to \alpha x + \beta$ ($\alpha > 0$, $\beta > 0$) is amenable.

To show that $\mathcal{A}$ is a *proper* subset of $\mathcal{S}$ is easy, and worthwhile. As was noted in Remark 6, the set $(0, T] \subset R^+$ has no scaled measure, since there exist scale-invariant measures δ_1 and δ_2 in $\mathcal{S}$ such that $\delta_1(0, T] = 0$ and $\delta_2(0, T] = 1$. But if $\delta \in \mathcal{A}$, it is easy to see that $\delta(0, T] = 0$; thus not all members of $\mathcal{S}$ are members of $\mathcal{A}$.

For,

$$\begin{aligned} \delta(0, T] &= \delta(0, 2T] \text{ by scale invariance} \\ &= \delta(0, T] + \delta(T, 2T] \text{ by finite additivity} \\ &= \delta(0, T] + \delta(0, T] \text{ by } (5') \\ &= 2\delta(0, T]. \end{aligned}$$

Hence $\delta(0, T] = 0$.

Combining (5′) with (1′) we derive the phrase "affine-invariance." The relevance of such measures to the first-digit problem is this: Certain tables of physical constants depend for their values not only on the size of the units involved, but on the choice of zero. A typical example would be a table of freezing points (Example (c) of Section 1), where conversion from Fahrenheit to centigrade (or Kelvin) should preserve the statistical regularity of the table.

It seems at first as if the universe R^+ is insufficient once translations are allowed; some freezing points, in some systems of units, may be negative. However, the fact that $\delta(0, T]=0$ for any T implies that the lower end of the scale doesn't count anyhow. The technical details involved in extending the measures $\mathcal{A}$ to half-lines beginning outside of R^+ (e.g., temperatures centigrade begin at -273) are not really worth going into here.

Since $\mathcal{A}\subset\mathcal{S}$, any set $A\subset R^+$ which is scale-measurable is also affine-measurable, i.e., $\{\delta(A):\delta\in\mathcal{A}\}$ is a single point. And of course the $\mathcal{A}$-measure of D_p remains $\log(p+1)$.

8. Rational and decimal models. For those who do not enjoy the idea of R^+ as the model for physical constants, I have a reassuring comment. If, in all the development above, R^+ is replaced by Q^+, the set of positive *rational* numbers, or by T^+, the set of all positive terminating decimals, nothing need be changed. T^+ is probably the best model, for actual tables of constants are generally written (or approximated) as decimal fractions. This is why Flehinger, choosing to suppress the decimal point, could regard them as integers. Changes of scale and translations will, in these two models, have to be made via numbers in the system in question, and the set

$$K_a = \bigcup_{n=-\infty}^{\infty} [n, n + a)$$

of Section 2 has to be interpreted as the intersection of the genuine K_a with Q^+ or T^+, and it must be shown that Banach measures on Q^+ and T^+ exist and give value a to K_a. None of this is trivial, but it all can be done. My proofs employ theorems from [**4**, **5**, **8**, **11**].

9. Final remarks. Of the three types of tables (a), (b), (c) of Section 1, the scale-invariance hypothesis serves (b), the areas of rivers, and the affine-invariance hypothesis serves (c), the freezing points of chemicals. The problem of (a) remains: why should the street addresses of a thousand famous men obey the logarithm law? I know no answer to this question.

Actually, invariance principles alone do not really answer this question for (b) and (c) either. All that has been proved mathematically is this: *If* there is a "frequency of occurrence" of D_p in R^+ (or Q^+ or T^+) in tables of type (b) and (c), then this frequency, since it must be scale invariant, is $\log(p+1)$. Scientific observation tells us there is such a frequency, but there is nothing philosophically *a priori* about this. In the case of street addresses the situation is worse. Even the belief, from observation, that a frequency does exist doesn't

help, since scale-invariance is quite meaningless. In fact, one would have to be a numerologist to make sense out of the operation of multiplying every entry in *Who's Who* by some positive constant.

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LINEARIZATION IN RINGS AND ALGEBRAS

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1. Introduction. If R is a ring in which every element satisfies the identity $x^2=0$, then it is easy to show that R is anticommutative, i.e., that $xy=-yx$ for all x, y in R. This is so because in $xy+yx=(x+y)^2-x^2-y^2$ each of the terms $(x+y)^2$, $-x^2$, and $-y^2$ is zero and, therefore, so is their sum. The method by which the anticommutativity of R is deduced from $x^2=0$ is a simple illustration of a process known as "linearization" or "polarization," a technique by means of which an identity of high degree in one variable is made to yield an identity of lower degree in several variables. Linearization is an important technique in associative and nonassociative rings and algebras, but it usually appears in the literature in a form that fails to reveal its general features. Although the linearization process has been studied in [4] and [7], to our knowledge an elementary exposition of this technique has never been made.

In this article we shall describe a technique for linearizing polynomial identities systematically. To demonstrate its utility, in Section 4 we use linearization to prove the new result that a strictly power-associative p-ring is commutative.

2. Preliminaries. Let A be an algebra over a field F. Let $\mathcal{P}$ be the algebra of all polynomials (not necessarily associative or commutative) in indeterminates $x_1, x_2, \cdots$ over F. Let $P(x_1, \cdots, x_n) \in \mathcal{P}$. We say that A satisfies P if P vanishes when the $x_1, \cdots, x_n$ are replaced by any $a_1, \cdots, a_n$ from A. Also, we identify the polynomial $P(x_1, \cdots, x_n)$ with the identity $P(x_1, \cdots, x_n) = 0$ in order to say that A satisfies $P = 0$.

We will need a special case of the class of linear operators studied by Gerstenhaber in [4]. Let α be a monomial in $\mathcal{P}$. Corresponding to each variable $x_i \in \mathcal{P}$ we define an operator $\delta_i(\alpha)$ from the monomials of $\mathcal{P}$ to $\mathcal{P}$ as follows: $m(x_1, \cdots, x_n)\delta_i(\alpha) = 0$ if x_i does not appear in m; otherwise, $m(x_1, \cdots, x_n)\delta_i(\alpha)$ = the polynomial that is obtained by making *all possible* replacements of the arguments x_i one at a time by α and summing the resulting monomials. The number of monomials obtained is the degree of x_i in $m(x_1, \cdots, x_n)$. For example:

$$[(xx)(yx)]\delta_x(\alpha) = (\alpha x)(yx) + (x\alpha)(yx) + (xx)(y\alpha).$$

The operator $\delta_i(\alpha)$ is then extended to all of $\mathcal{P}$ by linearity. We denote the rth iterate of $\delta_x(\alpha)$ by $\delta_x^r(\alpha)$. Thus

$$(*) \qquad [(xx)(yx)]\delta_x^2(\alpha) = 2(\alpha\alpha)(yx) + 2(\alpha x)(y\alpha) + 2(x\alpha)(y\alpha).$$

In general, if m is a monomial of degree n in x and if $i \leqq n$, then $m\delta_x^i(\alpha)$ is a polynomial with $\binom{n}{i}$ *distinct* terms, each of which occurs with multiplicity $i!$. We define

$$m\frac{\delta_x^i(\alpha)}{i!}$$

to denote the sum of the $\binom{n}{i}$ distinct terms. Thus, the $i!$ occurring in the operator $(\delta_x^i(\alpha))/i!$ is not to be construed as a field element. In (*), if the characteristic of F is 2, then

$$[(xx)(yx)]\delta_x^2(\alpha) = 0,$$

but

$$[(xx)(yx)]\frac{\delta_x^2(\alpha)}{2!} = (\alpha\alpha)(yx) + (\alpha x)(y\alpha) + (x\alpha)(y\alpha).$$

The operator $\delta_x^i(\alpha)/i!$ is extended to all of $\mathcal{P}$ by linearity.

We will need to use the fact that $\delta_x(\alpha)$ is a *derivation* on $\mathcal{P}$.

PROPOSITION. *Let $P_1, P_2 \in \mathcal{P}$. Then*

$$(1) \qquad (P_1P_2)\delta_x(\alpha) = P_1\delta_x(\alpha)P_2 + P_1\cdot P_2\delta_x(\alpha).$$

Proof. Without loss of generality we may assume that P_1 and P_2 are mo-

nomials. Suppose x occurs m times in P_1 and n times in P_2. There are $m+n$ occurrences of x in P_1P_2; therefore, $(P_1P_2)\delta_x(\alpha)$ is a sum of $m+n$ monomials. We divide this set of monomials into two disjoint classes. The first consists of all terms arising from a replacement of the x's in P_1 by α; the second consists of terms arising from a replacement of the x's in P_2 by α. Every monomial in $P_1\delta_x(\alpha)\cdot P_2$ belongs to the first class; every monomial in $P_1\cdot P_2\delta_x(\alpha)$ belongs to the second class. Since $P_1\delta_x(\alpha)\cdot P_2$ has m terms and $P_1\cdot P_2\delta_x(\alpha)$ has n terms, we must have equality in (1).

3. Linearization and identities. We define $P(x_1, \cdots, x_n)=0$ to be a *homogeneous* identity if P is homogeneous in each variable singly. That is, each x_i occurs the same number, k_i, of times in each monomial of P. We then say that the degree of x_i is k_i and assign the degree $k_1+ \cdots +k_n$ to P. For example, if $P(x_1, x_2) = (x_1^2x_2)x_1-x_1(x_2x_1^2)$, then the degree of x_1 is 3, the degree of x_2 is 1, and P is homogeneous of degree 4.

It is known [5] that if A satisfies a multilinear identity (every monomial is of degree 1 in each variable) $P(x_1, \cdots, x_n)=0$, then A will satisfy a homogeneous identity regardless of the cardinality of F. For if x_1 does not appear in some monomial of P, then $P(0, x_2, \cdots, x_n)=0$ is a multilinear identity of degree not greater than P satisfied by A. Continuation of this process will yield a homogeneous identity satisfied by A.

Let A be an algebra over a field F which satisfies a homogeneous polynomial identity $P(x_1, \cdots, x_m)=0$. It is natural to ask what new identities also satisfied by A can be derived from P. It is clear that the set $\mathcal{S}$ of all identities satisfied by A is a subalgebra, in fact, an ideal of $\mathcal{P}$.

Let $\mathcal{P}_m$ be defined as the subalgebra of $\mathcal{P}$ of all polynomials in m indeterminates. Then, for any elements $a_1, \cdots, a_m$ of A, we can define a natural homomorphism, $P(x_1, \cdots, x_m)\rightarrow P(a_1, \cdots, a_m)$, of $\mathcal{P}_m$ onto A. The kernel $\mathcal{S}(a_1, \cdots, a_m)$ of this homomorphism is merely the collection of all polynomials of $\mathcal{P}_m$ which are satisfied by $a_1, \cdots, a_m$. Thus, the collection of all elements of $\mathcal{P}_m$ which are satisfied by A is

$$\bigcap_a \mathcal{S}(a_1, \cdots, a_m),$$

the intersection being taken over all sequences, $a=(a_1, \cdots, a_m)$, of m elements of A.

It is easy to see from the above that $\mathcal{S}=\bigcup_m\bigcap_a \mathcal{S}(a_1, \cdots, a_m)$.

For any indeterminates x, y in $\mathcal{P}$ and λ in F, we have the following

LEMMA.

$$(x+\lambda y)^n = \sum_{i=0}^{n} \lambda^i x^n \frac{\delta_x^i(y)}{i!}. \tag{2}$$

Proof. We proceed by induction on n. If $n=1$, (2) is clear. We suppose that (2) holds and will show that

$$(x+\lambda y)^{n+1} = \sum_{i=0}^{n+1} \lambda^i x^{n+1} \frac{\delta_x^i(y)}{i!}. \tag{3}$$

We have

$$\begin{aligned}(x+\lambda y)^{n+1} &= (x+\lambda y)^n (x+\lambda y) \\ &= x^{n+1} + \sum_{i=1}^{n} \lambda^i \left[x^n \frac{\delta_x^i(y)}{i!}\right] x + \sum_{i=0}^{n-1} \lambda^{i+1} \left[x^n \frac{\delta_x^i(y)}{i!}\right] y + \lambda^{n+1} y^{n+1}.\end{aligned} \tag{4}$$

By shifting the index in the third term of (4), we find that

$$(x+\lambda y)^{n+1} = x^{n+1} + \sum_{i=1}^{n} \lambda^i \left\{ \left[x^n \frac{\delta_x^i(y)}{i!}\right] x + \left[x^n \frac{\delta_x^{i-1}(y)}{(i-1)!}\right] y \right\} + \lambda^{n+1} y^{n+1}.$$

The proof will be completed by showing that

$$x^n \frac{\delta_x^i(y)}{i!} \cdot x + x^n \frac{\delta_x^{i-1}(y)}{(i-1)!} \cdot y = x^{n+1} \frac{\delta_x^i(y)}{i!} \qquad \text{for } i = 1, \cdots, n. \tag{5}$$

However, (5) merely states that the $\binom{n+1}{i}$ possible distinct terms resulting by replacing y, i-times, in x^{n+1} and summing, can be alternatively gotten by adding the results of the following two operations:

(i) summing the $\binom{n}{i}$ terms resulting from replacing y, i-times, in the first n factors of x^{n+1},

(ii) summing the $\binom{n}{i-1}$ terms resulting from replacing y, $(i-1)$ times, in $x^n y$.

The equality of these two alternate procedures for computing $x^{n+1}\delta_x^i(y)/i!$ follows directly from the definition. Thus, (5) follows and the lemma is proved.

When we want to fix our attention upon some one variable, say $x_i = x$ of degree n, we write $P(x)$ instead of $P(x_1, \cdots, x_m)$. We must have

$$P(x+\lambda y) = \sum_{i=0}^{n} \lambda^i S_i(x, y)$$

for some polynomials $S_i(x, y)$, where y is chosen as an indeterminate independent of x and $\lambda \in F$.

THEOREM 1. $S_i(x, y) = P(x)[\delta_x^i(y)/i!]$.

Proof. It suffices to prove the theorem for P a monomial. We just note that since $S_i(x, y)$ is the "coefficient" of λ^i, we must have i occurrences of y. Thus, as in the lemma, $S_i(x, y)$ is equal to the sum of all $\binom{n}{i}$ possible distinct monomials formed from $P(x)$ by substituting y for x, i-times, in all possible ways. This sum is precisely $P(x)[\delta_x^i(y)/i!]$.

THEOREM 2. *Let A be an algebra over the field F. If the homogeneous polynomial $P(x_1, \cdots, x_m)$ belongs to $\mathcal{S}$ and F has at least n elements, where n is the degree of x in $P(x)$, then $P(x)[\delta_x^i(y)/i!]$ belongs to $\mathcal{S}$.*

That is, application of the δ-operator to a homogeneous identity $P=0$, satisfied by such an A, yields a new identity, $P(x)[\delta_x^i(y)/i!]=0$, also satisfied by A. Thus, the original identity of high degree in x yields a set of new identities, each in more indeterminates, but of lower degree in x. The process by means of which the S_i are obtained from P is called *partial linearization*. Of course, S_1 may be further linearized by repeating the process and introducing new variables, until finally a multilinear polynomial, called the *linearized form* of P, is obtained. If the new variables introduced are set equal to one another, the new result is a scalar multiple of P.

Proof of Theorem 2. $P(x)=0$ is satisfied by A; thus, A satisfies $P(x+\lambda y)=0$ for $\lambda\in F$. Therefore, $\sum_{i=0}^{n}\lambda^i S_i(x, y)=0$. We note that $S_0(x, y)=P(x)=0$ and that $S_n(x, y)=P(y)=0$. Thus, $\sum_{i=1}^{n-1}\lambda^i S_i(x, y)=0$.

Now successively replace λ in this last equation by the $n-1$ distinct nonzero scalars $\lambda_1, \cdots, \lambda_{n-1}$ to obtain the system:

$$\begin{array}{l} \lambda_1 S_1 + \lambda_1^2 S_2 + \cdots + \lambda_1^{n-1} S_{n-1} = 0, \\ \vdots \\ \lambda_{n-1} S_1 + \lambda_{n-1}^2 S_2 + \cdots + \lambda_{n-1}^{n-1} S_{n-1} = 0. \end{array}$$

Let the (Vandermonde) determinant of the system be $V=\prod(r-s)$ for $1\leqq s<r\leqq n-1$, and let V_j denote the cofactor of the element in column one and row j of the coefficient matrix. Multiply the jth equation by V_j for each $j=1, \cdots, n-1$, and add the resulting equations. By elementary determinant theory, $VS_1=0$, and because $V\neq 0$, we have $S_1=0$. Similarly, each $S_i=0$, and the theorem is proved.

Theorem 2 may be restated to say that $\mathcal{S}$ is invariant under $\delta_x^i(y)$ if F is of high enough characteristic. Certainly a field F of characteristic zero will do. We give some examples.

The ring R, mentioned at the beginning of Section 1, satisfies $P(x)=x^2=0$. Using Theorem 2, we see that $0=P(x)\delta_x(y)=xy+yx$. Thus, R is anticommutative.

Next, just as the commutator $[x, y]=xy-yx$ measures departure of a ring from commutativity, so the associator $(x, y, z)=(xy)z-x(yz)$ measures departure from associativity. If an algebra satisfies $P(x)=(x, x, x)=0$ (3rd power-associativity), we have

$$S_1(x, y) = (x, x, x)\delta_x(y) = (y, x, x) + (x, y, x) + (x, x, y) = 0.$$

We may linearize further to obtain the multilinear identity

$$\begin{aligned} P(x)\delta_x(y)\delta_x(z) &= S_1(x, y)\delta_x(z) \\ &= (y, z, x) + (y, x, z) + (z, y, x) + (x, y, z) + (z, x, y) + (x, z, y) = 0. \end{aligned}$$

We mention that an algebra A over F which satisfies a multilinear identity $P=0$ will also satisfy $P=0$ over any scalar extension of F. Thus, in matters

involving an enlarging of a field of scalars, it is of obvious utility to try to reduce a given polynomial identity to an equivalent multilinear identity. An important illustration of this is given by power-associativity.

An algebra A over F is said to be *power-associative* if it satisfies $x^n x^m = x^{n+m}$ for all positive integers n and m, where powers are defined inductively by $x^1 = x$ and $x^{n+1} = x^n x$ for $n = 1, 2, 3, \cdots$.

An algebra A which is power-associative over every scalar extension of F is called *strictly* power-associative. Kokoris [6] has shown that a power-associative algebra need not be strictly power-associative, but that the two concepts coincide for characteristic $\neq 2, 3, 5$. One insures strict power-associativity in a power-associative algebra by requiring that all partial linearizations of identities equivalent to $x^n x^m - x^{n+m} = 0$ also be identities satisfied by A.

The theory of power-associative algebras was created by A. A. Albert. Under the aegis of Albert and Jacobson the subject has been developed extensively in the past twenty-five years (see the bibliography in [8] for references to their work).

Jordan algebras are among the more important power-associative algebras. A Jordan algebra A is a commutative algebra over a field F which satisfies the identity

$$(x^2y)x - x^2(yx) = 0.$$

A successful structure theory exists for these algebras which depends upon a Pierce decomposition analogous to the classical Pierce decomposition for associative algebras. As another example of the use of linearization, we obtain the Pierce decomposition for Jordan algebras.

We assume here that the characteristic of F is not 2 or 3. Apply the operator $\delta_x(u)$ to the Jordan identity to get

$$[(ux)y]x + [(xu)y]x + (x^2y)u - [(ux)(yx) + (xu)(yx) + x^2(yu)] = 0.$$

Or, since A is commutative, $2[(ux)y]x + (x^2y)u - 2(ux)(yx) - x^2(yu) = 0$.

Now we apply $\delta_x(v)$ to this last equation and obtain the linearized form of the Jordan identity:

$$(6) \quad [(uv)y]x + [(ux)y]v + [(xv)y]u - (uv)(yx) - (ux)(yv) - (xv)(yu) = 0.$$

(Setting $u = v = x$, we see that this reduces to $3(x^2y)x - 3x^2(yx) = 0$. Thus, the Jordan identity and its linearized form are equivalent for our choice of F.)

For any $a \in A$ define the mapping R_a: $A \to A$ by $xR_a = xa$ for every $x \in A$. It is easy to see that each R_a is a linear transformation on A. If A has an idempotent e (an element e such that $e^2 = e$), we can set $x = y = v = e$ in (6) to get $u(2R_e^3 - 3R_e^2 + R_e) = 0$ for any $u \in A$. Thus,

$$2R_e^3 - 3R_e^2 + R_e = R_e(2R_e - 1)(R_e - 1) = 0.$$

Hence, the characteristic roots of R_e are in the set $\{0, \frac{1}{2}, 1\}$. A can now be decomposed as a vector space direct sum

$$A = A_e(0) + A_e(\tfrac{1}{2}) + A_e(1),$$

where $A_e(\lambda)=\{x\in A \mid xR_e=\lambda x\}$ are R_e-invariant subspaces for $\lambda=0, \frac{1}{2}$, or 1. Each $a\in A$ can be written $a=a_0+a_{1/2}+a_1$ where $a_0=a+2(ae)e-3ae$, $a_{1/2}=4[ae-(ae)e]$, and $a_1=2(ae)e-ae$, with $a_\lambda\in A_e(\lambda)$. This sum is known as the Pierce decomposition of A with respect to the idempotent e. Properties of this decomposition are fundamental in Albert's structure theory for Jordan algebras [1, 2].

4. **An application.** A p-ring is a ring R with the property that for all $a\in R$, $a^p=a$ and $pa=0$, where p is a fixed prime. An elementary proof that associative p-rings are commutative was given by Forsythe and McCoy in [3]. We will now extend this result to power-associative p-rings. The proof is made by essentially rewriting the proof in [3] in terms of δ-operators and then using the fact that $\delta_x(y)$ is a derivation.

THEOREM 3. *A strictly power-associative p-ring is commutative.*

Proof. Let a and b be any two elements of R. We may regard R as an algebra over the field J_p, the integers modulo p. Let $\mathcal{P}_2$ be the algebra in two indeterminates x and y over J_p as defined in Section 3. The correspondence $x\to a$ and $y\to b$ can be extended in a natural way to a homomorphism of $\mathcal{P}_2$ into R. Denote the kernel by K. Clearly, $z^p-z\in K$ for any z in $\mathcal{P}_2$. In particular, $(x+\lambda y)^p-(x+\lambda y)\in K$ for every $\lambda\in J_p$. Using (2) we see that

$$\sum_{i=1}^{p-1}\lambda^i x^p\frac{\delta_x^i(y)}{i!}\in K.$$

Recalling that $x^p(\delta_x^i(y)/i!)$ is just S_i of Theorem 2, and replacing λ by each of its nonzero values in turn, we obtain

$$\begin{array}{l}
S_1+\quad S_2 \quad +\cdots+ \qquad\qquad S_{p-1}\in K,\\
2S_1+2^2S_2 \quad +\cdots+ \qquad 2^{p-1}S_{p-1}\in K,\\
\quad\vdots \qquad\quad \vdots \qquad\qquad\qquad\qquad\quad \vdots\\
(p-1)S_1+(p-1)^2S_2+\cdots+(p-1)^{p-1}S_{p-1}\in K.
\end{array}$$

If we operate on each of these elements by cofactors of the coefficient matrix as in the proof of Theorem 2, we reach the conclusion that $VS_1\in K$, where V is the (nonzero) determinant of the coefficient matrix. Thus, $S_1\equiv x^p\delta_x(y)\in K$. Now $xx^p-x^px\in K$, and because R is *strictly* power-associative, we know that $(xx^p-x^px)\delta_x(y)\in K$. Since $\delta_x(y)$ is a derivation, we have

$$K\ni(xx^p-x^px)\delta_x(y)=x\delta_x(y)x^p+x(x^p\delta_x(y))$$
$$-[x^p\delta_x(y)x+x^p(x\delta_x(y))]=yx^p+x(x^p\delta_x(y))-x^p\delta_x(y)x-x^py.$$

Because $x^p\delta_x(y)\in K$ and K is an ideal, $x(x^p\delta_x(y))\in K$ and $x^p\delta_x(y)x\in K$. Thus, $yx^p-x^py\in K$, so that $ba^p-a^pb=0$, or $ba-ab=0$ and Theorem 3 is proved.

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GENERALIZATIONS OF THE THEOREMS OF PAPPUS

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1. Introduction. The Theorem of Pappus concerning the volume of a solid of revolution can be found in any book on the calculus and runs as follows:

THEOREM A. *Let $\mathfrak{D}$ be a region in the plane and let $\mathfrak{L}$ be a line in the plane of $\mathfrak{D}$. If $\mathfrak{L}$ does not meet $\mathfrak{D}$, then the volume of the solid generated when $\mathfrak{D}$ is rotated around $\mathfrak{L}$ is given by*

$$V = AL, \tag{1}$$

where A is the area of $\mathfrak{D}$ and L is the perimeter of the circle described by the centroid of $\mathfrak{D}$.

It turns out that this theorem is capable of wide generalization. Indeed if the region $\mathfrak{D}$ is allowed to move in a natural manner on any sufficiently smooth simple closed curve, then formula (1) still gives the volume of the solid generated. However the direct generalization of the corresponding theorem for the area of the surface is *false*. We shall make these statements more precise when we give the proofs of Theorems 1, 2, and 3.

2. History of the problem. According to Eves [3, p. 179], Pappus stated Theorem A (also Theorem B noted in Section 5 below) in book VII of his Mathematical Collection about 300 A.D. As far as we know he did not supply a proof. The theorems were rediscovered independently by Guldin [4] and are frequently called by his name. Neither the proofs given by Guldin (1640) nor those by Cavalieri who attacked them (1647) are very satisfactory. For such an ancient

problem, one may expect many generalizations. Professor Dirk J. Struik kindly pointed out that Euler [2] and Richter [6] had already generalized the two theorems of Pappus, but neither of these authors go quite as far as we do. (We are deeply indebted to Professor Struik for many of the historical notes.)

At first glance it may seem that certain generalizations are intuitively obvious. In the first place if $\mathfrak{D}$ is rotated only part way around the axis $\mathfrak{L}$, formula (1) still holds when L is the length of the path described by the centroid. Then we can consider an arbitrary path for the centroid as approximated by a sequence of circular arcs. Here it is of interest to quote precisely the words of Williamson [7, p. 264].

"Again Guldin's Theorems are still true if we suppose the rotation to take place around a number of different axes in succession; in which case the center of gravity, instead of describing a single circle would describe a number of arcs of circles consecutively; and the whole area of the surface generated will still be measured by the product of the length of the generating curve into the path of its center of gravity; for this result holds for the part of the surface corresponding to each axis of revolution separately, and therefore holds for the sum."

"Again, in the limit, when we suppose each separate rotation indefinitely small, we deduce the following theorem. If any plane curve moves so that the path of its center of gravity is at each instant perpendicular to the moving plane, then the surface generated by the curve is equal to the length of the curve into the path described by its center of gravity."

"The corresponding theorem holds for the volume of the surface generated."

Ball [1] and Williamson [7] ascribe these statements to Leibniz [5] but Leibniz only considered the extensions to figures generated by moving the center of gravity along a plane curve.

Unfortunately, the attractive argument presented by Williamson (and probably many others) leads to a true result for volumes, but to a *false result for surfaces*, as we shall show in Section 5. Hence the argument cannot be regarded as valid.

According to the French summary of the paper by Euler [2], he makes the same erroneous claim that Williamson does with respect to the surface area. However, the summary seems to be erroneous because in the text Euler imposes additional restrictions under which his generalization is valid.

In Sections 3 and 4 we give an analytic treatment which (we hope) separates the correct generalizations from the false ones.

3. The first generalization of the theorem of Pappus. We shall be concerned with two simple closed curves: $\mathfrak{C}_S$, a twisted curve in space; and $\mathfrak{C}_P$, a curve lying in the xy-plane. We let $\mathfrak{D}$ be the domain inside $\mathfrak{C}_P$.

To simplify the computations we use the arc length as a parameter for each of these curves. Let the space curve $\mathfrak{C}_S$ have total length L and let

$$\mathbf{R} \equiv \mathbf{R}(s) \equiv \overrightarrow{OR} = [x(s), y(s), z(s)], \qquad 0 \leqq s \leqq L, \tag{2}$$

be the vector equation of $\mathfrak{C}_S$, where s is the arc length on the curve measured

from the point

$$R_0 \equiv (x(0), y(0), z(0)) = (x(L), y(L), z(L)). \tag{3}$$

Without loss of generality, we let R_0 be a point on the positive x-axis, so that $x(0)>0$, and $y(0)=z(0)=0$. We then use R_0 as the origin for describing $\mathcal{C}_P$. Thus by definition, the points R_1 on $\mathcal{C}_P$ are given by the vector equation

$$\mathbf{R}_1 \equiv \mathbf{R}_1(t) \equiv \overrightarrow{R_0R_1} = [f(t), g(t), 0], \qquad 0 \leqq t \leqq M, \tag{4}$$

where t is the arc length along the curve $\mathcal{C}_P$, and the curve $\mathcal{C}_P$ has total length M. The situation is illustrated in Figure 1, where we use X, Y for the coordinate axes for the plane curve in order to avoid confusion with the x, y, z-axes for the space curve.

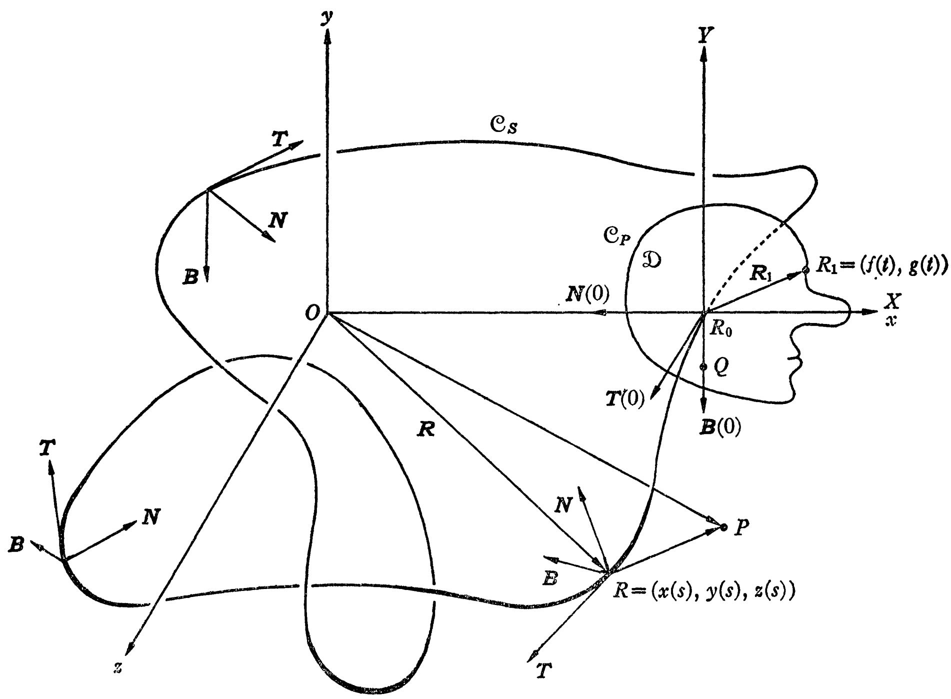

FIG. 1

As is customary we assume that the functions involved have the number of derivatives necessary to carry out the proof. However, it turns out that the requirements are different for the two curves. For $\mathcal{C}_S$ we require that the third derivative $\mathbf{R}'''(s)$ be continuous in the interval $0 \leqq s \leqq L$, and in order that this simple closed curve be smooth at the end points we require that

$$\mathbf{R}^{(k)}(0) = \mathbf{R}^{(k)}(L), \qquad k = 0, 1, 2, 3, \tag{5}$$

where $\mathbf{R}^{(k)}$ denotes the kth derivative with respect to the arc length s. We will

say that $\mathcal{C}_S$ is sufficiently smooth if and only if these two conditions are met.

Similarly $\mathcal{C}_P$ will be called sufficiently smooth if and only if $\boldsymbol{R}_1'(t)$ is continuous in the interval $0 \leqq t \leqq M$ and

$$R_1^{(k)}(0) = R_1^{(k)}(M), \qquad k = 0, 1. \tag{6}$$

With these hypotheses we can introduce the unit vectors $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ associated with the space curve $\mathcal{C}_S$. Here, of course, $\boldsymbol{T}$ is the tangent, $\boldsymbol{N}$ is the normal, and $\boldsymbol{B}$ is the binormal for $\mathcal{C}_S$ and these vectors are related by the equations

$$\frac{d\boldsymbol{R}}{ds} = \boldsymbol{T}, \tag{7}$$

$$\frac{d\boldsymbol{T}}{ds} = \kappa \boldsymbol{N}, \tag{8}$$

$$\frac{d\boldsymbol{N}}{ds} = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}, \tag{9}$$

$$\frac{d\boldsymbol{B}}{ds} = -\tau \boldsymbol{N}, \tag{10}$$

where κ and τ are the curvature and torsion of $\mathcal{C}_S$, and $\boldsymbol{B} = \boldsymbol{N} \times \boldsymbol{T}$. We must observe that κ and $\boldsymbol{N}$ are defined by equation (8) and the condition that $\kappa > 0$. Hence at any point of zero curvature, $\boldsymbol{N}$ fails to be defined. To avoid this difficulty we require that $\kappa \neq 0$ on $\mathcal{C}_S$. With this additional hypothesis, it is clear that $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$, are continuous vector functions of s.

We are now in a position to define rigorously just what is meant by a "natural" motion of the region $\mathfrak{D}$ along the curve $\mathcal{C}_S$. We select our coordinate system (or the curve $\mathcal{C}_S$) so that at $s = 0$, the vectors $\boldsymbol{T}$ and $\boldsymbol{N}$ lie in the xz-plane, and $\boldsymbol{N}$ lies on the x-axis pointed toward the origin. Let R_0Q be a line segment in the plane of $\mathfrak{D}$ that coincides with a portion of $\boldsymbol{B}(0)$. As the point R moves along the curve $\mathcal{C}_S$, starting at R_0, we move the region $\mathfrak{D}$ in such a way that: R_0 always coincides with $\mathcal{C}_S$, the segment R_0Q always coincides with $\boldsymbol{B} \equiv \boldsymbol{B}(s)$ and $\mathfrak{D}$ always lies in the plane of $\boldsymbol{N}(s)$ and $\boldsymbol{B}(s)$. Clearly, in the special case that $\mathcal{C}_S$ is a circle about the z-axis, this motion of $\mathfrak{D}$ is just the old familiar rotation of $\mathfrak{D}$ about the z-axis. Consequently the motion of $\mathfrak{D}$ described above will be called the *natural motion* of $\mathfrak{D}$ along the curve $\mathcal{C}_S$.

Although a description of the natural motion was awkward, it is an easy matter to describe the surface of the solid $\mathcal{V}$ generated by the motion. Indeed if we write

$$\boldsymbol{P} = \overrightarrow{OP} = \overrightarrow{OR} + \overrightarrow{RP} \tag{11}$$

or

$$\boldsymbol{P} = \boldsymbol{P}(s, t) = \boldsymbol{R}(s) - f(t)\boldsymbol{N}(s) - g(t)\boldsymbol{B}(s), \tag{12}$$

it is obvious that $\boldsymbol{P}$ is the position vector for points on the surface of $\mathcal{V}$. This means that P is a point on the surface of $\mathcal{V}$ if and only if there is a point (s_0, t_0) in the rectangle

$$0 \leqq s \leqq L, \qquad 0 \leqq t \leqq M, \tag{13}$$

such that

$$\boldsymbol{P}(s_0, t_0) = \overrightarrow{OP}. \tag{14}$$

With these preparations it is now an easy matter to compute V, the volume of $\mathcal{V}$, and to show that formula (1) still holds. We apply the divergence theorem

$$\iiint_{\mathcal{V}} \nabla \cdot \boldsymbol{P} dv = \iint_{\mathcal{S}} \boldsymbol{P} \cdot \boldsymbol{n} d\sigma, \tag{15}$$

where $\mathcal{S}$ denotes the surface of $\mathcal{V}$, $\boldsymbol{n}$ is an outward unit normal and $d\sigma$ is a scalar element of surface area. Of course the curves $\mathcal{C}_S$ and $\mathcal{C}_P$ must be such that the solid $\mathcal{V}$ is not self-intersecting but this is again a natural condition for the problem under consideration.

Since $\boldsymbol{P}$ is merely a position vector we have $\nabla \cdot \boldsymbol{P} = 3$ so (15) gives

$$V = \frac{1}{3} \iint_{\mathcal{S}} \boldsymbol{P} \cdot \boldsymbol{n} d\sigma. \tag{16}$$

To evaluate this integral we must select our parameterization of $\mathcal{C}_P$ and $\mathcal{C}_S$ so that $\boldsymbol{n}$ is the outward normal from the solid $\mathcal{V}$. This will occur if (a) we select the parameterization in (4) so that $\mathcal{C}_P$ is described counterclockwise as t increases, (b) we select the parameterization in (2) so that at $s=0$ the tangent vector $\boldsymbol{T}$ points in the direction of the positive z-axis, and (c) we set

$$\boldsymbol{n} d\sigma = \frac{\partial \boldsymbol{P}}{\partial t} \times \frac{\partial \boldsymbol{P}}{\partial s} dt ds. \tag{17}$$

Using equations (12), (7), (9), and (10), and the fact that $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ form a righthanded orthonormal system we find that:

$$\frac{\partial \boldsymbol{P}}{\partial t} = -f'(t)\boldsymbol{N}(s) - g'(t)\boldsymbol{B}(s), \tag{18}$$

$$\frac{\partial \boldsymbol{P}}{\partial s} = (1 + f(t)\kappa(s))\boldsymbol{T}(s) + g(t)\tau(s)\boldsymbol{N}(s) - f(t)\tau(s)\boldsymbol{B}(s), \tag{19}$$

and

$$\frac{\partial \boldsymbol{P}}{\partial t} \times \frac{\partial \boldsymbol{P}}{\partial s} = (ff' + gg')\tau\boldsymbol{T} - g'(1 + f\kappa)\boldsymbol{N} + f'(1 + f\kappa)\boldsymbol{B}, \tag{20}$$

where we have dropped the arguments t and s in equation (20). Using (12), (17)

and (20) in (16) we find that

$$V = \frac{1}{3}\int\int_{\mathcal{S}} \boldsymbol{P}\cdot\boldsymbol{n}d\sigma = \frac{1}{3}(I_1 - I_2 + I_3 + I_4 - I_5) \tag{21}$$

where

$$I_1 \equiv \int\int \boldsymbol{R}\cdot\boldsymbol{T}(ff' + gg')\tau dtds,$$

$$I_2 \equiv \int\int \boldsymbol{R}\cdot\boldsymbol{N}g'(1 + f\kappa)dtds,$$

$$I_3 \equiv \int\int \boldsymbol{R}\cdot\boldsymbol{B}f'(1 + f\kappa)dtds,$$

$$I_4 \equiv \int\int fg'(1 + f\kappa)dtds,$$

$$I_5 \equiv \int\int gf'(1 + f\kappa)dtds,$$

and each double integral (here and hereafter) is taken over $\mathcal{S}$.

Each of these integrals contains terms of the form $H(t)J(s)$, and in each such case we can write

$$\int\int H(t)J(s)dtds = \int_0^L\int_0^M H(t)J(s)dtds = \left(\int_0^M H(t)dt\right)\left(\int_0^L J(s)ds\right). \tag{22}$$

Furthermore it frequently happens that the factor on the right side of (22) is zero, because of the periodicity and smoothness of the curve $\mathcal{C}_P$. The following results are the useful ones in simplifying (21):

$$\int_0^M (ff' + gg')dt = \frac{1}{2}(f^2(t) + g^2(t))\Big|_0^M = 0, \tag{23}$$

$$\int_0^M g'dt = g(M) - g(0) = 0, \tag{24}$$

$$\int_0^M f'dt = f(M) - f(0) = 0, \tag{25}$$

$$\int_0^M fg'dt = \oint_{\mathcal{C}_p} XdY = A, \tag{26}$$

$$\int_0^M gf'dt = \oint_{\mathcal{C}_p} YdX = -A, \tag{27}$$

$$\int_0^M f^2 g' dt = \oint_{\mathcal{C}_p} X^2 dY = \iint_{\mathfrak{D}} 2X dX dY = 2M_Y, \tag{28}$$

$$\int_0^M f g f' dt = \oint_{\mathcal{C}_p} XY dX = \iint_{\mathfrak{D}} - X dX dY = - M_Y, \tag{29}$$

where M_Y is the moment of $\mathfrak{D}$ about the Y-axis. Further we will need the obvious

$$\int_0^L ds = L, \tag{30}$$

and the less obvious

$$\begin{aligned}\int_0^L \boldsymbol{R}\cdot \boldsymbol{N}\kappa ds &= \int_0^L \boldsymbol{R}\cdot\frac{d\boldsymbol{T}}{ds} ds = \int_0^L \boldsymbol{R}\cdot\frac{d^2\boldsymbol{R}}{ds^2} ds \\ &= \int_0^L \left\{\frac{d}{ds}\left(\boldsymbol{R}\cdot\frac{d\boldsymbol{R}}{ds}\right) - \frac{d\boldsymbol{R}}{ds}\cdot\frac{d\boldsymbol{R}}{ds}\right\} ds \\ &= \boldsymbol{R}(L)\cdot\boldsymbol{R}'(L) - \boldsymbol{R}(0)\cdot\boldsymbol{R}'(0) - \int_0^L \boldsymbol{T}\cdot\boldsymbol{T} ds = - L.\end{aligned} \tag{31}$$

When equations (22) through (31) are used in (21) we find that $I_1 = I_3 = 0$ and

$$\boxed{V = AL + M_Y \int_0^L \kappa(s) ds.} \tag{32}$$

Consequently we have proved the following generalization of the Theorem of Pappus.

THEOREM 1. *Let $\mathcal{C}_P$ be a sufficiently smooth simple closed curve that bounds a region $\mathfrak{D}$. Let $\mathcal{C}_S$ be a sufficiently smooth space curve on which the curvature $\kappa(s)$ is never zero. Let R_0 be a fixed point in the plane of $\mathfrak{D}$ (but not necessarily contained in $\mathfrak{D}$). Let the plane of $\mathfrak{D}$ move on $\mathcal{C}_S$ in such a way that (a) the point R_0 is always on $\mathcal{C}_S$, (b) $\mathfrak{D}$ is normal to $\mathcal{C}_S$, and (c) a certain fixed line R_0Q in the plane of $\mathfrak{D}$ always coincides with the binormal $\boldsymbol{B}$ of $\mathcal{C}_S$. If $\mathcal{V}$, the solid generated by this motion of $\mathfrak{D}$, is not self-intersecting, then the volume of $\mathcal{V}$ is given by equation (32) where L is the length of $\mathcal{C}_S$, A is the area of $\mathfrak{D}$, and M_Y is the moment of $\mathfrak{D}$ about $\boldsymbol{B}(0)$.*

COROLLARY. *If in addition to the hypotheses of Theorem 1, the centroid of $\mathfrak{D}$ lies on $\boldsymbol{B}(0)$ (or its extension) then $V = AL$.*

Proof. Under these conditions $M_Y = 0$ and the last term in (32) drops out.

Clearly both Theorem 1 and its corollary are natural generalizations of the Theorem of Pappus. To appreciate the scope of these results, it should be noted that R_0 need not lie in $\mathfrak{D}$. Further the curve $\mathcal{C}_S$ may even have knots in it, as we have tried to indicate in Figure 1.

Clearly our form is not the most general because some of the smoothness conditions on the curves can be relaxed. For example, if $\mathcal{C}_P$ is only piecewise smooth, it could be approximated by a sequence of smooth curves, and it is obvious that (32) would still hold.

Another type of generalization, in which $\mathfrak{D}$ is allowed to spin about R_0 during the motion, is considered in the next section.

4. Motion with a spin. We begin with the same two curves used in Section 3, and again let the point R_0 move along $\mathcal{C}_S$, but we no longer restrict $\mathfrak{D}$ to rotating with $\boldsymbol{B}$. This time we allow $\mathfrak{D}$ to go through any smooth spin about R_0. This can be specified by prescribing $\theta=\theta(s)$, the angle from $\boldsymbol{B}(s)$ to the line P_0Q. To obtain a smooth spin we require that $\theta'(s)$ be continuous for $0\leqq s\leqq L$, and that

$$\theta^{(k)}(0) = \theta^{(k)}(L), \qquad k = 0, 1.$$

This spin takes the vector

$$\boldsymbol{R}_1(s, t) = -f(t)\boldsymbol{N}(s) - g(t)\boldsymbol{B}(s)$$

into the new vector $\boldsymbol{R}_2(s, t)$ where

$$\boldsymbol{R}_2(s,t) = -(f(t)\cos\theta(s) - g(t)\sin\theta(s))\boldsymbol{N}(s) - (f(t)\sin\theta(s) + g(t)\cos\theta(s))\boldsymbol{B}(s). \tag{33}$$

Consequently the vector equation for the surface generated by this motion of $\mathcal{C}_P$ along $\mathcal{C}_S$ is

$$\boldsymbol{P}(s, t) = \boldsymbol{R}(s) + \boldsymbol{R}_2(s, t), \tag{34}$$

for (s, t) in the rectangle (13). We follow the same method of analysis used in Section 3, except that now the expressions are a little more complex.

The left side of (15) still gives $3V$. For the right side, we first observe that

$$\frac{\partial \boldsymbol{P}}{\partial t} \times \frac{\partial \boldsymbol{P}}{\partial s} = a_1\boldsymbol{T} + a_2\boldsymbol{N} + a_3\boldsymbol{B}, \tag{35}$$

where

$$a_1 = (\tau + \theta')(ff' + gg'), \tag{36}$$

$$a_2 = -(f'\sin\theta + g'\cos\theta)(1 + \kappa f\cos\theta - \kappa g\sin\theta) \tag{37}$$

and

$$a_3 = (f'\cos\theta - g'\sin\theta)(1 + \kappa f\cos\theta - \kappa g\sin\theta). \tag{38}$$

Hence

$$V = \frac{1}{3}\iint \boldsymbol{P}\cdot\boldsymbol{n}d\sigma = \frac{1}{3}\iint (\boldsymbol{R} + \boldsymbol{R}_2)\cdot(a_1\boldsymbol{T} + a_2\boldsymbol{N} + a_3\boldsymbol{B})dtds.$$

$$V = \frac{1}{3}(I_1 + I_2 + I_3 + I_4), \tag{39}$$

where

$$I_1 \equiv \int\int \boldsymbol{R}\cdot\boldsymbol{T}a_1 dtds = 0, \tag{40}$$

$$I_2 \equiv \int\int \boldsymbol{R}\cdot\boldsymbol{N}a_2 dtds = -A\int_0^L \boldsymbol{R}\cdot\boldsymbol{N}\kappa ds = AL, \tag{41}$$

$$I_3 \equiv \int\int \boldsymbol{R}\cdot\boldsymbol{B}a_3 dtds = 0, \tag{42}$$

and, from equations (33), (36), (37) and (38),

$$\begin{aligned} I_4 &= \int\int \boldsymbol{R}_2\cdot(a_1\boldsymbol{T} + a_2\boldsymbol{N} + a_3\boldsymbol{B})dtds \\ &= \int\int(1 + \kappa f \cos\theta - \kappa g \sin\theta)(fg' - gf')dtds \\ &= 2AL + 3M_Y\int_0^L \kappa(s)\cos\theta(s)ds - 3M_X\int_0^L \kappa(s)\sin\theta(s)ds. \end{aligned} \tag{43}$$

Combining the results of equation (39), (40), (41), (42), and (43) we have proved

THEOREM 2. *Suppose that the conditions of Theorem 1 are satisfied, except that as the region $\mathfrak{D}$ is moved in a natural manner along the curve $\mathcal{C}_S$, it is given an additional smooth spin defined by $\theta = \theta(s)$. If the region $\mathcal{V}$ that is generated by $\mathfrak{D}$ does not intersect itself, then the volume of $\mathcal{V}$ is given by*

$$\boxed{V = AL + M_Y\int_0^L \kappa(s)\cos\theta(s)ds - M_X\int_0^L \kappa(s)\sin\theta(s)ds,} \tag{44}$$

where M_X and M_Y are respectively the moments of $\mathfrak{D}$ about the X and Y-axes through R_0.

COROLLARY 2. *Suppose that the conditions of Theorem 2 are satisfied, and in addition R_0 is the centroid of $\mathfrak{D}$. Then $V = AL$.*

5. The area of a surface. A second theorem, also due to Pappus, gives the area of the surface of a figure of revolution.

THEOREM B. *If a plane curve $\mathcal{C}_P$ of length M is rotated about an axis that does not meet $\mathcal{C}_P$, then S, the area of the surface generated, is given by*

$$S = ML, \tag{45}$$

where L is the length of the curve described by the centroid of $\mathcal{C}_P$ during the rotation.

We cannot prove a generalization of Theorem B similar in form to Theorems

1 and 2, because such a generalization is false. However if the centroid of $\mathcal{C}_P$ moves on a *plane* curve in a natural manner then we can again obtain (45). The situation is pictured in Figure 2. We observe that in this problem, the curves $\mathcal{C}_P$ and $\mathcal{C}_S$ need not be closed. However if $\mathcal{C}_S$ is a plane curve and $\kappa > 0$, then $\mathcal{C}_S$ must be a strictly convex curve.

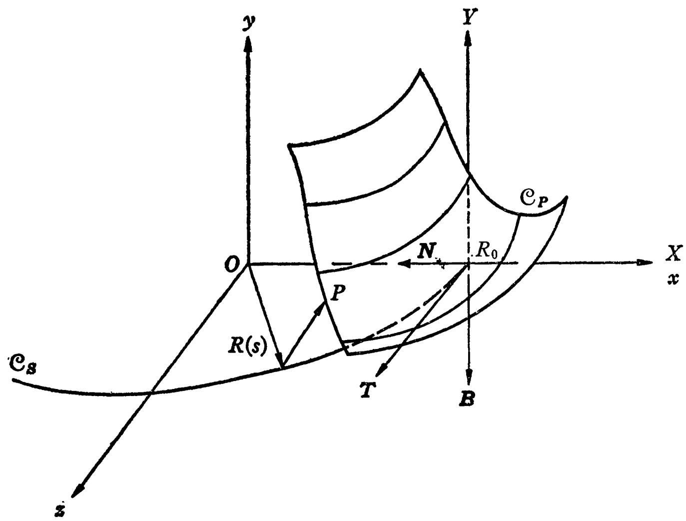

FIG. 2

We select our coordinate system so that $\mathcal{C}_S$ lies in the xz-plane, starts at $R_0 = (x(0), 0, 0)$ with $x(0) > 0$, $T(0)$ is parallel to the positive z-axis, and $\mathcal{C}_P$ lies in the xy-plane.

Just as before, if the curve $\mathcal{C}_P$ is moved in a natural manner along $\mathcal{C}_S$, the surface generated is given by equation (12). Equation (20) still holds except that now $\tau = 0$ because $\mathcal{C}_S$ is a plane curve. Consequently

$$\begin{aligned}\left|\frac{\partial P}{\partial t} \times \frac{\partial P}{\partial s}\right| &= \left| - g'(1 + f\kappa)N + f'(1 + f\kappa)B \right| \\ &= \left| 1 + f\kappa \right| \sqrt{(f')^2 + (g')^2} = \left| 1 + f\kappa \right|,\end{aligned}$$

since t is arc length on $\mathcal{C}_P$. In order to drop the absolute value sign, we need the additional hypothesis that

$$f(t) > -1/\kappa(s) \tag{46}$$

in the rectangle (13). Then the area S is given by

$$S = \iint \left|\frac{\partial P}{\partial t} \times \frac{\partial P}{\partial s}\right| dtds = \iint (1 + f\kappa)dtds \quad \text{or}$$

$$S = ML + M_Y \int_0^L \kappa(s)ds. \tag{47}$$

THEOREM 3. *Let $\mathcal{S}$ be the surface obtained by moving a sufficiently smooth plane curve $\mathcal{C}_P$ in a natural manner along $\mathcal{C}_S$ (see equation (12)). If $\mathcal{C}_S$ is a sufficiently smooth plane convex curve, and if the condition (46) is satisfied, then (47) gives the area of $\mathcal{S}$. In particular if the centroid of $\mathcal{C}_P$ lies on $\boldsymbol{B}(0)$ or its extension, then $S = ML$.*

We leave the case of a nonconvex plane curve for the reader.

We return briefly to the general case considered in Theorem 2, in which the curve $\mathcal{C}_P$ is moved along a twisted curve in space and $\mathcal{C}_P$ is given a spin during the motion. The surface area is still given by

$$S = \iint \left| \frac{\partial P}{\partial t} \times \frac{\partial P}{\partial s} \right| dtds. \tag{48}$$

A brief computation using (35), (36), (37), (38), and $(f')^2 + (g')^2 = 1$, gives

$$S = \iint [(\tau + \theta')^2(ff' + gg')^2 + (1 + \kappa f \cos\theta - \kappa g \sin\theta)^2]^{1/2} dtds. \tag{49}$$

Suppose now that $\mathcal{C}_P$ is a line segment of length $2H$ lying on the x-axis with center at $(a, 0, 0)$ in the xyz-system, where $a > H > 0$. Let $\theta(s) \equiv 0$ (no spin), but suppose that $\tau \not\equiv 0$. For this line segment $f = t$, $g = 0$ and (49) gives

$$S = \iint [\tau^2 t^2 + (1 + \kappa t)^2]^{1/2} dtds.$$

If $\kappa > -1/H$, on the curve $\mathcal{C}_S$, then

$$S > \int_0^L \int_{-H}^H (1 + \kappa t) dtds = 2HL. \tag{50}$$

Consequently if $\mathcal{C}_S$ is not a plane curve, then the surface area is not ML, so a strict analogue of Theorem 1 does not hold.

Suppose finally that $\mathcal{C}_S$ is a plane curve, but we give $\mathcal{C}_P$ a spin. To be specific suppose that $\mathcal{C}_S$ is a circle of radius a, and that $\mathcal{C}_P$ is the line segment of the preceding example. We generate a Möbius strip if we give the line segment a uniform spin of total amount π. Under these conditions $\tau = 0$, $\kappa = 1/a$, $\theta = s/2a$ and (49) gives

$$\begin{aligned} S &= \iint [(t/2a)^2 + (1 + (t/a)\cos(s/2a))^2]^{1/2} dtds \\ &> \int_0^{2\pi a} \int_{-H}^H \left(1 + \frac{t}{a}\cos\frac{s}{2a}\right) dtds = ML. \end{aligned}$$

Consequently $S \neq ML$.

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ALGEBRAIC THEORY FOR DIFFERENCE AND DIFFERENTIAL EQUATIONS

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Introduction. At one time the elementary theory of differential equations was presented so that the subject appeared to be a branch of analysis where the distinctive feature is a limit process. Today an attempt is being made to separate theorems from analysis which can be proved in an algebraic setting without the use of a limit process. In this paper we give a very elementary exposition of a portion of the algebraic theory of linear homogeneous differential equations with constant coefficients, and at the same time we illustrate a connection between these equations and linear difference equations. In the next section we review the algebra of power series and show how the solution of a linear difference equation can be found by using this theory. In the last section we introduce two operators which are usually defined only for power series which have a positive radius of convergence; this analytic information is actually irrelevant if we only require a solution in terms of formal power series. Using these operators we show how to solve a homogeneous differential equation with constant coefficients in terms of the formal power series which defines e^x; the analytic behavior of the power series solution belongs properly to analysis, but this seldom is a topic for discussion in elementary courses.

None of the theorems in this paper are new; the fact that difference equations and differential equations with constant coefficients can be solved in a formal setting has been understood for some time. It is our hope that the limited scope of this paper and the elementary nature of the exposition will make these ideas easier for beginning students. A brief list of books treating differential equations at various levels of sophistication appears at the end of the paper.

The algebra of formal power series and the theory of linear difference equations. Let S be the set of all *power series*

$$\sum_{n=0}^{\infty} a_n x^n, \tag{1}$$

where a_n is a complex number for $n=0, 1, \cdots$. (The development that follows could be given in terms of the sequences $\{a_n\}$ only; however, the power series notation is very convenient for defining certain operations on these sequences, so we retain this notation. Thus, $x^0, x^1, \cdots$ merely indicate the term numbers in a sequence and have no other significance.) Two elements of S, $A=\sum a_n x^n$ and $B=\sum b_n x^n$ are *equal* if and only if $a_n=b_n$, for $n=0, 1, \cdots$; also, if A and B are arbitrary elements of S the *sum* $A+B$ and (Cauchy) *product* $A\times B$ is defined in the usual way; namely,

$$A+B=\sum_{n=0}^{\infty}(a_n+b_n)x^n \quad \text{and} \quad A\times B=\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} x^n. \tag{2}$$

In particular, if $A=a_0$, $A\times B=a_0 B=\sum a_0 b_n x^n$, which defines a *scalar multiplication*. The *quotient* A/B is defined if there exists a power series C such that $A=B\times C$.

THEOREM 1. *S is a commutative ring with a unit under sum and product of power series.*

Proof. The elements of S form an Abelian group under sum; the zero of S is $\mathbb{O}=0+0x+0x^2+\cdots$, and the inverse of A is $(-1)A=-A$. The associative and commutative laws for sum in S are consequences of the associative and commutative laws for the ordinary sum of complex numbers. The elements of $S-\{\mathbb{O}\}$ form an Abelian semi-group under product; the unit is $I=1+0x+0x^2+\cdots$. Again the associative and commutative laws for product are consequences of the associative and commutative laws for complex numbers. Finally, the distributive law holds in S because it holds for complex numbers.

THEOREM 2. *Suppose $A=a_0+a_1x+\cdots$, then there exists an element I/A in S if and only if $a_0\neq 0$.*

Proof. Suppose $A=\sum_{n=0}^{\infty} a_n x^n$ and $a_0\neq 0$. We show by induction that there exists a unique sequence of complex numbers $\{b_n\}$ such that $b_0=1/a_0$ and

$$\sum_{k=0}^{n} b_k a_{n-k}=0, \tag{3}$$

for $n=1, 2, \cdots$. The relation (3) is satisfied for $n=1$ if and only if $b_1=-a_1/a_0^2$. Suppose $b_0, b_1, \cdots, b_{N-1} (N>1)$ are uniquely defined in terms of A so that (3) is satisfied for $n=1, 2, \cdots, N-1$, then (3) is satisfied for $n=N$ if and only if

$$b_N=(-1/a_0)\sum_{k=0}^{N-1} b_k a_{N-k}; \tag{4}$$

thus, b_N is also uniquely defined in terms of A. Evidently $B=\sum_{n=0}^{\infty} b_n x^n$ is uniquely defined and

$$A \times B = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_k a_{n-k} x^n = I. \tag{5}$$

Thus, $I/A = B$ is an element of S. If $a_0 = 0$, $A \times B = 0x + a_1 b_0 x + \cdots$ for every B in S, so $A \times B \neq I$ for every B in S.

Let P be the set of *polynomials* in S; that is, $A \in P$ implies $A = a_0 + a_1 x + \cdots + a_n x^n$, where n is a nonnegative integer; P^* is the subset of P having elements $A = a_0 + a_1 x + \cdots + a_n x^n$ with $a_0 \neq 0$. Two polynomials A and B are *relatively prime* if $A = A'K$, $B = B'K$ with A', B', $K \in S$ implies $K = cI$ where c is a complex number. Finally, R is the set of all *rational power series* A/B, where $A \in P$, $B \in P^*$, and A and B are relatively prime.

THEOREM 3. *R is a subring of S.*

Proof. R is a subset of S since if $A/B \in R$, then $A \in P$, $B \in P^*$, and so A and I/B are elements of S; hence, $A \times I/B = A/B \in S$. The sum and product of an arbitrary pair of elements in R are elements in R; also, I and $\mathfrak{O}$ are contained in R so R is a subring of S.

THEOREM 4. *$\sum c_n x^n \in R$ if and only if there exist complex numbers b_0, b_1, $\cdots$, b_k, with b_0, $b_k \neq 0$, and there is a nonnegative integer N such that*

$$b_0 c_{n+k} + b_1 c_{n+k-1} + \cdots + b_k c_n = 0 \tag{6}$$

for all $n > N$.

Proof. If $C = \sum c_n x^n \in R$, then there exist polynomials $A = a_0 + \cdots + a_N x^N$, and $B = b_0 + \cdots + b_k x^k$ with b_0, $b_k \neq 0$, such that A and B are relatively prime and $A/B = C$. Thus,

$$\sum_{n=0}^{N} a_n x^n = A = B \times C = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_k c_{n-k} x^n. \tag{7}$$

By definition of equality of power series (7) implies (6). Conversely, if (6) holds, (7) is true for some polynomial A with degree less than $N+1$, so $C = A/B$ is an element of R. Clearly, for a given polynomial $B = b_0 + \cdots + b_k x^k$ and any polynomial A with degree less than $N+1$, one obtains $\{c_n\}$ satisfying (6) from $A/B = \sum c_n x^n$, and all $\{c_n\}$ are obtained in this way.

THEOREM 5. *Suppose $C = A/(1-\theta_1 x)^{k_1} \cdots (1-\theta_i x)^{k_i} \in R$, where A is a polynomial with degree less than $k_1 + \cdots + k_i$, and $\theta_1, \cdots, \theta_i$ are distinct complex numbers, then there exist unique polynomials $A_1, \cdots, A_i$ with the degree of A_j less than $k_j (j = 1, \cdots, i)$ such that*

$$C = \frac{A_1}{(1-\theta_1 x)^{k_1}} + \cdots + \frac{A_i}{(1-\theta_i x)^{k_i}} \cdot \tag{8}$$

Proof. This result is very well known (see for example, Birkhoff and MacLane [1], page 84) so we dispense with the proof here; also, it should be pointed out that the fact that C can be expressed uniquely in the form given by (8) is a consequence of the Fundamental Theorem of Algebra (every polynomial equation with degree greater than 0 has at least one complex number as a root).

THEOREM 6. *Suppose k and t are non-negative integers, and let θ be an arbitrary complex number; then*

$$\frac{x^t}{(1-\theta x)^{k+1}} = \sum_{n=t}^{\infty} \theta^{n-t} \binom{n+k-t}{k} x^n. \tag{9}$$

Proof. It is enough to prove (9) for the case $\theta = 1$, $t = 0$, and k an arbitrary non-negative integer since other cases can be obtained from this one by substitution. When $k = 0$ we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \binom{n}{0} x^n, \tag{10}$$

since $1/(1-x)$ is a quotient of two power series and by using the definitions for quotient, product, and sum of power series respectively we obtain

$$1 = (1-x) \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} x^{n+1}, \tag{11}$$

which is equivalent to (10).

Now suppose for some $k \geqq 0$

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n, \tag{12}$$

then

$$\frac{1}{(1-x)^{k+2}} = \frac{1}{(1-x)} \times \frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{i+k}{k} x^n, \tag{13}$$

by definition of product of power series, (10), and (12). But it can easily be shown by induction that

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{k+n}{k} = \binom{n+k+1}{k+1},$$

for $n = 0, 1, \cdots$. Thus, (12) holds for $k = 0, 1, \cdots$ and the proof of the theorem is complete.

Theorems 4, 5, and 6 can be combined to give a theory for the solution of a linear homogeneous difference equation with constant coefficients such as in (6). In this case we can write $b_0 + b_1 x + \cdots + b_k x^k = (1-\theta_1 x)^{k_1} \cdots (1-\theta_i x)^{k_i}$, where $\theta_1, \cdots, \theta_i$ are distinct complex numbers, then for arbitrary polynomials

$A_1, \cdots, A_i$ with the degree of A_j less than k_j for $j=1, \cdots, i$, we know that

$$\frac{A_1}{(1-\theta_1 x)^{k_1}} + \cdots + \frac{A_i}{(1-\theta_1 x)^{k_i}} = \sum_{n=0}^{\infty} c_n x^n \tag{14}$$

defines a sequence $\{c_n\}$ which satisfies (6) for $n=0, 1, \cdots$. Writing each term in the left member of (14) as a power series and combining these power series yields the following result.

THEOREM 7. *If $b_0, \cdots, b_k$ are given complex numbers and $b_0+b_1x+\cdots+b_kx^k = (1-\theta_1x)^{k_1}\cdots(1-\theta_ix)^{k_i}$, where $\theta_1, \cdots, \theta_k$ are distinct complex numbers, then every sequence of complex numbers $\{c_n\}$ which satisfies*

$$b_0c_{n+k} + b_1c_{n+k-1} + \cdots + b_kc_n = 0 \tag{15}$$

for $n=0, 1, \cdots$ is given by

$$c_n = p_1(n)\theta_1^n + \cdots + p_i(n)\theta_i^n, \tag{16}$$

where $p_1(n), \cdots, p_i(n)$ are polynomials in n with the degree of $p_j(n)$ less than k_j for $j=1, \cdots, i$. If $c_0, \cdots, c_{k-1}$ are specified numbers the polynomials $A_1, \cdots, A_i$ are uniquely defined so that the polynomials $p_1(n), \cdots, p_i(n)$ are also uniquely defined.

The operators D^k and the theory of linear differential equations. Suppose $A = \sum a_nx^n \in S$ and define

$$LA = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \tag{17}$$

$$D^kA = \sum_{n=0}^{\infty} (n+k)_k a_{n+k} x^n, \tag{18}$$

for $k=0, 1, \cdots$, where $(n)_0=1$ and $(n+k)_k=(n+k)(n+k-1)\cdots(n+1)$ for $k=1, 2, \cdots$, and $n=0, 1, \cdots$. Also, we *define* the special power series

$$E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = L[1/(1-x)]. \tag{19}$$

THEOREM 8. *L is a linear operator; that is, for arbitrary complex numbers a and b, and power series A and B*

$$L(aA+bB) = aLA + bLB. \tag{20}$$

Also, LA is a polynomial if and only if A is a polynomial.

THEOREM 9. *$D^0A=A$, and $D^kA=D(D^{k-1}A)$ for all power series A and $k=1, 2, \cdots$. D^k is a linear operator for $k=0, 1, \cdots$. If A and B are arbitrary power series*

$$D^k(A \times B) = \sum_{i=0}^{k} \binom{k}{i} D^i A \times D^{k-i} B. \tag{21}$$

Finally, $D^k A = \Theta$ *if and only if* A *is a polynomial with degree less than* k.

Proof. All of the proofs are verifications involving an arbitrary power series and the definition of the operator. The proof of (21) can be carried out by induction in the same way one proves the binomial theorem for positive exponents; of course, we use the fact that $D^k(A \times B) = D(D^{k-1}(A \times B))$.

THEOREM 10. $D^k L x^r A = D^{k-r} L A$, *for* $r = 0, \cdots, k$, A *a power series and* $k = 0, 1, \cdots$.

Proof. $D^k L x^r A = D^k \sum_{n=r}^{\infty} (a_{n-r}/n!)\, x^n = \sum_{n=0}^{\infty} (a_{n+k-r}/n!)\, x^n = D^{k-r} \sum_{n=0}^{\infty} \cdot (a_n/n!) x^n = D^{k-r} L A$.

THEOREM 11. *Suppose* $F \in S$, *and* $b_0, \cdots, b_k$ *are complex numbers then*

$$b_0 D^k F + b_1 D^{k-1} F + \cdots + b_k F = \Theta \tag{22}$$

if and only if $F = LC$, *where* $C = A/(b_0 + b_1 x + \cdots + b_k x^k)$ *and* A *is a polynomial with degree less than* k.

Proof. Suppose $C = A/(b_0 + b_1 x + \cdots + b_k x^k)$ where A is a polynomial with degree less than k, then using the definition of quotient and Theorems, 8, 9, and 10 we have:

$$\begin{aligned} b_0 C + b_1 x C + \cdots + b_k x^k C &= A, \\ b_0 L C + b_1 L x C + \cdots + b_k L x^k C &= L A, \\ b_0 D^k L C + b_1 D^k L x C + \cdots + b_k D^k L x^k C &= D^k L A, \\ b_0 D^k L C + b_1 D^{k-1} L C + \cdots + b_k L C &= \Theta. \end{aligned} \tag{23}$$

Thus, $F = LC$ is a solution of (22). Now suppose $F = \sum (f_n/n!) x^n$ satisfies (22), then

$$\sum_{n=0}^{\infty} (b_0 f_{n+k} + b_1 f_{n+k-1} + \cdots + b_k f_n)(x^n/n!) = \Theta. \tag{24}$$

But (24) holds if and only if

$$b_0 f_{n+k} + b_1 f_{n+k-1} + \cdots + b_k f_n = 0 \tag{25}$$

for $n = 0, 1, \cdots$, and according to Theorem 4 this means $\sum_{n=0}^{\infty} f_n x^n \in R$. In particular,

$$\sum_{n=0}^{\infty} f_n x^n = A/(b_0 + b_1 x + \cdots + b_k x^k) = C, \tag{26}$$

where A is an arbitrary polynomial with degree less than k. So $F = LC$ if F is a solution of (22).

Now we are equipped with a theory for finding the solution of equation (22), a linear homogeneous differential equation with constant coefficients. Usually the solution of this equation is given in terms of the special power series $E(x)$ and polynomials. Evidently we can also give the solution of (22) in these terms if we can describe LC in terms of $E(x)$ and polynomials for any rational power series $C=A/B$, where the degree of A is less than the degree of B.

THEOREM 12. *Suppose k and t are nonnegative integers and θ is an arbitrary complex number, then*

$$L\left\{\frac{x^t}{(1-\theta x)^{k+1}}\right\} = \frac{E(\theta x)}{\theta^t}\sum_{r=0}^{t}\sum_{i=0}^{k-r}\frac{(-1)^r}{i!}\binom{t}{r}\binom{k-r}{i}\theta^i x^i. \tag{27}$$

Proof. We need only consider the case when $\theta=1$ since the other cases can be obtained from this one by substituting θx for x. When $t=0$, we can verify directly that

$$L\left\{\frac{1}{(1-x)^{k+1}}\right\} = E(x)\sum_{i=0}^{k}\binom{k}{i}\frac{x^i}{i!} \tag{28}$$

for $k=0, 1, \cdots$; thus, for $0\leqq r\leqq k$ we have

$$L\left\{\frac{(1-x)^r}{(1-x)^{k+1}}\right\} = E(x)\sum_{i=0}^{k-r}\binom{k-r}{i}\frac{x^i}{i!}\,. \tag{29}$$

But

$$\frac{x^t}{(1-x)^{k+1}} = \frac{(1-(1-x))^t}{(1-x)^{k+1}} = \sum_{r=0}^{t}(-1)^r\binom{t}{r}\frac{(1-x)^r}{(1-x)^{k+1}}, \tag{30}$$

so using the fact that L is a linear operator and (29) we have

$$L\left\{\frac{x^t}{(1-x)^{k+1}}\right\} = E(x)\sum_{r=0}^{t}\sum_{i=0}^{k-r}(-1)^r\binom{t}{r}\binom{k-r}{i}\frac{x^i}{i!}, \tag{31}$$

and this completes the proof.

Now if A is a polynomial with degree less than k we can use the linearity of L and Theorem 12 to find $L\{A/(1-x)^k\}=A'E(x)$, where A' is a polynomial with degree less than k; also, the correspondence $A\leftrightarrow A'$ is one to one. Using the factorization for C given by Theorem 5, we can apply L to C and use Theorem 12 to obtain the following result.

THEOREM 13. *Suppose $b_0+b_1x+\cdots+b_kx^k=(1-\theta_1x)^{k_1}\cdots(1-\theta_ix)^{k_i}$, where $\theta_1,\cdots,\theta_i$ are distinct complex numbers, then every solution $F\in S$ to the equation*

$$b_0D^kF+b_1D^{k-1}F+\cdots+b_kF=\mathfrak{O} \tag{32}$$

is given by

$$F = A_1E(\theta_1x)+\cdots+A_iE(\theta_ix), \tag{33}$$

where $A_1, \cdots, A_i$ *are arbitrary polynomials with the degree of* A_j *less than* k_j *for* $j=1, \cdots, i$.

This paper was written while the author was a post-doctoral fellow at McMaster University, Hamilton, Canada.

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RELATIONS IN POLYGONS

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Introduction. Consider n arbitrary lines in a plane, $n>3$, denoted by the numbers $0, 1, 2, 3, \cdots, n-1$. The point of intersection of lines p and q will be represented by (p, q). The points $(0, 1)$, $(1, 2)$, $(2, 3)$, $\cdots$, $(n-1, 0)$ will be taken as the vertices of a polygon with sides on $0, 1, 2, \cdots, n-1$ in that order.

A segment on the line p, intercepted by the lines q and r, as well as its (positive) length shall be denoted by $p(q:r)$ or by $p(r:q)$. The endpoints of this segment are (p, q) and (p, r). If one of these points is a vertex of the n-gon, the segment will be called an end-segment. In [1] only the end-segments incident with the sides or diagonals of a cyclic n-gon have been considered. In this paper we propose to find relations for any segments, be they end-segments or not, incident with the sides of the n-gon.

Triangles. A triangle with sides on lines p, q, r will be denoted by (p, q, r); the order in which the sides are written is evidently immaterial. Each segment $p(q:r)$ has one and only one triangle (p, q, r) associated with it. Any triangle having one vertex incident with a vertex of the n-gon will be called an end-triangle; obviously it has two end-segments as sides.

Sets of corresponding segments. The n segments $\overline{p+\lambda}(\overline{q+\lambda}:\overline{r+\lambda})$, where $\lambda=0, 1, 2, \cdots, n-1$, and all quantities are to be taken as the least nonnegative residues modulo n, form a set L of corresponding segments. Any segment of the set is representative of the entire group.

For two segments $p(q:r)$ and $p'(q':r')$ to belong to the same set a necessary condition is that $p-p' \equiv q-q' \equiv r-r'$ (modulo n); or since $p'(q':r')$ is the same as $p'(r':q')$, $p-p' \equiv q-r' \equiv r-q'$(mod n). For the sake of uniformity of expression, the segment on the side 0 will often be used to represent the set; clearly, the segment $0(\overline{q-p}:\overline{r-p})$ generates the same set as $p(q:r)$.

Conjugate segments. The n segments $\overline{\lambda-p}(\overline{\lambda-q}:\overline{\lambda-r})$, where $\lambda=0, 1, 2, \cdots, n-1$, form a set L', which will be defined to be the conjugate set of L. Any member of the one set L is said to be conjugate to any member of the other set L'.

Given a segment $\overline{p+\lambda}(\overline{q+\lambda}:\overline{r+\lambda})$ of L, a conjugate segment of L' might be designated either $\overline{\lambda'-p}(\overline{\lambda'-q}:\overline{\lambda'-r})$, or $\overline{\lambda'-p}(\overline{\lambda'-r}:\overline{\lambda'-q})$. Hence if $p(q:r)$ is a segment of L and $p'(q':r')$ is a segment of L', one or the other of two conditions must hold. Either

$$p + p' \equiv q + q' \equiv r + r' \text{ (modulo } n\text{)}, \qquad \text{or} \tag{1}$$

$$p + p' \equiv q + r' \equiv r + q' \text{ (modulo } n\text{)}. \tag{2}$$

Self-conjugate segments. If a segment belongs to both L and L', it is said to be self-conjugate. As shown below, if there is one such segment, all the segments of L and L' are self-conjugate in pairs which makes the two sets identical.

For a segment $p(q:r)$ to be self-conjugate, it follows from (1) and (2) that either

$$p + p \equiv q + q \equiv r + r \text{ (modulo } n\text{)}, \qquad \text{or} \tag{3}$$

$$p + p \equiv q + r \equiv r + q \text{ (modulo } n\text{)}. \tag{4}$$

Since p, q, and r must be distinct modulo n, the first condition cannot be fulfilled since at most two distinct nonnegative integers less than n can satisfy the relation $2p \equiv 2q$ (modulo n). Hence the condition for a self-conjugate segment must be

$$2p \equiv q + r \text{ (modulo } n\text{)}. \tag{5}$$

If $p(q:r)$ is self-conjugate, then any segment $\overline{p+\lambda}(\overline{q+\lambda}:\overline{r+\lambda})$ is self-conjugate since $2(p+\lambda) \equiv (q+\lambda)+(r+\lambda)$ follows from (5).

Since any side of an n-gon is of the form $m(\overline{m-1}:\overline{m+1})$ or $m(\overline{m+1}:\overline{m-1})$, it follows that such sides are self-conjugate segments.

A set of self-conjugate segments shall be denoted by L_s. Evidently the set may also be indicated by L_s'.

On the "0" side of the n-gon, the self-conjugate segments take the form $0(q: -q)$ where q has the values $1, 2, 3, \cdots, n-1$. In giving q this succession of values, segments are repeated in pairs with one inadmissible segment corresponding to $q \equiv -q$ when n is even. Thus the number of self-conjugate segments on line "0" is

$$\left[\frac{n-1}{2}\right], \tag{6}$$

where the square brackets mean the greatest integer function. By symmetry the same can be said of any other side of the polygon.

Conjugate triangles. Two sets of triangles associated with the segments of L and L' shall be denoted T and T' respectively. It can be shown that each vertex of a triangle of T is also a vertex of some triangle of T' and vice versa. For if (p, q, r) is a triangle of T, then $(\overline{\lambda-p}, \overline{\lambda-q}, \overline{\lambda-r})$ are the triangles of T'. Giving λ the values $\overline{p+q}$, $\overline{q+r}$, $\overline{r+p}$ successively results in three triangles of T', namely, $(q, p, \overline{p+q-r})$, $(\overline{q+r-p}, r, q)$, and $(r, \overline{r+p-q}, p)$ which have respectively the vertices (q, p), (r, q), or (r, p) in common with triangle (p, q, r).

THEOREM 1. *Three products of the corresponding sides of the triangles of T are proportional to the corresponding products of the sides of T', namely,*

$$\frac{\prod [p(q:r)]}{\prod [-p(-q:-r)]} = \frac{\prod [q(p:r)]}{\prod [-q(-p:-r)]} = \frac{\prod [r(q:p)]}{\prod [-r(-q:-p)]} \tag{7}$$

where

$$\prod [p(q:r)] = \prod_{\lambda=0}^{n-1} [\overline{p+\lambda}(\overline{q+\lambda}:\overline{r+\lambda})].$$

Proof. Write the sine theorem for each triangle and multiply the results together. Then

$$\frac{\prod [p(q:r)]}{\prod \sin(q, r)} = \frac{\prod [q(p:r)]}{\prod \sin(p, r)} = \frac{\prod [r(p:q)]}{\prod \sin(p, q)}, \tag{8}$$

$$\frac{\prod [-p(-q:-r)]}{\prod \sin(-q, -r)} = \frac{\prod [-q(-p:-r)]}{\prod \sin(-p, -r)} = \frac{\prod [-r(-p:-q)]}{\prod \sin(-p, -q)}. \tag{9}$$

But the set of sines of angles $(q+\lambda, r+\lambda)$ is the same as that of angles $(\lambda-q, \lambda-r)$, which may be seen by subtracting $q+r$ from the elements of the first set. Therefore the numerators of (8) are proportional to those of (9). Therefore (7) follows.

THEOREM 2. *If in a triangle (p, q, r) two sides q and r are conjugate to each other, the side p is self-conjugate; conversely, if p is self-conjugate, then either q and r are conjugate to each other, or both are self-conjugate.*

Proof. If $q(p:r)$ and $r(p:q)$ are conjugate to each other, then either $q+r\equiv 2p$ or $q+r\equiv p+q\equiv r+p$. The second condition cannot hold since the three quantities p, q, and r are distinct. But the first indicates that p is self-conjugate.

Vice versa, $q+r\equiv 2p$ shows that $q(p:r)$ and $r(p:q)$ are conjugate. Should

one of them, say $q(p:r)$ be self-conjugate, the condition would be that $2q \equiv p+r$. Combining this with the condition $q+r \equiv 2p$ leads to $2r \equiv p+q$ showing that $r(p:q)$ is also self-conjugate.

A triangle with at least one side self-conjugate is said to be a self-conjugate triangle.

THEOREM 3. *In a chain of self-conjugate triangles with the sides belonging to L, L' and L_s, the product of the sides of L equals the product of the sides of L', that is, if $q+r \equiv 2p$, which means that the side $p(q: r)$ belongs to L_s, then*

$$\prod q(p: r) = \prod r(p: q). \tag{10}$$

Proof. By relation (8)

$$\frac{\prod [q(p: r)]}{\prod \sin(p, r)} = \frac{\prod [r(p: q)]}{\prod \sin(p, q)}. \tag{11}$$

But

$$\prod \sin(p, r) = \prod \sin[\overline{p - (p - q)}, \overline{r - (p - q)}] = \prod \sin[q, \overline{r - p + q}].$$

From the condition $2p \equiv r+q$, it follows that $p \equiv r-p+q$. Hence $\prod \sin (p, r) = \prod \sin (q, p)$. Accordingly the relation (10) of Theorem 3 is seen to hold.

Quadrilateral. Since the sides of the n-gon itself are self-conjugate, it follows from Theorem 3 that

$$\prod [\overline{p-1}(p: \overline{p+1})] = \prod [\overline{p+1}(p: \overline{p-1})], \quad p = 0, 1, 2, 3, \cdots, n-1. \tag{12}$$

For $n=4$, this relation becomes

$$0(1: 2)\cdot 1(2: 3)\cdot 2(3: 0)\cdot 3(0: 1) = 2(1: 0)\cdot 3(2: 1)\cdot 0(3: 2)\cdot 1(0: 3),$$

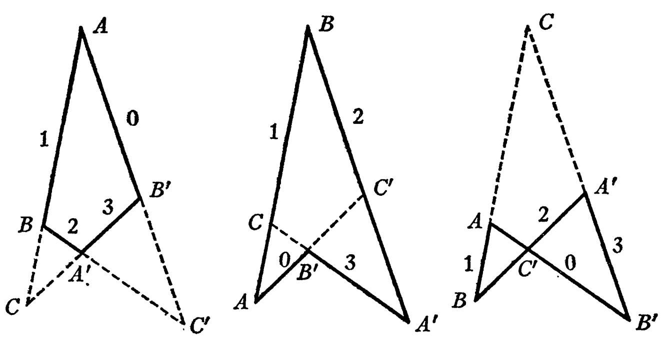

FIG. 1.

or in terms of the lettered segments of Figure 1, a, b, c,

$$AC' \cdot BC \cdot A'C' \cdot B'C = BC' \cdot A'C \cdot B'C' \cdot AC,$$

which in a more symmetric form can be written

$$\frac{CA\cdot CA'}{C'A\cdot C'A'}=\frac{CB\cdot CB'}{C'B\cdot C'B'}\,. \tag{13}$$

This is a three-in-one relation which incidentally may be derived from Menelaus' Theorem. It may be found also in [2].

Pentagon. For the pentagon, there are thirty segments, ten of which (including the five sides) are self-conjugate, while the remaining twenty are associated with these ten to form self-conjugate triangles. Every triangle in a pentagon is self-conjugate and the triangles form two distinct sets. Therefore for any product of five corresponding segments that are not self-conjugate, it is always possible to find an equal product of five corresponding segments by Theorem 3. For the cyclic pentagon this was proved in [1], p. 537 (see also [3]).

Cyclic polygons. If a polygon is inscribable in a circle, it is said to be a cyclic polygon.

THEOREM 4. *In a cyclic n-gon, the product of a set of corresponding end-segments, L, is equal to the product of the conjugate set of corresponding end-segments, L'.*

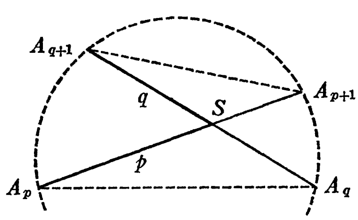

FIG. 2.

Proof. Let $A_p=(p,p-1)$, $A_{q+1}=(q,q+1)$. Then in Figure 2, $p(\overline{p-1:q})=A_pS$ and $q(\overline{q+1:p})=A_{q+1}S$. Triangles $SA_{q+1}A_{p+1}$ and SA_qA_p are similar. Hence

$$\frac{p(\overline{p-1:q})}{q(\overline{q+1:p})}=\frac{A_pA_q}{A_{p+1}A_{q+1}}\,.$$

In taking the product of all such n ratios, the numerators and denominators on the right are found to be equal. Therefore

$$\prod p(\overline{p-1:q}) = \prod q(\overline{q+1:p}) \tag{14}$$

and the theorem is proved.

THEOREM 5. *In a cyclic n-gon, in the end-triangles of a set T, the product of the sides opposite to the vertices of the n-gon equals the product of the corresponding sides in the conjugate set of end-triangles T'.*

Proof. Let the end triangles of a set T be given by $(\overline{q+\lambda}, \overline{p+1+\lambda}, \overline{p+\lambda})$. Then the end triangles of the set T' are $(\overline{\lambda-q}, \overline{\lambda-p-1}, \overline{\lambda-p})$. By Theorem 1,

$$\frac{\prod [q(p: \overline{p+1})]}{\prod [-q(-p: \overline{-p-1})]} = \frac{\prod [p(q: \overline{p+1})]}{\prod [-p(-q: \overline{-p-1})]}.$$

On the right hand side according to Theorem 4, the numerator and denominator are equal, so that the same must be true on the left. This proves the proposition.

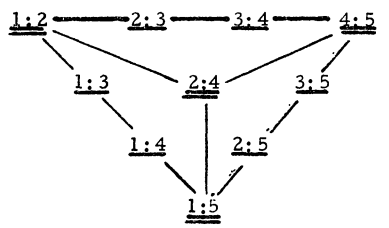

FIG. 3. Application of Theorems 3, 4 and 5 to the "0" Line of the Hexagon.

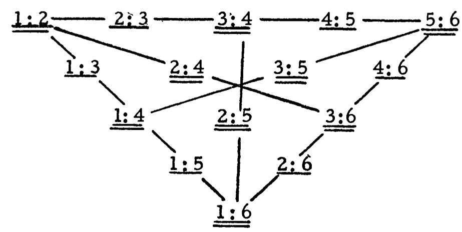

FIG. 4. Application of Theorems 3, 4 and 5 to the "0" Line of the Heptagon.

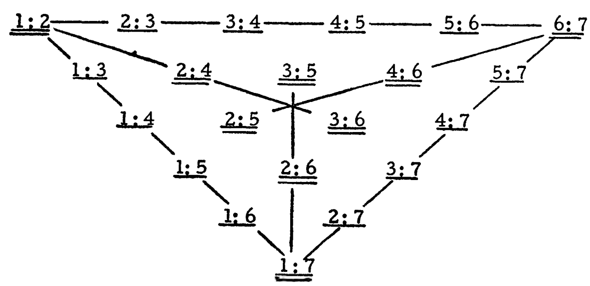

FIG. 5. Application of Theorems 3, 4 and 5 to the "0" Line of the Octagon.

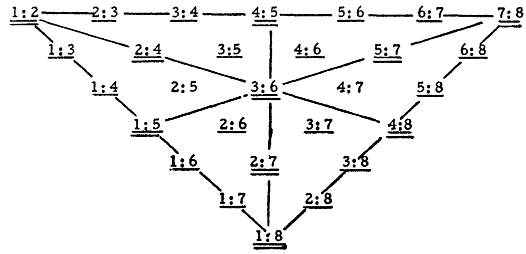

FIG. 6. Application of Theorems 3, 4 and 5 to the "0" Line of the Nonagon.

Relations in other polygons. In Figures 3, 4, 5, and 6, segments are shown on the "0" line. Double underlining indicates that the segment is covered by Theorem 3 which applies to all polygons; single underlining shows that the segment is covered by Theorems 4 and 5 for cyclic polygons. The segments on the central vertical line show self-conjugate segments, while the two diagonal lines

indicate segments belonging to self-conjugate triangles. The segments along the border diagonal lines consist of end-segments. The segments in the top horizontal row are those which are enclosed between two consecutive sides of the n-gon.

Note that for the cyclic 6-gon, 7-gon and 8-gon, each segment is covered by at least one of the last three theorems (3, 4, 5) while for the 9-gon, the triangle (0, 2, 5) of the set T and its conjugate (0, 7, 4) of set T' are not involved in any of our theorems.

Completeness considerations. The Theorems 3, 4, and 5 cover all cases when the products of segments and their conjugates are equal.

To prove this, consider an n-gon with sides arbitrarily drawn except that side p has been drawn through the point (q, r). Then the length of $p(q:r)$ is zero, which makes the product of the entire set L equal to zero. But the corresponding product for L' is not zero, since it does not include the zero segments $q(p:r)$ or $r(p:q)$, unless one of these segments should happen to be self-conjugate which would necessarily involve Theorem 3.

In the cyclic polygon a zero segment would appear in L' only if it belonged to an end-triangle which would thus involve Theorems 4 and 5.

Additional equalities of the products could be found by changing the order of the sides of the n-gon or by reducing the number of sides.

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INTEGRAL POWER RESIDUES AS PERMUTATIONS

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1. Introduction. Since, and even before, Gauss gave his proof of the reciprocity law which depended upon the number of numbers among

$$D, 2D, \cdots, tD \tag{1}$$

with negative least absolute residues $\bmod(2t+1)$, where $2t+1$ is a prime, there has been a steady and recurrent interest in the least residues (in one or another sense) of such a sequence. One very interesting theorem proved by Zolotareff [10] (see also [2], and [9]) asserts, for the case of $t+1$ an odd prime, that when the least positive residues $\bmod(t+1)$ of (1) are considered as a permutation of $1, 2, \cdots, t$ then this permutation is even or odd precisely when D is or is not a quadratic residue of $t+1$.

In this paper we investigate in some detail permutations of type (1) for t an arbitrary positive integer. The results will be seen to provide a simultaneous generalization of the so-called Monge card shuffle, discussed by Gaspard Monge in 1773 [8] (see also [1, 3, 4, 7]), and of the work of Bouniakowsky in 1870 [5] (see also [6] Chapter 7) on solutions of binomial congruences of the form $q\cdot 3^x \equiv \pm r \pmod{M}$. The discussion provides a workable algorithm for the determination of all solutions of such a congruence in which the 3 is replaced by any number prime to M. Finally, as corollaries, we prove the lemma of Gauss in the theory of quadratic residues and two theorems about primitive roots which quickly yield a number of known numerical results.

2. The general permutation P. Throughout, n and M are to be relatively prime positive integers, k is to be the largest integer less than or equal to $\frac{1}{2}M$, and K is the set of the first k positive integers. We split K into the n subsets $I_0, \cdots, I_{n-1}$, where

$$m \in I_j \quad \text{if and only if } \frac{jM}{2n} < m \leqq \frac{(j+1)M}{2n}. \tag{2}$$

Using square brackets for the largest integer function we define the function f on K by

$$f(m) = (-1)^j\left(nm - \left[\frac{j+1}{2}\right]M\right) \qquad \text{for } m \in I_j. \tag{3}$$

Since, for $m \in I_j$,

$$nm - \left[\frac{j+1}{2}\right]M = \begin{cases} nm - \dfrac{jM}{2} > 0 & \text{for } j \text{ even;} \\ nm - \dfrac{(j+1)M}{2} < 0 & \text{for } j \text{ odd,} \end{cases} \tag{4}$$

we see that $f(m) > 0$ for all $m \in K$. Also, from (2) and (4), it is clear that $f(m) \leqq k$ as well. If $f(m_1) = f(m_2)$ then, from (3), $m_1 \equiv \pm m_2 \pmod{M}$ so that, when m_1 and m_2 are in K, we must have $m_1 = m_2$. Therefore f is a one-to-one map of K onto itself; i.e., $f(1), \cdots, f(k)$ is a permutation of the integers $1, \cdots, k$. We denote the disjoint cyclic factorization of this permutation by P (or by $P(n, M)$ if necessary). When two elements of K are in the same cycle of P we shall say that these elements are *P-equivalent.* It is clear that P-equivalence is an equivalence relation and that a and b in K are P-equivalent if and only if each can be obtained from the other by repeated application of f. We study P by studying the results of successive applications of f, writing $f^0(m) = m$ and $f^{s+1}(m) = f(f^s(m))$.

From (3)

$$f(m) \equiv (-1)^j nm \pmod{M} \qquad \text{for } m \in I_j. \tag{5}$$

But, for $m \in I_j$, it is easy to see that $j = [2nm/M] - \delta$, where δ is 0 or 1 depending

on whether $M \neq 2m$ or $M=2m$. Writing $\epsilon(m) = (-1)^j$, (5) becomes

$$f(m) \equiv \epsilon(m)nm \pmod{M} \qquad \text{for } all\ m \in K. \tag{6}$$

Iteration of (6) yields

$$f^s(m) = Q_s n^s m \pmod{M} \qquad \text{for } s \geqq 1, \tag{7}$$

where

$$Q_s = \epsilon(m)\epsilon(f(m)) \cdots \epsilon(f^{s-1}(m)). \tag{8}$$

As a function defined over all integers, $\epsilon(m)$ is clearly periodic with period M. Further, an elementary calculation shows $\epsilon(-x)=\epsilon(x)$ if M divides $2nx$ and $\epsilon(-x)=-\epsilon(x)$ otherwise. Noting that for $1 \leqq m \leqq k$ the number M does not divide $2Q_s n^{s+1}m$ unless $2m=2k=M$, we may write

$$\begin{aligned} Q_s &= Q_{s-1}\epsilon(f^{s-1}(m)) = Q_{s-1}\epsilon(Q_{s-1}n^{s-1}m) \\ &= \begin{cases} \epsilon(n^{s-1}m) & \text{for } 1 \leqq m < k \quad \text{or} \quad M \text{ odd;} \\ Q_{s-1}\epsilon(n^{s-1}m) & \text{for } 2m = 2k = M. \end{cases} \end{aligned} \tag{9}$$

When $2m=2k=M$, then

$$Q_s = \prod_{j=0}^{s-1} (-1)^{[2n^{j+1}m/M]-\delta} = (-1)^s \prod_{j=0}^{s-1} (-1)^{n^{j+1}} = 1. \tag{10}$$

Since $Q_1 = \epsilon(m)$ we therefore have

$$f^s(m) \equiv \begin{cases} n^s m & \text{if } M = 2k = 2m; \\ & \pmod{M} \\ \epsilon(n^{s-1}m)n^s m & \text{otherwise.} \end{cases} \tag{11}$$

Denoting the number of terms of $f(a), f^2(a), \cdots, f^s(a)$ which lie in an I_j with j odd by $\alpha(a, s)$ we see, directly from (5), that the coefficient of $n^s m$ on the right side of (11) is $(-1)^{\alpha(m,s)}$. Hence we can write (11) as

$$f^s(m) \equiv (-1)^{\alpha(m,s)} n^s m \pmod{M}. \tag{12}$$

3. Cyclic structure of P and binomial congruences. Using (11), or (12), we see that when a and b are P-equivalent then $a \equiv \pm n^s b \pmod{M}$ for some positive integer s. On the other hand, if one of these congruences is satisfied by a and b in K then, since $f^s(b) \equiv \pm n^s b \pmod{M}$ we have $f^s(b) \equiv \pm a \pmod{M}$, and, since both $f^s(b)$ and a are in K, this implies $f^s(b) \equiv a \pmod{M}$. This proves:

a is P-equivalent to b if and only if there is a positive integer s for which one of the congruences $a \equiv \pm n^s b \pmod{M}$ is true.

We digress momentarily to define a number theoretic function which we shall find useful. Since $n^{\phi(a)} \equiv 1 \pmod{a}$ whenever $(n, a)=1$, there exists, in this case, a smallest integer s for which either $n^s \equiv 1 \pmod{a}$, or $n^s \equiv -1 \pmod{a}$.

We denote this smallest integer by $g_n(a)$. It is a simple calculation to verify the following assertions.

$$\text{(13)}\qquad \begin{cases} \text{(i) } n^t \equiv \pm 1 \pmod{a} \Rightarrow g_n(a) \text{ divides } t; \\ \text{(ii) } b \text{ divides } a \Rightarrow g_n(b) \text{ divides } g_n(a). \end{cases}$$

The number of elements P-equivalent to a given element m of K is equal to the number of elements in that cycle of P which contains m; we denote this number by m'. To determine m' we determine the number of distinct x (mod $[M/2]$) for which $x \equiv \pm n^s m \pmod{M}$ for some positive integer s. If an x satisfies both congruences then $M = 2m$, or $m = k$, and we see, directly from (3), that $f(k) = k$, $k' = 1$. Otherwise, only one of the two congruences is possible. Dividing by $d = (m, M)$ yields $(x/d) \equiv \pm n^s(m/d) \pmod{M/d}$ and in this congruence $\pm n^s$ takes on exactly $2g_n(M/d)$ distinct values. Consequently there are at most $g_n(M/d)$ distinct values for x. But, if $1 \leqq s < t \leqq g_n(M/d)$ then $n^s(m/d) \equiv \pm n^t(m/d) \pmod{M/d}$ would imply $n^{t-s} \equiv \pm 1 \pmod{M/d}$, which leads to the false consequence, via (13 (i)), that $g_n(M/d)$ divides $t - s$. Hence there are exactly $g_n(M/d)$ possible values of x. This proves

$$\text{(14)}\qquad m' = g_n\left(\frac{M}{(m, M)}\right).$$

From (14) we see that the maximum cycle length is $g_n(M)$ and every m in K which is prime to M is in a cycle of this length. Further, using (13) (ii), we see that all cycle lengths divide the maximum cycle length and that the order of the permutation P itself is $g_n(M)$.

If d is a divisor of M and $d < k$ then there are $\frac{1}{2}\phi(M/d)$ numbers m in K for which $(m, M) = d$. Each of these has order $g_n(M/d)$. Thus they form $\phi(M/d)/2g_n(M/d)$ cycles of length $g_n(M/d)$. Since it is possible for $g_n(a) = g_n(b)$ for $a \neq b$ we cannot conclude that there are no more than this number of such cycles. When M is even, the divisor $d = k$ of M leads to a cycle of length $g_n(2) = 1$. If we put $\eta = 0$ or 1 according to whether M is odd or even we may assert that the number of cycles in P is $\eta + \sum \phi(d)/2g_n(d)$, where the sum is taken over all divisors d of M with $d > 2$.

We gather together those of the above results that relate to the binomial congruences

$$\text{(15)}\qquad x \equiv n^s y \pmod{M}, \qquad x \equiv -n^s y \pmod{M},$$

where $1 \leqq x \leqq [M/2]$, $1 \leqq y \leqq [M/2]$. (It is clear that knowledge of the solutions x, y, $s(\geqq 0)$ of these congruences is equivalent to knowledge of the solutions of

$$\text{(16)}\qquad x \equiv n^s y \pmod{M}, \qquad 1 \leqq x < M, \qquad 1 \leqq y < M.$$

Therefore the algorithm contained in the following summary yields a method of finding all solutions of (16)).

The congruence $b \equiv (-1)^{\alpha} n^s a \pmod{M}$, *where* $(n, M)=1$, *is solvable in non-negative integers* α *and* s *if and only if* a *and* b *are in the same cycle of* $P(n, m)$. *When* a *and* b *are in the same cycle then* s *is the number (inclusive) of elements of this cycle starting with* $f(a)$ *and ending with* b, *while* α *is the number of these elements which lie in an* I_j *with* j *odd. All cycle lengths in* P *divide the length of the cycle containing* 1 *and this length is* $g_n(M)$. *Finally,* $n^{g_n(M)} \equiv (-1)^{\beta} \pmod{M}$, *where* β *is the number of elements in the cycle containing* 1 *which lie in an* I_j *with* j *odd.*

For $n=3$ and M odd most of these results were given by Bouniakowsky [**5**]. For $n>3$ they are reminiscent of some very general remarks stated near the end of a very elegant paper of Marcel Riesz [**9**] and of Problem #3 in Chapter VI of Vinogradoff [**11**, pp. 121–122, 202–3].

4. The Monge card shuffle. We turn now, momentarily, to the special case $n=2$.

In this case there are only two subsets of K,

$$I_0 = \{m \mid 0 < m < (M/4)\}, \qquad I_1 = \{m \mid (M/4) < m < (M/2)\}$$

and we have $f(1), \cdots, f(k)$ just

$$2, 4, 6, \cdots, 2[(k-1)/2], k, 2[k/2]-1, \cdots, 5, 3, 1. \tag{17}$$

If one numbers a deck of k cards from top to bottom with 1 to k and then shuffles them so that, when k is even (odd), the card $k-1$ is placed below (above) the card k, the card $k-2$ is placed above (below) these two, the card $k-3$ is placed below (above) these three, the card $k-4$ is placed above (below) these four, etc., then one finishes with the order (17). Such a shuffle is called a *Monge shuffle* after G. Monge who investigated it in 1773. (See [**1, 3, 4, 7, 8**]. We note however that we have numbered the cards in the opposite direction to these other authors in order to facilitate the calculations.)

After a single Monge shuffle the card that was in the $f(m)$th spot occupies the mth spot; after two Monge shuffles the card that was in the $f(f(m))$th spot occupies the mth spot, etc. In order to determine the number of shuffles to return all cards to their original positions we need only find s such that $f^s(m)=m$ for all m. But, from our general discussion just following (14), this is $g_2(2k+1)$.

In the case of an ordinary deck of 52 cards this gives $g_2(105)=12$ shuffles. In this case the divisors of $2k+1=105$ are 1, 3, 5, 7, 15, 21, 35, 105 and from $g_2(3)=1$, $g_2(5)=2$, $g_2(7)=3$, $g_2(15)=4$, $g_2(21)=6$, $g_2(35)=g_2(105)=12$, we see that there are cycles of lengths 1, 2, 3, 4, 6, 12. There is only one cycle of each of the first five lengths while there are

$$\frac{\phi(35)}{2g_2(35)} + \frac{\phi(105)}{2g_2(105)} = 1 + 2 = 3$$

cycles of length 12. Writing these cycles yields:

$$
\begin{aligned}
P(2,\ 105) = {} & (35)(21,\ 42)(15,\ 30,\ 45)(7,\ 14,\ 28,\ 49)(5,\ 10,\ 20,\ 40,\ 25,\ 40) \\
& (11,\ 22,\ 44,\ 17,\ 34,\ 37,\ 31,\ 43,\ 19,\ 38,\ 29,\ 47) \\
& (3,\ 6,\ 12,\ 24,\ 48,\ 9,\ 18,\ 36,\ 33,\ 39,\ 27,\ 51) \\
& (1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 41,\ 23,\ 46,\ 13,\ 26,\ 52).
\end{aligned}
$$

As a final remark in connection with the Monge shuffle we observe that it is not difficult, using our f, to prove the following (see [4] and [8]):

(i) There is a cycle of length 1 if and only if $k\equiv 1 \pmod 3$ and in this case the cycle is $((2k+1)/3)$;

(ii) there is a cycle of length 2 if and only if $k\equiv 2 \pmod 5$ and in this case the cycle is $((2k+1)/5,\ (4k+2)/5)$;

(iii) there is the cycle $((2k+1)/7,\ (4k+2)/7,\ (6k+3)/7)$ when $k\equiv 3 \pmod 7$ and the cycle $((2k+1)/9,\ (4k+2)/9,\ (8k+4)/9)$ when $k\equiv 4 \pmod 9$.

There are never other cycles of length three.

Example. From the factorization of $P(2,\ 105)$ displayed and our results on binomial congruences in Section 3 we see, for example, that $7\equiv(-1)^{\alpha}\cdot 2^{s}\cdot 17 \pmod{105}$ is not solvable for nonnegative α, s while $11\equiv(-1)^{\alpha}\cdot 2^{s}\cdot 17 \pmod{105}$ is solvable and $s=9$, $\alpha=7$. Further, $g_2(105)=12$ and $2^{12}\equiv(-1)^{\beta} \pmod{105}$, where $\beta=4$.

5. Further applications.

(a) The lemma of Gauss: If in the congruence (12) we let $s=m'$, the order of m (i.e., m' is the smallest positive integer for which $f^{s}(m)=m$), then we have $m\equiv f^{m'}(m)\equiv(-1)^{\alpha(m,m')}n^{m'}m \pmod M$. If we let A be a maximal subset of K no two elements of which are P-equivalent then multiplying these last congruences over A yields

$$
\prod_{m\in A} m \equiv (-1)^{\alpha}n^{k}\prod_{m\in A} m \pmod M, \tag{18}
$$

where α is the number of elements of K lying in an I_j with j odd. In the special case where M is an odd prime each $(m,\ M)=1$ and this congruence is equivalent to $n^{(M-1)/2}\equiv(-1)^{\alpha} \pmod M$. Since, by Euler's criterion, $(n/M)\equiv n^{(M-1)/2} \pmod M$ this is just the lemma of Gauss.

(b) Primitive roots: If we let B be the set of elements in K which are prime to M and multiply congruences (5) over all $m\in B$ we find

$$
\prod_{m\in B} f(m) \equiv (-1)^{Q}n^{(\phi(M)/2)}\prod_{m\in B} m \pmod M, \quad \text{where} \quad Q=\sum_{m\in B}[2nm/M].
$$

Since $\Pi_{m\in B}f(m)=\Pi_{m\in B}m$ we may divide out to obtain

$$
n^{\phi(M)/2} \equiv (-1)^{Q} \pmod M. \tag{19}
$$

When M is an odd prime this result, combined with the last remarks in (a) above, yields $n^{(M-1)/2}\equiv(-1)^{\alpha}\equiv(-1)^{Q} \pmod M$. Thus, α and Q have the

same parity. If, in addition, n is a primitive root mod M then, since $n^{(M-1)/2} \equiv -1 \pmod{M}$ in that case, Q must be odd. Therefore

(20) *if M is an odd prime then a necessary (but not sufficient) condition that n be a primitive root of M is that $\sum_{m=1}^{(M-1)/2}[2nm/M]$ be odd.*

We now prove the following result:

(21) *if p and $M=4p+1$ are primes and if $\sum_{m=1}^{2p}[2nm/M]$ is odd and M does not divide n^2+1 then n is a primitive root of M.*

We note first that (*) $n^{(1/2)\phi(M)} = n^{2p} \equiv (-1)^Q \equiv -1 \pmod{M}$, and therefore, $g_n(M)$ divides $2p$. If $g_n(M)=2$ then $n^{g_n(M)} = n^2 \equiv \pm 1 \pmod{M}$. Since M does not divide n^2+1 this means $n^2 \equiv 1 \pmod{M}$ which yields $n^{2p} \equiv 1 \pmod{M}$ which contradicts (*). If $g_n(M)=p$ then $n^{2p} \equiv 1 \pmod{M}$ giving the same contradiction. Thus $g_n(M)=2p$ and, since $n^{2p} \equiv -1 \pmod{M}$, n is a primitive root of M.

The results (20) and (21) lead readily to various known numerical results only a small number of which are stated here. (See [5] and [6].)

(i) 3 is not a primitive root of any prime of the form $12q+1$;

(ii) 5 is not a primitive root of any prime of the form $20q+9$;

(iii) If p and $M=4p+1$ are primes then (a) 2 is a primitive root of M; (b) 3 is a primitive root of M when $p>3$; (c) 5 is a primitive root of M when $p \equiv 3$ or 4 (mod 5), *i.e.*, 5 is a primitive root of $M=40q+13$ or $40q+37$ when M and $(M-1)/4$ are primes.

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MATHEMATICAL NOTES

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ON CERTAIN SEQUENCES

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Let $\{a_n\}$ be a monotonic nonincreasing sequence of nonnegative numbers;

$$(1) \qquad a_n \geqq a_{n+1} \geqq 0 \quad \text{for } n = 1, 2, 3, \cdots.$$

We consider the condition

$$(2) \qquad \lim_{u\to\infty} \sum_{u<n\leqq ux} a_n \text{ exists for each real } x > 1, \quad u \text{ real},$$

and ask the question: how well are the a_n determined if $\{a_n\}$ satisfies both (1) and (2)?

It is clear that the a_n are not too explicitly determined. For one thing, if $\{a_n\}$ satisfies (1) and (2) and we alter a finite number of the a_n, then the new sequence will clearly still satisfy (2). It is easy to make the changes in such a way that (1) is not violated.

The following two theorems also show that the a_n are not too explicitly determined.

THEOREM 1. *If $\sum_{n=1}^{\infty} a_n$ converges then $\{a_n\}$ satisfies* (2).

Proof. This follows directly from Cauchy's convergence criterion. In fact' given $\epsilon>0$ then there is a U such that $\left|\sum_{u<n\leqq ux} a_n\right| <\epsilon$ for $u\geqq U$, $x>1$, and hence $\lim_{u\to\infty} \sum_{u<n\leqq ux} a_n=0$.

THEOREM 2. *If $\{a_n\}$ and $\{b_n\}$ both satisfy* (1) *and* (2) *then so does $\{a_n+b_n\}$.*

Proof. Since $a_n\geqq a_{n+1}\geqq 0$ and $b_n\geqq b_{n+1}\geqq 0$ we have $a_n+b_n\geqq a_{n+1}+b_{n+1}\geqq 0$ and (1) is satisfied by $\{a_n+b_n\}$. Also

$$\lim_{u\to\infty} \sum_{u<n\leqq ux} (a_n + b_n) = \lim_{u\to\infty} \sum_{u<n\leqq ux} a_n + \lim_{u\to\infty} \sum_{u<n\leqq ux} b_n$$

since the limits on the right exist. Thus $\{a_n+b_n\}$ satisfies (2).

We now come to our main result.

THEOREM 3. *Suppose that $\{a_n\}$ satisfies* (1). *Then $\{a_n\}$ also satisfies* (2) *if and only if $\lim_{n\to\infty} na_n$ exists.*

Proof. We write $f_u(x)=\sum_{u<n\leqq ux} a_n$ and $f(x)=\lim_{u\to\infty} f_u(x)$ if it exists.

Suppose $\{a_n\}$ satisfies (2). Then $f(x)$ exists for each $x>1$. For positive integers m we have

$$f_m(x) = \sum_{n=m+1}^{[mx]} a_n. \tag{3}$$

Then

$$f_j(x) \leqq ([jx] - j)a_j \leqq (jx - j)a_j, \quad ja_j \geqq \frac{f_j(x)}{x-1}. \tag{4}$$

Also for each integer $j > x$ we take $m = [j/x]$ in (3) and let $r = j - [mx]$. Then we have

$$0 = j - \frac{j}{x}x \leqq j - mx \leqq j - [mx] < j - (mx - 1) < j - \left(\left(\frac{j}{x} - 1\right)x - 1\right)$$
$$= x + 1,$$

from which we obtain

$$0 \leqq r < x + 1, \quad \text{and} \quad j \geqq mx \geqq [mx].$$

We then have

$$f_m(x) = \sum_{n=m+1}^{[mx]} a_n \geqq ([mx] - m)a_{[mx]} \geqq ([mx] - m)a_j,$$

$$ja_j \leqq \frac{[mx] + r}{[mx] - m} f_m(x) \leqq \frac{mx + x + 1}{mx - 1 - m} f_m(x) \leqq \frac{j + x + 1}{((j/x) - 1)x - 1 - m} f_m(x),$$

$$ja_j \leqq \frac{j + x + 1}{j - x - 1 - (j/x)} f_{[j/x]}(x). \tag{5}$$

Now

$$\lim_{j\to\infty} \frac{f_j(x)}{x-1} = \frac{f(x)}{x-1}$$

and

$$\lim_{j\to\infty} \frac{j + x + 1}{j - x - 1 - (j/x)} f_{[j/x]}(x) = \frac{1}{1 - (1/x)} f(x) = \frac{x}{x-1} f(x)$$

and hence, by (4) and (5),

$$\frac{f(x)}{x-1} \leqq \liminf_{j\to\infty} ja_j \leqq \limsup_{j\to\infty} ja_j \leqq \frac{x}{x-1} f(x). \tag{6}$$

Next we obtain a certain property of $f(x)$. For $x > 1$, $y > 1$, $n > 1$, n an integer, we have

$$f_n(x) + f_{[nx]}(y) - f_n(xy) = \sum_{k=n+1}^{[nx]} a_k + \sum_{k=[nx]+1}^{[[nx]y]} a_k - \sum_{k=n+1}^{[nxy]} a_k = \sum_{k=[[nx]y]+1}^{[nxy]} a_k$$

where $[[nx]y] \leqq [nxy]$ and $[[nx]y] > [nxy-y] > nxy-y-1 \geqq [nxy]-y-1$. Therefore

$$0 \leqq f_n(x) + f_{[nx]}(y) - f_n(xy) \leqq (y+1)a_{[[nx]y]} = \frac{y+1}{[[nx]y]}[[nx]y]a_{[[nx]y]},$$

$$(7) \qquad 0 \leqq f_n(x) + f_{[nx]}(y) - f_n(xy) \leqq \frac{y+1}{nxy-y-1}[[nx]y]a_{[[nx]y]}.$$

We let n approach infinity in (7) noting that

$$\limsup_{n\to\infty} [[nx]y]a_{[[nx]y]}$$

is finite by (6). Then the right side of (7) approaches zero and we have

$$(8) \qquad f(x) + f(y) - f(xy) = 0.$$

Let k denote a positive integer. Repeated applications of (8) shows that

$$kf(e^{1/k}) = f(e).$$

Taking $x = e^{1/k}$ in (6) we find

$$\frac{f(e)}{k(x-1)} \leqq \liminf_{j\to\infty} ja_j \leqq \limsup_{j\to\infty} ja_j \leqq \frac{x}{x-1}\cdot\frac{f(e)}{k}$$

for $k = 1, 2, 3, \cdots$. But $\lim_{k\to\infty} k(x-1) = \lim_{k\to\infty} k(e^{1/k}-1) = 1$ and hence we see that $\lim_{j\to\infty} ja_j$ exists, in fact $\lim_{j\to\infty} ja_j = f(e)$. Clearly $f(e) \geqq 0$.

We must now prove the converse. We suppose that $\lim_{n\to\infty} na_n = c \geqq 0$. We take a fixed $x > 1$ and arbitrary $\epsilon > 0$. There is an N such that $|na_n - c| < \epsilon$ for $n \geqq N$. Then for $u \geqq N$ we have

$$(9) \qquad \left| \sum_{u<n\leqq ux} a_n - c \sum_{u<n\leqq ux} \frac{1}{n} \right| < \epsilon \sum_{u<n\leqq ux} \frac{1}{n}.$$

We also have

$$\int_n^{n+1} \frac{dt}{t} < \frac{1}{n} < \int_{n-1}^{n} \frac{dt}{t}$$

and hence

$$\int_{u+1}^{[ux]+1} \frac{dt}{t} \leqq \sum_{u<n\leqq ux} \frac{1}{n} < \int_{u-1}^{ux} \frac{dt}{t}, \quad \log\frac{[ux]+1}{u+1} \leqq \sum_{u<n\leqq ux} \frac{1}{n} < \log\frac{ux}{u-1}$$

which implies

$$\lim_{u\to\infty} \sum_{u<n\leqq ux} \frac{1}{n} = \log x.$$

Thus there is an N_1 such that

$$\left| \sum_{u<n\leqq ux} \frac{1}{n} - \log x \right| < \epsilon$$

if $n \geqq N_1$. For $u \geqq N+N_1$, using (9) we find

$$\left| \sum_{u<n\leqq ux} a_n - c \log x \right| < \epsilon \sum_{u<n\leqq ux} \frac{1}{n} + c\epsilon < \epsilon(\log x + \epsilon) + \epsilon,$$

and this implies

$$\lim_{u\to\infty} \sum_{u<n\leqq ux} a_n = c \log x.$$

As an immediate consequence of this proof we have:

COROLLARY. *If* $\{a_n\}$ *satisfies* (1) *and* (2) *then* $\lim_{u\to\infty} \sum_{u<n\leqq ux} a_n = c \log x$ *for some constant* $c \geqq 0$.

The monotonicity (1) of $\{a_n\}$ is actually needed for Theorem 3. To show this we consider the example

$$a_n = \begin{cases} \dfrac{2}{n}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

We have

$$\sum_{u<n\leqq ux} a_n = \sum_{(u/2)<k\leqq (u/2)x} \frac{2}{2k} = \sum_{(u/2)<k\leqq (u/2)x} b_k \tag{10}$$

where $b_k = (1/k)$. Applying Theorem 3 to $\{b_k\}$ we see that the right side of (10) has the limit $\log x$. Therefore $\lim \sum_{u<n\leqq ux} a_n = \log x$, but $\lim_{n\to\infty} na_n$ does not exist.

The $\{a_n\}$ mentioned in Theorem 1 all have $\lim_{u\to\infty} \sum_{u<n\leqq ux} a_n = 0$. There also exist sequences $\{a_n\}$ such that $\lim_{u\to\infty} \sum_{u<n\leqq ux} a_n = 0$ but for which $\sum_{n=1}^{\infty} a_n$ diverges. One such example is given by $a_1 = 1$, $a_n = 1/n \log n$ for $n \geqq 2$.

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AN INTEGRAL REPRESENTATION FOR THE EULER NUMBERS

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As is well known, [3], the Euler numbers satisfy the symbolic equation

$$(E+1)^n + (E-1)^n = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{*}$$

In particular, $E_0=1$, $E_1=0$, $E_2=-1$, $E_3=0$, $E_4=5, \cdots$. Although it can be deduced rather easily from a known integral ([1] p. 555, [5]), the fact that, for each nonnegative integer n,

$$E_n = (i)^{-n}\left(\frac{2}{\pi}\right)^{n+1}\int_0^\infty\left(\frac{(\log(x))^n}{1+x^2}\right)dx \tag{1}$$

seems not to be well known. It is evident that this formula gives the desired zero value whenever n is odd. For $n=0$, evaluation of the integral is a familiar calculus problem and, with $n=2$, it has appeared as a higher level problem [4], but the connection with the Euler numbers is hardly discernible from these special cases. In the interest of making this note self-contained, we give a derivation which is, in part, the same as that used by Glasser [2] in establishing a more general recursion formula.

If $r>0$, $0\leqq\theta\leqq\pi$ and $z=r\exp(i\theta)$, we let $\log z=\log r+i\theta$; and we let C represent the contour consisting of the real axis from r to R, the upper half of the circle $|z|=R$, the real axis from $-R$ to $-r$ and the upper half of the circle $|z|=r$. Then $\log i=i\pi/2$ and, if r is sufficiently small and R is sufficiently large, the Cauchy integral formula yields

$$\int_r^R\frac{(\log x)^n}{1+x^2}dx+\int_R^{-R}\frac{(\log z)^n}{1+z^2}dz+\int_{-R}^{-r}\frac{(\log(-x)+i\pi)^n}{1+x^2}dx+\int_{-r}^{r}\frac{(\log z)^n}{1+z^2}dz$$
$$=\int_C\frac{(\log z)^n}{z^2+1}dz = 2\pi i(i\pi/2)^n/2i.$$

Using a direct attack, one shows easily that the integrals on the semicircles go to zero as $r\to 0$ and $R\to\infty$. Hence, by using "$-x$ for x" in one integral and then the binomial theorem, we obtain

$$\int_0^\infty\frac{(\log x)^n}{1+x^2}dx+\sum_{k=0}^{n}\binom{n}{k}(i\pi)^{n-k}\int_0^\infty\frac{(\log x)^k}{1+x^2}dx = \pi\left(\frac{i\pi}{2}\right)^n.$$

If we multiply both sides by $i^{-n}(2/\pi)^{n+1}$, we have

$$\begin{aligned}(i)^{-n}\left(\frac{2}{\pi}\right)^{n+1}&\int_0^\infty\frac{(\log x)^n}{1+x^2}dx\\ &+\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(i)^{-k}\left(\frac{2}{\pi}\right)^{k+1}\int_0^\infty\frac{(\log x)^k}{1+x^2}dx = 2.\end{aligned} \tag{2}$$

Upon setting $E_k'=(i)^{-k}(2/\pi)^{k+1}\int_0^\infty(\log x)^k dx/(1+x^2)$, equation (2) can be written symbolically

$$(E')^n+(E'+2)^n = 2. \tag{3}$$

We now notice that the fundamental identity (*) generalizes at once to $f(E+1)+f(E-1)=2f(0)$ where f is any polynomial function. Taking $f(x)=(x+1)^n$ we have $(E+2)^n+(E)^n=2$.

By comparing this with (3) and noticing that $E_0'=1=E_0$ we see that $E_n=E_n'$ for each nonnegative n and that (1) is established.

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CONVEXITY IN THE THEORY OF EQUATIONS

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H. W. Milnes [1] proved a theorem which characterizes all polynomials with zeros only on the interval $[-1, 1]$ among polynomials of degree n with all zeros real. Simply stated, it is that $S_h(n)\geqq 0(h=0, \cdots, n)$ where $S_h(n)$ are certain symmetric functions of the zeros. We shall give a different treatment, using convexity, and find formulas for the $S_h(n)$ in terms of the polynomial coefficients; see (5).

To the real polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \qquad a_n = 1,$$

of degree n, associate the vector

$$V(P) = (a_0, \cdots, a_{n-1}).$$

Let $R_4=R_4(n)$ be the region of all vectors $V(P)$ for which $P(x)$ has no real zeros outside the interval $[-1, 1]$. Such a $P(x)$ is characterized by $P(x)>0$ for $x>1$ and $(-1)^nP(x)>0$ for $x<-1$. Since these inequalities are satisfied by the convex combination $c_1P_1(x)+c_2P_2(x)$ with $c_1+c_2=1$, c_1 and c_2 positive, it follows that R_4 is convex.

The subset R_1 of R_4 corresponding to $P(x)$ with all roots real, as well as their being on the interval $[-1, 1]$, is closed. It is also bounded since the coefficients of $P(x)$ are elementary symmetric functions of the zeros, and the zeros are bounded. The convex hull $R_2=R_2(n)$ of R_1 is therefore bounded, closed, and is contained in R_4; $R_4\supset R_2\supset R_1$; i.e., only points corresponding to polynomials with no real zeros outside $[-1, 1]$ are in R_2, and all such polynomials with n

real zeros are in R_2. The region described in Milnes' Theorem is R_2. Figure 1 shows these three regions for $n=2$.

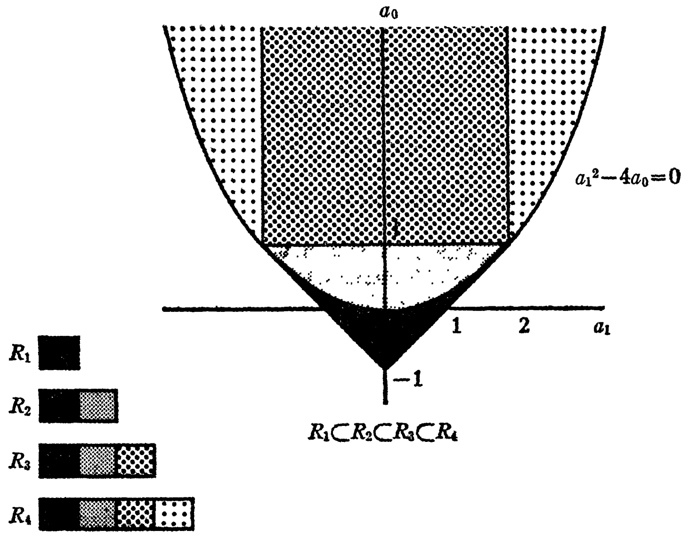

FIG. 1.

$$R_1 = \{(a_1, a_0) \mid a_1^2 - 4a_0 \geqq 0\} \cap R_3,$$
$$R_2 = \{(a_1, a_0) \mid a_0 \leqq 1,\ a_0 + a_1 \geqq -1,\ a_1 - a_0 \leqq 1\},$$
$$R_3 = \{(a_1, a_0) \mid |a_1| \leqq 2,\ a_1 + a_0 \geqq -1,\ a_1 - a_0 \leqq 1\},$$
$$R_4 = \{(a_1, a_0) \mid a_1^2 - 4a_0 < 0\} \cup R_2.$$

If in $P(x)$ the variable x is translated by h, i.e., $y=x+h$, then a new polynomial $Q(y)=b_0+b_1y+\cdots+y^n$ is obtained with $V(Q)$ related to $V(P)$ by a linear transformation

$$b_i = \sum_{j=i}^{n} C_{ji}(-h)^{j-i}a_j, \qquad C_{ji} = j!/i!(j-i)! \qquad (i = 0, \cdots, n-1).$$

If all $b_i \geqq 0$, then all real zeros of $Q(y)$ are $\leqq 0$ since $Q(y)>0$ for $y>0$. Conversely, if all the zeros $r_1, \cdots, r_n$ of $Q(y)$ are real and $\leqq 0$, then all $b_i \geqq 0$, as can be seen readily by expanding

$$Q(y) = (y - r_1) \cdots (y - r_n).$$

Thus the conditions $b_i \geqq 0$ $(i=0, 1, \cdots, n-1)$ characterize all $Q(y)$ with only negative zeros among all polynomials having all real zeros. Putting $h=-1$ we get the conditions

$$\sum_{j=i}^{n} C_{ji}a_j \geqq 0 \qquad (i = 0, \cdots, n-1)$$

for all zeros of $P(x)$ to be $\leqq 1$ if $P(x)$ has all real zeros. A condition for the zeros to be $\geqq -1$, found by replacing x by its negative, is

$$\sum_{j=i}^{n} C_{ji}(-1)^j a_j \geqq 0 \qquad (i = 0, 1, \cdots, n-1).$$

The region R_3 described by these $2n$ inequalities is convex since it is the intersection of half spaces; also $R_4 \supset R_3 \supset R_2$ (see Figure 1).

We shall prove that R_2 has exactly $n+1$ extreme points and will describe them; hence R_2 is an n-simplex and the inequalities found by Milnes give the $n+1$ faces. Knowing the extreme points, we find the inequalities explicitly.

LEMMA. *The set R_2 is an n-simplex with vertices $V(T_i)$ $(i=0, 1, \cdots, n)$ where*

$$T_i(x) = (x+1)^i(x-1)^{n-i}.$$

Let $P(x)$ correspond to a point $V(P)$ in R_1. We first show that if $P(x)$ has a zero r such that $|r| < 1$, then $V(P)$ is not an extreme point of R_2. For setting $P(x) = (x-r)S(x)$ and letting $e = 1-|r|$ we see that the polynomials $P_i(x) = (x-r+(-1)^i e)S(x)$ $(i=1, 2)$ are in R_1 and distinct and $P(x) = \frac{1}{2}P_1(x) + \frac{1}{2}P_2(x)$.

Since R_2 is the convex hull of R_1, all the extreme points of R_2 are in R_1 and hence are a subset of the set of $V(T_i)$ $(i=0, \cdots, n)$.

We will show later [see (3)] that the $n+1$ points $V(T_i)$ are linearly independent. Therefore R_2 is not in a hyperplane and, being bounded, has at least $n+1$ extreme points. Consequently the $V(T_i)$ $(i=0, \cdots, n)$ are the extreme points of R_2, and thus R_2 is the n-simplex that is the convex hull of the $V(T_i)$.

We want to find the equations of the $n+1$ bounding hyperplanes of R_2; these are described by the fact they each contain all but one of the $V(T_i)$. It is easier to treat the polynomials

$$U_i(y) = y^{n-i}(y+1)^i = \sum_{j=n-i}^{n} C_{i,n-j}y^j = \sum b_{ij}y^j \qquad (i = 0, \cdots, n) \tag{1}$$

and apply the transformation

$$y = \tfrac{1}{2}(x-1) \tag{2}$$

which maps T_i onto $2^n U_i$. This induces a linear transformation which maps R_2 onto R_5 defined as the convex closure of the $V(U_i)$. The region R_5 is the convex hull of all $V(Q)$ for which Q has all zeros real and lying on the interval $[-1, 0]$.

The matrix $B = (b_{ij})$ $(i, j = 0, \cdots, n)$ has an inverse $G = (g_{ij})$ with

$$g_{ij} = (-1)^{i-j-n}b_{n-i,n-j} = (-1)^{i+j+n}C_{n-i,j}$$

since

$$\sum_h b_{ih}g_{hj} = \sum C_{i,n-h}(-1)^{h+j+n}C_{n-h,j}$$
$$(3) \qquad = (-1)^{j+n}\sum(-1)^h C_{ij}C_{i-j,i+h-n}$$
$$= (-1)^{j+i}C_{ij}(1-1)^{i-j} \qquad (\text{when } i \neq j)$$
$$= \delta_{ij}.$$

Since we have $\delta_{ij}=0$, $i\neq j$, the equation of the hyperplane H_j through $V(U_i) = (b_{i0}, \cdots, b_{in})$ $(i=0, \cdots, n; i\neq j)$ is $\sum b_h g_{hj}=0$. Also, because $\delta_{jj}>0$, the half-space bounded by H_j and containing $V(U_j)$ is given by the inequality

$$(4) \qquad \sum b_h g_{hj} \geqq 0.$$

Finally, the transformation (2), $x=2y+1$, is used to obtain the $n+1$ inequalities describing R_2. If $2^nQ(y)$ is obtained from $P(x)$ by means of (2), then $b_j=2^{j-n}\sum_{i=j}^n a_iC_{ij}$. Hence R_2 is the intersection of the half-spaces satisfying

$$\sum f_{hi}\, a_i \geqq 0,$$
$$(5) \qquad f_{hi} = (-1)^{n+h}\sum_{j=0}^{i}(-2)^jC_{ij}C_{n-j,h}.$$

These expressions are the $S_h(n)$ of [1].

Some of the comments made in [1] can now be easily proved. For example, the Lemma 2 follows easily from $\sum_h f_{hi}=(-1)^n\sum_j C_{ij}(-2)^j\sum_h(-1)^hC_{n-j,h}=0$ if $n-j>0$; otherwise $j=n$ and hence $i=n$ and $\sum_h f_{hi}=(-1)^nC_{nn}(-2)^n=2^n$.

Also, the region R_6 of points corresponding to the $P(x)$ having all zeros with absolute value $\leqq 1$ is contained in R_2. This will be proved by noticing that R_6 is bounded and showing that the extreme points of the convex hull of R_6 are the same as those of R_2. In the lemma above it was shown that $V(P)$ is not an extreme point in R_6 if P has a real zero r with $|r|<1$. Next, suppose that P has a complex zero $r=s+it$, $t\neq 0$, and $|r|=s^2+t^2\leqq 1$. Then $F(x)=x^2-2sx+(s^2+t^2)$ is a factor of $P(x)=F(x)R(x)$. But $P(x)=\frac{1}{2}P_1(x)+\frac{1}{2}P_2(x)$ with

$$P_i(x) = (x^2-[s+(-1)^iv]2x+s^2+t^2)R(x), \qquad (i=1, 2),$$
$$v = (s^2+t^2)^{1/2}-|s|,$$

and $V(P_1)$, $V(P_2)$ both lie in R_6 and are distinct. For $n=2$, $R_6=R_2$.

Finally, suppose $P^*(x)$ is the polynomial of degree $n^*=n(n-1)/2$ constructed to have for zeros all possible products $r_\alpha r_\beta$ of the zeros of $P(x)$. In particular, $P^*(x)$ has as zeros the norms of all nonreal zeros of $P(x)$. Thus all the zeros of $P(x)$ will have absolute value $\leqq 1$ whenever $V(P)$ is in $R_2(n)$ and $V(P^*)$ is in $R_2(n^*)$.

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SOME CONGRUENCES FOR THE ELEMENTARY DIVISOR FUNCTIONS

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Congruence properties of the elementary divisor function $\sigma_k(n)$, the sum of the kth powers of the divisors of n, have been studied by a number of authors including the present writer [1]. Still, as far as is known to the author, the following simple, but nonetheless interesting, theorems—where $\phi(n)$ stands as usual for Euler's function—are not included among the published results.

THEOREM 1. *If* $q=\frac{1}{2}\phi(p^\lambda)=\frac{1}{2}(p-1)p^{\lambda-1}$ *where* p *is an odd prime,* $\lambda\geqq 1$, $p\nmid n$, *and* α *is any nonnegative integer, then*

$$\sigma_{q+\alpha}(n) \equiv (n/p)n^\alpha\sigma_{q-\alpha}(n) \pmod{p^\lambda}$$

where (n/p) *is Legendre symbol of quadratic character.*

The above theorem also holds for the even prime 2 provided $\lambda>1$. In this case, however, the relation can be expressed in the neater form given below.

THEOREM 2. *If* $q=\frac{1}{2}\phi(2^\lambda)=2^{\lambda-2}$, $\lambda>1$, n *is odd, and* α *is any nonnegative integer, then*

$$\sigma_{q+\alpha}(n) \equiv n^\alpha\sigma_{q-\alpha}(n) \pmod{2^\lambda}.$$

The particular case of Theorem 1 corresponding to $\alpha=0$ leads immediately to the following interesting corollary.

COROLLARY. *If* $q=\frac{1}{2}\phi(p^\lambda)=\frac{1}{2}(p-1)p^{\lambda-1}$ *where* p *is an odd prime,* $\lambda\geqq 1$, *and* n *is a quadratic nonresidue of* p, *then*

$$\sigma_q(n) \equiv 0 \pmod{p^\lambda}.$$

The special case of the corollary corresponding to $\lambda=1$, unlike the theorems or the (general) corollary is, however, known, [2], [1].

The proof of Theorem 1 rests upon the very well-known result which holds for any odd prime p, viz.,

$$n^{\frac{1}{2}(p-1)} \equiv \left(\frac{n}{p}\right) \pmod{p}, \qquad p \nmid n; \tag{1}$$

or rather upon the not so well catered generalization given in the following lemma:

LEMMA 1. *If* p *is an odd prime, then* $n^{\frac{1}{2}\phi(p^\lambda)}\equiv(n/p) \pmod{p^\lambda}$, $p\nmid n$.

This lemma is easily established from (1) by the method of induction when one remembers that with $\lambda\geqq 1$

$$\left[\left(\frac{n}{p}\right) + ap^\lambda\right]^p \equiv \left(\frac{n}{p}\right) \pmod{p^{\lambda+1}}, \tag{2}$$

since

$$\left[\left(\frac{n}{p}\right) + ap^\lambda\right]^p = \left(\frac{n}{p}\right)^p + \binom{p}{1}\left(\frac{n}{p}\right)^{p-1}\cdot ap^\lambda + \binom{p}{2}\left(\frac{n}{p}\right)^{p-2}\cdot a^2p^{2\lambda} + \cdots + a^pp^{\lambda p}$$
$$= \left(\frac{n}{p}\right) + ap^{\lambda+1} + \text{terms involving higher powers of } p.$$

For Theorem 2 we require the simpler lemma given below.

LEMMA 2. $n^{2^{\lambda-2}} \equiv 1 \pmod{2^\lambda}$, $2\nmid n$, $\lambda > 2$.

This lemma which is very similar to the previous one follows also by the method of induction since the lemma is true for $\lambda = 3$, which is easily seen when one remembers that n must belong to one of the forms $4m \pm 1$, and further

$$(2^\lambda a + 1)^2 \equiv 1 \pmod{2^{\lambda+1}}. \tag{3}$$

We shall now prove the theorems in two parts: Case (i) when $\alpha \leqq q$, and Case (ii) when $\alpha > q$. Remembering that

$$n^s\sigma_{-s}(n) = \sigma_s(n), \tag{4}$$

it is readily seen that the theorems are meaningful even when $\sigma_{q-\alpha}(n)$ is fractional, as it happens in Case (ii) where $\alpha > q$. Each of these cases of Theorem 2 will be considered in two parts: $\lambda > 2$, and $\lambda = 2$.

Our first step lies in establishing that

$$\sigma_{q+\alpha}(n) \equiv n^{q+\alpha}\cdot\sigma_{q-\alpha}(n) \pmod{N}, \tag{5}$$

where $\alpha \leqq q = \frac{1}{2}\phi(N)$ and $(n, N) = 1$. Now using (4) we get

$$\begin{aligned} n^{q+\alpha}\cdot\sigma_{q-\alpha}(n) - \sigma_{q+\alpha}(n) &= n^{q+\alpha}[\sigma_{q-\alpha}(n) - \sigma_{-q-\alpha}(n)] \\ &= n^{q+\alpha}\left[\sum_{d|n} d^{q-\alpha} - \sum_{d|n} d^{-q-\alpha}\right] \\ &= n^{q+\alpha}\sum_{d|n} d^{-q-\alpha}(d^{2q} - 1) \\ &= \sum_{d|n} d'^{\,q+\alpha}(d^{2q} - 1), \end{aligned} \tag{6}$$

where $dd' = n$. Now, by virtue of Euler's Theorem

$$d^{2q} - 1 = d^{\phi(N)} - 1 \equiv 0 \pmod{N}, \qquad (d, N) = 1; \tag{7}$$

we conclude the validity of (5) from the identity (6). Putting respectively $N = p^\lambda$ and 2^λ—so that n^q can be replaced by $n^{\frac{1}{2}\phi(p^\lambda)}$ and $n^{2^{\lambda-2}}$ respectively—and remembering Lemmas 1 and 2, we get Case (i) of Theorems 1 and 2, with $\lambda > 2$ for the second theorem.

For Case (ii) of Theorem 2 where $\alpha > q = 2^{\lambda-2}$ we can, by using (4) and Lemma

2, rewrite the congruence of Theorem 2 in the neater form (if $\lambda > 2$)

$$\sigma_{\alpha+q}(n) \equiv \sigma_{\alpha-q}(n) \pmod{2^\lambda}; \tag{8}$$

and by using (4) and Lemma 1 and Euler's congruence (7), Theorem 1 can be rewritten with essentially the same form of congruence, viz.

$$\sigma_{\alpha+q}(n) \equiv \sigma_{\alpha-q}(n) \pmod{p^\lambda}. \tag{9}$$

Now, if d is any positive integer prime to an arbitrary number N and $q=\frac{1}{2}\phi(N)$ then by Euler's Theorem (7)

$$d^{2q} \equiv 1 \pmod{N}. \tag{10}$$

Multiplying both sides of (10) by $d^{\alpha-q}$ we get

$$d^{\alpha+q} \equiv d^{\alpha-q} \pmod{N}. \tag{11}$$

It follows therefore that if $(n, N)=1$ then $\sum_{d|n} d^{\alpha+q} \equiv \sum_{d|n} d^{\alpha-q} \pmod{N}$, and in other words,

$$\sigma_{\alpha+q}(n) \equiv \sigma_{\alpha-q}(n) \pmod{N}, \tag{12}$$

a result which covers (8) and (9) as particular cases as can be seen by putting $N=2^\lambda$ and p^λ respectively. We have thus proved Theorem 1; and also Theorem 2 excepting for the situation $\lambda=2$ (Lemma 2 which was used above not being valid for $\lambda=2$).

When $\lambda=2$ the congruence of Theorem 2 reduces to

$$\sigma_{\alpha+1}(n) \equiv n\sigma_{\alpha-1}(n) \pmod{2^2}. \tag{13}$$

We may suppose $\alpha \geqq 1$, $\alpha=0$ being trivial. Now,

$$\sigma_{\alpha+1}(n) - n\sigma_{\alpha-1}(n) = \sum_{d|n} d^\alpha(d - d'), \tag{14}$$

where $dd'=n$. But n being odd it must be of the form $4m+1$ or $4m-1$. In the first case d and d' must be both of the form $4m+1$, or both of the form $4m-1$. In either case $d-d' \equiv 0 \pmod{4}$ and (14) establishes the validity of (13), when n is of the form $4m+1$. When n is of the form $4m-1$ it cannot be a square, and therefore d and d' cannot be equal. We can now write (14) as

$$\sigma_{\alpha+1}(n) - n\sigma_{\alpha-1}(n) = \sum_{d>d'} (d - d')(d^\alpha - d'^\alpha). \tag{15}$$

Now d and d' are both odd, hence $d-d'$ and $d^\alpha-d'^\alpha$ are both even, hence $(d-d')(d^\alpha-d'^\alpha)$ is divisible by 4. As a consequence (15) establishes the validity of (13) when n is of the form $4m-1$. We have thus completed the proof of Theorem 2.

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NOTE ON PROPER AND GENERALIZED EIGENVECTORS

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In [1], Brand gives a method for obtaining proper and generalized eigenvectors. To find the eigenvectors corresponding to a given eigenvalue λ_1 of an n by n matrix A, it is necessary to find a vector $e(\lambda)$ which satisfies an equation of the form

$$(A - \lambda I)e(\lambda) = (f(\lambda), 0, \cdots, 0)^{\tau}$$

and evaluate

$$e_1 = e(\lambda_1), \cdots, e_k = \frac{1}{(k-1)!} e^{k-1}(\lambda_1).$$

Since the components of $e(\lambda)$ are polynomials in λ, for large matrices it is frequently difficult to find $e(\lambda)$.

The purpose of this note is to give an alternative computational method which is an easy consequence of the work of Brand.

THEOREM. *For an n by n matrix A, assume λ_1 is an eigenvalue of multiplicity k with a proper eigenvector e_1 and $k-1$ generalized eigenvectors $e_2, e_3, \cdots, e_k$, then for any nonzero vector x in the span of $\{e_1, e_2, \cdots, e_k\}$, $(A-\lambda_1 I)^{k-1}x = e_1$ where any scalar multiple of e_1 is regarded as the same eigenvector.*

Proof. The eigenvectors of λ_1 are linearly independent; therefore, x can be expressed in the form

$$x = c_1e_1 + c_2e_2 + \cdots + c_ke_k.$$

Since $(A-\lambda_1 I)e_i = e_{i-1}$, $i=2, 3, \cdots, k$, and $(A-\lambda_1 I)^i e_i = 0$, $i=1, 2, \cdots, k$, it follows that

$$\begin{aligned} (A - \lambda_1 I)x &= c_2e_1 + c_3e_2 + \cdots + c_ke_{k-1}, \\ &\cdots\cdots\cdots\cdots \\ (A - \lambda_1 I)^{k-2}x &= c_{k-1}e_1 + c_ke_2, \\ (A - \lambda_1 I)^{k-1}x &= c_ke_1. \end{aligned} \tag{1}$$

However, since a proper eigenvector e_1 is arbitrary to a multiplicative constant, and generalized eigenvectors are arbitrary to additive multiples of all eigenvectors of lower index, we can take $e_k = x$ and the left members of equations (1) as $e_{k-1}, \cdots, e_2, e_1$. With this choice,

$$\begin{aligned} x &= e_k, \\ (A - \lambda_1 I)x &= e_{k-1}, \\ &\cdots\cdots \\ (A - \lambda_1 I)^{k-2}x &= e_2, \\ (A - \lambda_1 I)^{k-1}x &= e_1. \end{aligned} \tag{2}$$

Equations (2) imply $(A-\lambda_1 I)e_i=e_{i-1}$, $i=2, 3, \cdots, k$, $(A-\lambda_1 I)e_1=0$.

In practice, we do not usually have a vector x in the span of $e_1, e_2, \cdots, e_k$. In this case, we may use the orthonormal basis $x_1=[1, 0, \cdots, 0]$, $x_2=[0, 1, 0, \cdots, 0]$, $\cdots x_n=[0, 0, \cdots, 1]$ and compute

$$(A-\lambda_1 I)^r x_i,$$

$r=1, 2, \cdots, k$, until an x_i yields

$$(A-\lambda_1 I)^k x_i = 0.$$

The set of eigenvectors corresponding to λ_1 is

$$e_r = (A-\lambda_1 I)^{k-r} x_i,$$

$r=1, 2, \cdots, k$, and $e_k=x_i$.

COROLLARY. *If $k=n$, then for any nonzero n-component vector x, $(A-\lambda_1 I)^{n-1}x=e_1$.*

As an example, we consider Example 2 in [1] where

$$(A-\lambda_1 I) = \begin{bmatrix} 2 & 2 & -1 \\ -3 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}, \qquad \lambda_1=\lambda_2=\lambda_3=1.$$

If we choose $x=[1, 0, 0]$, then $e_3=[1, 0, 0]$, $e_2=(A-I)e_3=[2, -3, -1]$, $e_1=(A-I)e_2=[-1, 1, 0]$.

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SOME COMPUTER SOLUTIONS TO THE REFLECTING QUEENS PROBLEM

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Klarner [1] considers the following problem: For which n is it possible to form pairs $(1, b_1), (2, b_2), \cdots, (n, b_n)$ with $\{b_1, b_2, \cdots, b_n\}=\{n+1, n+2, \cdots, 2n\}$, so that all of the numbers b_i-i, b_i+i, $i=1, 2, \cdots, n$ are distinct? He gives a geometric interpretation of the problem in terms of the n queens problem and lists all solutions for $n\leqq 8$. In this note we give solutions for values of n from 9 through 27. A solution for $n=9$ different from the one given here has been given by Slater [2].

The algorithm used to obtain a solution was a form of the backtrack method with preclusion [3] developed by the author in collaboration with D. Ewing

HYPERGEOMETRIC DIFFERENTIAL EQUATION WITH UNUSUAL PROPERTIES

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The equation that will be discussed is the following special case of a hypergeometric equation

$$x(x-1)y'' + (2\nu+1)xy' + \nu^2 y = 0. \tag{1}$$

One solution contains the term $\ln x$ while the other, analytic at the origin, may be written

$$\begin{aligned} xF(\nu+1, \nu+1; 2; x) &= \int_0^x (1-x)^{-1-\nu} P_\nu\left(\frac{1+x}{1-x}\right) dx \\ &= x\nu^{-2} \frac{d}{dx}\left[(1-x)^{-\nu} P_{\nu-1}\left(\frac{1+x}{1-x}\right)\right], \end{aligned} \tag{2}$$

where the standard hypergeometric function is given by [1, p. 466] or [2, p. 556] as

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!} x^2 + \cdots \tag{3}$$

and P_ν is the Legendre function of arbitrary degree.

For ν an integer $\pm n \neq 0$, we can also write this solution of (1) as

$$[(1-x)^{-n-1} - (1-x)^{-n}] F\left(1+n, 1-n; 1; \frac{1}{1-x}\right) \tag{4}$$

or

$$x^{-n} F(1+n, n; 1; x^{-1}). \tag{5}$$

If n is a negative integer then these solutions of (1) reduce to a polynomial of order $|n|$, as of course do all hypergeometric functions. However, if n is a positive integer this solution still reduces to a finite number of terms, now of the form $(1-x)^{-2n+m}$ so as to have a pole of order $2n$ at $x=1$. The reason for this is clearly shown by applying the transformation

$$y(x) = (1-x)^{-1/2-\nu} Y(x) \tag{6}$$

which reduces (1) to the form

$$x(1-x)^2 Y'' + ((x/4) - \nu^2) Y = 0. \tag{7}$$

The solutions for $Y(x)$ are linearly dependent for $\pm\nu$; consequently the solutions of (1) are related, for any positive or negative values of ν, in the following manner:

$$(1 - x)^{2\nu} y_\nu(x) = y_{-\nu}(x). \tag{8}$$

Therefore a very useful property of (1) is that various relations can be obtained for the hypergeometric functions involved in (2), (4) and (5), because either (1) can be solved directly, or a more appropriate form for the infinite series can be obtained very easily by the method of Frobenius. For example

$$y = x \sum_{m=0}^{\infty} C_m x^m; \qquad \frac{C_{m+1}}{C_m} = \frac{(\nu + m + 1)^2}{(m + 1)(m + 2)} \tag{9}$$

for any values of ν, while for ν an integer $n > 0$ we obtain the rational expression

$$y = (x - 1)^{-2n} \sum_{m=0}^{m=n} B_m (x - 1)^m; \qquad \frac{B_{m+1}}{B_m} = \frac{(n - m)^2}{(2n - 1 - m)(m + 1)} . \tag{10}$$

Similarly for ν a negative integer $n < 0$ we obtain the polynomial

$$y = x^{-n} \sum_{m=0}^{m=-n} A_m x^{-m}; \qquad \frac{A_m}{A_{m-1}} = \frac{(n - 1 + m)(n + m)}{m^2} . \tag{11}$$

From a pedagogical viewpoint these relations provide good exercises for the student, and they verify (2), (4), and (5). As a matter of fact when n is a positive integer, (10) gives the $(n+1)$ terms which provide the explicit solution of (1) in the form

$$y/B_0 = (x - 1)^{-2n} \left[1 + \sum_{m=1}^{n} (B_m/B_0)(x - 1)^m \right], \tag{12}$$

$$\frac{B_m}{B_0} = \frac{n^2(n - 1)^2 \cdots (n - m + 1)^2}{m!(2n - 1) \cdots (2n - m)}$$

which is more useful than the equivalent form given directly by (4). It is also of interest to note that (11) is identical to (5), and gives the asymptotic expansion of (12) for $n > 0$.

Other interesting relations for n a positive integer can be obtained from the differentiation formulae of the hypergeometric functions [2, p. 557] by writing (2) as

$$y = \frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}} [x^n (1 - x)^{-n-1}] = \int_0^x \frac{P_n((1 + x)/(1 - x))}{(1 - x)^{n+1}} dx \tag{13}$$

and (5) as

$$x^n y = F(1 + n, n; 1; x^{-1}) = \left\{ \frac{d^{n-1}}{dz^{n-1}} \left[\frac{z^{n-1}}{(1 - z)^{n+1}} \right] \right\}_{z=x^{-1}} \frac{1}{(n - 1)!} \tag{14}$$

for n a positive integer. These relations are a special case of the generalized Rodrigues' formula for hypergeometric functions, and resemble that for Jacobi polynomials [2, p. 773], and somewhat similar recurrence relations can also be

obtained. However the polynomial solutions of (1) are not even special cases of the Jacobi polynomial, and they do not have this type of orthogonality property in the intervals $0 \leqq x \leqq 1$ or $-1 \leqq x \leqq 1$.

The linear transformation defined by $x = (\xi+1)/2$ transforms (1) into

$$(\xi^2 - 1)y'' + (2\nu + 1)(\xi + 1)y' + \nu^2 y = 0, \tag{15}$$

so all of the preceding solutions can be added to those given by Kamke [1, p. 462] after carefully noting that the polynomial solutions of (1) and (15) are neither ultraspherical polynomials nor Jacobi polynomials.

The solutions of (1) are mainly useful because different values of ν provide various relations for the hypergeometric functions in (2), (4) and (5). Also non-integer values of ν provide a good variety of differential equations with solutions given by (2) or (9). For example if we let $\nu = -1/2$ we obtain the following from (1) and (2):

$$x(x - 1)y'' + \tfrac{1}{4}y = 0, \tag{16}$$

$$\begin{aligned} y &= xF(\tfrac{1}{2}, \tfrac{1}{2}; 2; x) = 2/\pi \int_0^x K(x)dx \\ &= \frac{8}{\pi} x(1 - x)\frac{dK(x)}{dx} = \int_0^x \frac{P_{-1/2}((1 + x)/(1 - x))}{(1 - x)^{1/2}} dx, \end{aligned} \tag{17}$$

where K is the complete elliptic integral of the first kind. Then if we change variables by substituting $x = \xi^2$ we can write (16) as

$$\xi(\xi^2 - 1)y'' - (\xi^2 - 1)y' + \xi y = 0, \tag{18}$$

so the solution $y(\xi)$ provided by (17) is an interesting addition to the corresponding equations given by Kamke [1, p. 483, (2.316)].

The usefulness of (8) in obtaining new relations for the hypergeometric functions can be illustrated by obtaining the solution of (1) for $\nu = 1/2$ directly from (8) and (17), the solution for $\nu = -1/2$, as

$$y = \frac{2}{\pi(1 - x)}\int_0^x K(x)dx = \frac{8}{\pi} x \frac{dK(x)}{dx} = \int_0^x \frac{P_{1/2}((1 + x)/(1 - x))}{(1 - x)^{3/2}} dx. \tag{19}$$

Consequently we obtain

$$F(\tfrac{3}{2}, \tfrac{3}{2}; 2; x) = (8/\pi)(dK(x)/dx), \tag{20}$$

where

$$\begin{aligned} K(x) &= \int_0^{\pi/2} \frac{d\theta}{(1 - x\sin^2\theta)^{1/2}} = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) \\ &= \frac{\pi}{2}\, \frac{P_{-1/2}((1 + x)/(1 - x))}{(1 - x)^{1/2}}. \end{aligned} \tag{21}$$

Finally, since the Laplace transform of (1) is easily obtained, and the solutions (2) and (9) are valid for all ν, therefore we may be able to obtain some valuable inverse Laplace transform formulae. For example if we consider the case of $\nu=-1/2$, the Laplace transform of (16) is

$$Ly=\int_0^\infty y(x)e^{-sx}dx=s^{-3/2}e^{-s/2}J_0(\pm is/2). \tag{22}$$

Consequently the inverse Laplace transform that can be obtained from (2), (17) and (22) is

$$L^{-1}\left\{s^{-3/2}e^{-s/2}K_0\left(\frac{s}{2}\right)\right\}=\frac{2}{\pi}\int_0^x K(x)dx=\int_0^x\frac{P_{-1/2}((1+x)/(1-x))}{(1-x)^{1/2}}dx, \tag{23}$$

where K_0 is the modified Bessel function of the second kind.

TABLE 1

ν	$y(x)=$Solution of (1)
-3	$x+2x^2+\frac{1}{3}x^3$
-2	$x+\frac{1}{2}x^2$
-1	x
$-\frac{1}{2}$	$x(1-x)\ (dK/dx)$
0	$\ln(1-x)$
$\frac{1}{2}$	$x(dK/dx)$
1	$1/(x-1)^2+1/(x-1)$
2	$1/(x-1)^4+(4/3)/(x-1)^3+(1/3)/(x-1)^2$
3	$1/(x-1)^6+(18/10)/(x-1)^5+(9/10)/(x-1)^4+(1/10)/(x-1)^3$

The pedagogical usefulness of (1) is now clearly evident. It not only provides a multitude of problem assignments for students, but also leads them very easily into a discussion of the solutions, and their series expansions about the regular singular points, of the hypergeometric equations. Table 1 gives some of the solutions of (1), and a subsequent paper will give some physical applications of (1).

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SOME RESULTS ON FIXED POINTS—II

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Let E be a complete metric space with ρ as metric. The well-known Banach fixed point theorem ([4], p. 27) runs as follows: *Let there be given in a complete metric space E a map T of E into itself. Suppose further that*

$$\text{(A)} \qquad \rho[T(x), T(y)] \leqq \alpha\rho(x, y) \quad \textit{where} \quad 0 < \alpha < 1 \quad \textit{and} \quad x, y \in E.$$

Then there exists exactly one point x_0 such that $T(x_0) = x_0$.

For extension of Banach's contraction principle and certain other related results, see [1], [2].

For the unique common fixed point of two maps T_1 and T_2 each mapping E into itself, see [3].

The purpose of the present paper is to prove three theorems in each of which we have omitted the completeness of the space and we have obtained the same conclusion as in Banach's Theorem but with different sufficient conditions.

THEOREM 1. *Let E be a metric space with ρ as metric. Let T be a map of E into itself such that*

(i) $\rho[T(p), T(q)] \leqq \alpha\{\rho[p, T(p)] + \rho[q, T(q)]\}$, $0 < \alpha < \frac{1}{2}$, $p, q \in E$.

(ii) *T is continuous at a point $\xi \in E$.*

(iii) *There exists a point $x \in E$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to ξ.*

Then ξ is the unique fixed point of T.

Proof. Continuity at ξ of T implies that $\{T^{n_i+1}(x)\}$ converges to $T(\xi)$. Suppose $\xi \neq T(\xi)$. We consider two open discs $S_1 = S_1(\xi, \eta)$ and $S_2 = S_2(T(\xi), \eta)$ centered at ξ and $T(\xi)$ respectively and of radius $\eta > 0$ where $\eta < \frac{1}{3}\rho[\xi, T(\xi)]$. Since $\{T^{n_i}(x)\}$ converges to ξ and $\{T^{n_i+1}(x)\}$ converges to $T(\xi)$, there exists a positive integer N_1 such that $i > N_1$ implies

$$T^{n_i}(x) \in S_1, \qquad T^{n_i+1}(x) \in S_2.$$

Hence

$$\text{(1)} \qquad \rho[T^{n_i}(x), T^{n_i+1}(x)] > \eta, \qquad (i > N_1).$$

On the other hand,

$$\rho[T^{n_i+1}(x), T^{n_i+2}(x)] \leqq \alpha\{\rho[T^{n_i}(x), T^{n_i+1}(x)] + \rho[T^{n_i+1}(x), T^{n_i+2}(x)]\}.$$

Hence

$$\rho[T^{n_i+1}(x), T^{n_i+2}(x)] \leqq \frac{\alpha}{1-\alpha}\rho[T^{n_i}(x), T^{n_i+1}(x)].$$

For $l > j > N_1$ we have

$$\begin{aligned}\rho[T^{n_l}(x), T^{n_l+1}(x)] &\leqq \frac{\alpha}{1-\alpha}\,\rho[T^{n_l-1}(x), T^{n_l}(x)]\\ &\leqq \left(\frac{\alpha}{1-\alpha}\right)^2 \rho[T^{n_l-2}(x), T^{n_l-1}(x)]\\ &\leqq \left(\frac{\alpha}{1-\alpha}\right)^{n_l-n_j} \rho[T^{n_j}(x), T^{n_j+1}(x)].\end{aligned}$$

But this last expression approaches 0 as l approaches ∞ and we would get a result contradicting (1). Hence $T(\xi)=\xi$. Hence ξ is a fixed point of T. If δ is an element of E such that $T(\delta)=\delta$ then

$$\rho(\delta, \xi) = \rho[T(\delta), T(\xi)] \leqq \alpha\{\rho[\xi, T(\xi)] + \rho[\delta, T(\delta)]\} = 0.$$

Hence $\xi=\delta$ and thus the theorem is proved.

In [3] we have proved the following theorem:

THEOREM. *If T is a map of the complete metric space E into itself and if*

(B) $\rho[T(x), T(y)] \leqq \alpha\{\rho[x, T(x)] + \rho[y, T(y)]\}$ *where* $x, y \in E$ *and* $0 < \alpha < \frac{1}{2}$

then T has the unique fixed point in E.

If we compare this theorem with Theorem 1, we see that we have omitted the completeness of the space and instead, we have assumed conditions (ii) and (iii). The conditions (ii) and (iii) in Theorem 1 together do not guarantee the completeness of the space. The easy example in support of this is the following:

Example 1. $E=[0, 1)$, $T(x)=x/2$ and the distance function ρ is the ordinary euclidean distance on the line.

Comparing Banach fixed point theorem with the above theorem proved in [3] one might seek the relationship between conditions (A) and (B). It is clear at a first glance that the condition (A) implies the continuity of the map in the whole space but condition (B) does not. Moreover we provide two examples which show that the conditions (A) and (B) are independent.

Example 2. Let $E = [0, 1]$, $T(x) = x/4$ for $x \in [0, 1/2)$, $T(x) = x/5$ for $x\in[1/2, 1]$ and the distance function ρ is the ordinary euclidean distance on the line.

Here T is discontinuous at $x=1/2$; consequently, condition (A) is not satisfied. But it is easily seen that condition (B) is satisfied by taking $\alpha=4/9$.

Example 3. Let $E=[0, 1]$, $T(x)=x/3$ for $x\in[0, 1]$ and the distance function is the ordinary euclidean distance. Here condition (A) is satisfied but it is easily seen that condition (B) is not satisfied if we take $x=\frac{1}{3}$, $y=0$.

THEOREM 2. *Let E be a metric space with ρ as metric and T be a continuous map of E into itself. Suppose*

(i) $\rho[T(x), T(y)| \leqq \alpha\{\rho[x, T(x)]+\rho[y, T(y)]\}$, $0<\alpha<1/2$, *x and y belonging to an everywhere dense subset M of E.*

(ii) *There exists a point $x\in E$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to a point $\xi\in E$. Then ξ is the unique fixed point of T.*

Proof. The proof will follow from Theorem 1, if we can show that (i) holds for any pair of points $x, y\in E$.

Let x, y be any two elements of E. If $x\in M$, $y\in E-M$, let $\{z_n\}$ be a sequence in M such that $z_n\to y$. Then

$$\begin{aligned}\rho[T(x), T(y)] &\leqq \rho[T(x), T(z_n)] + \rho[T(z_n), T(y)]\\ &\leqq \alpha\{\rho[x, T(x)] + \rho[z_n, T(z_n)]\} + \rho[T(z_n), T(y)]\\ &\leqq \alpha\{\rho[x, T(x)] + \rho[y, T(y)]\}\\ &\quad + (1+\alpha)\rho[T(z_n), T(y)] + \alpha\rho(z_n, y).\end{aligned}$$

Now $z_n\to y$ and so letting $n\to\infty$ in the above inequality, we get

$$\rho[T(x), T(y)] \leqq \alpha\{\rho[x, T(x)] + \rho[y, T(y)]\}.$$

Now, consider the case when $x\in E-M$, $y\in E-M$. Let $\{x_n\}$ be a sequence of elements in M such that $x_n\to y$. Then

$$\begin{aligned}\rho[T(x), T(y)] &\leqq \rho[T(x), T(x_n)] + \rho[T(x_n), T(y)]\\ &\leqq \alpha\{\rho[x, T(x)] + \rho[x_n, T(x_n)]\} + \rho(T(x_n), T(y))\end{aligned}$$

from the preceding case. Hence,

$$\begin{aligned}\rho[T(x), T(y)] &\leqq \alpha\{\rho[x, T(x)] + \rho[y, T(y)]\}\\ &\quad + (1+\alpha)\rho[T(x_n), T(y)] + \alpha\rho(x_n, y).\end{aligned}$$

As before, letting $n\to\infty$, we get

$$\rho[T(x), T(y)] \leqq \alpha\{\rho[x, T(x)] + \rho[y, T(y)]\}.$$

The theorem follows.

THEOREM 3. *Let E be a metric space with ρ as metric and let T be a map of E into itself. Suppose that T is continuous at a point $x_0\in E$. If there exists a point $x\in E$ such that the sequence of iterates $\{T^n(x)\}$ converges to x_0 then $T(x_0)=x_0$. If in addition,*

$$\rho[T(x_0), T(\xi)] \leqq \alpha\rho(x_0, \xi), \qquad \xi\in E, \qquad 0<\alpha<1,$$

then x_0 is the unique fixed point of T.

Proof. Let $x_n=T^n(x)$; then $x_n\to x_0$ as $n\to\infty$. Now

$$\begin{aligned}\rho[T(x_0), x_0] &\leqq \rho[T(x_0), x_n] + \rho(x_n, x_0)\\ &= \rho[T(x_0), T(x_{n-1})] + \rho(x_n, x_0).\end{aligned}$$

The left hand side is independent of n and the right hand side tends to zero as $n \to \infty$, so letting $n \to \infty$ we obtain $T(x_0)=x_0$.

If there exists $\delta \neq x_0$ such that $T(\delta)=\delta$ then $\rho(x_0, \delta)=\rho[T(x_0), T(\delta)] \leqq \alpha\rho(x_0, \delta)$ gives $1 \leqq \alpha$, which is a contradiction. This proves the theorem.

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

CAN THE BOUNDARY OF A d-DIMENSIONAL CONVEX BODY CONTAIN SEGMENTS IN ALL DIRECTIONS?

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Let E^d denote the d-dimensional Euclidean space and S^{d-1} its unit sphere $\{u \in E^d: \|u\|=1\}$. For any set X in E^d let $D(X)$ denote the set of all directions of line segments contained in X. Representing directions by unit vectors, $D(X)$ is a subset of S^{d-1}; specifically, $u \in D(X)$ if and only if $\|u\|=1$ and the segment $[0, u]$ is parallel to a segment contained in X. For example, if $d=2$, Q is the square $\{(\alpha, \beta): \max(|\alpha|, |\beta|) \leqq 1\}$, and ∂Q is the boundary of Q, then $D(Q)=S^1$ and $D(\partial Q)$ consists of the four points $(\pm 1, 0)$, $(0, \pm 1)$. By an infinite sequence of truncations, one can obtain from Q a 2-dimensional convex body B such that ∂B is a countable dense subset of S^1. However, any simple closed curve J contains at most countably many maximal segments and hence $D(J)$ is countable. Thus the answer to the title question is negative when $d=2$. The general question was first raised in 1957 [4]. For $d=3$ McMinn [5] and Besicovitch [1] established the expected negative answer by showing that for any convex body B in E^3 the set $D(\partial B)$ is the union of countably many sets of finite 1-dimensional measure. However, the problem is open even for $d=4$. (Ewald [2] claimed to settle a more general problem but overlooked the essential measure-theoretic difficulties.)

For any convex body B in E^d and for any $\epsilon > 0$, let B_ϵ denote the closed ϵ-neighborhood of B; that is, $x \in B_\epsilon$ if and only if $\|x - b\| \leqq \epsilon$ for some $b \in B$. As $D(\partial B) = D(\partial B_\epsilon)$ and B_ϵ admits a unique supporting hyperplane at each boundary point, the title problem reduces to the case of smooth convex bodies. B. Grünbaum and the author have proved for $d \geqq 3$ that if B is a d-dimensional convex body of class $C^{(d-2)}$ (that is, one whose boundary can be locally parametrized by means of functions that are $d-2$ times continuously differentiable) then $D(\partial B)$ is not all of S^{d-1}. (In fact, $D(\partial B)$ is the union of countably many sets of finite $(d-2)$-dimensional measure.) Hence in the 4-dimensional case the title question has been answered for bodies of class $C^{(2)}$ but not for those of class $C^{(1)}$.

The title question may be viewed as a problem in abstract approximation theory. Recall Chebyshev's theorem asserting that for any n and for any continuous real function ϕ on $[0, 1]$, ϕ admits a unique best uniform approximation by polynomials of degree $\leqq n$. With $C[0, 1]$ denoting the space of all continuous real functions on $[0, 1]$, L_n the linear subspace of $C[0, 1]$ consisting of all polynomials of degree $\leqq n$, and with the distance between two functions ϕ and ψ given by $\sup\{|\phi(\alpha) - \psi(\alpha)| : \alpha \in [0, 1]\}$, Chebyshev's theorem asserts that each point of $C[0, 1]$ admits for each n a unique nearest point in L_n. Accordingly, a subset S of a metric space M is often called a *Chebyshev set* provided that each point of M admits a unique nearest point in S. (Such sets are also called *Motzkin sets*. See Valentine [7].) A negative answer to the title question is equivalent to an affirmative answer to the following:

(*) *Does every finite-dimensional normed linear space admit a one-dimensional Chebyshev subspace?*

If the answer to (*) is affirmative, an induction on d shows that any d-dimensional normed linear space E admits a system $L_1 \subset L_2 \subset \cdots \subset L_{d-1} \subset E$ of Chebyshev subspaces such that L_n is of dimension n (noted by B. Grünbaum and the author).

In closing, we note that a dual equivalent of (*) is concerned with norm-preserving extensions of linear functionals (see Phelps [6]), and that there are infinite-dimensional spaces in which the answer to (*) is negative (Phelps [6], Garkavi [3]).

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

STABILITY IN LINEAR SYSTEMS

D. B. HILL, University of Vermont

In teaching stability theory one often begins with a discussion of linear systems. While Bellman's lemma, [1], can be used to establish most of the theorems, it is believed that the following approach gives the students a deeper insight into the nature of the problem.

Consider the linear system

$$\dot{U}(t) = A(t)U(t), \qquad U(0) = I. \tag{1}$$

Here $U(t)$ and $A(t)$ are $n\times n$ matrices and I is the $n\times n$ identity matrix. The solution to equation (1) will be denoted by $S(A; t)$. The fundamental problem in linear stability theory can then be stated as: Find conditions on B in order that $S(A+B; t)=S(A; t)V(t)$ where $\lim_{t\to\infty} V(t)$ exists.

Let $G(t)$ be the $n\times n$ matrix defined by

$$G(t) = S(A; t)^{-1}B(t)S(A; t). \tag{2}$$

It is then trivial to verify that

$$S(A + B; t) = S(A; t)S(G; t). \tag{3}$$

Hence $S(A+B; t)$ will behave very much like $S(A; t)$ if $G(t)$ is "small" enough. The following theorem gives a condition on $G(t)$ which is sufficient to insure that $\lim_{t\to\infty} S(G; t)$ exists.

THEOREM. *Given $\dot{U}(t)=\{A(t)+B(t)\}U(t)$ where $B(t)$ is such that*

$$\int_0^\infty \|S(A; t)^{-1}B(t)S(A; t)\|dt = \int_0^\infty \|G(t)\|dt < \infty$$

($\|\ \|$ denotes any convenient matrix norm), then every fundamental matrix is of the form $S(A; t)S(G; t)$ where $\lim_{t\to\infty} S(G; t)$ exists.

Proof. From equation (3) and a well known theorem concerning solutions of

linear systems, $S(A+B;t)$ can be written

$$S(A+B;t) = S(A;t)\left\{I + \int_0^t G(\tau)d\tau + \int_0^t G(\tau)\int_0^\tau G(\sigma)d\sigma d\tau + \cdots\right\}$$
$$= S(A;t)\{I + W(t)\}.$$

We show that $\lim_{t\to\infty} W(t)$ exists. The series for $W(t)$, i.e.,

$$\left\{\int_0^t G(\tau)d\tau + \int_0^t G(\tau)\int_0^\tau G(\sigma)d\sigma d\tau + \cdots\right\}$$

is bounded term by term by the series for $[\exp \int_0^t \|G(\tau)\| d\tau] - 1$ which in turn is bounded by the series for $[\exp \int_0^\infty \|G(\tau)\| d\tau] - 1$. Hence, the series for $W(t)$ converges uniformly in $0 \leqq t < \infty$. It is clear that each term in the series for $W(t)$ tends to a limit as $t \to \infty$, since all the improper integrals converge absolutely. Now the following theorem holds, [2]:

Let $\sum_1^\infty W_n(x)$ *converge to* $f(x)$ *uniformly for all* x *with* $c \leqq x < \infty$. *Let* $\lim_{x\to\infty} W_n(x) = b_n$ *for* $n = 1, 2, \cdots$. *Then* $\sum_1^\infty b_n$ *converges and* $\lim_{x\to\infty} f(x) = \sum_1^\infty b_n$.

Applying this component-wise to the series for $W(t)$ establishes the theorem.

It is not difficult to show that the matrix, $\lim_{t\to\infty} S(G;t)$, which we shall denote by $S(G;\infty)$, is nonsingular if $\int_0^\infty \|G(\tau)\| d\tau < \infty$. Hence if V_0 is any prescribed matrix there exists an initial condition W_0, namely $W_0 = S(G;\infty)^{-1} V_0$, such that the solution to

$$\dot{U}(t) = \{A(t) + B(t)\} U(t), \qquad U(0) = W_0, \tag{4}$$

is $S(A;t)V(t)$, where $V(t) \to V_0$ as $t \to \infty$.

Many of the standard theorems concerning linear stability are rather immediate corollaries of this result. Questions concerning the stability of solutions to single second order equations can often be resolved by converting the equation to a first order system and applying the theorem.

References

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NOTE ON THE BETA AND GAMMA FUNCTIONS

T. S. NANJUNDIAH, University of Mysore, India

In N. Bourbaki, Éléments de Mathématique, Fonctions d'une variable réelle, Ch. 1, 2, 3, 2me éd., 1958, p. 127, one finds a simple and interesting method of evaluating the Euler-Poisson integral

$$\int_0^\infty e^{-x^2} dx$$

without the use of double integrals or an inversion of limit operations. We wish to point out that it can easily be generalized to yield the more inclusive result expressing the beta function in terms of the gamma function and the identity of Euler's and Gauss's definitions of the gamma function.

The gamma and the beta functions, defined for positive arguments by the Eulerian integrals

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \qquad B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx,$$

have values in $(0, \infty)$ and satisfy the functional equations

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad \Gamma(1) = 1, \tag{1}$$

$$B(\alpha, \beta+1) = \frac{\beta}{\alpha+\beta}B(\alpha, \beta), \qquad B(\alpha, 1) = \frac{1}{\alpha}\cdot \tag{2}$$

It follows from (2) that the identity

$$B(\alpha, \beta) = \frac{B(\alpha, \beta+n)B(\beta, n)}{B(\alpha+\beta, n)} \tag{*}$$

holds for $n=1$ and, by an easy induction on n, for every natural number n. We now employ the well-known inequality

$$e^x \geqq 1 + x$$

valid for every real x with equality only for $x=0$, and establish bounds above and below for $B(\alpha, t)$, $t>1$:

$$\frac{\Gamma(\alpha)}{(t-1)^\alpha} = \int_0^\infty x^{\alpha-1}e^{-(t-1)x}dx > \int_0^1 x^{\alpha-1}(1-x)^{t-1}dx = B(\alpha, t),$$

$$\frac{\Gamma(\alpha)}{(\alpha+t)^\alpha} = \int_0^\infty x^{\alpha-1}e^{-(\alpha+t)x}dx < \int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+t}}dx = B(\alpha, t).$$

Hence the asymptotic relation

$$B(\alpha, t) \sim \frac{\Gamma(\alpha)}{t^\alpha}, \qquad t \to \infty. \tag{**}$$

Letting $n \to \infty$ in (*) and utilizing (**), we obtain the expression for the beta function in terms of the gamma function:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot$$

This converts (**) into the familiar formula $\Gamma(\alpha+t) \sim t^\alpha\Gamma(t)$, $t \to \infty$, whose special case $t=n$ reduces by (1) to the limit

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n^{\alpha-1}n!}{\alpha(\alpha+1)\cdots(\alpha+n-1)},$$

identifying Euler's definition of the gamma function with Gauss's definition.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before August 31, 1969. Contributors (in North America) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2143 [1969, 82]. *Proposed by Peter Kornya, University of British Columbia*

Corrected Statement. In a triangle with sides a, b, c the line joining the centroid and the incenter is perpendicular to the bisector of the angle opposite side c. Show that the arithmetic mean of a, b, c equals the harmonic mean of a and b.

E 2165. *Proposed by J. L. Kazdan, University of Pennsylvania*

If $f\in C(R)$ can be uniformly approximated throughout R by polynomials, then f is itself a polynomial. Supply a proof, or a counterexample.

E 2166. *Proposed by Michael Stolnicki, Oakland Community College, Auburn Heights, Michigan*

Let a checkerboard consist of squares with sides of length 4. A regular $4n$-gon with radius 1 is tossed on the board. Determine the probability that the polygon will cross a line of the checkerboard.

E 2167. *Proposed by Steven Feigelstock, Polytechnic Institute of Brooklyn*

Which positive integers of the form p^n-1, p a prime, have all their divisors of the same form?

E 2168. *Proposed by Rosta János, Central Research Institute for Physics, Budapest, Hungary*

Let G denote a group and K a subset of G. Prove the following theorem: If for a given natural number $n \geqq 2$, the relations $K^{-1} \subseteq K$ and $K^n \subseteq K$ hold, then K^{n-1} is a subgroup of G and K^{n-2} is identical with some (e.g. left) coset of K^{n-1} in G. (Here K^i denotes as usual the set of all products of the form $k_1k_2 \cdots k_i$, with $k_j \in K$, $j=1, 2, \cdots, i$.)

E 2169. *Proposed by J. J. Hirstein, Illinois State University*

Let U be a unit circle. Let C_1 be a circle whose diameter is half the diameter of U. Recursively, let C_{k+1} be a circle whose diameter is half the diameter of C_k. Put C_1 tangent to U on the inside (at a_1). Put C_2 tangent to C_1 on the outside and tangent to U on the inside (at a_2). Continue in this manner, putting C_n tangent to C_{n-1} on the outside and tangent to U on the inside (at a_n). Let s_i be the arc length $a_i\, a_{i+1}$. Does $\sum_{i=1}^{\infty} s_i$ converge? If so, what is the limit?

E 2170. *Proposed by M. Slater, University of Bristol, England*

Let $\{a_n\}$ be an increasing sequence of positive reals. Suppose $\lim_{n\to\infty} a_n = \infty$. Show that $\sum_{n=1}^{\infty} \cos^{-1}(a_n/a_{n+1}) = \infty$.

E 2171. *Proposed by Kenneth Jackman, Federal Electric Corp., Fairbanks, Alaska*

Given N, what is the smallest W for which $B_1+B_2+ \cdots +B_c=W$, and $B_1B_2 \cdots B_c \geqq N$, with all B_k positive integers.

SOLUTIONS OF ELEMENTARY PROBLEMS

Tetrahedral Numbers

E 2076 [1968, 403]. *Proposed by Gregory Wulczyn, Bucknell University*

It has been proved that the only tetrahedral numbers $T_n = n(n+1)(n+2)/6$ which are perfect squares are T_1, T_2, T_{48}. Show that each of these tetrahedral numbers is the first term of an infinite series of tetrahedral numbers such that each partial sum is a square integer.

Solution by J. F. Golightly, Jacksonville University. For each $s \in I$,

$$s^2 + T_{6s^2-1} = s^2 + s^2(6s^2+1)(6s^2-1) = (6s^3)^2.$$

This identity provides a means for writing a series with the desired property. However, the series is not uniquely determined, e.g., $T_1+T_5=6^2$ and $T_1+T_8 = 11^2$.

Also solved by Terence Daniel, Robert Heller, D. C. B. Marsh, Norman Miller, Steven Minsker, E. S. Rosenthal, Chanchal Singh, P. D. Thomas, J. S. Vigder, Julius Vogel, and the proposer.

An n-leaved Rose Problem

E 2077 [1968, 403]. *Proposed by Alvin Hausner, City College, New York*

Consider the curves given in polar coordinates by $r^n = a^n \sin n\theta$, where a is a positive constant and $n = 1, 2, \cdots$. Let L_n, A_n denote the length of, and the area enclosed by, all the n leaves of these curves. Find expressions for $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} A_n$.

Solution by D. A. Hejhal, University of Chicago. We must look at locus $\{(r, \theta) \mid r = a(\sin n\theta)^{1/n},\ 2m\pi/n \leqq \theta \leqq (2m+1)\pi/n,\ m = 0, 1, \cdots, n-1\}$. Via a rotation, at once each of the n leaves is congruent to all the others. By the usual formula, then,

$$A_n = n \cdot \tfrac{1}{2} \int_0^{\pi/n} [a(\sin n\theta)^{1/n}]^2 d\theta = \tfrac{1}{2} a^2 \int_0^{\pi} (\sin \xi)^{2/n} d\xi, \qquad [\xi = n\theta].$$

Now, for large n,

$$\left| A_n - \tfrac{1}{2} a^2 \int_{\epsilon}^{\pi-\epsilon} (\sin \xi)^{2/n} d\xi \right| \leqq \tfrac{1}{2} a^2 \int_0^{\epsilon} d\xi + \tfrac{1}{2} a^2 \int_{\pi-\epsilon}^{\pi} d\xi \leqq a^2 \epsilon.$$

Let $n \to \infty$. Note $(\sin \xi)^{2/n} \to 1$ along $[\epsilon, \pi - \epsilon]$. Then, with $c = \lim\inf A_n$, $d = \lim\sup A_n$,

$$\left| c - \tfrac{1}{2} a^2 \int_{\epsilon}^{\pi-\epsilon} d\xi \right| \leqq a^2 \epsilon, \qquad \left| d - \tfrac{1}{2} a^2 \int_{\epsilon}^{\pi-\epsilon} d\xi \right| \leqq a^2 \epsilon.$$

Let $\epsilon \to 0^+$. Hence $c = d$ and $\lim_{n\to\infty} A_n = \frac{1}{2} a^2 \pi$.

Next, note that $r = a(\sin n\theta)^{1/n}$, $0 \leqq \theta \leqq \pi/n$, traces a curve (smooth) from $(0, 0)$ to $(a, \pi/2n)$ to $(0, 0)$. Hence its arc length is $\geqq 2a$. Then $L_n \geqq 2na$ and $\lim_{n\to\infty} L_n = \infty$.

We can improve this result by finding $\lim_{n\to\infty} L_n/n$. Note L_n/n is the arc length of one leaf. Thus, as usual,

$$\begin{aligned} L_n/n &= \int_0^{\pi/n} \sqrt{[a(\sin n\theta)^{1/n}]^2 + \left[a \frac{d}{d\theta} (\sin n\theta)^{1/n} \right]^2}\, d\theta \\ &= \int_0^{\pi/n} a\sqrt{(\sin n\theta)^{2/n} + (\sin n\theta)^{(2/n)-2} \cos^2 n\theta}\, d\theta \\ &= \frac{a}{n} \int_0^{\pi} (\sin \xi)^{(1/n)-1} d\xi \qquad (\xi = n\theta) \\ &= \frac{2a}{n} \int_0^{\pi/2} (\sin \xi)^{(1/n)-1} d\xi. \end{aligned}$$

Now

$$\left|\frac{L_n}{n} - 2a\right| \leqq \left|\frac{2a}{n}\int_\delta^{\pi/2} (\sin \xi)^{(1/n)-1} d\xi\right| + 2a\left|\frac{1}{n}\int_0^\delta (\sin \xi)^{(1/n)-1} d\xi - 1\right|.$$

Let $\epsilon(>0)$ be arbitrarily chosen and keep $\delta(>0)$ so small that $0 \leqq x \leqq \delta$ implies $\sin x = (1-\rho(x))x$, $0 \leqq \rho(x) \leqq \epsilon$. Hence

$$(\sin x)^{(1/n)-1} = (1-\rho)^{(1/n)-1} x^{(1/n)-1}, \qquad 0 \leqq x \leqq \delta.$$

As $n \to \infty$, and for ϵ sufficiently small

$$\left| (1-\rho)^{(1/n)-1} - 1 \right| \leqq 2\epsilon.$$

Hence,

$$\begin{aligned}\left|\frac{1}{n}\int_0^\delta (\sin x)^{(1/n)-1} dx - 1\right| &\leqq 1 - (1-2\epsilon)\int_0^\delta \frac{1}{n} x^{(1/n)-1} dx \\ &= 1 - (1-2\epsilon)\delta^{1/n}.\end{aligned}$$

Let $n \to \infty$. Then let δ (and ϵ)$\to 0^+$. At once,

$$\left|\frac{L_n}{n} - 2a\right| < \epsilon \qquad \text{for all } n \text{ sufficiently large.}$$

That is $\lim_{n\to\infty} L_n/n = 2a$.

Also solved by M. J. Brown, Michael Goldberg, David Gootkind, D. C. B. Marsh, Simeon Reich (Israel), P. A. Scheinok, and the proposer.

Totatives

E 2078 [1968, 403]. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

Let n be an integer $\geqq 210$. Prove or disprove: there are always at least 5 positive composite integers less than and relatively prime to n.

Solution by Heiko Harborth, Braunschweig, Germany. Let p_k be the kth prime number. Then $n_k = 2\cdot 3 \cdots p_k$ is the smallest number with k different prime divisors. p_{k+1}^2, $p_{k+1}p_{k+2}$, p_{k+2}^2, $p_{k+1}p_{k+3}$, $p_{k+2}p_{k+3}$ are five numbers being composite and prime to every n with exactly k different prime divisors, if one in addition replaces p_{k+1}, p_{k+2} or p_{k+3} by a prime $\leqq p_k$ not dividing n, in case one of these three primes divides n. So the assertion is proved, if the greatest of the five composite numbers $p_{k+2}p_{k+3}$ is smaller than n_k for every $k \geqq 4$. By Bertrand's postulate there is at least one prime number between every $m > 1$ and $2m$, so that

$$p_{k+2}p_{k+3} < 2p_{k+1}2p_{k+2} < 64p_{k-1}p_k < 2\cdot 3\cdot 5\cdot 7 \cdots p_{k-1}p_k = n_k.$$

The last inequality holds for $k \geqq 6$. For $k=5$ one has $17\cdot 19 < n_5 = 2\cdot 3\cdot 5\cdot 7\cdot 11 = 2310$. The five numbers for $k=4$ are 11^2, $11\cdot 13$, 13^2, $11\cdot 17$, $11\cdot 19$. In addition one sees easily that the assertion holds also for $n \geqq 121$.

Also solved by J. C. Abad, Neal Felsinger, Emil Grosswald, Donald Jeffords, C. G. Khatri & A. M. Vaidya (India), Eric Langford, Norman Miller, Frances Poley, Simeon Reich (Israel), and A. Zujus.

A Summation Problem

E 2079 [1968, 403]. *Proposed by D. S. Levine, Harvard University*

Let n be any integer $\geqq 2$. Prove that $\sum(1/pq)=1/2$, where the summation is over all integers p, q which satisfy $0<p<q\leqq n$, $p+q>n$, $(p, q)=1$.

Solution by D. M. Bloom, Brooklyn College. Let $f(n)$ be the given sum. The summands which appear in $f(n)$ but not in $f(n-1)$ are those of the form $a_p=1/pn$ where $1\leqq p<n$, $(p, n)=1$; the summands in $f(n-1)$ but not in $f(n)$ are those of the form $b_p=1/p(n-p)$ where $1\leqq p<n-p$, $(p, n-p)=1$ (equivalently, $(p, n)=1$). Hence summing only over values of p such that $(p, n)=1$, we have

$$(*)\qquad f(n)-f(n-1)=\sum_{p<n} a_p-\sum_{2p<n} b_p=\sum_{2p<n}(a_p+a_{n-p}-b_p).$$

But $a_p+a_{n-p}-b_p=0$; hence $f(n)=f(n-1)$ for all $n\geqq 3$, and the result follows. (Note: for $n\geqq 3$, there is no term in (*) when $p=n/2$, since $(p, n)=1$.)

Also solved by Bruce Berndt, L. Carlitz, David Carlson, G. J. Ford, E. J. Freebrook, David Fried & Daniel Shapiro, M. G. Greening (Australia), Emil Grosswald, Heiko Harborth (Germany), D. A. Hejhal, R. A. Jacobson, Donald Jeffords, B. Litov, J. A. Long, D. C. B. Marsh, Norman Miller, G. B. Parrish, Simeon Reich (Israel), Klaus Steffen (Germany), A. M. Vaidya (India), J. W. Wilson, S. B. Wineberg, A. Zujus, and the proposer.

The Product of Units in $I/(n)$

E 2080 [1968, 404]. *Proposed by B. D. Wick, San Diego State College*

Prove: The product of the units of the ring $I/(n)$ is -1 if and only if the units form a cyclic group under multiplication.

Solution by Bernard Jacobson, Franklin and Marshall College. Let G_n be the group of units of $I/(n)$ under multiplication and P_n be the product of the elements of G_n. We apply the following known results.

THEOREM 1. *G_n is cyclic if and only if $n=2$, 4, p^a or $2p^a$ where p is an odd prime.* (W. J. LeVeque, *Topics in Number Theory*, vol. I, Theorem 4.11.)

THEOREM 2. *If $n>2$, then $\phi(n)$ is even.*

THEOREM 3. *If G is a finite cyclic group of even order, then G contains exactly one element of order 2.*

THEOREM 4. *Let G be a finite abelian group and P be the product of the elements of G. If G has one element y of order 2, then $P=y$, and P is the identity otherwise. (I. N. Herstein, Topics in Algebra, p. 80.)*

If $n=1$ or 2, the proposition is seen to be true by inspection. If $n>2$ and G_n is cyclic, then $P_n=-1$ by Theorems 1–4. If G_n is not cyclic then $n=p^am$, where

p is an odd prime, $a \geqq 1$, $m > 2$ and $(p^a, m) = 1$ or $n = 2^b$ where $b \geqq 3$. If $n = p^a m$, there exists x, by the Chinese remainder theorem, such that $x \equiv 1 \bmod p^a$ and $x \equiv -1 \bmod m$. It follows that $x^2 \equiv 1 \bmod n$ and $x \not\equiv \pm 1 \bmod n$. If $n = 2^b$, then $(2^{b-1}-1)^2 \equiv 1 \bmod n$ and $2^{b-1} \not\equiv \pm 1 \bmod n$. In either case G_n has more than one element of order 2 and $P_n = 1$ by Theorem 4.

Also solved by Anders Bager (Denmark), David Carlson, H. M. Edgar, M. G. Greening (Australia), Robert Gilmer, Donald Jeffords, Douglas Lind, Simeon Reich (Israel), V. V. Subrahmanyasastri (India), and the proposer.

Umbugio's Vanishing Triangle

E 2081 [1968, 404]. *Proposed by Leon Bankoff, Los Angeles, California*

With characteristic tenacity, Professor E. P. B. Umbugio has been struggling to solve the following paraphrased version of Problem 9, page 201 of Hobson's *Plane and Advanced Trigonometry* (Dover reprint):

"If the orthocenter H, the incenter I, and the circumcenter O of a triangle ABC are the vertices of an equilateral triangle, show that $\cos A + \cos B + \cos C = 3/2$."

Help terminate the professor's futile floundering by showing that (a) the triangle HIO can never be equilateral and (b) when $\cos A + \cos B + \cos C = 3/2$, the triangle HIO is nonexistent.

Solution by Simeon Reich, Israel Institute of Technology.

(a) Were HIO equilateral, we would have

$$(1) \qquad OI^2 = OH^2, \qquad\qquad (2) \qquad OI^2 = IH^2.$$

Let R, ρ, r denote the circumradius of ABC, the inradius of ABC, and the inradius of the pedal triangle $H_1H_2H_3$, respectively. It is known (Johnson, *Advanced Euclidean Geometry*, p. 205) that

$$OI^2 = R^2 - 2R\rho, \qquad IH^2 = 2\rho^2 - 2Rr, \qquad OH^2 = R^2 - 4Rr.$$

Therefore (1) implies $2r = \rho$ and (2) implies $R^2 = 2R\rho + 2\rho^2 - R\rho = R\rho + 2\rho^2$. But $\rho \leqq R/2$ with equality if and only if ABC is equilateral. Hence

$$R\rho + 2\rho^2 \leqq R^2/2 + 2R^2/4 = R^2.$$

Thus (2) shows that ABC is equilateral. In this case HIO is not a triangle at all.

(b) Since $\sum \cos A = 1 + \rho/R$, we have $\rho = R/2$. That is, ABC is equilateral and $H \equiv I \equiv O$.

Also solved by A. N. Aheart, L. Carlitz, Ragnar Dybvik (Norway), Michael Goldberg, J. F. Golightly, M. G. Greening (Australia), K. R. S. Sastry (Ethiopia), R. E. Shafer, P. D. Thomas, C. S. Venkataraman (India), Gregory Wulczyn, and the proposer.

Similar Matrices over a Field of Characteristic p

E 2082 [1968, 404]. *Proposed by Ih-Ching Hsu, Fordham University*

F is a field of characteristic p $(p \neq 0)$; T is a $p \times p$ nonsingular matrix over F.

Prove that the only matrix of the form λT $(\lambda \in F)$ similar to T is T itself.

Solution by Geoffrey Kandall, Boston University. More generally, we suppose that T is of size $p^n \times p^n$. If λT is similar to T, $\lambda T = P^{-1}TP$ for a suitable matrix P. Taking determinants:

$$\lambda^{p^n} |T| = |T|,$$

and hence $\lambda^{p^n} = 1$ since $|T| \neq 0$. Thus $(\lambda - 1)^{p^n} = \lambda^{p^n} - 1 = 0$, so $\lambda = 1$.

Also solved by L. Carlitz, David Carlson, M. G. Greening (Australia), D. A. Hejhal, J. V. Michalowicz, W. G. Roughead, Jr., Klaus Steffen (Germany), R. C. Thompson, and the proposer.

A Ballot Problem

E 2083 [1968, 404]. *Proposed by Erwin Just, Bronx Community College, New York*

Find the total number of ways of arranging in a row the $2n$ integers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ with the restriction that for each i, a_i precede b_i, a_i precede a_{i+1} and b_i precede b_{i+1}.

Solution by Einar Andresen, University of Oslo, Norway. There are $\binom{2n}{n}$ arrangements with a_i preceding a_{i+1}, b_i preceding b_{i+1}, namely, the number of ways of choosing n elements among $2n$ different elements. First we choose where to put the n a's, then we index the a's and the b's. Among these choices, we will denote as *permissible* all those with a_i preceding b_i.

Let $ip(n, j)$ be the number of nonpermissible arrangements with b_1 in the jth place. We have obviously $ip(n, 1) = \binom{2n-1}{n-1}$. We shall prove by induction,

$$ip(n,j) = \binom{2n-j}{n-j}. \tag{1}$$

When $n=1$, (1) is correct. ($\binom{a}{b} \neq 0$ only when $a \geqq b \geqq 0$.) Suppose that (1) is correct when $n=k$. If we put b_1 in the jth place, there are exactly as many impermissible arrangements as if we had only $2n-2$ integers to arrange in a row, and we had put a's in the first $j-2$ places and had laid no restriction on the arrangement of the rest. Hence, if $j \geqq 2$, $n \geqq 2$,

$$ip(n,j) = \sum_{i=j-1}^{n-1} ip(n-1, i).$$

By the induction hypothesis,

$$ip(k+1,j) = \sum_{i=j-1}^{k} \binom{2k-i}{k-i} = \sum_{p=0}^{p+1-j} \binom{k+p}{p} = \binom{2(k+1)-j}{k+1-j},$$

which completes the proof of (1). (If $j=1$, the formula is already established.)

Let $ip(n)$ denote the total number of nonpermissible arrangements. Then $ip(n) = ip(n+1, 2) = \binom{2n}{n-1}$. Let $p(n)$ denote the number of permissible arrangements. We have

$$p(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

This is the desired result.

Also solved by J. C. and P. R. Abad, W. D. Bouwsma, L. Carlitz, Ted Cullen, R. D. Fray, Michael Goodman, David Gootkind, M. G. Greening (Australia), R. E. Greenwood, Heiko Harborth (Germany), Steve Hartman, D. A. Hejhal, Eric Langford, Dan Marcus, Steven Minsker, C. B. A. Peck, Pierre Robillard, R. E. Shafer, Klaus Steffen (Germany), and D. P. Sumner.

Editorial Note. This problem is equivalent to E 2054 [1969, 192] and is one of the class of ballot problems discussed by W. Feller (*An Introduction to Probability Theory and Its Applications*, 2nd ed., vol. 1, pp. 70–71), and earlier by others. Numbers of the form $\binom{2n}{n}/(n+1)$ are Catalan numbers and, according to R. E. Greenwood, were studied by Cayley in 1859.

A Goode Identity

E 2084 [1968, 404]. *Proposed by J. J. Goode, Georgia Institute of Technology*

For all choices of integers k, n, $0 \leqq k \leqq n$, prove the identity

$$\sum_{i=k}^{n} \binom{i}{k} \frac{1}{n+1-i} = \binom{n+1}{k} \sum_{i=k}^{n} \frac{1}{i+1}.$$

Solution by David Zeitlin, Minneapolis, Minnesota. The proposed identity is trivially true for $k \geqq n+1$. Assuming the identity is true for some $n > 0$, we have

$$\begin{aligned}
\binom{n+2}{k} \sum_{i=k}^{n+1} \frac{1}{i+1} &= \left\{\binom{n+1}{k-1} + \binom{n+1}{k}\right\} \sum_{i=k}^{n+1} \frac{1}{i+1} \\
&= \binom{n+1}{k} \sum_{i=k}^{n} \frac{1}{i+1} + \binom{n+1}{k-1} \sum_{i=k-1}^{n} \frac{1}{i+1} \\
&= \sum_{i=k}^{n} \binom{i}{k} \frac{1}{n+1-i} + \sum_{i=k-1}^{n} \binom{i}{k-1} \frac{1}{n+1-i} \\
&= \sum_{i=k}^{n+1} \binom{i}{k} \frac{1}{n+2-i},
\end{aligned}$$

and the result follows now by mathematical induction.

Also solved by Einar Andresen (Norway), M. T. L. Bizley (England), Leonard Carlitz, David Gootkind, H. W. Gould, M. G. Greening (Australia), D. A. Hejhal, Robert Heller, B. W. King, J. A. Long, W. D. Markel, D. C. B. Marsh, Norman Miller, G. B. Parrish, K. R. Penrose, Simeon Reich (Israel), R. E. Shafer, Klaus Steffen (Germany), and the proposer.

A Trio of Constant Ratios

E 2085 [1968, 541]. *Proposed by J. H. Butchart, Northern Arizona University*

Find the unifying feature and prove the following three statements:

(a) The area bounded by two normals of one arch of a cycloid, the curve itself, and its evolute is divided in the ratio of 1:3 by the line on which the circle rolls.

(b) The area bounded by two normals to the right half of the catenary $y=\cosh x$, the x-axis, and the evolute is divided in the ratio 1:3 by the catenary.

(c) The area bounded by two normals to the upper half of the parabola $y^2=4ax$, the directrix, and the evolute is divided in the ratio 4:5 by the parabola.

Solution by Michael Goldberg, Washington, D. C. The unifying feature of the three statements is the fact that the point N, which traces a straight line locus, divides the normal radius of curvature into portions in a constant ratio.

(a) If P is a point on the cycloid and C is its center of curvature, then N, which is the midpoint of PC, lies on the base of the cycloid (E. H. Lockwood, *A Book of Curves*, 1961, p. 83). Hence, an infinitesimal shift of the normal will make a triangle, with C as its vertex, whose two long sides are bisected by the base of the cycloid. Hence the area of this infinitesimal triangle is divided in the ratio 1:3. In general, this ratio holds for any two normals.

(b) The point P on the catenary $y=\cosh x$ is the midpoint of the line CN, where N is on the x-axis. This follows from the fact that the slope of the normal is $-1/\sinh x=\tan\theta$, $\cos\theta=1/\cosh x$, and the radius of curvature is $\rho=y^2$ (Lockwood, p. 123). Then the ordinate Y of N is given by

$$Y = y - \rho\cos\theta = y - \rho/\cosh x = y - y^2/y = 0.$$

Again, the area of the infinitesimal triangle with vertex C is divided into two parts of ratio 1:3.

(c) The point P on the parabola $y^2=4ax$ divides the normal CP into lengths so that $PC/PN=2$. This follows from the fact that the slope of the normal is $-y/2a=\tan\theta$, $\cos\theta=2a/(y^2+4a^2)^{1/2}$, and $\rho=(y^2+4a^2)^{3/2}/4a^2$. Hence the abscissa X of N is given by

$$\begin{aligned} X &= x - (\rho\cos\theta)/2 = x - (y^2+4a^2)^{3/2}/4a(y^2+4a^2)^{1/2} \\ &= x - (4ax+4a^2)/4a = -a. \end{aligned}$$

Hence, the locus of N is the directrix of the parabola. Since the long sides of the infinitesimal triangle whose vertex is C, are divided into parts having the ratio 2:1, the area of the small part of the triangle is $(2/3)^2$ of the whole triangle. Hence, the ratio of the areas of the parts of the triangle is $4/9$: $(1-4/9)=4:5$.

Also solved by J. M. Quoniam (France), L. E. Ward, Sr., and the proposer.

An Increasing Function

E 2086 [1968, 542]. *Proposed by Beatriz Margolis, Fundacion Bariloche, Buenos Aires, Argentina*

Show that the following is an increasing function:

$$f(t) = \ln\frac{t+2}{t+1}\bigg/\ln\frac{t+1}{t} \qquad (t>1).$$

Solution by D. A. Hejhal, University of Chicago. Keep $t>1$.

$$f'(t)=\frac{\left(\ln\frac{t+1}{t}\right)\left[\frac{1}{t+2}-\frac{1}{t+1}\right]-\left(\ln\frac{t+2}{t+1}\right)\left[\frac{1}{t+1}-\frac{1}{t}\right]}{(\ln(t+1)-\ln t)^2}.$$

It suffices to show

$$\left(\ln\frac{t+1}{t}\right)\left[\frac{1}{t+2}-\frac{1}{t+1}\right]>\left(\ln\frac{t+2}{t+1}\right)\left[\frac{1}{t+1}-\frac{1}{t}\right] \tag{1}$$

which is equivalent in turn to (2), (3) and (4).

$$\left(\ln\frac{t+1}{t}\right)\frac{-1}{(t+1)(t+2)}>\left(\ln\frac{t+2}{t+1}\right)\frac{-1}{(t+1)t}, \tag{2}$$

$$\left(\ln\frac{t+1}{t}\right)\frac{1}{(t+1)(t+2)}<\left(\ln\frac{t+2}{t+1}\right)\frac{1}{(t+1)t}, \tag{3}$$

$$\left(\ln\frac{t+1}{t}\right)\frac{1}{t+2}<\left(\ln\frac{t+2}{t+1}\right)\frac{1}{t}. \tag{4}$$

Now $\ln(t+2)-\ln(t+1)=\int_{t+1}^{t+2}dx/x>1(t+2)$. Thus

$$\left(\ln\frac{t+2}{t+1}\right)\frac{1}{t}>\frac{1}{t+2}\,\frac{1}{t}. \tag{5}$$

Next

$$\left(\ln\frac{t+1}{t}\right)\frac{1}{t+2}=\frac{1}{t+2}\int_t^{t+1}\frac{dx}{x}<\frac{1}{t+2}\,\frac{1}{t}. \tag{6}$$

From (5) and (6), (4) now follows.

Also solved by seventy-six other readers, many of whom noted that the hypothesis "$t>1$" is redundant.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before September 30, 1969. Contributors (in North America) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

Correction. 5642 [1968, 1125] *Proposed by Raymond Redheffer, University of California at Los Angeles.*

If x is real, show that $(\sin \pi x)/\pi x \geqq (1-x^2)/(1+x^2)$.

5665. *Proposed by Peter Ungar, New York University*

A student claimed that an arbitrary set of measure zero can for any $\epsilon > 0$, be covered by a family of intervals $I_1, I_2, \cdots$ such that the length of I_n is $< \epsilon/2^n$. Is this true?

5666. *Proposed by Alexandru Lupaş, Cluj, Romania*

Let Φ be a linear space of real valued functions defined on a set X, assume that $1 \in \Phi$. Let F be a positive linear functional on Φ such that $F(1) = 1$. For $0 < m \leqq f_i \leqq M$, $i = 1, 2$, prove the inequality

$$F\left(\frac{1}{mf_1 + Mf_2}\right) F\left(\frac{f_1 f_2}{mf_1 + Mf_2}\right) \leqq \frac{1}{4mM}$$

with equality if and only if $f_1 = M, f_2 = m$.

5667. *Proposed by Ray Glenn, Asheville-Biltmore College*

Prove that a function f which has a finite limit at each point of the closed interval $[a, b]$ is Riemann integrable on $[a, b]$.

5668. *Proposed by P. M. Perdew, University of Hawaii*

Prove the following generalization of the Theorem of the Primitive Element. Let $K = F(\alpha, \beta)$ where α and β are algebraic over F. Denote by r and s the multiplicities of α and β, and set $t = \min(r, s)$. Then $(t, \text{char}(F)) = 1$ implies K/F is a simple extension.

5669. *Proposed by M. E. Harris, University of Illinois at Chicago*

A *left Ore domain* R is an associative ring without zero divisors such that the intersection of any two nonzero left ideals is nonzero.

Prove: if R is a left Ore domain then so is $R[x]$ (the polynomial ring in one variable over R). How about the formal power series case?

5670. *Proposed by Stephan Silverman, University of British Columbia*

Let $M = \{A \times B \mid A$ and B are subsets of the reals$\}$.

Then is the σ-algebra generated by M all subsets of the plane?

SOLUTIONS OF ADVANCED PROBLEMS

Sums of Divisors of Certain Integers

5590 [1968, 551]. *Proposed by E. C. Milner and A. Oppenheim, University of Reading, England*

For any integer $n \geqq 1$, let $f(n)$ denote the largest integer m such that $\sigma(m) = \sum_{d|m} d \leqq n$. Prove that, for fixed $k \geqq 1$, the equation $n - f(n) = k$ has infinitely many solutions.

Solution by R. L. Vogt, University of Nebraska. Fix $k \geqq 1$. For any integer

$n \geqq 1$, let $g(n)$ denote the largest prime integer p_n such that $p_n < (k+n)!+2$. Clearly $g(n)$ takes on an infinite number of distinct values and all but finitely many of them are greater than k^2. For such n, $\sigma(g(n)) = g(n)+1 \leqq g(n)+k$, while for $g(n) < m \leqq g(n)+k$, m is composite, so $\sigma(m) > m+\sqrt{m} > g(n)+k$. Thus, for such $n, f(g(n)+k) = g(n)$, so $g(n)+k-f(g(n)+k) = k$.

Also solved by Einar Andresen (Norway), L. Carlitz, G. J. Foschini, A. S. Fraenkel (Israel), M. G. Greening (Australia), Emil Grosswald, Heiko Harborth (Germany), R. Kochendörffer (Tasmania), J. L. Mishra (India), S. Srinivasan (India), C. F. Stephens, Jr., A. M. Vaidya (India), J. H. van Lint (Netherlands), M. B. Villarino, W. J. Woan, R. L. Zarling, and the proposers.

A Bound for Euler's Totient

5591 [1968, 552]. *Proposed by A. Oppenheim, University of Reading, England*

Suppose that $n > 1$, $\sigma(n)$ is the sum of the divisors of n, $\phi(n)$ the totient of n (Euler's function). Prove the inequality

$$\phi\left(n\left[\frac{\sigma(n)}{n}\right]\right) < n.$$

Solution by Neal Felsinger, Yale University. First note that $\phi(ab) \leqq b\phi(a)$ whenever $a, b \geqq 1$. This follows from the formula $\phi(n) = n\prod(1-1/p)$ where the product is over all primes which divide n. Using the bound $\phi(n)\sigma(n) < n^2$, it follows at once that

$$\phi\left(n\left[\frac{\sigma(n)}{n}\right]\right) \leqq \phi(n)\left[\frac{\sigma(n)}{n}\right] \leqq \phi(n)\,\frac{\sigma(n)}{n} < n.$$

Also solved by U. Annapurna (India), M. A. Bershad, L. Carlitz, D. L. Carlson, P. A. Catlin, M. A. Ettrick, A. S. Fraenkel (Israel), M. G. Greening (Australia), Emil Grosswald, Heiko Harborth (Germany), D. A. Hejhal, C. G. Khatri & A. M. Vaidya (India), J. L. Mishra (India), Ivan Niven, R. C. Orr, Simeon Reich (Israel), S. Srinivasan (India), C. F. Stephens, Jr., R. H.Toliver, J. H. van Lint (Netherlands), C. S. Venkataraman (India), M. B. Villarino, R. L. Vogt, and the proposer.

Niven and the proposer note that the given bound holds for $\phi(jn)$ for all $j \leqq \sigma(n)/n$.

Weak Convergence in L

5592 [1968, 552]. *Proposed by W. M. Myers and D. R. Arterburn, University of Montana*

Let E be a measurable subset of the line and $\{f_n\}$ a sequence of functions summable on E with the property that $\int_F f_n \to 0$ for all measurable subsets F of E. Prove or disprove that $\int_E |f_n| \to 0$.

Solution by D. A. Hejhal, University of Chicago. It does not follow that $\int |f_n| \to 0$ as may be seen by setting $f_n(x) = \sin nx$, $E = [0, 2\pi]$ and using the Riemann-Lebesgue lemma. Using the characteristic function of the measurable set F we have $\int_F \sin nx\, dx \to 0$ but $\int_E |\sin nx|\, dx = 2$.

Also solved by Arlen Brown, E. D. J. Buckley & R. S. C. Wong, J. T. Burnham, P. R. Chernoff, the Echols

Mathematics Club of the University of Virginia, N. A. Fava, N. J. Fine, G. J. Foschini, Steve Hartman, D. A. Herrero, R. A. Horn, J. L. Leonard & Stoddart Smith, Jr., O. P. Lossers (Netherlands), Dan Marcus, D. A. Mavinkurve (India), Bernard McCabe, Frank Meyer, Hugh Noland, J. C. Morgan II, Nicholas Passell, N. T. Peck, Charles Riley, J. T. Rosenbaum, Ron Weger, and the proposers.

Many solvers used the well-known Rademacher functions to establish the result. Brown and McCabe observed that the problem constitutes an example of a Banach space in which weak convergence to zero does not imply convergence in the strong topology.

Uniqueness Problem for Scalar Multiplication

5594 [1968, 552]. *Proposed by W. G. Dotson, Jr., North Carolina State University*

Suppose F is a field with a rational subfield R. Suppose V is an Abelian group and that $\cdot$ and $*$ are functions from $F\times V$ into V which satisfy the usual scalar multiplication axioms. (1) Show that if F is a prime field, then $\cdot$ and $*$ coincide. (2) Give an example in which $\cdot$ and $*$ do not coincide.

Solution by Charles Riley, Keene (N. H.) State College. $F=R$, since F is prime. $1\cdot\alpha=\alpha=1*\alpha$. If $n\cdot\alpha=n*\alpha$, then

$$(n+1)\cdot\alpha = n\cdot\alpha + 1\cdot\alpha = n*\alpha + 1*\alpha = (n+1)*\alpha,$$

so that $n\cdot\alpha=n*\alpha$ for each positive integer n. If n is a positive integer,

$$(-n)\cdot\alpha = ((-1)n)\cdot\alpha = (-1)\cdot(n\cdot\alpha) = -(n\cdot\alpha).$$

Similarly, $(-n)*\alpha=-(n*\alpha)$. Thus $n\cdot\alpha=n*\alpha$ for all integers n. For integers n, m with $m\neq 0$,

$$n*\alpha = n\cdot\alpha = m\cdot\left(\frac{n}{m}\cdot\alpha\right) = m*\left(\frac{n}{m}\cdot\alpha\right)$$

and

$$\frac{n}{m}*\alpha = \frac{1}{m}*(n*\alpha) = \frac{1}{m}*\left(m*\left(\frac{n}{m}\cdot\alpha\right)\right) = \left(\frac{1}{m}m\right)*\left(\frac{n}{m}\cdot\alpha\right) = \frac{n}{m}\cdot\alpha.$$

Thus $\cdot$ and $*$ are the same operation.

If F is the field of complex numbers, and V the additive group of complex numbers, define functions on $F\times V$ by $z\cdot w=zw$, $z*w=\bar{z}w$. These operations satisfy the scalar multiplication axioms.

Also solved by P. R. Chernoff, R. Kochendörffer (Tasmania), Kwangil Koh, Miguel L. Laplaza (Puerto Rico), W. G. McArthur, and the proposer.

Maximum Modulus of Derivative of an Entire Function

5595 [1968, 552]. *Proposed by Fred Gross, Bell Communications, Inc., New York*

Let $f(z)$ be an entire function and let $M(r)$ and $M'(r)$ denote the maximum

modulus function of f and f' respectively. If for some $\sigma > 0$, $M(r) < M'(r) = e^{\sigma r}$ for sufficiently large r, then $\sigma > e^{-1}$.

Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands. If $|z| = r$ is sufficiently large and C is the circle $|w - z| = e$, then

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z)^2}\, dw \right| < \frac{M(r + e)}{e},$$

i.e., $e^{\sigma r} < e^{\sigma(r+e)-1}$, which implies $\sigma > e^{-1}$.

Also solved by R. Goldstein (England), and the proposer.

Analytic Periodic Functions

5596 [1968, 552]. *Proposed by J. D. Dixon, University of New South Wales, Australia*

Let f be a real-valued function defined on a real interval I. We shall call f *quasi-periodic* on I if, for each $x \in I$, there exists a rational number $r \neq 0$, depending on x, such that $x + r \in I$ and $f(x+r) = f(x)$. Prove that if f is analytic and quasi-periodic on I, then f is periodic on I.

Is this result still true if we define quasi-periodic as above except that we permit r to be irrational?

Solution by M. L. J. Hautus, Technological University, Eindhoven, Netherlands. Since the rationals are countable, there exists a rational number r such that $f(x+r) = f(x)$ holds for uncountably many numbers x in I. The set of x's must have an accumulation point in I, and since f is analytic the relation $f(x+r) = f(x)$ must hold for all x. Therefore f is periodic.

If r is allowed to be irrational the result does not hold, e.g., for $f(x) = x^2$ we have $f(x - 2x) = f(x)$.

Also solved by P. R. Chernoff, M. A. Ettrick, G. J. Foschini, D. A. Hejhal, M. L. Laplaza (Puerto Rico), M. A. Mavinkurve (India), Nicholas Passell, Charles Riley, and the proposer.

Simultaneous Pythagorean Relations

5597 [1968, 552]. *Proposed by Joseph Arkin, Suffern, New York.*

If t, x, y, u, v and w are nonzero distinct integers, solve the simultaneous Diophantine equations

$$v^2 + w^2 = (u^2 - t^2)(y^2 + x^2)$$

$$v^2 - w^2 = (u^2 + t^2)(y^2 - x^2).$$

Solution by D. A. Klarner, McMaster University, Canada, and the Technical University, Eindhoven, Netherlands. Combining the two equations in an obvious way gives rise to the equivalent system

$$(*) \qquad v^2 + t^2x^2 = u^2y^2, \qquad w^2 + t^2y^2 = u^2x^2.$$

Every solution of $A^2+B^2=C^2$ is given by $A=a^2-b^2$, $B=2ab$, $C=a^2+b^2$, or $A=2ab$, $B=a^2-b^2$, $C=a^2+b^2$, for arbitrary integers a and b. Thus, we can solve the two equations in the system (*) separately in terms of four parameters giving rise to four cases, but we will consider only one:

$$(**)\qquad \begin{aligned} v &= 2a_1b_1, & tx &= a_1^2 - b_1^2, & uy &= a_1^2 + b_1^2 \\ w &= 2a_2b_2, & ty &= a_2^2 - b_2^2, & ux &= a_2^2 + b_2^2. \end{aligned}$$

Solving (**) for u, x, y in terms of a_1, a_2, b_1, b_2 and t, we find that

$$u = (a_2^2 + b_2^2)t/(a_1^2 + b_1^2), \qquad x = (a_1^2 - b_1^2)/t, \qquad y = (a_1^4 - b_1^4)/(a_2^2 + b_2^2)t$$

and a_1, b_1, a_2, b_2 must satisfy

$$(***)\qquad a_2^4 + b_1^4 = a_1^4 + b_2^4.$$

Thus a complete solution of the system (*) depends on finding a complete solution of (***), but this problem has remained unsolved since the time of Euler. However, parametric solutions of (***) are known and these solutions can be used to find integers a_1, a_2, b_1, b_2 satisfying (***), then we select $t\neq 0$ and obtain rational numbers u, v, t, w, x and y which satisfy (*). Multiplying through the relations by the square of the least common multiple of t, $a_1^2+b_1^2$, and $a_2^2+b_2^2$ gives a solution in integers.

Also solved by Marcia Ascher, Merrill Barnebey, C. V. Bitterli, John Leech, Edna M. Pratt, Hugo Sun, and the proposer.

Discontinuities of a Real Function

5598 [1968, 552]. *Proposed by S. J. Metz, Fort MacArthur, California*

Does there exist a real function whose set of discontinuities has measure zero but has an uncountable intersection with every open interval?

I. *Solution by B. A. Fusaro, Queens College, N. C.* We can obtain such a function. Let A_1 denote the Cantor set formed by the successive deletion of middle thirds of a given interval I; let the rejected set of intervals be denoted by B_1. Let A_2 denote the union of Cantor sets constructed in each of the intervals of set B_1; let the new set of rejected intervals be denoted by B_2. Construct A_3, A_4, $\cdots$ in a similar manner and put $A=\bigcup A_i$. Each A_i is an uncountable set of measure zero and therefore so is A. Moreover, every subinterval (a, b) of I includes at least one of the constructed Cantor sets and hence its intersection with A is an uncountable set. Now put $f(x)=2^{-n}$ if $x\in A_n$, and put $f(x)=0$ if $x\notin A$ (cf. Gelbaum and Olmstead, *Counterexamples in Analysis*, pp. 30, 44). Evidently f is discontinuous on A. If $x\notin A$, and given an $\epsilon>0$, choose an N such that $2^{-N}<\epsilon$; then choose a neighborhood of x that excludes $A_1, A_2, \cdots, A_N$ and in which, therefore, $f(x)<2^{-N}<\epsilon$.

II. *Solution by J. G. Mauldon, Amherst College.* For each real number x, let

$N=N(x)$ denote the number of zeros and nines after the decimal point in the ordinary decimal representation of x, so that $N=\infty$ for terminating decimals. Now define $f(x)=10^{-N(x)}$. Then the discontinuities of f are precisely the points x at which $N(x)$ is finite, and this set has measure zero but has an uncountable intersection with every open interval.

Also solved by Einar Andresen (Norway), Arlen Brown, Sung-cheng Chang (Taiwan), Theodore Chang, P. R. Chernoff, L. D. Crowson, Jerry Fischer, G. J. Foschini, R. Goldstein (England), M. L. J. Hautus (Netherlands), Tony Haworth, D. A. Hejhal, J. L. Leonard, M. A. Mavinkurve (India), Frank Meyer, Steven Minsker, J. C. Morgan II, Hugh Noland, E. A. Nordgren, Warren Page, Nicholas Passell, Stephen Puckette, Daniel Putnam, Charles Riley, A. C. Segal, W. J. Woan, and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY, University of Toronto

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Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital C in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.

Evolution of Mathematical Concepts: An Elementary Study. By R. L. Wilder. Wiley, New York, 1968. xiii+224 pp. $8.00. (Telegraphic Review, Feb. 1969)

The status of mathematics and its development inside a given culture and its variation from culture to culture is still little studied, and we can therefore welcome this book by the emeritus professor of the University of Michigan. He confines himself to two concepts, number and geometry. Mathematics, he writes, did not develop independently of cultural forces (some peculiar to its own nature), any more than did physics, art, or other cultural components. The treatment is therefore rather from the cultural than from the psychological point of view; and indeed, although we know something about the way of thinking of some particular modern mathematicians such as Poincaré or Hadamard, most of the development of number and geometry has been anonymous or semianonymous (what do we know of Euclid as a person?). But we do know something about the culture, say, of ancient Babylonia, ancient Egypt, or Renaissance Europe, and we can try to place mathematics inside these cultures.

The author addresses himself to a wide audience, so that he does a good deal of explaining about early and present number systems and numerals, Euclidean and non-Euclidean geometry, and illustrates it with pretty pictures. The book will therefore be of great interest to teachers of mathematics, especially in secondary instruction, but also to historians and cultural anthropologists. As a matter of fact, a whole section is devoted to the explanation of the approach

and terminology of the cultural anthropologists, referring, for example, to the *Anthropology* of A. L. Kroeber (N. Y., 1948). The principal forces of mathematical evolution, according to Professor Wilder, can then be classified as follows: environmental stress (physical, cultural), hereditary stress, symbolization, diffusion, abstraction, generalization, consolidation, diversification, cultural lag, cultural resistance (why not also cultural impetus?) and selection (p. 169). He gives examples of all these factors. Diffusion, for instance, took place when ancient oriental mathematics met Greek philosophy, consolidation took place when algebra and geometry met in analytic geometry, selection when various alternative concepts evolved, all directed toward the same mathematical objective, but with the eventual survival of one or a few (as recently in vector analysis, or in the past, when the present decimal system with the symbols $0, 1, 2, \cdots, 9$ emerged). There are also stages through which a concept passes, as in the case of the evolution of numbers from the primitive one-two differentiation through tallying, numeral systems, etc. to the present logical definitions and axiomatics (p. 100). The reader can follow it all in detail and learn not only some good history but also some good mathematics.

I myself have once tried, on a more modest scale, but not confining myself to number and geometry, to analyze the principal elements of mathematical evolution. It is in the chapter "Mathematics" of the book with the somewhat ambitious title *Philosophy of the Future* (New York, 1949), and I have found it pleasant to compare my analysis with that of Professor Wilder. In what he calls environmental stress I have laid more emphasis on socio-economic and political factors. Some of the trends I have specified, as Hankel's permanence of formal laws, may well find place under Wilder's "generalization," another, as "passing from the discrete to the continuous," under his "abstraction." But we both agree on what Wilder writes (p. 13): "Although the humanistic aspects of mathematics may be more important from the standpoint of the individual mathematician, it is as a basic science that mathematics functions within our culture—as witness its support together with the other sciences by the National Science Foundation and other agencies."

All in all, a very useful and well written book, recommended to all who like to know what mathematicians have done in the past and why they did it—with a lesson for the present.

D. J. STRUIK, Massachusetts Institute of Technology.

C *Introduction to Complex Analysis*. Rev. ed. By Zeev Nehari. Allyn and Bacon, Boston, 1968. xi+272 pp. $8.50. (Telegraphic Review, Nov. 1968).

The response to the book by the reviewer's students was very warm and enthusiastic. They found it readable and the examples solvable. Some of the better students commented that definitions and theorems were not stated in a very rigorous manner. The author sometimes introduces a new concept by informal observations and remarks and then states that a theorem has been proved in these remarks.

The book is just long enough to be covered in a one semester class meeting for three hours a week. The presentation is very lucid, the level of rigor is not very high, and the notations are well chosen. It is a welcome addition to the present literature and I recommend it for a first course in complex analysis.

NARENDRA SHETH, State Univ. College at Oswego

C *Fundamental Concepts of Analysis.* By Alton H. Smith and Walter A. Albrecht, Jr. Prentice-Hall, Englewood Cliffs, N. J., 1966. 190 pp. $6.95.

This short, concise book is very well written. Its purpose, according to the authors' preface, is, ". . . to provide a text which is both rigorous and thoroughly understandable, dealing with the *fundamental concepts* (reviewer's italics) of analysis. We have considered it important to ease the difficulty that many students experience in making the transition from problem solving to theory." The authors have been extremely successful in providing just such a text.

The book is addressed to students with some background in calculus. The reviewer, for example, used portions of this book in an honors calculus course and found it to be admirably suited for this purpose.

Chapter 1, "Introductory Set Theory," considers, among other things, operations with sets, Venn diagrams, and some simple set-theoretic theorems. Chapter 2 presents a thorough study of the real numbers, including the field axioms, theorems such as $(-1)(-1)=1$, the axioms of order, and completeness. In Chapter 3, Cartesian products and binary relations are used in giving a rigorous discussion of functions. Cardinality, the composition of functions, and monotone functions are also included. In Chapter 4, metric spaces are introduced, along with neighborhoods, interior points and boundary points of a set, open and closed sets, and limit points of a set. Next, Cauchy and monotone sequences are studied, and the Bolzano-Weierstrass Theorem and the theorem that every Cauchy sequence in E^n converges are proved. Thus, the theory of limits for sequences is developed before the theory of limits of functions is considered.

The last two chapters, "Continuity and Differentiation" and "Integration," present a careful treatment of these subjects. It is proved, for example, that the inverse image of an open set is open under a continuous function, that if $f: E^n \to E^1$ is continuous on a closed and bounded subset A of E^n, then f assumes its maximum and minimum on A, and that a function continuous on a closed and bounded subset of E^n is uniformly continuous there. Also, the mean value and Cauchy mean value theorems are proved. The chapter ends with a brief introduction to partial differentiation. In Chapter 6, the theory of the integral of a step function is developed, and this is then used to study the integral of a bounded function. The fundamental theorem of calculus and the theorem concerned with a change of variable in an integral are, for example, proved. The book concludes with an introduction to integrals in R^2.

The book contains a generous number of examples and exercises, the latter including many proofs. Answers and hints to some of the exercises are provided. The book has a good index but no bibliography.

This book is modern in its approach. Its strongest point is its proofs; they are models of conciseness and clarity. At a time when many such books tend to be monolithic treatises weighing in somewhere around ten pounds, this trim little book comes as a welcome relief. But therein lies its major fault. In keeping it small, the authors have omitted many topics whose inclusion would have made the book considerably more suitable for, say, an honors calculus course or an introductory advanced calculus course. For example, the book contains no mention of Taylor's Theorem or integration by parts. Infinite series are not studied, nor are sequences of functions or uniform convergence. The authors should be encouraged to write a second volume containing some of these topics.

C. A. GROBE, JR., Bowdoin College

TELEGRAPHIC REVIEWS

The following abbreviations indicate possible uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)—18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Analysis

Entire Functions and Related Parts of Analysis. Edited by Jacob Korevaar, S. S. Chern, Leon Ehrenpreis, W. H. J. Fuchs, and L. A. Rubel. Proceedings of Symposia in Pure Mathematics. Vol. XI. AMS, Providence, 1968. vi+554 pp. $12.00. Although an outgrowth of the Society's summer research institute, this is not a full proceedings but a collection of articles by participants, or stimulated by discussion at the research institute. The description of the institute in the preface suggests that it would be well worthwhile to publish also some of the expository presentations that were made to inform participants of recent developments. P, L.

An Introduction to Linear Difference Equations. By Paul M. Batchelder. Dover, New York, 1967. xi+209 pp. $2.00 (paper). This is a reprint of the original published by Harvard University Press in 1927. It treats with some thoroughness first order linear difference equations and second order equations with constant coefficients, presupposing some knowledge of complex analysis and making use of 20th century methods. T, S, P.

Analytic Functions of a Complex Variable. By David Raymond Curtiss. Dover, New York, 1967. ix+173 pp. $1.75 (paper). A reprint of the Carus Monograph No. 2, published in 1926 by Open Court for the MAA. A good candidate for inexpensive supplementary reading in calculus and analysis courses. S, P.

★*Some Problems in Real and Complex Analysis.* By John E. Littlewood (Cambridge Univ.). Heath, Boston, 1968. viii+57 pp. $7.95. This is the first formal publication of a list of unsolved problems mimeographed by Littlewood for his students. It realizes one of the purposes of the *Heath Mathematical Monographs* by helping incipient researchers to get started. It is also precious as a heuristic document, of which there are very few in mathematics. As D. V. Widder, the editor of the monographs, says in his preface, "any glimpse into the mind of a master can only be esteemed." Incidentally, in a footnote Littlewood informs us that he believes the Riemann hypothesis to be false! S, P, L.

Fourier and Laplace Transforms. By P. D. Robinson (Univ. of York). Routledge & Kegan Paul Ltd., London; Dover, New York, 1968. vii+96 pp. $1.25. (Distributed

in Canada by General Publishing Co. Ltd., Toronto). Basic theory and applications to linear differential equations and linear integral equations. There is in an appendix a sketch proof of Fourier's integral formula, references and tables of transforms. There are exercises with answers and an index. This is the latest member of the *Library of Mathematics* series edited by Walter Ledermann of the University of Sussex. Because of their quality, convenience, and reasonable price they should be seriously considered as supplementary material in appropriate courses. Other titles in series (all about the same size and at the same price) are: Sequences and Series, Elementary Differential Equations and Operators, Partial Derivatives, Complex Numbers, Principles of Dynamics, Electrical and Mechanical Oscillations, Vibrating Systems, Vibrating Strings, Fourier Series, Solutions of Laplace's Equation, Numerical Approximation, Differential Geometry, and Multiple Integrals. S.

Introduction to Analysis. By Maxwell Rosenlicht (Univ. of California at Berkeley)· Scott, Foresman, Glenview, Illinois, 1968. 254 pp. $10.75. "The object is to redo calculus correctly in a setting of sufficient generality to provide a reasonable foundation for advanced work . . . " Topics include set theory, the real number system, metric space, continuous functions, differentiation, Riemann integration, interchange of limit operations, successive approximation, partial differentiation, and multiple Riemann integrals. T (15).

Applications

Mathematics of Finance. 3rd ed. By Robert Cissell and Helen Cissell. Houghton Mifflin, Boston, 1969. xviii + 346+89 pp. $7.50. "New material is largely additional explanations with many examples and diagrams." The tables are printed on blue paper. T (13).

Nonlinear Differential Equations of Chemically Reacting Systems. By George R. Gavalas. Springer Tracts in Natural Philosophy, Vol. 17. Springer Verlag, New York, 1968. viii+106 pp. $8.50. Main topics are *a priori* bounds and existence, uniqueness, stability, and asymptotic behavior. Fixed point methods are used. S, P.

Priority Queues. By N. K. Jaiswal. Academic, New York, 1968. xiii+240 pp. $12.00. An exposition of recent work on this special aspect of queuing theory, in which "priority" refers to the rules according to which units in the queue are selected and serviced. There is a bibliography of 111 items. P.

Mathematical Models of Arms Control and Disarmament. Application of Mathematical Structures in Politics. By Thomas L. Saaty (U. S. Arms Control and Disarmament Agency). Publications in Operations Research of the Operations Research Society of America, No. 14. Wiley, New York, 1968. ix+190 pp. $10.95. The analysis proceeds with emphasis on qualitative models and deals with such problems as consistency of objectives, stability of policies, effectiveness of enforcement actions, and the growth of conflicts. Its publication might be a significant event in the invasion of political science and history by mathematical methods. P, L.

Fields and/or Particles. By D. K. Sen (Univ. of Toronto). Academic Press, New York and Ryerson Press, Toronto, 1968. x+139 pp. $7.50. A brief survey of fundamental theories of physics, from a mathematical viewpoint whose underlying theme is the problem of field-particle duality. P.

Calculus

An Introduction to Calculus. By Robert G. Bartle (Univ. of Illinois) and C. Ionescu Tulcea (Northwestern Univ.). Scott, Foresman, Glenview, Illinois, 1969. ix+290 pp. $8.50. Designed for a short course presenting the "basic ideas of calculus and its ap-

plication" without sacrificing mathematical rectitude, this book begins with 48 pages of prerequisites and ends with a discussion of infinite series. T (13).

Solutions of Ordinary Linear Differential Equations with Constant Coefficients. By Everard M. Williams (Carnegie-Mellon Univ.) and Asok K. Mukhopadhyay (Jet Propulsion Lab., Pasadena). Wiley, New York, 1968. xvii+134 pp. $5.95. This is a programmed book intended for self instruction designed to achieve skill. It is notable in the field of programmed books in several ways. It is the first program above the level of elementary calculus. Its objectives are carefully described in a good preface. There is a detailed table of contents describing each portion of the program. There are closing remarks and a detailed index, so that users can redo appropriate portions of the program. There is a test of minimal mathematical prerequisites, but there is no final test covering the entire program. The cost is high per page, and it would appear possible to achieve the results by a carefully designed problem set taking much less space. By writing answers on separate sheets, the students could use the book and recover part of their expense on the used book market. Even so, if very many topics during a year were to be dealt with in this way, the cost would be prohibitive. I invite correspondence from professors who may use this as an auxilliary. S.

Some Exercises in Pure Mathematics, with Expository Comments. By J. D. Weston and H. J. Godwin (both of Univ. College of Swansea, Wales). Cambridge Univ. Press, New York, 1968. vii+136 pp. $1.95 (paper). Designed to help the student bridge the gap between manipulative and traditional mathematics and the more rigorous modern approach to sets, the number system, and the basic ideas of calculus, this book contains two hundred exercises followed by fifty-eight pages of solutions and expositions, a list of topics, an essay on elementary analysis, and some selected definitions. This should be very useful. T (13), S.

Computers etc.

Computer Evaluation of Mathematical Functions. By C. T. Fike (IBM Systems Research Inst.). Prentice-Hall, Englewood Cliffs, N.J. 1968. xii+227 pp. $10.50. Topics include error, square and cube root, reducing the argument range, polynomials, series, rational approximation, and asymptotic expansions. There are bibliographies at the ends of chapters. T, P.

A First Course in ALGOL 60. By Eric Foxley and Henry R. Neave, assisted by Matthew C. Grayshon (all of Univ. of Nottingham). Edited by Peter Lambert. Addison-Wesley, Reading, 1968. viii+246 pp. $4.95 (paper). This is a programmed book with questions on the right hand page and answers on the back. About one half the available paper is used. T, S.

Education

Numbers and Such: A Lively Guide to the New Math for Parents and Other Perplexed Adults. By A. N. Feldzamen. Illustrations by Richard Erdoes. Prentice-Hall, Englewood Cliffs, N.J. 1968. 294 pp. $6.95. A sprightly and amusingly illustrated *tour de force.* TT, S.

Mathematical Spectrum. A Magazine of Contemporary Mathematics. Published by Oxford Univ. Press. Managing Editor: J. Gani, Hicks Building, The University, Sheffield S3 7RH, England. Vol. I. 1968–1969. 64 pp. Annual subscription $1.20 ($1.00 in bulk). This is a magazine "for the instruction and entertainment of student mathematicians in schools, colleges and universities. It is published by Oxford University Press on behalf of the Applied Probability Trust, a nonprofit making organization established in 1963 with the support of the London Mathematical Society." Articles

of an expository and historical character relating to mathematics and its applications in the broadest sense are to be included, as well as student research and information on educational opportunities and careers. There will be also a problem section. The editorial committee and advisory board is a distinguished group from many parts of the English speaking world. Its American participants are R. L. Ackoff, P. R. Halmos, and G. Polya. This is a promising venture which might become the much needed international journal for students of mathematics. P, L.

General

Elements of Mathematics. 3rd ed. By J. Houston Banks. Allyn and Bacon, Boston, 1969. x+470 pp. $8.50. Subjects are sets, logic, number systems, measurements, statistics, geometric systems and mathematical functions. The new edition gives more space to set theory, defines cardinal numbers as equivalence classes, gives more space to probability, and has sections on convex sets and linear programming. Each chapter is preceded by questions to stimulate interest. The first one is as follows; "Do you know: Two sets cannot be equal unless they are really two names for the same set?"

The Surprise Attack in Mathematical Problems. By L. A. Graham. Dover, New York, 1968. vii+125 pp. $1.75 (paper). By "surprise attack" the author means "the unexpected approach that not only brings simplicity to the solution but often broadens the scope of the problem and adds an esoteric touch dear to the mathematician's heart." The problems are all from elementary algebra and geometry, but some are interesting. S.

Mathematics for the Million. By Lancelot Hogben. 4th ed., revised, reset, and reillustrated. Norton, New York, 1968. 660 pp. $8.95. This is the 42nd printing of a book whose first edition appeared in 1937. It is not a reliable history of mathematics, or an accurate picture of mathematics today or even yesterday, but it is Hogben and that means interesting and provocative.

The Mind of a Mnemonist. A Little Book About a Vast Memory. By A. R. Luria. Translated from the Russian by Lynn Solotaroff with a foreword by Jerome S. Bruner. Basic Books, New York, 1968. xi+160 pp. $4.95. This fascinating book about a man with a pathologically "good" memory, contains many passages that will intrigue mathematicians, especially those passages describing the manner in which the mnemonist solves simple mathematical problems. P.

Dialogues on Mathematics. By Alfred Renyi (Hungarian Acad. of Sciences). Holden-Day, San Francisco, 1967. 100 pp. $4.95 (cloth), $2.50 (paper). Three dialogues, the first between Socrates and Hippocrates on the nature of mathematics, a second between Archimedes and Hieron on applications of mathematics, and a third on the same topic between Galileo and Torricelli. There is no preface, or rather the preface is disguised as a postscript. S, P, L.

Geometry and Topology

Éléments de Géométrie Algébrique. By A. Grothendieck. Edited with the collaboration of J. Dieudonné. IV. *Étude Locale des Schémas et des Morphismes de Schémas. Publications Mathématiques*, No. 32 Institut des Hautes Études Scientifiques, France. Distributed outside of France by W. A. Benjamin, New York, 1967. 361 pp. $15.75.

Uniform Spaces and Transformation Groups. By Hidegoro Nakano (Wayne State Univ.). Wayne State University Press, Detroit, 1968. xv+253 pp. $14.50. This is an exposition of the author's original generalization of the theory of topological groups to transformation groups on uniform spaces. P.

Topology. By Horst Schubert (Univ. of Kiel). Translated from German by Siegfried Moran (Univ. of Kent at Canterbury). Allyn & Bacon, Boston, 1968. 358 pp. \$11.65. Topological spaces, uniform spaces, homotopy, singular homology, and an appendix on fundamental concepts of set theory. The original was published in 1964. T (16–17).

Einführung in die Graphentheorie. By Jiri Selacek. Translated from the original Czech. Teubner, Leipzig, 1968. 171 pp. 6.40 M. (paper). An elementary, but up-to-date treatment, originally published in 1964 and translated into Bulgarian in 1967. S, P.

Logic and Foundations

★*Elements of Mathematics. Theory of Sets.* By Nicolas Bourbaki. Hermann, Paris, and Addison-Wesley, Reading, 1968. viii+414 pp. \$18.50. This is a translation of Éléments de Mathématique, Théorie des Ensembles, the first and basic part of the Bourbaki series. It is the second part to be translated, the first being volume three on general topology, and it will be followed by translations of the rest of the Bourbaki series. Chapter headings are Description of formal mathematics, Theory of sets, Ordered sets, Cardinals, Integers, and Structures. The summary of results, indexes, list of axioms and schemes, and the substantial historical notes are included. S, P, L. (!)

The Art of Philosophizing and Other Essays. By Bertrand Russell. Philosophical Library, New York, 1968. 119 pp. \$3.95. This little book contains three essays none of which have the title "The Art of Philosophizing." The third essay, "The Art of Reckoning," is on mathematics, and there are scattered references to mathematics throughout. The book is a shoddy publishing job with an inadequate preface and no information as to where the apparently reprinted essays first appeared. S, P.

Probability and Statistics

Normal Centroids, Medians and Scores for Ordinal Data. By F. N. David, D. E. Barton, S. Ganeshalingham, H. L. Harter, P. J. Kim, M. Merrington, D. Walley. Cambridge University Press, New York, 1968. 201 pp. \$7.00. This is volume 29 of the series *Tracts for Computers*, P, L.

Stochastic Convergence. By Eugene Lukacs (Catholic Univ. of America). Heath, Boston, Mass., 1968. viii+142 pp. \$7.95. The subject is "the simplest kind of infinite families" of random variables, "namely random sequences." S, P.

Statistics: Uncertainty and Behavior. By I. Richard Savage (Florida State Univ.). Houghton Mifflin, Boston, 1968. xiii+344 pp. \$8.95. An elementary introduction not presupposing calculus and emphasizing the "behavior viewpoint" rather than the mathematical. The approach is Bayesian. Chapter headings are Probability concepts, Bayes's Theorem, Random variables and expectations, Two states of nature, Statistics without probabilities for states of nature, Statistical problems, and Normal probability function. T (13).

The Future of Statistics. Proceedings of a Conference on the Future of Statistics Held at the Univ. of Wisconsin, Madison, Wisconsin, June 1967. Edited by Donald G. Watts. (Univ. of Wisconsin). Academic, New York, 1968. xvi+315 pp. \$12.50. There are three dedications ceremony talks (by A. G. Oettinger, J. W. Tukey and G. A. Barnard), three theme papers (relating to bio-medical sciences, engineering, and economics), two panel discussions (on departments of statistics and on statistical inference), and seven technical papers. A very interesting volume, but it seems strange that in attempting to analyze the future of statistics, the participants did not feel it necessary to look at the past, which experience shows is often helpful in such considerations. P, L.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Rufus Oldenburger, Director of the School of Mechanical Engineering at Purdue University, was the first recipient of the Rufus Oldenburger Award, established by the American Society of Mechanical Engineers, in recognition of his outstanding service to the field of Automatic Control. He also received a certificate for being the "most Honored Member of the Automatic Control Division."

University of Colorado: Assistant Professor Neville Williams, University of West Florida, has been appointed Assistant Professor; Assistant Professors George Clements, W. B. Jones and Richard Roth have been promoted to Associate Professors; Associate Professor Burnett Meyer has been promoted to Professor.

Idaho State University: Dr. S. G. Crossley, Texas Technological College, has been appointed Assistant Professor; Associate Professor John Hilzman has been promoted to Professor.

Illinois State University: Dr. S. R. Clemens, University of North Carolina at Chapel Hill, has been appointed Assistant Professor; Associate Professor L. C. Eggan, Pacific Lutheran University, has been appointed Associate Professor; Dr. T. W. Laetsch, California Institute of Technology, has been appointed Assistant Professor; Assistant Professor P. G. O'Daffer, Ball State University, has been appointed Associate Professor.

Northwestern University: Dr. R. O. Hamel, University of Oregon, has been appointed Assistant Professor; Associate Professor Marvin Shinbrot has been promoted to Professor.

Rutgers, The State University: Dr. Glen Bredon, University of California at Berkeley, has been appointed Professor; Dr. Bertram Walsh, UCLA, has been appointed Associate Professor.

Assistant Professor David Ballard, Adrian College, has been appointed Assistant Professor at Albion College.

Assistant Professor P. M. Curran, Fordham University, has been promoted to Associate Professor.

Professor R. A. Dobyns, McNeese State College, has been appointed Professor and Chairman of the Mathematics Department at Georgetown College.

Dr. E. W. Ellers, Flinders University of South Australia, has been appointed Associate Professor at the University of New Brunswick.

Assistant Professor Harold Heie, King's College, has been promoted to Associate Professor.

Assistant Professor H. R. Lutz, Asbury College, has been appointed Associate Professor and Chairman of the Mathematics Department at Azusa Pacific College.

Assistant Professor J. E. Morrill, DePauw University, has been promoted to Associate Professor.

Dr. R. A. Whiteman, IIT Research Institute, has been promoted to the position of Scientific Advisor with the Electromagnetic Compatibility Analysis Center, Annapolis, Maryland.

Assistant Professor G. K. Williams, University of Notre Dame, has been appointed Associate Professor at Southwestern at Memphis.

Mr. Jesse Williams, Washington Township High School, has been appointed Assistant Professor at Cheyney State College.

Dr. Vernon Zander, Catholic University of America, has been appointed Assistant Professor at West Georgia College.

CANADIAN MATHEMATICAL CONGRESS—TWENTY-THIRD SUMMER MEETING

The twenty-third summer meeting of the Canadian Mathematical Congress, a Symposium on Graduate Training of Mathematics Teachers, will be held at Sir George Williams University, Montreal, on June 5 and 6, 1969. Symposium addresses by Professors Z. P. Dienes, K. E. Iverson, K. May, C. Moser, G. Paquette, G. Polya, S. Schuster and W. W. Sawyer have been arranged; a panel discussion on the theme of the symposium will take place on June 6. The Jeffery-Williams Lecture, "Whither Statistics and Probability," will be delivered by Professor R. Pyke, of the University of Washington. Members of the Congress are earnestly invited to support the meeting by their attendance. Mathematicians in industry and teachers of mathematics are cordially invited to attend whether or not they are members.

1969 IEEE INTERNATIONAL SYMPOSIUM ON CIRCUIT THEORY

The 1969 IEEE International Symposium on Circuit Theory will be held at Mark Hopkins Hotel, San Francisco, California on December 8–10, 1969. This annual symposium features the presentation of original research papers and invited papers by distinguished researchers from universities and industry, and provides a forum for discussion of topics in circuit and system theory. The theme which is to be continued to the second symposium is that of the work at the interface between theory and practice. Of course high quality papers from throughout the broad spectrum of the field are to be welcome as always. Deadline for submission of papers is July 1, 1969. The Technical Program Chairman will notify authors of accepted symposium papers by October 1, 1969. All manuscripts are to be submitted directly to the editor, IEEE Transactions on Circuit Theory: Professor B. J. Leon, School of Electrical Engineering, Cornell University, Ithaca, New York 14850.

EIGHTH SYMPOSIUM OF THE NATIONAL GAMING COUNCIL

The Eighth Symposium of the National Gaming Council will be held on June 23 and June 24, 1969. Booz, Allen Applied Research, Inc., will host the meeting at the Sheraton-Elms Hotel, Excelsior Springs, Missouri (25 miles Northeast of Kansas City). Invited and contributed papers will be presented in the areas of business, military and social gaming. A special feature for interested participants will be a tour of the USACDC Institute of Combined Arms and Support War Game Facility at Fort Leavenworth. The National Gaming Council is an informal organization of individuals interested in gaming. Participants in previous symposia have included members from the business, research, university, government and military communities. For further information contact: Dr. Richard L. Crawford, Booz, Allen Applied Research, Inc., 911 Walnut Street, Kansas City, Missouri 64106.

SUMMER INSTITUTES FOR JUNIOR COLLEGE TEACHERS OF MATHEMATICS

In 1967 Professor Leonard Gillman, Chairman of the CUPM Panel on College Teacher Preparation, and Dr. D. L. Thomsen, Jr., Chairman of the MAA Committee on Institutes, made a strong plea to the mathematical community to submit many more proposals for summer institutes for college teachers of mathematics. The Association's Committee on Institutes is repeating this request. In particular, they are asking mathematicians to submit proposals for institutes for junior college teachers to be held in the summer of 1970. Many teachers of calculus in the junior colleges urgently require continuing education in real analysis and linear algebra.

Both the National Science Foundation and the Office of Education of the U. S. Department of Health, Education and Welfare conduct programs to train and re-train teachers for junior colleges. The NSF deadline for submitting applications for 1970 institutes is June 1. Guides for preparing NSF proposals may be obtained from their College Teacher Program Division. The HEW deadline for applications is September 1.

V. O. McBrien, Chairman, Committee on Institutes

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FIFTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The Fifty-second Annual Meeting of the Mathematical Association of America was held at the Jung Hotel, New Orleans, Louisiana, from Saturday to Monday, January 25 to 27, 1969, in conjunction with the Annual Meeting of the American Mathematical Society and meetings of the National Council of Teachers of Mathematics, the Association for Symbolic Logic, and Mu Alpha Theta. The sessions of the Association on Saturday and Sunday were joint sessions with the National Council of Teachers of Mathematics. There were registered 4811 persons, including 2328 members of the Association.

Sessions of the Association were held on Saturday morning, on Sunday morning, and on Monday morning and afternoon in the Grand Ballroom. Presiding officers were Dr. J. H. Hlavaty, President of the National Council of Teachers of Mathematics, for the first hour on Saturday morning, and Professor Dorothy L. Bernstein for the remainder of the session on Saturday morning, Mrs. Sarah Greenholz on Sunday morning for the lecture by Professor Hardgrove, and Professor G. S. Young for the lecture by Professor Greenberg, Professor J. R. Dorroh on Monday morning, and Professor R. L. Pendleton on Monday afternoon. The Program Committee for the meeting consisted of R. D. Anderson, Chairman; A. H. Clifford, J. R. Dorroh, Mrs. Sarah Greenholz, R. L. Pendleton, and F. D. Quigley.

FIRST SESSION OF THE ASSOCIATION

Joint Session with the National Council of Teachers of Mathematics

Report on the Joint Committee of the American Statistical Association and NCTM, by Professor Frederick Mosteller, Harvard University.

The Joint Committee on the Curriculum in Probability and Statistics has been at work for one year. It plans two books. The first, a source book in statistics and probability, offers many real world examples, together with solutions. Although not a textbook, topics in the source book could be adapted by textbook writers and teachers as units in a curriculum. Illustrative examples were given of exploring data for structure through graphical and tabular devices. The second is a book of readings on applications of probability and statistics to problems of importance to the individual and the nation. The purpose of the latter work, to be written for the educated layman, is to make clear the broad range of uses of these fields, rather than to teach probability and statistics. The schedule calls for completion of both works within three years. The Committee members are W. H. Kruskal, R. F. Link, R. S. Pieters, G. R. Rising, and Frederick Mosteller, Chairman.

Reactions to the Report, by Professor G. R. Rising, SUNY at Buffalo.

The speaker discussed the concerns and goals of the NCTM in connection with the work of the Joint Committee.

Panel Discussion: Secondary School Preparation of Students of Freshman Calculus.

A panel discussion with Professor A. B. Clarke, Western Michigan University, Dr. W. E. Ferguson, Newton High School, Professor C. B. Allendoerfer, University of Washington, and Mr. W. K. McNabb, St. Mark's School of Texas.

Professor Clarke expressed the feeling that experienced college teachers continue to have a rather negative opinion concerning the preparation of the majority of their *average* entering students. This opinion merits serious consideration. To get full benefit from "modern" programs of high school mathematics, the question of *feasibility* for both students and teachers deserves as much attention as the mathematical correctness of the programs. As an example of such a feasibility emphasis, the mathematical community in Michigan has almost unanimously recommended the introduction of a year of analytic geometry in the senior year of most high schools, instead of topics in modern mathematics or calculus.

Dr. Ferguson noted that schools that have a modern kindergarten to grade twelve mathematics program prepare students for several different kinds of freshman calculus courses. Some students take calculus in high school, and the College Entrance Examination Board has a BC and an AB advanced placement calculus program designed for these students. Much more mathematics is being done in high school today.

The material in the courses once called college algebra, trigonometry, and analytic geometry is now scattered through the grades 7–12 mathematics program. Analytic geometry is now a high school subject for the best, capable mathematics students. There are now available courses in computers, matrix algebra, probability and statistics.

During the last thirteen (13) years many schools have shown that calculus is a bona fide high school subject for well-qualified students. The preparation given to students for college freshman calculus is the same as that given to the juniors or seniors taking calculus in high school. In the senior year the students get what might be called pre-calculus topics: logarithmic and exponential functions, circular and trigonometric functions, polynomial functions, complex numbers, analytic geometry (one, two, and some three dimensional), polar coordinates, some algebra of vectors, mathematical induction, sequences and series, limits, continuity, slopes, and an introduction to the derivative and some of its applications.

Professor Allendoerfer emphasized that the most serious difficulty with the high school preparation of students for calculus is not with the high school courses themselves but with the fact that students enter the colleges without having taken the available high school courses. Proper preparation for calculus requires four years of high school mathematics, but many students appear in college with two or three years of this and still expect the colleges to teach them calculus. Proper advising of college bound students by high school counselors is essential.

The high school course should not include calculus except for students who have completed the four years of precalculus by the end of the 11th grade. Then an Advanced Placement type course is appropriate. Emphasis should be placed on the ideas of calculus, the standard manipulations, and applications. The presentation of serious rigor should be postponed to a second calculus course, and then given only to those students with the requisite mathematical talent.

Mr. McNabb observed that, from personal knowledge of a number of independent schools and applications for Mu Alpha Theta chapters, two general pre-calculus courses seem to appear most commonly in this type of school: elementary functions and analytic geometry. Other courses frequently involve special topics directly related to calculus.

The Advanced Placement Program of the College Entrance Examination Board affects the pre-calculus program of many of these schools by forcing work from grade 12 to grade 11, and even grade 10 in some instances. With the new AB and BC sequences, it has been estimated that the 12,000-odd papers of last year may double this year.

SECOND SESSION OF THE ASSOCIATION

Joint Session with the National Council of Teachers of Mathematics

A Further Look at Teacher Training, by Professor Clarence Ethel Hardgrove, Northern Illinois University.

Some of the major issues facing the Teacher Training Panel of CUPM, as it reconsiders recommendations for the pre-service mathematics education of teachers for Levels I, II, III and IV, were identified. For example, should the Panel recommend a course or courses in synthetic geometry *or* a combination of synthetic geometry and transformation geometry *or* some other combination. Some of the issues identified were those discussed at the June 1968 conference on trends in school mathematics and their implications for teacher training.

Annual Business Meeting of the Association; the Association's Eighth Award for Distinguished Service to Mathematics.

Computing and the Mathematics Teacher, by Professor H. J. Greenberg, University of Denver.

The anticipated effect in the next few years of the use of computers and computer related

material in the mathematics classrooms was surveyed with particular reference to the demands on the teacher. For grades K–12, changes both in content and classroom procedure were described with specific reference to new texts and curricula. Based on these, recommendations were made for teacher training.

THIRD SESSION OF THE ASSOCIATION

Session on the Theory of Distributions

An Introduction to Distributions, by Professor John Horvath, University of Maryland.

The distributions of Laurent Schwartz were introduced to generalize the concept of Radon measure, i.e., of mass or charge distributions. They are defined as continuous linear forms on the vector space of all infinitely differentiable functions with compact support, equipped with an appropriate topology. Many operations, usually performed on functions, can be extended to distributions, e.g., differentiation, multiplication by a function, convolution and Fourier transform. In particular, the concept of a fundamental solution of a partial differential operator with constant coefficients can be defined precisely, and the calculations with "Dirac's delta-function" and its derivatives can be given a rigorous basis.

Connections between Distributions and Boundary Values of Analytic Functions with Applications, by Professor H. J. Bremerman, University of California, Berkeley.

For every distribution T in $(\mathfrak{D}')$ there exists an analytic representation $f(z)$, analytic in $\prod_{j=1}^{n} (\mathfrak{C}_j - \mathfrak{R}_j)$, such that if $f^*(z) = f(z_1, \cdots, z_n) - f(\bar{z}_1, \cdots, z_n) + \cdots + (-1)^n f(\bar{z}_1, \cdots, \bar{z}_n)$, then $f^*(x_1 + i\epsilon, \cdots, x_n + i\epsilon) \to T$ in the topology of $(\mathfrak{D}')$. Distributional Fourier transforms have analytic representations that are obtainable by partitioning of the Fourier integral (Carleman-Fourier transform). For dimension one this method is useful for solving systems of linear differential equations with constant coefficients, it has advantages over the Laplace transform and operational calculus. For higher dimension, the method is important for quantum field theory. Distributions are a special case of Sato's hyperfunctions. This theory, which depends strongly upon complex variables theory, has been greatly improved by Reese Harvey in his thesis (Stanford 1966).

Applications of Distributions and Other Generalized Functions to Partial Differential Equations, by Professor François Treves, Purdue University.

About 1930, Hadamard represented fundamental solutions of hyperbolic equations as "finite parts" of divergent integrals, thus showing that their study requires more general objects than functions. Shortly afterwards, Sobolev introduced "weak derivatives" of square-integrable functions in the study of elliptic boundary problems. Today similar or much more complex operations are completely "automatized" in the framework of distributions (cf. pseudo-differential operators). Another feature added to PDE theory by distributions is the very general use of topological vector spaces and duality. A striking example is Malgrange's existence and approximation theorems (a sketch of the proof was given). Even more general objects than distributions turn out to be useful: generalized functions à la Gelfand-Shilov, "analytic functionals" in relation with PDE in complex space, or with the Cauchy problem for equations with analytic coefficients.

FOURTH SESSION OF THE ASSOCIATION

Session on Multivariate Calculus

Introductory Multivariate Calculus, by Professor A. P. Mattuck, Massachusetts Institute of Technology.

A summary was given of the first course in calculus of several variables as it is now offered by a number of textbooks. The coordinate-free treatment of partial differentiation was described. The formulas are elegant, but for science students the confusion of point with vector and the absence

of variables are drawbacks. Redeeming features of multiple integration were described and an argument made for regarding Green's and Stokes' theorems as natural culminations for such a course on the grounds of both intrinsic mathematical worth and social utility.

Advanced Multivariate Calculus, by Professor W. H. Fleming, Brown University.

This lecture described "serious" advanced calculus, the kind of rigorous course taken by students who continue into graduate school in mathematics. The traditional course is now often split into two parts, one dealing with point set topology and baby real analysis, the other multivariate calculus. Essential topics which should be included in the latter course are: differentials of vector-valued functions, implicit and inverse function theorems, integrals over subsets of n-dimensional space, R^n, submanifolds of R^n, differential forms, Green's and Stokes' formulas, and a few applications.

Undergraduate Differential Geometry, by Professor Harley Flanders, Purdue University.

The speaker outlined a one or two semester geometry course presupposing some linear algebra and some advanced calculus. Objectives: presenting some valid modern geometric topics, reinforcing previous training by concrete visual applications, introducing new tools after their need is clear, e.g., index of a vector field, surface topology, convex sets, exterior forms. Details were discussed of the global and local structure of the rigid motion group, the exterior calculus in two variables, the structure and integrability equations for a surface, parallel surfaces, invariants, and other topics.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held on Friday in the Tulane Room as follows:

Film of the College Geometry Project of the University of Minnesota (in color):

7:30– 7:38 P.M. CAROMS, by Chandler Davis.
7:39– 7:50 P.M. EQUIDECOMPOSABLE POLYGONS, by J. D. E. Konhauser.
7:51– 8:04 P.M. PROJECTIVE GENERATION OF CONICS, by S. Schuster.

CEM Animated Calculus Films (in color):

8:15– 8:23 P.M. I MAXIMIZE, by Chandler Davis.
8:24– 8:32 P.M. CONTINUITY OF MAPPINGS, by Albert Fadell.
8:33– 8:45 P.M. THE THEOREM OF THE MEAN, by Felix P. Welch.
8:46– 9:06 P.M. NEWTON'S METHOD, by Herbert Wilf.
9:07– 9:17 P.M. LIMIT, by Robert C. Fisher.

Film showings were *scheduled* for Saturday and Sunday in the Grand Ballroom as follows:

Saturday
7:30– 7:45 P.M. SURFACE AREA WITH BLOCKS, a first grade taught by Miss Phyllis Klein, (A Film of the University of Illinois Arithmetic Project).
7:55– 8:56 P.M. LET US TEACH GUESSING: A DEMONSTRATION WITH GEORGE POLYA, (A CEM Individual Lectures Film in color).
9:10– 9:43 P.M. A FIRST CLASS WITH NUMBER LINE RULES AND LOWER BRACKETS, a fifth grade taught by Mr. Lee Osborn, (A film of the University of Illinois Arithmetic Project).

Sunday
7:30– 8:05 P.M. INTRODUCTION TO COMPOSITION WITH JUMPING RULES, a fifth grade taught by Mrs. Marie Herman, (A film of the University of Illinois Arithmetic Project).
8:15– 9:10 P.M. CHALLENGE IN THE CLASSROOM: THE METHODS OF R. L. MOORE, (A CEM Individual Lectures Film in color).

9:20–10:07 P.M. CAN YOU HEAR THE SHAPE OF A DRUM? A Lecture by Mark Kac, (A CEM Individual Lectures Film in color).

Due to a jamming of the projector on Saturday evening half-way through the film LET US TEACH GUESSING, the remainder of the program that evening could not be shown and was rescheduled for Sunday evening as follows: the second half of the film LET US TEACH GUESSING for 7:00 P.M., the last film originally scheduled for Saturday evening for 8:05 P.M., to be followed immediately by the last two regularly scheduled films for Sunday evening.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Friday morning and afternoon in Meeting Room 1 of Howard Johnson's with 42 members present.

The Board approved the appointment by President Moise of the following Nominating Committee for 1969: R. L. Wilder, Chairman; J. M. H. Olmsted, G. B. Price.

The Board elected Professor S. A. Jennings as Second Vice-President of the Association for the two-year term 1969–70.

The Board elected the following additional Associate Editors of the MATHEMATICS MAGAZINE: Professor L. C. Eggan, Professor D. Elizabeth Kennedy, Dr. B. L. Schwartz, and Professor H. A. Thurston.

The Board approved a proposal prepared by the Committee on Assistance to Developing Colleges designed to strengthen such colleges by providing improved curricula and instructional materials for developing colleges, with particular concern for students with inadequate mathematical preparation, by establishing visiting professorships for mathematicians at the developing colleges, by further training for mathematics faculty members at the developing colleges, perhaps through a series of imaginatively conceived institutes, and by organizing regional seminars involving mathematics staff members of the developing colleges.

The Board approved a revision of the By-Laws of the Association to be submitted for a vote by the membership at its meeting on August 26, 1969, at the University of Oregon.

The Board approved the following schedule of future meetings of the Association: University of Oregon, August 25–27, 1969; Miami, Florida, January 24–26, 1970; University of Wyoming, August 24–26, 1970; Atlantic City, New Jersey, January 23–25, 1971; San Francisco, California, January 17–19, 1974; Shoreham Hotel, Washington, D.C., January 1975.

The Executive Director reported the membership of the Association as 18,082 individual members, an increase of 223 since the corresponding date last year, 3 corporate members, and 246 academic members.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting was held on Sunday, January 26, 1969, in the Grand Ballroom with President Moise presiding. The Association's Eighth Award for Distinguished Service to Mathematics was made to Professor E. G. Begle of Stanford University. The citation (which appears on pages 1–2 of the January issue of this MONTHLY) was prepared and read by Professor B. J. Pettis of the University of North Carolina; the Award was presented by President Moise. Mrs. Begle was presented with a specially-bound reprint of the citation from the January issue of this MONTHLY.

In accepting the Award, Professor Begle said: "Some years ago, when classroom teachers were first being asked to learn something about the symbolism and terminology of set theory, they had a little difficulty in adjusting to the fact that you could name the element of a set in many different ways, and it was still just one element. I could think of n different ways of saying thank you but, if I did, I would be saying thank you n times,

and $n=1$ seems sufficient to me. I am very flattered and pleased at this honor, and I do thank you."

The Secretary then announced the results of the balloting for Governors of the Association in which 2349 votes were cast: Professor R. C. Buck of the University of Wisconsin, Madison, and Professor Mary P. Dolciani of the City University of New York were elected Governors for the three-year term 1969–71.

The Secretary reported on four major items of significance in the work of the Association during the past few months:

1. The Association's Washington, D.C. office at 1225 Connecticut Avenue, N.W. was opened in October and is now in full operation under the direction of the Association's Executive Director, Dr. A. B. Willcox. He would be very pleased to meet members of the Association when they are in the Washington area.

2. Steps have been taken by the new Editor of the MONTHLY, Professor Harley Flanders, to bring its publication policies more in line with the original purpose of the MONTHLY. A letter has been sent earlier in the month to the chairmen of all departments of mathematics in the United States and Canada acquainting them with these new policies, in particular, stressing that the main goal of the new Editor will be the publication of outstanding expository and survey articles covering all important aspects of current mathematics.

3. The Association is taking numerous steps to improve its service to junior and community colleges. The Association feels that it has an obligation to do this because of the rapidly rising number of junior colleges. One evidence of this increasing interest in the junior colleges by the Association is a revamping of the MATHEMATICS MAGAZINE under the editorship of Professor Jennings, who is anxious to make this publication particularly appealing to teachers of mathematics in junior colleges. The Association is also trying to make its programs more interesting to junior college teachers. Thus, at the summer meeting at the University of Oregon, there will be a session devoted to the particular interests of junior college teachers of mathematics.

4. The Association has established a program of assistance to developing colleges organized by a committee under the chairmanship of Professor George Springer. This Committee has been active in providing liaison between developing colleges and mathematicians interested in temporary or permanent appointment in these colleges. To facilitate such liaison, the Committee has maintained a table at this meeting for the exchange of information. The Committee is greatly encouraged by the responses received from both sides.

The Secretary noted that the organization of this meeting—by far the largest meeting ever held by the Association—had presented special problems, which had all been successfully and imaginatively solved by the local Committee on Arrangements under the able leadership of its Chairman, Professor Z. L. Loflin, to whom he expressed the deepest gratitude on behalf of the Association.

The Secretary then moved to amend Article VII, Section 2, of the By-Laws of the Association to read as follows: "The Board shall establish the annual dues and privileges of membership for ordinary and institutional members. The dues of ordinary members shall include a subscription to the official journal", and to delete Article VII, Section 3, and to renumber Sections 4 through 6 as 3 through 5. The motion was approved without dissent.

Under new business, Professor J. A. Ernest of the University of California, Santa Barbara, presented the following resolution for adoption: "Resolved that the Mathematical Association of America shall not hold any meetings in Chicago until such time as the Board of Governors shall determine that the situation has sufficiently improved so as to make such a meeting site appropriate."

Professor Ernest noted that a number of professional organizations have decided, on the basis of the situation which occurred in Chicago last August, to hold their meetings

in other cities. Most recently the AAUP decided to move its annual meeting from Chicago, and on the previous day the AMS had passed a resolution requesting that no more meetings be held in Chicago for the time being.

The motion was seconded and a discussion followed in which the question was raised whether the wording of the resolution would also include meetings of the Sections and whether the Association has authority to restrict the Sections in their decisions on where to meet. The Secretary replied that he would interpret the motion to apply only to national meetings, as the authority to set the time and place of sectional meetings rests, in accordance with the By-Laws of the Association, with the Sections.

It was then suggested that, since the AMS had already passed a resolution on this matter and since all national meetings of the MAA are held jointly with those of the AMS, there was no need for the MAA to pass a resolution of its own. The Secretary of the AMS, Professor Everett Pitcher, then reviewed the actions taken at the Business Meeting of the AMS the previous day, when the Executive Committee was requested to take all possible steps to remove the April 1969 meeting (a meeting of the AMS alone and not a joint meeting with the MAA) from Chicago. He noted further that it was true that the Joint Committee on Places of Meetings of the AMS and MAA have already made plans for joint meetings of the two organizations for the next eight years, and for this period no meetings in Chicago are planned. He pointed out that this was an accident, but nevertheless a fact.

Professor Morris Kline felt that, since this matter was brought up for approval of the entire membership, the decision to resume meetings in Chicago should also be approved by the entire membership and not by the Board of Governors. Accordingly, he moved that the motion be amended by replacing "Board of Governors" by "membership of the Association at a business meeting".

A number of speakers spoke against both the amendment and the original motion on the grounds that the political considerations involved in the motion have little relevance to the scientific and educational purposes for which the Association was founded and which it pursues. One speaker put it this way: "To involve the Association in this would weaken the Association in its scientific endeavour, and I would hate to see this take place."

Professor Melvin Henriksen then stated his opinion that everyone in the room had made up his mind on how he would vote and he doubted that anyone would be swayed by further debate. He asked that a vote be taken promptly.

The President then asked for a vote on the amendment. It was defeated by voice vote. There followed a vote on the original motion which was also defeated by voice vote.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Thursday, January 23, to Sunday, January 26. The forty-second Josiah Willard Gibbs Lecture was delivered by Professor R. L. Wilder of the University of Michigan on Thursday at 8:00 P.M. in the Grand Ballroom on "Trends and Social Implications of Research." Professor C. B. Morrey, Jr. of the University of California, Berkeley, gave the Retiring Presidential Address on Friday at 1:30 P.M. in the Grand Ballroom on "Differentiability Theorems for Weak Solutions of Differential Equations." Invited addresses were given by Professor J. K. Moser of the Courant Institute of Mathematical Sciences on Thursday at 11:30 A.M. on "Stability Theory and Invariant Manifolds for Dynamical Systems", and by Professor C. C. Moore of the University of California, Berkeley, on Friday at 11:00 A.M. on "Geometric Ergodic Theory," both in the Grand Ballroom.

The Bôcher Memorial Prize was awarded on Thursday at 1:30 P.M. to Professor I. M. Singer of the Massachusetts Institute of Technology.

The Association for Symbolic Logic met on Wednesday and Thursday, January 22 and 23. Professor C. C. Chang of the University of California, Los Angeles, gave an invited address "On the Uses of Saturated and Special Models" on Wednesday at 2:00

P.M., Professor H. J. Keisler of the University of Wisconsin, Madison, one on "Model Theory" on Thursday at 9:00 A.M., and Professor M. O. Rabin of the Hebrew University one on "Definability and Decidability in Second-order Theories" on Thursday at 2:00 P.M.

The Governing Council of Mu Alpha Theta, the National High School and Junior College Mathematics Club, met on Saturday at 2:00 P.M. in Room 205 of the Jung Hotel.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of Z. L. Loflin, Chairman; H. L. Alder, R. D. Anderson, Mrs. Beverly L. Brechner, A. H. Clifford, J. E. Diem, L. J. Derr, E. L. Dubinsky, J. W. Ellis, J. K. Feibleman, O. G. Harrold, Jr., Brother Charles Klein, R. J. Newman, Sister Mary Robert von Wolff, G. L. Walker.

Registration headquarters were located in the Convention Lobby on the ground floor of the Jung Hotel. The Mathematical Sciences Employment Register was maintained in the Tulane Room from 9:00 A.M. to 5:00 P.M. from Friday through Monday, and book and educational media exhibits were displayed in the Exhibition Hall (Hall of the Americas) of the Jung Hotel from 9:00 A.M. to 5:00 P.M. on Friday through Sunday.

A dinner meeting was held on Friday at 6:00 P.M. in Terrace Suite 4 of the Jung Hotel for the participants in the 1965 and 1966 MAA Cooperative Summer Seminars held at Bowdoin, their wives, and other invited guests.

HENRY L. ALDER, *Secretary*

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1969

General Offices: 1225 Connecticut Avenue, N.W., Washington, D.C. 20036
Executive Director: A. B. WILLCOX
Executive Director Emeritus: H. M. GEHMAN

OFFICERS

President, G. S. YOUNG, Tulane University (1969–70)
Past-President, E. E. MOISE, Harvard University (1969)
First Vice-President, VICTOR KLEE, University of Washington (1968–69)
Second Vice-President, S. A. JENNINGS, University of Victoria (1969–70)
Editor, HARLEY FLANDERS, Purdue University (1968–71)
Secretary, H. L. ALDER, University of California, Davis (1965–69)
Treasurer, E. A. CAMERON, University of North Carolina (1968–72)
Associate Secretary, RAOUL HAILPERN, SUNY at Buffalo (1968–72)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

R. H. BING, University of Wisconsin, Madison (1965–70)
E. E. MOISE, Harvard University (1969–74)
R. L. WILDER, University of Michigan (1967–72)

Elected Members of the Finance Committee

H. M. GEHMAN, SUNY at Buffalo (1968–69)
G. B. PRICE, University of Kansas (1968–71)

Governors at Large

R. C. BUCK, University of Wisconsin, Madison (1969–71)
MARY P. DOLCIANI, City University of New York (1969–71)
S. A. JENNINGS, University of Victoria (1968–69)
N. H. MCCOY, Smith College (1967–69)
LEO MOSER, University of Hawaii (1968–70)
WALTER RUDIN, University of Wisconsin, Madison (1968–70)

Sectional Governors (July 1, 1966–June 30, 1969)

Allegheny Mountain, A. F. STREHLER, Carnegie-Mellon University
Indiana, G. N. WOLLAN, Purdue University
Kentucky, L. L. SCOTT, University of Louisville
Metropolitan New York, ABRAHAM SCHWARTZ, City University of New York
Nebraska, D. W. MILLER, University of Nebraska
Northern California, D. W. BLAKESLEE, San Francisco State College
Oklahoma-Arkansas, R. B. DEAL, University of Oklahoma Medical School, Oklahoma City
Rocky Mountain, F. M. STEIN, Colorado State University
Wisconsin, E. F. WILDE, Beloit College

Sectional Governors (July 1, 1967–June 30, 1970)

Kansas, R. N. BRADT, University of Kansas
Missouri, P. B. BURCHAM, University of Missouri
New Jersey, JOSHUA BARLAZ, Rutgers–The State University
Northeastern, GRACE E. BATES, Mount Holyoke College
Ohio, H. M. MACNEILLE, Case Western Reserve University
Pacific Northwest, JOSEPH HASHISAKI, Western Washington State College
Southeastern, R. W. BALL, Auburn University
Southwestern, A. P. HILLMAN, University of New Mexico
Upper New York State, F. D. PARKER, St. Lawrence University

Sectional Governors (July 1, 1968–June 30, 1971)

Florida, D. B. GOODNER, Florida State University
Illinois, F. E. HOHN, University of Illinois
Iowa, D. E. SANDERSON, Iowa State University
Louisiana-Mississippi, L. VIRGINIA CARLTON, Centenary College
Maryland-D.C.-Virginia, E. E. FLOYD, University of Virginia
Michigan, L. M. KELLY, Michigan State University
Minnesota, F. L. WOLF, Carleton College
Philadelphia, B. H. BISSINGER, Pennsylvania State University, Middletown
Southern California, R. B. HERRERA, Univesity of California, Los Angeles
Texas, D. E. EDMONDSON, University of Texas, Austin

COMMITTEES OF THE ASSOCIATION

Terms of office of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms of office are listed, since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January 1971.

EXECUTIVE COMMITTEE

G. S. YOUNG, *Chairman* (1969–70); H. L. ALDER (1965–69), E. A. CAMERON (1968–72), HARLEY FLANDERS (1968–71), S. A. JENNINGS (1969–70), VICTOR KLEE (1968–69), E. E. MOISE (1966–69), all *ex officio*.

FINANCE COMMITTEE

G. S. YOUNG, *Chairman* (1969–70), *ex officio*; H. M. GEHMAN (1968–69), G. B. PRICE (1968–71), H. L. ALDER (1965–69), *ex officio*, E. A. CAMERON (1968–72), *ex officio*.

COMMITTEE ON ADVISEMENT AND PERSONNEL

K. J. ARNOLD, *Chairman* (1969–71); E. A. DAVIS (1967–69), J. C. EAVES (1969–71), K. W. FOLLEY (1968–70), Mrs. F. J. MACWILLIAMS (1967–69), B. E. RHOADES (1967–69), W. H. SCHMIDT (1969–71).

Committee on Assistance to Developing Colleges

George Springer, *Chairman*; R. D. Anderson, David Blackwell, C. M. Braden, L. L. Clarkson, C. O. Oakley, B. E. Rhoades, M. B. Smith, Jr., F. M. Stewart.

Committee on Earle Raymond Hedrick Lectures

N. J. Fine, *Chairman* (1967–69); Harley Flanders (1969–71), Gian-Carlo Rota (1968–70).

Committee on Educational Media

C. O. Oakley, *Chairman* (1966–69); A. A. Blank (1969–71), L. W. Cohen (1967–69), B. R. Gelbaum (1969–71), P. S. Jones (1967–69), Robert Kalin (1969–71), P. E. Miles (1968–70), Seymour Schuster (1967–69).

Advisory Committee for Individual Lecture Films: A. A. Blank, *Chairman* (1967–69); R. G. Long, *Director* (1967–69), R. C. Buck (1968–70), L. W. Cohen (1968–70), R. D. James (1967–69), Mark Kac (1967–69), N. D. Kazarinoff (1967–70), P. D. Lax (1968–70), B. J. Pettis (1967–69), H. O. Pollak (1967–69), L. B. Williams (1968–70).

Subcommittee on Television: P. S. Jones, *Chairman*; C. B. Allendoerfer, R. C. Fisher.

Committee on High School Contests

J. M. Earl, *Chairman* (1969–71); W. H. Fagerstrom, *Director*, Pamela Ames (1967–69), L. C. Dalton (1969–71), Harrison Givens, Jr. (1966–69), S. L. Greitzer (1969–71), N. S. Mendelsohn (1969–71), Hans Sagan (1967–70), E. E. Strock (1967–70).

Committee on Institutes

V. O. McBrien, *Chairman* (1966–69); D. B. Goodner (1969–71), D. K. Harrison (1969–71), Mark Kac (1967–69), M. W. Pownall (1969–71), D. L. Thomsen, Jr. (1967–69).

Committee on Publications

Ivan Niven, *Chairman* (1969–71); Joshua Barlaz (1967–69), R. G. Bartle (1969–71), Dorothy L. Bernstein (1969–71), R. P. Boas (1967–69), D. T. Finkbeiner, II (1969–71), David Gale (1967–69), P. R. Halmos (1968–70), Alex Rosenberg (1968–70), E. A. Cameron (1968–72), *ex officio*, Harley Flanders (1968–71), *ex officio*, S. A. Jennings (1969–73), *ex officio*.

Editorial Committee for a Volume of Collected Notes on Calculus: T. M. Apostol, *Chairman;* H. E. Chrestenson, C. S. Ogilvy, D. E. Richmond, N. J. Schoonmaker.

Subcommittee on Carus Monographs: R. G. Bartle, *Chairman* (1969–71); R. P. Boas (1967–69), D. T. Finkbeiner II (1969–71).

Subcommittee on Lester R. Ford Awards: Ivan Niven, *Chairman* (1969–71), *ex officio;* Edwin Hewitt (1968–70), D. E. Richmond (1969–71).

Subcommittee on MAA Studies in Mathematics: David Gale, *Chairman* (1967–69); P. R. Halmos (1968–70), Alex Rosenberg (1968–70).

Subcommittee on Slaught Papers: Joshua Barlaz, *Chairman* (1967–69); Dorothy L. Bernstein (1969–71), Harley Flanders (1968–71), *ex officio*.

Subcommittee on Miscellaneous Publications: Ivan Niven, *Chairman* (1969–71); H. L. Alder (1966–69), E. A. Cameron (1968–72), all *ex officio*.

Committee on Secondary School Lecturers

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CALENDAR OF FUTURE MEETINGS

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

Fifty-Third Annual Meeting, Miami, Florida, January 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, May 3, 1969.
FLORIDA
ILLINOIS, Western Illinois University, Macomb, May 9–10, 1969.
INDIANA, Purdue University, Indianapolis, May 10, 1969.
IOWA
KANSAS
KENTUCKY, Morehead State University, Morehead, Spring 1969.
LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
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NEW JERSEY, Drew University, Madison, May 3, 1969.
NORTH CENTRAL
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PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.
ROCKY MOUNTAIN, University of Colorado, Boulder, May 9–10, 1969.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Western Ontario, London, Ontario, Canada, May 10, 1969.
WISCONSIN, Oshkosh, Wisconsin, May 2–3, 1969.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.
AMERICAN MATHEMATICAL SOCIETY, University of Oregon, Eugene, Oregon, August 26–29, 1969.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION
ASSOCIATION FOR COMPUTING MACHINERY, Statler-Hilton Hotel, Washington, D. C., May 7–9, 1969.
ASSOCIATION FOR SYMBOLIC LOGIC
CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Milwaukee, Wisconsin, November 27–29, 1969.
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS, New York City, August 19–22, 1969.
MU ALPHA THETA, University of Oregon, Eugene, Oregon, August 27, 1969.
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, Colorado, June 17–20, 1969.
PI MU EPSILON, University of Oregon, Eugene, Oregon, August 26–27, 1969.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.

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NOTICE

At its January meeting, the Board of Governors approved 112 additional pages for the current volume of the MONTHLY. This space will be used in remaining issues to exhaust the large backlog of articles accepted prior to my editorship.

Articles submitted henceforth should conform to the Statement of Policy published in the last January issue.

HARLEY FLANDERS, *Editor*

THE EARLY DEVELOPMENT OF ALGEBRAIC GEOMETRY

SOLOMON LEFSCHETZ, Brown University and Princeton University

1. As I am neither a historian nor an archeologist of mathematics, I shall merely present my idea of the major contributions to the subject, say by the end of the last century.

By "algebraic geometry" one really understands what I have described as *bi-rational geometry.* A simple example will clarify this point. Take the irreducible curve

$$C: f(x, y) = y^2 - (x^3 - 1) = 0.$$

Let any complex rational function

$$R(x, y) = P(x, y)/Q(x, y),$$

Prof. Lefschetz continues an astonishingly productive career. His profound influence in the development of topology and of algebraic geometry is expounded at length in articles by W. V. D. Hodge and Norman E. Steenrod in the Princeton Symposium volume in honor of S. Lefschetz, *Algebraic Geometry and Topology* (1957) edited by R. H. Fox, D. C. Spencer, and A. W. Tucker. His numerous publications in these fields include the books *L'Analyse Situs et la Géométrie Algébrique* (1924), *Géométrie sur les Surfaces et les Variétés Algébriques* (1929), *Topology* (1930), *Algebraic Topology* (1942), *Topics in Topology* (1942), *Introduction to Topology* (1949), and *Algebraic Geometry* (1953). In recent years he has produced fundamental research in ordinary differential equations, including the volumes *Differential Equations, Geometric Theory* (1957) and (with J. La Salle) *Stability by Liapunov's Direct Method with Applications* (1961).

Prof. Lefschetz began his mathematical career in 1911 with his Ph.D. under W. E. Story at Clark University. He held positions at the Univ. of Nebraska, Univ. of Kansas, then Princeton University until his retirement. At Princeton he was Research Professor, 1932–1953, and Department Chairman, 1945–1953. Since, he has been at the National University of Mexico, RIAS, and Brown University. His numerous awards include the Bardin Prize (Académie des Sciences 1919), the Bôcher Prize (AMS 1924), the Feltriaelli Prize (Accademia dei Lincei 1956), and foreign memberships in the Royal Society and the Académie des Sciences. He was Editor, Annals of Mathematics, President AMS (1935–1937), and is a member of the National Academy of Sciences and the American Philosophical Society. *Editor*

P and Q polynomials, be declared as null: $R=0$, whenever P but not Q is divisible by f. Place in one class S^* all the rational functions which differ by zero from a given one. The collection $\{S^*\}$ is a *field*, an overfield of the field K of all complex numbers. This is the *function field* $K(C)$ of the curve C. The *algebraic geometry* of C is the study of the properties of C which depend solely upon the function field $K(C)$.

All this applies, of course, to any plane curve

$$C: f(x, y) = 0,$$

where f is a complex irreducible polynomial. It applies in fact also to higher dimensional varieties (surfaces, $\cdots$) but I shall restrict my discussion to mere plane curves.

The particular choice of the complex field K (as field of constants) rather than any other field, is simply because with K one may freely utilize complex analysis and topology.

The justification of "birational" is this: Let D: $g(u, v)=0$ be a second curve like C and suppose that $K(C)$ is isomorphic over K with $K(D)$. One may identify the two fields under some fixed isomorphism T. As a consequence, say

$$Tx = S_1(u, v), \quad Ty = S_2(u, v); \qquad S_1, S_2 \in K(D)$$

and similarly

$$T^{-1}u = S_1'(x, y), \quad T^{-1}v = S_2'(x, y); \qquad S_1', S_2' \in K(C).$$

Thus T defines a *birational* transformation $D \to C$, which is almost a 1-1 correspondence between the points of the two curves.

Example: Let $f_h(x, y)$ denote a complex homogeneous polynomial of degree h. Let C be an irreducible curve with the origin as point of multiplicity h, so that its equation is

$$f(x, y) \equiv f_h(x, y) + f_{h+1}(x, y) = 0,$$

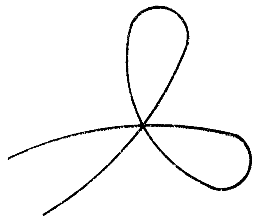
where f_h and f_{h+1} have no common factor. Let D be the line $v=0$ of the u, v plane. Then T^{-1}: $u=y/x$, $v=0$,

$$T: x = \frac{-f_h(1, u)}{f_{h+1}(1, u)}, \qquad y = \frac{-uf_h(1, u)}{f_{h+1}(1, u)}\cdot$$

Thus all the curves such as C are birationally equivalent to the u line, and hence to one another (see Fig. 1).

Incidentally this underscores the vast richness of birational as compared with projective or Euclidean geometries. For it has enabled us to place in one class a seemingly vastly dissimilar collection of curves.

It has always tacitly been understood by the savants of the 19th century dedicated to our subject that the curves considered were inhabitants of a complex projective plane. However, for convenience in the utilization of complex analysis, they were placed, as here, in a complex Euclidean plane complemented



TRIPLE POINT (h=3)

FIG. 1

for "infinity" by some such device as a projective transformation

$$x = \frac{1}{x'}, \qquad y = \frac{y'}{x'},$$

replacing the line at infinity by the line $x'=0$. This is to be understood in what follows.

2. The first vestige of algebraic geometry is found in the extensive theory of elliptic functions by Legendre (before 1830). The first step was the natural extension of trigonometric integrals to the type

$$u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^x \frac{dx}{y},$$

where $k^2 \neq 0$, 1 and $y^2=(1-x^2)(1-k^2x^2)$.

Legendre confined his attention to the real domain. He defined the inverse function $x = \operatorname{sn} u$ as *sine amplitude*, introduced other functions which I shall not describe, and was primarily interested in the applications, notably to the arc of an ellipse (hence elliptic functions) and to large oscillations of a pendulum.

The integral u, finite for all x, has evidently birational character. However, perhaps not as yet influenced by the fundamental work of Cauchy, Legendre failed to catch the double periodicity of sn u. It was brought out (around 1830) by Abel, Jacobi, and Gauss. A brilliant proof of the fact that both distinct periods of sn u could not be real is due to Jacobi. The proof is so simple that I cannot escape the temptation of outlining it. Suppose that there exist two rationally independent periods ω_1, ω_2, both real. Then for any $\epsilon>0$ there exist integers m, n such that $|m\omega_1+n\omega_2|<\epsilon$. Hence for real u we have $|\operatorname{sn} u|<\epsilon$, which is untrue.

3. By 1850 these basic steps had been taken:

ABEL. Generalization of elliptic integrals as follows: Given C, let $R\in K(C)$ be expressed in acceptable cartesian coordinates x, y. Then

$$u = \int_{(x_0,y_0)}^{(x,y)} R\,dx,$$

limits and path in C, is known as an *abelian* integral. There are 3 types: first kind $|u|$—bounded; 2nd kind—u has poles only; 3rd kind—remaining type.

ABEL'S THEOREM. Let the variable curve $\phi = \sum c_h \phi_h(x, y) = 0$, (the ϕ_h are linearly independent modulo f and are polynomials of the same degree) cut C in points $P_1, \cdots, P_n$. Then

$$\sum_k \int_A^{P_k} du = v$$

is a constant (independent of the c_h).

JACOBI. Let $u_1, \cdots, u_p$ be a maximal set of linearly independent integrals of the first kind modulo constants. Given p general values $v_1, \cdots, v_p$, the system in the unknowns P_k, $k \leqq p$

$$\sum_k \int_A^{P_k} du_h = v_h$$

has a unique solution.

WEIERSTRASS. The maximal number of linearly independent integrals of the second kind modulo $K(C)$ is $2p$. (That is K-linearly independent modulo elements of the function field $K(C)$.)

PUISEUX. Let C be *any* algebraic curve and $P(x_0, y_0)$ any (finite) point of C. The complete neighborhood of the point on C is represented by a finite number of fractional power series

$$y - y_0 = a_1(x - x_0)^{q_1/q_0} + a_2(x - x_0)^{q_2/q_0} + \cdots,$$

where $|x - x_0| < \sigma$, and the q_h have no common factor. These series come in collections of q_0, forming a system of roots $y_1, y_2, \cdots, y_{q_0}$ of $f(x, y) = 0$, which are circularly permuted as x turns around x_0 in its complex plane. This set of solutions is jointly represented parametrically in the form

$$x = x_0 + t^{q_0}, \quad y = y_0 + a_1 t^{q_1} + a_2 t^{q_2} + \cdots, \qquad 0 \leqq |t| < \sigma^{1/q_0}.$$

Such a pair of series defines what is now known as a *place of center P*. There are simple analogues for the points "at infinity" which I shall not describe.

One may therefore summarize Puiseux's fundamental result as follows: *Each point P of the curve C has a neighborhood on C consisting of a finite number of places of which P is the center.* (I am anticipating in this statement a formulation really due to Hermann Weyl.)

General observation. All the results described so far have strict birational character.

4. I come now to one of the greatest contributions ever made to mathematics. The author is Riemann, the period around 1860 and the topic:

RIEMANN SURFACES. The representation of an algebraic curve by a truly geometric, and possibly simple model was assuredly most desirable. This is exactly what Riemann accomplished.

Let C be our usual curve and let m be the degree of f in y. A *branch point* of y is a value $x=a$, marked on the sphere S of the complex variable x, where two or more roots $y(x)$ of $f=0$ are permuted as x turns around a. For instance if C is the curve

$$y^2 = (x - a_1)(x - a_2) \cdots (x - a_{2n})$$

the a_h (assumed all distinct) are the branch points.

Mark the positions a_j of the branch points on S and let α, β be two diametral points on the sphere such that no two branch points are on a great circle through α and β. Draw arcs of great circles αa_j none containing β. Cut S open along these arcs and let Ω be their complement in S. In Ω each of the roots $y_j(x)$, $j \leqq m$, of $f(x, y)=0$ is uniquely determined by the value $y_j(\beta)$.

Now choose for each k a copy S_k of S corresponding to $y_k(x)$, save that in S_k we only draw the a_j and cuts αa_j which permute $y_k(x)$. The complement Ω_k of the cuts in S_k is a 2-cell, i.e., a simply connected region of the sphere.

To simplify matters suppose that the cut αa_1 permutes y_1 and y_2. Thus this cut will be found in S_1 and S_2. The situation in both spheres is shown in Fig. 2 with proper orientations.

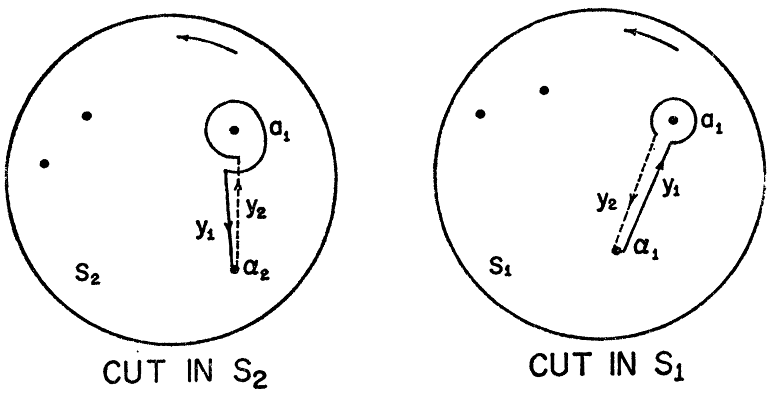


FIG. 2

Now match the two sides labelled y_1 and those labelled y_2 and repeat the process throughout all the spheres S_k. The result is a configuration consisting of m polygons which are all oriented in such a way that where a side belongs to two polygons (never more than two) the side is oppositely oriented relatively to the two polygons. (See Fig. 3.)

This configuration is precisely the Riemann surface $\Phi(C)$ of the curve C.

5. We now state a certain number of the major properties of $\Phi(C)$.

(a) Φ is a smooth surface which is decomposed into m simple polygons (closed

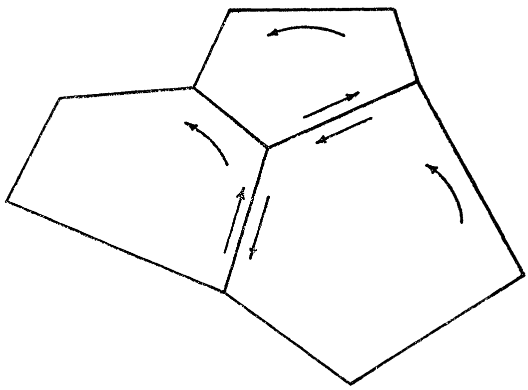


FIG. 3

2-cells). This is technically an orientable and oriented 2-manifold. ("2-manifold" = 2 dimensional manifold—less precisely a figure as smooth as a plane around each point. It is often called a surface.)

(b) Here I fall back upon well-known elementary properties of topology. If α_0, α_1, α_2 designate the number of vertices, arcs, and polygons in a decomposition of a surface like Φ, then the expression

$$\chi(\Phi) = \alpha_0 - \alpha_1 + \alpha_2 = 2 - 2p$$

is the *characteristic* of Φ. The integer p is the *genus* of the surface and is independent of the mode of decomposition of Φ. Its calculation is very simple. Let a place π_j result from the cyclic permutation of q_j roots. Set $N = \sum(q_j - 1)$. (When C has only multiple points with distinct tangents, N is the class of C: number of tangents issued from a general point of the plane.) The calculation of the number χ yields easily this result: If β_0, β_1, β_2 are the α_j for a sphere then it is known that

$$\beta_0 - \beta_1 + \beta_2 = 2.$$

Here $\alpha_1 = m\beta_1$, $\alpha_2 = m\beta_2$, $\alpha_0 = m\beta_0 - N$, hence

$$\chi(\Phi) = 2 - 2p = 2m - N.$$

This yields the following formula, due to Riemann:

$$N = 2(p + m - 1).$$

Since N and m are readily obtained from the equation of C, this expression yields the determination of p, the *genus* of C, from the equation of C.

I believe that p had already been found earlier by Plücker and even shown to be birationally invariant (at least under certain simple transformations).

I must underscore that essentially Riemann had constructed his surface as a *topological image* of the "space of places." In other words he had come to realize that if C has, say, a point A of multiplicity k with k distinct tangents (hence A is

the center of k *distinct* places), then in $\Phi(C)$ it is to be represented by k *distinct points*. That is, he already had the intuitive concept of *place*.

Of the following two properties Riemann did not know the first but in some manner knew of the second.

(c) Φ is homeomorphic to a two sided disk with p holes.

(d) Let a_h be an oriented circuit on Φ around the hole h and b_h one through the same hole, both initiated from some fixed point A of Φ. Then Φ may be reconstructed as follows: Draw a regular $4p$-sided plane polygon Π with sides labeled successively in positive orientation as

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \cdots, b_p^{-1}$$

(a_h^{-1}, b_h^{-1} are a_h, b_h oppositely oriented). Match all vertices with one point A, then a_h with a_h^{-1}, b_h with b_h^{-1}. The closed 2-manifold thus obtained is Φ.

(e) On the preceding manifold one may operate upon the functions on Φ, locally analytic in terms of the local place variables t, as with complex analytic functions on a complex plane. This means that the basic results of Cauchy extend to the Riemann surface.

Riemann's Equality and Inequality. These are two results whose importance could not be exaggerated. Let u, v be two integrals of the first kind and define their basic *periods* as

$$\int_{a_\mu} du = \omega_\mu, \quad \int_{b_\mu} du = \omega_{p+\mu}, \qquad 1 \leqq \mu \leqq p,$$

and let ω_μ', $\omega_{p+\mu}'$ be the same for v. Then by integrating $\int u\, dv$ along the polygon Π and applying Cauchy's result for holomorphic functions, we obtain

Riemann's Equality: $$\sum_{\mu=1}^{p} \begin{vmatrix} \omega_\mu & \omega_{p+\mu} \\ \omega_\mu' & \omega_{p+\mu}' \end{vmatrix} = 0.$$

Let now $u = u' + iu''$ and $\omega_\mu = a_\mu + ib_\mu$, $\mu \leqq 2p$. There follows from Cauchy's inequality and again from $\int_\pi u' du'' > 0$

Riemann's Inequality: $$\sum_{\mu=1}^{p} \begin{vmatrix} a_\mu & a_{p+\mu} \\ b_\mu & b_{p+\mu} \end{vmatrix} > 0,$$

(Jacobi for $p = 1$). By means of these properties plus difficult Dirichlet principle analysis Riemann proved:

Theorem 1. *There is exactly a maximum p of linearly independent differentials of the first kind and similarly a maximum of $2p$ linearly independent differentials of the second kind modulo $dK(C)$.*

Noteworthy also is:

Theorem 2. (Part of the Theorem of Riemann-Roch.) *Let $G = n_1\pi_1 + \cdots + n_s\pi_s$ be a set of places, with n_h as the multiplicity of π_h. Set $n = \sum n_h$. Then the*

elements of the function field $K(C)$ with G as set of poles and $n>2p$ form a linear system of dimension $n-p$.

Obvious implications: The Riemann surface as topological structure is a birational invariant of the curve C. This holds also for the genus p and for the space of places (homeomorphic to $\Phi(C)$).

I hope that the little I have said about Riemann's contribution will at least make clear its enormous importance for algebraic geometry.

6. If one had asked Riemann or any one of his great predecessors where he would place his contribution he would certainly have declared "in functions of a complex variable." The first, and all important, deviation towards more algebra was made in 1870 by *Max Noether*. His work marks definitely the beginning of a new epoch in algebraic geometry.

The starting point is a famous theorem due to Noether and usually referred to as the *$A\phi+B\psi$ Theorem*. Given two algebraic curves $\phi(x, y)=0$ and $\psi(x, y)=0$ intersecting only in isolated points, find n.a.s.c. in order that a polynomial $f(x, y)$ be representable in the form

$$f = A\phi + B\psi; \quad A, B \text{ polynomials.}$$

That is, under what condition is f an element of the ideal (ϕ, ψ)?

This question was completely solved by Noether (around 1870). When ϕ, ψ have no common tangent where they intersect, a sufficient condition is this: if an intersection P of $\phi=0$ and $\psi=0$ is of multiplicity p and q for the curves then it is sufficient that $f=0$ be of multiplicity $\geqq p+q-1$ at P.

Applications were made a couple of years later in a fundamental memoir of Brill and Noether. The basic theorem there is the *Remainder Theorem*.

Recall that a *Cremona transformation* of a projective plane P^2 is the result of a finite succession of quadratic and projective transformations of P^2. This is a general birational transformation of P^2 into itself.

Let me admit this very important result due to Noether: By a suitable Cremona transformation, an irreducible curve C is transformable (implicit: birationally) into a curve still called C, which has only multiple points with distinct tangents. An *adjoint* to C is a curve having at any p-tuple point of C at least multiplicity $p-1$. The Remainder Theorem asserts this: Let an adjoint of degree n to C intersect it besides the imposed intersections in two sets Q and R. Let two adjoints ϕ, ψ of degree n intersect C respectively in $R+Q'$ and $Q+R'$. Then there is an adjoint ω of degree n intersecting C in $R'+Q'$ (in all cases intersections besides those imposed).

The goal of Brill and Noether was this: First let the linear system $\psi=c_0\phi_0 + \cdots + c_r\phi_r=0$ (where the ϕ_h are of equal degree and linearly independent modulo f) intersect C in n points, some of which may be fixed. The collection of all such sets of n points is called a *linear series* of *degree n* and *dimension r*; the series is *complete* if not amplifiable with the same n but larger r. General notation g_n^r (usually complete series).

Let the curve C have only ordinary singularities, that is multiple points with distinct tangents.

I. Every complete g_n^r may be cut out by all the adjoints ϕ_μ of a suitable degree μ passing through a certain fixed set of points of C.

II. Let m be the degree of C. The adjoints ϕ_{m-3} of degree $m-3$ are called *canonical*. Their complete series is a unique g_{2p-2}^{p-1} without fixed points: the *canonical series*. The complete set of differentials

$$\left\{\frac{\phi_{m-3}dx}{f_y'}\right\}$$

is a complete set of differentials of the first kind. This implies the birational invariance of p and of the canonical series (by purely algebraic methods).

III. THEOREM OF RIEMANN-ROCH. *Let g_n^r be complete and let σ be the maximal number of linearly independent canonical curves through an element of g_n^r. Then $n-r=p-\sigma$. Hence if no canonical curve passes through any element of g_n^r then $n-r=p$.* (*Strict "Riemann part."*)

IV. If the complete g_n^r is generated by the linear system

$$\sum_0^r c_h\psi_h(x) = 0$$

then $\{\psi_h/\psi_0\}$ is a linear base for the complete system of rational functions on C (elements of $K(C)$) whose poles consist of the element of g_n^r cut out by $\psi_0=0$. This identifies the "rational function" problem and the complete g_n^r problem.

A constant tool of Brill and Noether was addition and subtraction of complete series $g\pm g'$.

Important observation: The Noether-Brill results are applicable to algebraic curves over any algebraically closed field of characteristic zero. This was in no sense underscored by Noether and Brill but is implicit in their results and methods.

7. The resonance from the Brill-Noether memoir was scarcely felt in Germany, but had very great effect in Italy. For assuredly the work of the very brilliant school of Italian geometers was a direct consequence of the Brill-Noether stimulus. It began in 1882 with Castelnuovo, associated quite soon with Enriques, and greatly enriched around 1900 by the addition of Severi. Briefly speaking, while Brill and Noether studied the effect of the addition of complete linear series on a curve, the Italian group dedicated much effort upon the far more complicated addition of algebraic curves on a surface.

During the same period Émile Picard in France developed entirely alone, and practically without topology, a considerable portion of Riemann's work for algebraic surfaces.

With these few observations I must stop, since beyond it would decidedly take me out of "early development."

This is an invited address given on August 26, 1968 at the Madison Meeting of the Mathematical Association.

Bibliography

The reader desiring fuller information upon the topics discussed in my lecture may profitably consult the following sources (chosen almost at random among the vast literature on the subject):

For the contributions up to and including the work of Riemann:

Émile Picard, Traité d'Analyse, 3rd edition, Vol. 2, Gauthier-Villars, Paris, 1922–28. Chs. 13–18.

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For the work of Max Noether and Noether-Brill:

Émile Picard and George Simart, Théorie des fonctions algébriques de deux variables, Vol. 2, Gauthier-Villars, Paris, 1906.

For elementary algebraic topology:

See, for example, my monograph: Introduction to Topology, Princeton University Press, 1949.

SOME TEACHING REMINISCENCES

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My first teaching was done at Williams College, 1906–11. I had just received my doctorate in mathematical astronomy at the University of Chicago, where I had also done my undergraduate work. After seven years in the intensely scholarly atmosphere of Chicago, my move to a small "Ivy League" college with the lively student activities of that era presented some great contrasts. I feel sure, however, that both types of institutions supplied basic needs in American life. A lot of fine scientific and other intellectual work was done at Williams long ago, and that small college has contributed many great men to our nation.

In 1906–11 our mathematics staff consisted of Dean Frederick C. Ferry (who brilliantly taught a one-semester senior class in modern geometry), Associate Professor James G. Hardy and three instructors, fresh from graduate schools. We also had the help of a noted professor of astronomy who taught one freshman section. Though Williams had only about 180 freshmen, we ran 12 sections of freshman mathematics with a maximum enrollment of 15 each. (The generosity of Williams trustees was also shown by a maximum enrollment of 8 students in each section of beginning French or German!)

Most members of our staff taught four sections, freshman or sophomore, each meeting four hours a week. We corrected our own home-work papers, which were not burdensome and helped us keep in touch with the progress of our students. (Naturally there was little time for research! Few faculty members did any at all.)

The freshman course consisted of several weeks of old-fashioned solid geometry, for students who had entered without it, then three months of college

algebra, and a like amount of plane trigonometry. Plane analytic geometry came in the first half of sophomore year, and differential calculus in the second half. Students heard nothing about integration until their third year, and nothing about a differential equation until the last half of senior year!

My first year of teaching discouraged me greatly. Intensive preoccupation at Chicago with the intricacies of my thesis (obtaining a periodic solution of a system of differential equations in the form of $2k$ infinite series in powers of a parameter, whose coefficients were Fourier series in the time) had effectually rendered me unable to explain elementary mathematics in language that freshmen could understand. Often I wished during that year that I had a job "where my success would depend only on what I might be able to do myself, and not on what I could get students to do." Fortunately, frequent long walks in the beautiful surroundings of Williamstown with Professor Hardy (the most remarkable classroom teacher of mathematics I have ever met) gave me some inklings of how to make matters clear to freshmen. In another year or two I became an *avid* teacher, and came to love the profession.

But I remained unhappy about the mathematical curriculum of those days, at Williams and many other colleges. Physical scientists, and soon some biologists, were clamoring for an early introduction of calculus. Before long some economists wanted it, too. Meanwhile our freshman work in solid geometry was relatively sterile. We had nothing to say about the nature of a deductive logical system based on postulates, or about postulates needed by, but missing from, Euclid's work. Instead we frittered away several weeks, having our freshmen memorize and recite defective proofs given in Wentworth's old-fashioned text. (Williams had an expert cabinet maker construct various wooden models to illustrate some three-dimensional relationships and thus provide mental crutches for some indolent thinkers.)

My conversations with colleagues about the possibility of revising our curriculum, so as to cover in the freshman year some elementary calculus, brought no results. There was skepticism as to the feasibility, and it was pointed out that students would have difficulty in switching from one of our 12 sections to another. I gave up the plan for a while.

I found time to continue some of my research begun at Chicago; and at Columbia in December 1906, I read a paper before a joint meeting of AMS, section A of AAAS, and the Astronomical Society, "On the law of gravitation in the binary systems" [1]. As a cub-instructor I was much over-awed by the presiding officer, the famous Simon Newcomb— "Who am I," thought I, "to be telling *him* anything about astronomy?" (But he seemed interested in my conclusions!)

That paper and my dissertation [2] were soon followed by some minor papers, mainly on central orbits, developing some ideas in E. J. Routh's Dynamics of a Particle [3]. None of my research helped me with my teaching difficulties—quite the contrary—but the aggregate of papers brought me several offers of jobs in prominent universities. I could not afford to accept any of these

and move my wife and small children away from lovely Williamstown. In early 1911, however, plans were made to open Reed College in September at Portland, Oregon, where my wife's family lived, and where stress was to be laid on curricular experimentation. It was a "natural," and of course I was delighted to come as a member of the first faculty. No outmoded old courses to get rid of. Naturally, I've had a wonderful time from the start!

In its first 41 years, while I served as head of the mathematics department, Reed never had occasion to offer elementary solid geometry, college algebra, or plane trigonometry. Any topics needed from one of those courses were incorporated in our analysis course, for freshmen or for sophomores. Our junior year was initially devoted to two parallel courses in modern geometry and modern higher algebra. In the senior year our early students took advanced calculus, history and foundations of mathematics, and wrote a senior thesis. That paper was originally supposed to be a critical essay; but our seniors soon sought to make it a substantial piece of research. A 1916 senior, later prominent in education, dealt with Euler's "36 officers problem" in her thesis and gave what I believe was the first proof of the impossibility of completing the marching program. Her conclusion was later confirmed by two other mathematicians, using other methods of analysis. Various other rather notable pieces of research were later carried out by numerous students of mathematics and science. Perhaps that is one reason why such an extraordinary percentage of Reed graduates in these fields have gone on to the doctorate.

Our Reed analysis course for freshmen frankly appealed considerably to intuition and mentioned many applications of the theory. Sophisticated logic came along in higher courses. In December 1915 I published in this MONTHLY a paper [4] explaining my ideas about freshman mathematics. I had read the paper in May 1914 before the AMS in Seattle. (The MAA was not yet set up.) Reed's first physics instructor was Karl T. Compton, later at Princeton and at MIT. He was at the Seattle meeting, and during the rather lively discussion of my paper, he expressed his pleasure that Reed had so many students majoring in mathematics. His remarks helped save me from drowning in a sea of dissent! For several years I worked on a text to cover our course, but was obliged to teach our students with only a few pages of mimeographed notes—formulas, basic principles, etc. Publication was delayed by shortages in World War I; but finally, in the late spring of 1921, my text appeared [5]. A few daredevil institutions used the book that fall, and a large number the next year.

About that time the Scientific American commented editorially at some length on my text, which brought me temporarily a lively correspondence to three other continents. In self-protection I composed three form letters and then had my typist send X, Y or Z, as seemed to me most appropriate. Meanwhile I was not sure that professors choosing a freshman text were really much pleased to be told editorially: "If we had our way, every teacher of mathematics would be obliged to read this book, and every person with responsibility for the laying out of mathematical instruction in any of our colleges would be obliged

to read a chapter from it every morning before breakfast."

In all humility and with real joy let me say this concerning my Introduction and its revised and somewhat enlarged edition [6]: some of the finest rewards I've received for the labor that went into the text have been personal letters of appreciation from employed persons studying calculus, or other topics, independently, and also such letters from individual educators in Britain and elsewhere abroad. Naturally the greatest reward of all has been the joy of teaching my own students at Reed. (I was especially happy to see our freshmen, in their first semester, calculate by single integration a variety of rather complicated volumes.)

In 1926 my sophomore text in analysis was published [7]. It contained a chapter on differential equations, with extensive applications, also a chapter on mean values and approximate integration, and one on curves and surfaces. Although it met our needs at Reed, it was, unfortunately, considerably too difficult for a typical sophomore class in many colleges. At a leading coast university a prominent professor once used this book with his class in advanced calculus. But it was not designed for such a use, either.

About the middle of the 1920's the Reed curriculum was changed so as to require for graduation some two out of five introductory scientific courses: Biology, chemistry, physics, mathematics, psychology. We then set up a new variety of freshman analysis, primarily for majors in literature, social science, and philosophy. These sections had three hours of credit, very light home work, and four meetings a week to provide adequate time for discussion and for practice under the instructor's guidance. I turned the two science sections of about 22 students each over to my colleague, Assistant Professor Jessie M. Short, and I taught the new "L" sections with somewhat larger enrollments. Each of us carried a heavy load. Including upper courses I usually had from 15 to 17 hours, besides directing from 3 to 7 senior theses. Miss Short carried substantially fewer hours, but too many, I think. She was also very active in civic affairs.

Gradually I modified the "L" program from pure analysis by giving a half-dozen lectures on non-Euclidean geometry, coupled with an hour-test and a brief paper. Later in the year I gave two talks on Huntington's system of postulates for algebra and on one of his remarkable independence proofs. The nonspecialist "L" freshmen were delighted with these excursions into abstract mathematics.

In 1929 I read a paper at the Summer Meeting of MAA [8].

My interest in mathematical economics was steadily rising; and from 1930 to 1932 I had the privilege of serving on a small committee of the Social Science Research Council to report on the collegiate mathematics needed by social scientists. Our chairman was H. R. Tolley, Director of the Gianinni Foundation of Agricultural Economics. Dr. E. B. Wilson, president of SSRC, in appointing the committee, made several suggestions, one of which was that we would "make it possible for students of social science to get the small amount of calculus that they do need without a frightful amount of calculus that they do not need." Our report pointed out one way that this could be done [9].

I obtained sabbatical leave from Reed for the spring semester of 1931 and spent six months in Britain and on the Continent sightseeing with my wife and interviewing some twenty or thirty men who had published extensively in the fields of biology, medicine, statistics, and economics, and who had used calculus or more advanced mathematics in their writings. (The SSRC gave me a grant-in-aid to facilitate this project.) Mainly I asked each scholar for his confidential opinion of the work of other important writers in his field. That helped me to estimate the validity of the initial assumptions upon which the other writer based the mathematical formulation of his theories. On the Continent I usually had to conduct my inquiries in French or German, but at that period my use of these languages was fairly fluent. (A few years later I wrote numerous French and German limericks, "just for fun.") A paper [**10**] that I sent to the 1931 Rome Congress on Population Problems was followed by my serving several years as a foreign correspondent of the *Comitato Italiano per lo studio dei problemi delle popolazioni.* About that time I became a charter member of the Econometric Society.

In 1932 Professor E. V. Huntington sent out a circular request for any suggestions as to unusual applications of mathematics that might well be mentioned in the Century of Progress Exposition at Chicago in 1933. Naturally my 1931 inquiries abroad provided a considerable number.

In Edinburgh in 1931 Professor E. T. Whittaker had told me of positions as "actuarial students" annually available by competitive examination in the home office of the Prudential Insurance Company of America. For about ten years thereafter one or more of our Reed seniors annually won places; and some of those men are now in charge of some regional home offices of Prudential. In the 1930s, with few and mostly small fellowships available for graduate study, those actuarial openings were extremely helpful to mathematics seniors; and the number of our majors increased considerably.

Back in 1917 I had given a talk [**11**] before the science section of the Oregon State Teachers Association in which I advocated a combination mathematics course for high school freshmen: elements of algebra, together with informal experimental geometry, some solving of simple interesting right triangles using a sine or tangent, and finally using 4-place logarithms. A committee of experienced teachers, with me as chairman, was appointed to report in 1918 on the feasibility of such a plan as I had recommended. Our very detailed report was approved, and one teacher, upstate, tried out the plan for a semester with good results; but when she moved to another city the experiment ended. In the fall of 1936, however, I had an opportunity in Portland to teach an adversely selected class of 24 high school freshmen using the 1918 outline. I gladly accepted the opportunity without pay or expense money, and was delighted with the enthusiasm evoked by the subject matter. Miss Lesta Hoel, mathematics supervisor, who cooperated in the project, remarked on the eagerness shown by the pupils, none of whom had done well in 8th grade arithmetic. Eleven times during the year as I was writing the brief home-work assignments on the board, the pupils

shouted: "Oh, give us some more of those." Several of the pupils went on successfully through third-year mathematics. (I had hoped to write a text for that course, but the coming of the "New Math" made that labor unnecessary.) This teaching was in addition to my college work and it was *real* fun! Gosh! Those kids!

In June 1936 I read before NCTM a paper on the nature of mathematics [**12**].

In 1937 I taught half the summer at the University of Southern California, giving graduate courses in the calculus of variations and in introductory Galois theory, and a senior course in elliptic integrals. The next summer I attended the Cowles Conference on Economics and Statistics, and read a paper [**13**].

In 1939 our Joint Commission on Mathematics in Secondary Education finished its work; and the chairman, Professor K. P. Williams, and I spent a couple of weeks in Chicago putting the report into final form for publication [**14**].

During the year 1939–40 Reed operated a Mathematics Teaching Seminar, a fore-runner of the now familiar faculty internships. Ours was financed by the General Education Board, and the four Fellows were these now well-known professors: Harry E. Goheen, L. Louise Johnson (now Mrs. R. A. Rosenbaum), Robert A. Rosenbaum, Henry Scheffé. Each of us taught one advanced class and one freshman or sophomore section. Our discussion topics and viewpoints are summarized in our Report [**15**]. I hardly need say that it was a wonderful year for me!

In 1941 at the Summer Meeting of MAA, I reported on undergraduate research in a group of colleges [**16**]. And in 1950 at the International Congress of Mathematicians I commented on Reed's further experience in this field [**17**].

As president of the Reed chapter of Phi Beta Kappa I substituted for President Keezer at the triennial conclave in 1940, speaking on "What should Phi Beta Kappa require of a students' program?" (unpublished).

When World War II came, I served on the War Preparedness Committee, of AMS and gave a talk in Berkeley on defense problems. My publishers asked me to write a pamphlet on spherical trigonometry to supplement my Introduction to Mathematical Analysis. We were under high pressure, but by working intensely over three weekends I got the job done. By first setting up three formulas valid for *all* spherical triangles and later dealing with right triangles the pamphlet was held to 32 printed pages, including illustrative calculations relating to the course between points near San Francisco and Tokyo [**18**].

Presently Reed obtained a pre-meteorology section of 250 men to train for the Air Force. Just after their arrival, a test from Chicago showed them to be an average group of midyear college freshmen. Besides myself we rounded up a staff of four good practical teachers, two of whom were rusty on their "calculus and beyond." I adapted textbook material to the term assignments of topics received from Chicago and devised numerous examples. (So did Professor A. A. Knowlton who directed the courses in physics and mechanics. He had come to Reed when Karl Compton left in 1915, and he gave his department brilliant leadership until his retirement in 1948.)

The Reed air force scores in these three fields were very close together. In

mathematics the median score of our men at the end of the first Quarter stood at the 70 percentile mark for the 10 colleges of which Reed was one. At the end of the year our median score was at the 80 percentile mark for the 10 colleges.

In the late 1930s I had become active in AAUP and had held a minor regional office. Then, in the term 1940–42, I served as a Council member from the Pacific Northwest, and in 1944–45 I served as First Vice President. Later, for a time I served on Committees A and B. I was also active in the Oregon Academy of Science, received a citation for services to science, and served as president of the Academy in 1950. I served also for 25 years on the board of regents of Multnomah College, and was vice-president of the board 1950–51. In 1941–44 I was on the board of governors of the City Club, and in 1942 as a committee chairman I submitted a report on a plan for teacher retirement in Portland. A few years later the alumni association of my alma mater (Chicago) awarded me a citation for civic services in Portland.

For a long time I had luckily enjoyed rugged health, not missing a day from work for almost 20 years. Then, just after our pre-meteorology work ended, I had an appendectomy, followed by other hospitalization. Reed generously gave me sabbatical leave until autumn. Returning, I gave the convocation address [**19**]: "What is a liberal education?" which I have partially repeated at several universities.

Finally the war was over and Reed got our Bob and Louise Rosenbaum back, a great day for Reed. Some of our senior thesis projects then became more diversified and more sophisticated.

I retired happily in 1952, a few weeks under 71. My early research had been recognized by Poggendorff; and I had supervised 114 senior theses, most of which broke some new ground. The Marquis people had listed me in Who's Who in America since 1914. (That continued eight years longer, while I was professionally active.) In 1952–53 I served as First Vice-President of MAA.

On retiring at Reed I served for a year each at Wesleyan and at Tulane-Sophie Newcomb. While at each of those universities I gave a number of public addresses and read some minor mathematical papers. On the way from Reed to Wesleyan I stopped at U. of Wisconsin and spoke on "Some liberal aspects of collegiate mathematical training" [**20**]. On the way home from Tulane in 1954 I stopped at U. of North Carolina to conduct an educational seminar in an institute whose principal lecturers were Professors E. Artin, T. Radó, E. A. Cameron and A. W. Tucker. I also lectured for a week at the U. of Washington on "Geometric constructibility by Euclidean methods."

Enthusiasts for trisecting an angle sometimes sent us at Reed an alleged trisection, by Euclidean methods, of any angle. Knowing the impossibility of constructing by Euclidean methods an angle of 20°, I would test the alleged trisection procedure on an angle of 60° or 120°. By getting exact coordinates for all essential intersections I would obtain an *exact* expression for a sine or tangent of the alleged 20° or 40°, approximate that function to many decimals and then see how far off the construction was. I would then inform the author and tell

him that if he wished to submit further plans, "I would have to charge him consultant rates." No one ever came back; and I saved the various initial proposals for detailed analysis in a thesis by some senior who planned to teach in high school.

In the fall of 1954 the Reed president suddenly resigned after long disagreement with the Faculty Council as to administrative procedures. The board of trustees made me president to clear up the disagreements and serve until a suitable young president could be found. After two years there was full agreement on procedures, and Dr. Richard H. Sullivan came as president. He gave Reed a notably fine administration for many years.

Reed gave me an honorary LL.D., with Professor W. L. Duren participating in the ceremony, along with my Fellows of 1939–40; the two Rosenbaums, Harry Goheen, and Henry Scheffé. A few months later the University of Oregon gave me a Citation for Services to Education at Reed.

After three years of relative idleness and some Asiatic travel with my wife, I returned to Wesleyan to teach a reduced load for a year, and then taught one course each term at Portland State College (now University) for six quarters. I also frequently lectured briefly at various Institutes or Seminars, at about a dozen universities, often on one of these three subjects:

(A) Some mathematical sectors of biology;
(B) Classical mathematics used in economic theory;
(C) Some advantages of abstract thinking.

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J. B. FOURIER—ON THE OCCASION OF HIS TWO HUNDREDTH BIRTHDAY

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Fourier's work on the conduction of heat has stimulated the most diverse developments in pure mathematics. The object of the pages which follow is to trace these developments in outline.

Fourier's other contributions to mathematics, such as his work on the theory of equations and linear inequalities, will not be discussed.

1. Convergence of Fourier series. The most original aspect of Fourier's work on trigonometric series, and the one which caused the greatest misgivings among his contemporaries, was his insistence that his expansion applied to *arbitrary* functions. In his *Théorie analytique de la Chaleur*, 1822, he says (Section 417): "In general the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary . . . We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity." This general concept of a function had appeared before Fourier, although more commonly "function" meant "function defined by an analytical expression." Fourier asserted that the two were the same. After giving what we would call today a plausibility argument rather than a proof he says (Section 418): "Thus there is no function $f(x)$, or part of a function, which cannot be expressed by a trigonometric series."

In this generality his statement is certainly not true. The first rigorous proof under wide conditions of the possibility of expanding a function in a Fourier series was given by Dirichlet (1829). In the extended form given it by C. Jordan (1881); his result reads: the Fourier series of a function f which is the difference of two increasing functions (i.e. a function of bounded variation) converges at any point x with sum $\frac{1}{2}[f(x+0)+f(x-0)]$.

Hamilton (1843), in a discussion of the convergence of Fourier series, proved that if f is continuous in an interval $[a, b]$ then

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$$\int_a^b f(x) \sin nx \, dx \to 0 \quad \text{as } n \to \infty.$$

This was extended from continuous to integrable functions by Riemann and Lebesgue and the result is now known as the Riemann-Lebesgue Lemma. Hamilton's contribution is forgotten. It follows from this lemma that the convergence of the Fourier series of a function at a particular point depends only on the behaviour of the function in an arbitrarily small neighbourhood of this point. Another almost immediate consequence is the convergence criterion of Dini (1880): the Fourier series of f converges at the point x with sum s if the integral

$$\int_0^{2\pi} | f(x+t) + f(x-t) - 2s | \, dt/t$$

exists.

Although the convergence tests of Dini and Dirichlet suffice for applications, a number of more refined tests have been given. Rather than describe them, I shall mention some results which show in what way a Fourier series may fail to converge. Du Bois Reymond (1876) gave an example of a continuous function whose Fourier series diverges on an everywhere dense set of points. Kolmogorov (1926) gave an example of a Lebesgue-integrable function whose Fourier series is everywhere divergent. Recently Carleson (1966) solved a long-standing problem by showing that the Fourier series of a function f in the space $L^2[0, 2\pi]$ (see Section 3) converges except on a *null-set* (i.e. a set which can be enclosed in a sequence of intervals whose total length is as small as one pleases). In particular, the Fourier series of a continuous function converges except possibly on a null-set. Finally, Kahane and Katznelson (1966) have shown that for any null set E there is a continuous function whose Fourier series diverges at each point of E.

Fejér (1904) showed that the situation is greatly simplified if instead of considering the convergence of the sequence of partial sums

$$s_N(x) = \sum_{n=-N}^{N} c_n e^{inx},$$

one considers the convergence of the sequence of arithmetic means

$$\bar{s}_N(x) = \frac{1}{N+1} [s_0(x) + s_1(x) + \cdots + s_N(x)].$$

If f is Lebesgue integrable, then $\bar{s}_N(x) \to f(x)$ for all x except possibly those in a null set; and if f is continuous at the point x, then $\bar{s}_N(x) \to f(x)$. Moreover if f is everywhere continuous the convergence is uniform. Fejér obtained in this way a simple proof of the Theorem of Weierstrass (1885) that each continuous function of period 2π can be uniformly approximated by trigonometric polynomials, i.e. by functions of the form

$$\sum_{n=-N}^{N} d_n e^{inx}.$$

A far-reaching generalization of Weierstrass' Theorem, and its analogue for ordinary polynomials, has been given by Stone (1948).

The possibilities of representing 'arbitrary' functions by Fourier series are illustrated by the first published example, due to Weierstrass (1875), of a function which is everywhere continuous and nowhere differentiable:

$$\sum_{n=0}^{\infty} a^n \cos (b^n x),$$

where $0<a<1$, b is an odd positive integer, and $ab>1+(3\pi/2)$.

2. Trigonometric series. To establish the convergence of the Fourier series

$$(1) \qquad f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,$$

for as wide a class of functions f as possible, one must be able to define the integral of such a function. It was for this reason that Riemann (1854) in his *Habilitationsschrift* "On the representation of a function by a trigonometric series" introduced what we now call the Riemann integral, generalizing the definition given by Cauchy for the integral of a continuous function. We shall return to the Riemann integral shortly. The main part of Riemann's paper was concerned with the representation of functions by general trigonometric series $\sum c_n e^{inx}$ in which the coefficients c_n are not necessarily given by the integral formulae (1). By an ingenious argument based on integrating twice term by term, he obtained necessary and sufficient conditions for the possibility of such a representation.

Riemann's work on trigonometric series was followed by that of Cantor. Cantor was concerned with the question whether the sum of a trigonometric series uniquely determines its coefficients. He showed first that if a trigonometric series converges to zero at every point of the interval $[0, 2\pi]$, then its coefficients must all be zero. In trying to extend this result he was led to the concept of *derived* set. Let E be a set of real numbers. The derived set E' of E consists of all real numbers x such that any neighbourhood of x contains a point of E distinct from x. One can then form the derived set E'' of E', and so on. Cantor (1872)* proved that any set E, whose nth derived set is empty for some positive integer n, is a *set of uniqueness*, i.e. a trigonometrical series which converges to zero at all points outside E must have all its coefficients zero. From the concept of derived set he was led to the concept of *closed* set (a set which contains its derived set as a subset) and thence to the general study of point set topology.

* Cantor's method of constructing the real numbers from the rationals by means of fundamental sequences appears at the beginning of this paper.

A set with empty nth derived set is either finite or *countable*, i.e. it can be put into 1-1 correspondence with the set of positive integers. Cantor then showed that the set of all algebraic numbers is countable, but the set of all real numbers is not countable. This led him to the general notion of 1-1 correspondence between two sets and the concept of cardinal number. Incidentally it was later shown by W. H. Young (1908) that any countable set is a set of uniqueness. However, not all sets of uniqueness are finite or countable.

3. Integration. We have seen that the discussion of convergence of Fourier series provoked a widening of the concept of "integral." The most satisfactory extension was found by Lebesgue (1902). It will be explained here in the alternative form due to Daniell (1917).

We are all agreed about what value the integral of a (real-valued) step function should have. If f has the constant value c_k of an interval J_k of length l_k ($k=1, \cdots, N$) and is everywhere else zero then the integral is defined by

$$I(f) = \sum_{k=1}^{N} c_k l_k.$$

The set S of all step-functions has the property that if f and g are in S then so are $|f|$, $f+g$ and cf for any real number c. Moreover,

$$I(f+g) = I(f) + I(g), \tag{2_1}$$

$$I(cf) = cI(f), \tag{2_2}$$

$$I(f) \geqq 0 \quad \text{if } f \geqq 0, \tag{2_3}$$

$$I(f_n) \to 0 \quad \text{if } f_1 \geqq f_2 \geqq \cdots \quad \text{and} \quad f_n \to 0. \tag{2_4}$$

The problem of integration is to extend the set of integrable functions so that these properties are preserved. Riemann solved this problem in the following way. Suppose there exists an increasing sequence of step functions $s_1 \leqq s_2 \leqq \cdots$ and a decreasing sequence of step functions $t_1 \geqq t_2 \geqq \cdots$ such that $s_n \leqq f \leqq t_n$ for all n and

$$\lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} I(t_n). \tag{$*$}$$

Then we say that f is Riemann integrable and we define $I(f)$ to be the common value of the limits ($*$).

Lebesgue's more general solution proceeds in two stages. Suppose first that we have an increasing sequence of step functions $s_1 \leqq s_2 \leqq \cdots$ whose integrals are bounded, $I(s_n) \leqq c$ for all n. Then $f(x) = \lim_{n\to\infty} s_n(x)$ exists for all x and $\lim_{n\to\infty} I(s_n)$ exists. We define $I(f) = \lim_{n\to\infty} I(s_n)$. The function f has values in the extended real number system, i.e. it can assume infinite values, although the boundedness of the integrals of the approximating step functions does restrict the set of points at which their limit is infinite (it must be a null set). It is not difficult to show that this definition does not depend on the approximating sequence of step functions, i.e. if $t_1 \leqq t_2 \leqq \cdots$ is another increasing sequence of

step functions such that $I(t_n) \leqq d$ for all n and if $f(x) = \lim_{n\to\infty} t_n(x)$ for all x then $\lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} I(t_n)$.

Let us call the functions f for which the integral is now defined "over" functions. Also let us call f an "under" function if $-f$ is an over function, and set $I(f) = -I(-f)$. There is no inconsistency in this if f is both "under" and "over." We now complete our definition by saying that a function f is Lebesgue integrable if there exists an increasing sequence of under functions $s_1 \leqq s_2 \leqq \cdots$ and a decreasing sequence of over functions $t_1 \geqq t_2 \geqq \cdots$ such that $s_n \leqq f \leqq t_n$ for all n and $\lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} I(t_n)$. Moreover we define $I(f)$ to be the common limit.

The integral thus defined has the properties $(2_1)-(2_4)$. Also it is easy to show that we get no further by repeating the process. If $f_1 \leqq f_2 \leqq \cdots$ is an increasing sequence of integrable functions such that $I(f_n) \leqq c$, then $f = \lim_{n\to\infty} f_n$ is already integrable and $I(f) = \lim_{n\to\infty} I(f_n)$.

Complex valued functions can be included by saying that f is integrable if its real and imaginary parts are integrable, and setting

$$I(f) = I(\Re f) + iI(\Im f).$$

It is customary to denote by $L^p(a, b)$, where $p \geqq 1$, the set of all functions f such that $|f|^p$ is Lebesgue integrable over the interval (a, b) and such that for any positive integer n there is a step function s_n with $I(|f-s_n|^p) < 1/n$.

4. Eigenfunction expansions. Fourier considered the conduction of heat in homogeneous bars. In seeking to extend his work to *inhomogeneous* bars, Sturm and Liouville (1836–37) were led to consider eigenfunction expansions defined by general second order linear differential equations. If we try to solve the inhomogeneous heat equation by the method of separation of variables, we obtain an ordinary differential equation

$$(k(x)y')' + [\lambda g(x) - l(x)]y = 0,$$

with boundary conditions

$$y'(a) - hy(a) = 0, \qquad y'(b) + Hy(b) = 0.$$

Here $k(x)$ and $g(x)$ are positive continuous functions representing the conductivity and specific heat, while the continuous nonnegative function $l(x)$ and the nonnegative constants h, H depend on the emissivity at the surface and ends of the bar respectively.

The values of λ for which there is a nontrivial solution y are called the *eigenvalues* of the boundary value problem and the corresponding solutions the *eigenfunctions*. Sturm and Liouville showed that there is an infinite sequence of positive eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with $\lambda_n \to \infty$. Moreover each eigenvalue λ_n is simple, i.e. the corresponding eigenfunction y_n is uniquely determined up to a constant factor, and eigenfunctions corresponding to different eigenvalues are *orthogonal* in the sense that

$$\int_a^b y_n(x)y_m(x)g(x)dx = 0 \quad \text{if } n \neq m.$$

They also obtained many results on the zeros of the eigenfunctions y_n.

With an "arbitrary" function f we associate the eigenfunction expansion

$$f(x) \sim \sum_{n=1}^{\infty} c_n y_n(x),$$

where

$$c_n = \int_a^b f(x)y_n(x)g(x)dx \Big/ \int_a^b y_n^2(x)g(x)dx.$$

This generalises the ordinary Fourier series, to which it reduces for $k(x) = \text{const.}$, $g(x) = \text{const.}$, $l(x) = 0$ and $h = H = 0$. Probably the simplest way of proving the convergence of the eigenfunction expansion is to introduce a Green's function. This replaces the boundary value problem by an equivalent integral equation, to which Hilbert's (1904) theory of integral equations with symmetric kernel can be applied. Indeed this was the first application which Hilbert made of his theory and it may be assumed that this was one of his motives for its construction.

De la Vallée Poussin (1893) proved that for each Riemann integrable function f with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{3}$$

This is usually known as Parseval's equation. It could with equal historical justification be attributed to Pythagoras, since it is an infinite-dimensional generalisation of the fact that in any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides. Parseval's equation was extended to functions f in $L^2[0, 2\pi]$ by Fatou (1906) and to Sturm-Liouville eigenfunction expansions by Steklov (1901). F. Riesz (1907) and Fischer (1907) independently found a converse to Parseval's equation: for any sequence $\{c_n\}$ of complex numbers for which the series $\sum |c_n|^2$ is convergent there exists a function f in L^2 with Fourier series $\sum c_n e^{inx}$ such that (3) holds. Fischer showed that this was a corollary of a much more general result. If $\{f_n\}$ is a sequence of functions in L^2 such that

$$\lim_{m,n\to\infty} \int_a^b |f_m(x) - f_n(x)|^2 dx = 0,$$

then there exists a function f in L^2 such that

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 dx = 0.$$

This is an analogue of Cauchy's general convergence principle in which the

norm of a function f in L^2 is defined by

$$\|f\| = \left[\int_a^b |f(x)|^2 dx \right]^{1/2}.$$

It is just such closure properties which make the Lebesgue integral more convenient than the Riemann integral.

One method of proving Parseval's equation for Sturm-Liouville eigenfunction expansions, due to G. D. Birkhoff (1917), may be mentioned here, not because it is simpler than the method of reduction to an integral equation, but because there has been a revival of interest in it recently. Liouville showed that the eigenvalues and eigenfunctions of his problem behave asymptotically for $n \to \infty$ like those of an ordinary Fourier series. On the other hand it can be shown that if $\{y_n\}$, $\{z_n\}$ are two orthogonal sequences in L^2 with $\|y_n\| = \|z_n\| = 1$ for all n, and if the sequences are close in the sense that the series $\sum \|y_n - z_n\|^2$ is convergent, then Parseval's equation holds for one sequence if it holds for the other. In this way the validity of Parseval's equation for general Sturm-Liouville expansions follows from its validity for the ordinary Fourier expansion.

5. The Fourier integral. Fourier series had to some extent been anticipated in the work of Clairaut, Euler, and Lagrange. The Fourier integral was Fourier's own. He obtained it from his series by a limiting process in the manner which is still given in textbooks. It is most simply stated as an inversion formula:

$$(4) \qquad \hat{f}(y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{-iyx} dx, \qquad f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \hat{f}(y) e^{iyx} dy$$

and is valid under conditions analogous to those for the convergence of Fourier series. Only the analogue of the Parseval equation, due to Plancherel (1910), will be mentioned here. It states that if f is in $L^2 = L^2(-\infty, \infty)$, the sequence

$$\hat{f}_n(y) = \frac{1}{(2\pi)^{1/2}} \int_{-n}^{n} f(x) e^{-iyx} dx$$

converges *in* L^2 to a function $\hat{f}$ such that

$$f_n(x) = \frac{1}{(2\pi)^{1/2}} \int_{-n}^{n} \hat{f}(y) e^{iyx} dy$$

converges *in* L^2 to f and $\|\hat{f}\| = \|f\|$.

The Fourier integral is associated with the differential equation $y'' + \lambda y = 0$ over the interval $(-\infty, \infty)$. The extension to general second order linear differential equations over an infinite interval was first made by H. Weyl (1910). The situation is complicated by the fact that the spectrum, instead of being discrete (viz. the sequence of eigenvalues λ_n) as in the ordinary Sturm-Liouville case, or continuous (viz. the whole line $-\infty < \lambda < \infty$) as in the case of the

Fourier integral, may be a combination of the two. The extension to differential equations of arbitrary order, which presents little difficulty for ordinary boundary value problems, was first achieved for singular boundary value problems by Kodaira (1950) and M. G. Krein (1950). The most elementary way of obtaining their results is to follow Fourier and apply a limiting process to the results for a finite interval.

We consider next the algebraic properties of the Fourier integral. Let $L=L^1(-\infty, \infty)$ denote the set of all complex-valued functions which are (Lebesgue) integrable over the interval $(-\infty, \infty)$. For any function f in L the Fourier transform $\hat{f}$ is defined and is a continuous function. The transformation $f\to\hat{f}$ is *linear*, i.e. the transform of $f+g$ is $\hat{f}+\hat{g}$ and the transform of cf is $c\hat{f}$, for any complex number c. Again, if f and g are in L their *convolution* product $f * g$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

is also in L and the transform of $f * g$ is the ordinary product $\hat{f}\hat{g}$. This important property seems to have been first observed by Čebyšev (1890/1) in the context of probability theory. Finally the transform of $f(x+a)$ is $e^{iya}\hat{f}(y)$ and, if the derivative $f'(x)$ is in L, its transform is $iy\hat{f}(y)$. It is the last property, which replaces differentiation by a simple algebraic process, that makes Fourier transforms especially useful in the solution of differential equations.

New applications of the Fourier integral were found by Wiener (1932), whose general 'Tauberian' theorem embraced a vast number of analytical results which previously had been obtained by different and quite special arguments. We state first the analogue of his theorem for series, which Wiener used as a stepping stone towards the corresponding result for integrals: Let $\hat{f}$ be a continuous complex valued function of period 2π with an absolutely convergent Fourier expansion:

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} f(n)e^{inx}, \qquad \sum_{n=-\infty}^{\infty} |f(n)| < \infty. \tag{5}$$

If $\hat{f}$ never vanishes then its reciprocal $1/\hat{f}$ also has an absolutely convergent Fourier expansion. A much clearer proof of this result has been given by Gelfand (1941) by means of the theory of Banach algebras, then in its infancy. It is a fine example of the application of algebraic ideas to problems in analysis.

Let $L(Z)$ denote the set of all functions f defined on the integers for which the series $\sum_{n=-\infty}^{\infty} |f(n)|$ is convergent. $L(Z)$ becomes an *algebra* if we define the sum $f+g$ and product $f * g$ of two functions by

$$(f+g)(n) = f(n) + g(n), \qquad (f * g)(n) = \sum_{m=-\infty}^{\infty} f(n-m)g(m).$$

The function **1** which takes the values 1 for $n=0$ and 0 for $n\neq 0$ is an identity

for multiplication. An *ideal* in this algebra is a subset I such that if f and g are in I and h is in $L(Z)$ then $f+g$ and $f*h$ are in I. For example, the set of all functions $f*h$. where h runs through $L(Z)$, is an ideal, the ideal *generated* by f. An ideal is *maximal* if it is not the whole algebra $L(Z)$ and is not a subset of any other ideal. Any ideal, apart from $L(Z)$ itself, is contained in a maximal ideal.

For any function f in $L(Z)$, let $\hat{f}$ denote the continuous function of period 2π defined by (5). It is readily seen that $(f*g)\hat{} = \hat{f}\hat{g}$ and hence that the set of all functions g in $L(Z)$ whose transforms $\hat{g}$ vanish at a particular point y is a maximal ideal. Gelfand showed that, conversely, each maximal ideal in $L(Z)$ is obtained in this way. Thus if f has the property that its transform $\hat{f}$ never vanishes, it is contained in no maximal ideal. Therefore the ideal generated by f is the whole of $L(Z)$. Thus f has an inverse f^{-1} in $L(Z)$ and the transform of f^{-1} is the reciprocal of $\hat{f}$.

Wiener's Tauberian Theorem says that if f is a function in $L=L'(-\infty, \infty)$, then each function in L can be approximated arbitrarily closely in L by finite linear combinations $\sum_{k=1}^{N} c_k f(x-x_k)$ of translates of f if and only if the Fourier transform $\hat{f}$ never vanishes. Since the convolution product $f*g$ is a limit of linear combinations of translates of f, the set of limits of such linear combinations is the same as the closed ideal in L generated by f. The argument is now similar to that in the series case, although there is no multiplicative identity.

Wiener's Theorem on absolutely convergent Fourier series was extended by Levy (1933). The analogue of this extension for integrals was stated by Paley and Wiener (1934) and may again be proved by the method of maximal ideals: If $\hat{f}(y)$ is the Fourier transform of a function $f(x)$ in L and if $\phi(u)$ is analytic over the range of values of $\hat{f}(y)$ for $-\infty \leqq y \leqq \infty$ (i.e. 0 is included), then $\phi[\hat{f}(y)]$ is also the Fourier transform of a function in L.

6. Almost periodic functions and positive definite functions. H. Bohr (1924–26) defined a continuous complex valued function f on $(-\infty, \infty)$ to be *almost periodic* if for each $\epsilon>0$ there is a corresponding $T=T(\epsilon)>0$ such that every interval of length T contains at least one point a with the property

$$|f(x+a)-f(x)| < \epsilon \qquad \text{for } -\infty < x < \infty.$$

Bochner (1927) showed that this was equivalent to requiring each sequence $\{a_n\}$ of real numbers to contain a subsequence $\{a_n'\}$ for which the sequence of translates $f(x+a_n')$ converges uniformly on $(-\infty, \infty)$. Bohr's main object was the construction of a theory of Fourier series for almost periodic functions. He showed that the limit

$$c(\lambda) = \lim_{X\to\infty} \frac{1}{2X}\int_{-X}^{X} f(x)e^{-i\lambda x}dx$$

exists for each real number λ and is different from zero for at most countably many values of λ. For the corresponding Fourier series

$$f(x) \sim \sum_{\lambda} c(\lambda)e^{i\lambda x}$$

the Parseval equation holds:

$$\lim_{X\to\infty} \frac{1}{2X}\int_{-X}^{X} |f(x)|^2 dx = \sum_{\lambda} |c(\lambda)|^2.$$

Moreover any almost periodic function can be approximated uniformly on $(-\infty, \infty)$ by generalised trigonometric polynomials $\sum_{k=1}^{N} d_k e^{i\lambda_k x}$, and conversely any function which can be uniformly approximated by generalised trigonometric polynomials is almost periodic.

Fourier integrals are an invaluable tool in the theory of probability. A random variable is described by its *distribution function*, a bounded nondecreasing function $\mu(y)$ with $\mu(y+0)=\mu(y)$ such that $\mu(-\infty)=0$ and $\mu(\infty)=1$. Its *characteristic function* is the Fourier-Stieltjes transform

$$f(x) = \int_{-\infty}^{\infty} e^{ixy} d\mu(y). \tag{6}$$

(Note: In the definition of the integral of a step function in Section 3 l_k no longer represents the length of the interval $J_k=(a_k, b_k)$ but the quantity $\mu(b_k)-\mu(a_k)$.) The convolution theorem states that to the sum of two independent random variables corresponds the product of their characteristic functions. In this way the Fourier-Stieltjes transform becomes the most powerful method for establishing the convergence of a sequence of random variables.

Bochner (1932) found an interesting intrinsic characterization of characteristic functions. A complex valued function f defined on $(-\infty, \infty)$ is said to be *positive definite* if for any finite set of real numbers $x_1, \cdots, x_n$ and any finite set of complex numbers $c_1, \cdots, c_n$,

$$\sum_{j,k=1}^{n} f(x_j - x_k) c_j \bar{c}_k \geqq 0.$$

Bochner showed that a function f could be represented in the form (6) for some bounded nondecreasing function μ, if and only if it was continuous and positive definite.

7. Fourier analysis on groups. Let G be a *locally compact topological group*, i.e. a group on which a topology is defined such that the group operations (multiplication and inversion) are continuous and such that each point has a compact neighbourhood. Haar (1933) showed how to define for all real-valued continuous functions on G which vanish outside compact sets, an integral I, not identically zero, with the properties $(2_1)-(2_4)$ and with the additional left invariance property

$$I(f_y) = I(f), \quad \text{where} \quad f_y(x) = f(yx).$$

Moreover this integral is uniquely determined apart from a positive constant factor. The domain of definition of the integral can then be extended by the process used in Section 3. We denote by $L=L(G)$ the set of all complex-valued

functions which are integrable on G.

For any two functions f, g in L the functions $f+g$ and $f*g$ defined by

$$(f+g)(x) = f(x) + g(x), \qquad (f*g)(x) = I[f(xy)g(y^{-1})]$$

are again in L. With these definitions of addition and multiplication L forms an algebra, the *group algebra* of G. If we define a norm by setting

$$\|f\| = I(|f|),$$

then L is actually a Banach algebra.

Suppose now that G, and hence also L, is commutative. Then a *character* of G is defined to be a continuous mapping γ of G into the complex numbers of absolute value 1 such that

$$\gamma(xy) = \gamma(x)\gamma(y) \qquad \text{for all } x, y \text{ in } G.$$

If we define the product of two characters γ_1, γ_2 by

$$(\gamma_1\gamma_2)(x) = \gamma_1(x)\gamma_2(x)$$

then the set Γ of all characters becomes a commutative group, the *dual* group of G. We give Γ a topology by defining the basic open sets to be the sets $H(C, \epsilon, \gamma_0)$ of all γ in Γ such that

$$|\gamma(x) - \gamma_0(x)| < \epsilon \qquad \text{if } x \text{ is in } C,$$

for some compact set C in G, some $\epsilon > 0$, and some γ_0 in Γ. With this topology Γ is also a locally compact topological group.

The Fourier transform of a function f in $L(G)$ is the continuous function $\hat{f}$ on Γ defined by

$$f(\gamma) = I_G[f(x)\overline{\gamma(x)}].$$

Then the Fourier transform of $f*g$ is $\hat{f}\hat{g}$. A function f on G is said to be positive definite if for any finite set $x_1, \cdots, x_n$ of elements of G and any finite set $c_1, \cdots, c_n$ of complex numbers

$$\sum_{j,k=1}^{n} f(x_j x_k^{-1}) c_j \bar{c}_k \geqq 0.$$

The Fourier Inversion Theorem holds in the following form: if f is a continuous positive definite function in $L(G)$ then $\hat{f}$ is a continuous function in $L(\Gamma)$ and

$$f(x) = I_\Gamma[\hat{f}(\gamma)\gamma(x)],$$

provided the invariant integral on Γ is suitably normalised.

A. Weil (1938) showed that Plancherel's Theorem and Bochner's Theorem on positive definite functions can be extended to this general situation, as can Wiener's Tauberian Theorem. The duality Theorem of Pontryagin (1939) says that conversely G is the dual of Γ.

Ordinary Fourier series and integrals both appear as special cases. In the

first case G is the additive group of all integers and its dual Γ is the multiplicative group of complex numbers of absolute value 1. In the second case G is the additive group of all real numbers and is its own dual. Seeing the two as special cases of the same phenomenon adds to our understanding of them.

The theory of almost periodic functions has been extended by von Neumann (1934) to arbitrary groups.

8. Singular integral equations. Numerous problems in mathematical physics lead to integral equations of the form

$$f(t) - \int_0^\infty k(t-s)f(s)ds = g(t) \qquad (0 \leqq t < \infty), \tag{7}$$

where f is the unknown function and k and g are given. The first explicit solutions of the corresponding homogeneous equation ($g=0$) were obtained by Wiener and Hopf (1931) for kernels k which are exponentially small at infinity. Their method depended on taking Fourier transforms and representing a function analytic in the strip $|Iz| < c$ as the product of two functions, one analytic in the half-plane $Iz > -c$ and the other analytic in the half-plane $Iz < c$. Rapoport (1948) made less stringent restrictions on the kernel k by reducing the integral equation (7) to Hilbert's problem on the boundary values of analytic functions. Then M. G. Krein (1958) treated the equation (7) under the sole conditions that k is in $L = L^1(-\infty, \infty)$ and that its Fourier transform

$$K(\lambda) = \int_{-\infty}^\infty k(t)e^{i\lambda t}dt \neq 1 \qquad \text{for } -\infty < \lambda < \infty.$$

(Note: The sign of the exponent in the integrand has been chosen to agree with Krein.) The basis of his method is the Theorem of Wiener and Levy mentioned at the end of Section 5.

Let

$$\nu = -\frac{1}{2\pi}\Delta \arg[1 - K(\lambda)],$$

where $\Delta \arg \phi(\lambda)$ denotes the net increase in $\arg \phi(\lambda)$ as λ increases from $-\infty$ to ∞. ν is an integer, since $K(\lambda) \to 0$ as $\lambda \to \pm\infty$ by the Riemann-Lebesgue Lemma.

Krein shows that the integral equation (7) has a unique solution f in L for every g in L if and only if $\nu = 0$. If $\nu > 0$ then (7) is always soluble but the solution is not unique, since the corresponding homogeneous equation

$$f(t) - \int_0^\infty k(t-s)f(s)ds = 0$$

has exactly ν linearly independent solutions.

If $\nu < 0$ then (7) either has no solution in L or a unique solution. The latter

case occurs if and only if

$$\int_0^\infty g(t)h_j(t)dt = 0 \qquad (j = 1, \cdots, |\nu|),$$

where $h_1, \cdots, h_{|\nu|}$ is a basis for the solutions of the adjoint homogeneous equation

$$h(t) - \int_0^\infty k(s-t)h(s)ds = 0.$$

Thus if $\nu \neq 0$, the homogeneous equation and its adjoint do not have the same number of linearly independent solutions, contrary to what occurs in the ordinary Fredholm theory of integral equations. These results have analogues for systems of linear equations of the form

$$\sum_{m=0}^{\infty} k_{n-m} f_m = g_n \qquad (n = 0, 1, 2, \cdots),$$

and extensions to the case where f and g in (7) are vector functions and k is a matrix function.

9. Generalized functions. The theory of distributions of L. Schwartz (1950–51) and the various "generalized functions" of Gelfand and Šilov (1958) are closely connected with the Fourier transform. Indeed this is the main feature which distinguishes Schwartz's theory from its precursors. We describe here the elementary approach used by Temple (1955).

An infinitely differentiable function on $(-\infty, \infty)$ is said to be *rapidly decreasing* if it, and its derivatives of all orders, tend to zero faster than any negative power of $|x|$ as $x \to \pm\infty$. For example, e^{-x^2} is rapidly decreasing. We denote the set of all rapidly decreasing functions by S. It is a linear space which contains $f(ax+b)$ for real $a \neq 0$ and b if it contains $f(x)$.

A sequence $\{f_n\}$ of functions in S is said to be *convergent* if for any function g in S the numerical sequence

$$(f_n, g) = \int_{-\infty}^{\infty} f_n(x)g(x)dx$$

converges. We call two convergent sequences *equivalent* if the corresponding limits are the same for every g in S. We then define a *generalized function* F to be an equivalence class of convergent sequences and we set

$$(F, g) = \lim_{n\to\infty} (f_n, g).$$

We can regard any rapidly decreasing function f as a generalized function by identifying it with the equivalence class containing the sequence $\{f_n\}$ in which $f_n = f$ for all n. The sequence $\{(n/\pi)^{1/2}e^{-nx^2}\}$ is easily seen to be convergent. The corresponding generalized function will be denoted by δ. It has the property

$(\delta, g) = g(0)$. Dirac's popular delta-function thus acquires a precise meaning.

The sum of two generalized functions, linear transformations of the independent variable, and the product of a generalized function by a constant or, more generally, a slowly increasing function are naturally defined. Here an infinitely differentiable function is said to be *slowly increasing* if it and all its derivatives are bounded by some power of $|x|$ as $x \to \pm\infty$. For example, $e^{i\lambda x}$ is slowly increasing for any real λ.

For any two functions f, g in S we have

$$(f', g) = -(f, g').$$

This enables us to define the derivative DF of a generalized function F: if the equivalence class F contains the convergent sequence $\{f_n\}$ then DF is the equivalence class containing the convergent sequence $\{f_n'\}$. Also the Fourier transform maps S onto itself. (It is this property and closure under differentiation that determine the choice of S.) Finally, for any two functions f, g in S we have

$$(f, g) = (\hat{f}, \hat{g}).$$

This enables us to define the Fourier transform $\hat{F}$ of a generalized function F: if the equivalence class F contains the convergent sequence $\{f_n\}$, then $\hat{F}$ is the equivalence class containing the convergent sequence $\{\hat{f}_n\}$. These definitions are easily shown to be consistent, i.e. they do not depend on the choice of sequence $\{f_n\}$ within an equivalence class. By the inversion theorem the Fourier transform of $\hat{F}(x)$ is $F(-x)$. Moreover the transform of δ is the constant $(2\pi)^{-1/2}$ and the transform of DF is the product of $\hat{F}$ by the slowly increasing function ix.

Finally, a sequence $\{F_n\}$ of generalized functions is said to *converge* to the generalized function F if

$$(F, g) = \lim_{n\to\infty} (F_n, g)$$

for any function g in S. If $F_n \to F$ then also $DF_n \to DF$ and $\hat{F}_n \to \hat{F}$.

The theory of trigonometric series is particularly simple within the domain of generalized functions. A trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to a generalized function F if and only if its coefficients c_n increase no faster than some power of $|n|$ as $n \to \pm\infty$. Moreover F is invariant under translation by 2π, and any generalized function which is invariant under translation by 2π can be uniquely represented as the sum of a convergent trigonometric series.

Generalized functions have found applications in several branches of mathematics, notably in the study of linear partial differential equations with constant coefficients, where they are now indispensable.

10. Miscellany. The summation formula of Poisson (1823) connects the values of a function f on a subgroup of the real line with the values of its Fourier transform $\hat{f}$ on another subgroup:

$$(2\pi)^{1/2} \sum_{n=-\infty}^{\infty} f(2n\pi) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

It holds for functions f in L which are of bounded variation and are normalised so that $f(x) = [f(x+0)+f(x-0)]/2$. Consequences of this formula include Jacobi's imaginary transformation of the theta functions, the reciprocity law for Gaussian sums, and Riemann's functional equation for the zeta-function.

Fejér and F. Riesz (1916) showed that any trigonometric polynomial

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

such that $f(x) \geqq 0$ for all real x can be expressed in the form $f(x) = |g(e^{ix})|^2$, where

$$g(w) = \sum_{n=0}^{N} a_n w^n.$$

Moreover g is uniquely determined if we require further that $g(0) > 0$ and $g(w) \neq 0$ for $|w| < 1$. This was extended by Szegö (1921): Let $f \neq 0$ be a nonnegative function in $L[0, 2\pi]$. Then there exists a function g in $L^2[0, 2\pi]$ such that $f = |g|^2$ and

$$\int_0^{2\pi} g(x) e^{inx} dx = 0 \qquad \text{for } n = 1, 2, \cdots$$

if and only if $\log f$ is in $L[0, 2\pi]$. Moreover there exists a unique g for which also

$$\frac{1}{2\pi} \int_0^{2\pi} g(x) dx = \exp\left[\frac{1}{4\pi} \int_0^{2\pi} \log f(x) dx\right].$$

Szegö's result has found applications to the prediction theory of stationary stochastic processes.

Paley and Wiener (1934) considered Fourier transforms in the complex domain. Only two of their results will be mentioned here:

A function $F(z)$ can be represented in the form

$$F(z) = \int_0^{\infty} f(t) e^{-zt} dt,$$

where f is in $L^2(0, \infty)$, if and only if it is analytic in the half-plane $Rz > 0$ and

$$\int_{-\infty}^{\infty} |F(x+iy)|^2 dy < \text{constant} \qquad \text{for } 0 < x < \infty.$$

A function $F(z)$ can be represented in the form

$$F(z) = \int_{-a}^{a} f(t) e^{itz} dt,$$

where f is in $L^2(-a, a)$, if and only if it is an entire function (i.e. the sum of an everywhere convergent power series), it is in L^2 on the real axis and

$$|F(z)| \leqq Ce^{a|z|},$$

for some positive constant C. It can be shown that any such function is uniquely determined by its values at a suitable sequence of equally spaced points on the real line, in fact

$$F(z) = \sum_{n=-\infty}^{\infty} F(n\pi/a) \sin(az - n\pi)/(az - n\pi).$$

11. Conclusion. Enough has been said to show how profoundly Fourier's work has influenced the development of mathematics, directly and indirectly.

An expanded version of a lecture given at the Australian National University, Canberra, on April 1, 1968. In addition Professor J. C. Jaeger spoke on the significance of Fourier's work for applied mathematics. Some of the material here also formed part of the third Behrend Memorial Lecture, given at the University of Melbourne on August 2, 1968.

I am grateful to Professor S. Izumi for the reference to Kahane and Katznelson, and to Drs. R. E. Edwards and P. Mandl for pointing out an error in the original treatment of the Lebesgue integral.

FUNCTIONAL ANALYSIS PROOFS OF SOME THEOREMS IN FUNCTION THEORY

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We present here functional analysis proofs of three theorems in function theory; the first two theorems are classical and the third is well known. The first theorem is Runge's Theorem on approximation by rational functions, which readily implies the Cauchy Integral Theorem. The second is the familiar theorem that there exists an analytic function that interpolates arbitrary values on any discrete subset of a given open set in the complex plane. This result readily implies the Mittag-Leffler Theorem, which in turn easily implies the Weierstrass Theorem about the existence of analytic functions with arbitrarily prescribed discrete zero set. The third result is that every closed ideal in the ring of functions analytic on a region is principal. The proofs are new, although their substance seems to be known to some workers in the field. For example, a closely related proof of Runge's Theorem appears in [2, pp. 47–48], which is relatively inaccessible.

Our proofs are based on the duality between the space $H(G)$ of all functions holomorphic on the region G in the complex plane, in the topology of uniform convergence on compact subsets of G, and the space $H_0(G')$ of germs of functions

holomorphic on G', the complement of G taken relative to the Riemann sphere, that vanish at ∞. This duality has been well studied, even for analytic functions with range in a locally convex topological vector space, by such authors as A. Grothendieck [2], G. Köthe [5], C. L. da Silva Dias [8], J. Sebastião et Silva [9] and H. G. Tillman [10], and also in an unpublished work by H. Royden.

We recall the definitions and properties of the spaces $H(G)$ and $H_0(G')$. If A is any subset of the Riemann sphere, then a complex valued function f which is holomorphic on an open set $B \supseteq A$ is said to be locally holomorphic on A. Two such functions f, g are said to be equivalent, written $f \sim g$, if there is an open set $B \supseteq A$ such that the restrictions of f and g to B are the same function. We denote by $[f]$ the class of functions equivalent to f, although we may sometimes, by abuse of notation, simply write f instead of $[f]$. These equivalence classes are called germs of holomorphic functions on A, and we denote by $H(A)$ the space of all such equivalence classes. Clearly, $H(A)$ is an algebra under the obvious pointwise operations of addition and multiplication of functions, and of scalar multiplication. If A is open, then $H(A)$ is simply the vector space of all functions holomorphic on A. In case $\infty \in A$, then by $H_0(A)$ we denote the subspace of $H(A)$ consisting of all germs $[f]$ for which $f(\infty)=0$.

We suppose from now on that G is an open subset of the complex plane $\mathbf{C}$, and that G' is the complement of G on the Riemann sphere, so that $\infty \in G'$. We put on $H(G)$ the topology of uniform convergence on compact sets; that is, the topology determined by the family of seminorms $\|f\|_K = \sup\{|f(z)| : z \in K\}$, where K ranges over the compact subsets of G. Then $H(G)$ is a Frechet algebra. The main duality theorem may then be stated as follows. (See [5], p. 38, Theorem 11 and p. 35, Theorem 6. For an explicit elementary construction of the cycle Γ, see [7], pp. 155–157. We remark that the proof of the theorem uses only the Cauchy integral formula for a system of rectangles, as given in [7, p. 113].)

Since $H(G)$ is a closed subspace of $C(G)$, the space of continuous complex valued functions on G in the same topology, every functional L in $H(G)'$ has an extension, still denoted by L, to $C(G)'$. By the Riesz representation theorem, we may write $L(f) = \int f d\mu$ where μ is a complex-valued Borel measure of compact support $K \subseteq G$, so that L depends only on the values f takes on K. In particular, $L(1/(z-w)) = \int_K (1/(z-w)) d\mu(z)$ is defined as an analytic function in some open set that contains G'.

THEOREM. *The dual space $H(G)'$ of the topological vector space $H(G)$ may be identified with $H_0(G')$, where the bilinear form pairing $H(G)$ and $H_0(G')$ is given by*

$$\langle f, [F] \rangle = \frac{1}{2\pi i} \int_\Gamma f(w) F(w) dw, \tag{1}$$

for $f \in H(G)$, $[F] \in H_0(G')$, where Γ is a finite sum of closed rectifiable contours contained in the intersection of G with the domain of analyticity of the representative function F of $[F]$, that has winding number -1 around every singular point of F. If L is a continuous linear functional on $H(G)$, then the corresponding function F is

given by $F(w)=L(1/(z-w))$ *for* w *in an open set containing* G', *where* L *denotes the extended functional.*

For certain computations, it is easier to use the fact that

$$\langle f, [F]\rangle = \int f(z)d\mu(z), \tag{2}$$

where μ is the measure mentioned earlier, such that $F(w)=\int(z-w)^{-1}d\mu(z)$. In particular, it follows that if $z\in G$ and $F(w)=(z-w)^{-(k+1)}$, then for $f\in H(G)$, we have

$$\langle f, [(z-w)^{-(k+1)}]\rangle = -\frac{1}{2\pi i}\int_{|\zeta-z|=\epsilon} f(\xi)\frac{d\xi}{(\zeta-z)^{k+1}} = -\frac{f^{(k)}(z)}{k!}. \tag{3}$$

We now prove our results in function theory. Let $W=\{w_n\}$ be a sequence of points in G'; the same point may occur more than once, or even infinitely many times. By $R(W)$ we denote the set of rational functions spanned by the functions $(w-w_n)^{-1}$, with the following conventions regarding multiplicities. If the point w_n occurs only finitely many times in the sequence W, then we include only $(w-w_n)^{-1}$, but if it occurs infinitely often, then we include $(w-w_n)^{-1}$, $(w-w_n)^{-2}$, $\cdots$. If $w_n=\infty$ then we include the constant function 1 as well as the function w (and possibly w^2, w^3, $\cdots$ in case ∞ has infinite multiplicity). Our version of Runge's Theorem seems a little stronger than what can be proved by the classical method of translation of poles.

RUNGE'S THEOREM. *If* W *has at least one limit point in each component of* G', *then* $R(W)$ *is dense in* $H(G)$.

In particular, if G' is connected and $W=\{\infty, \infty, \infty, \cdots\}$ this result asserts that every function holomorphic on a simply connected region can be uniformly approximated on compact subsets by polynomials. The Cauchy Integral Theorem follows immediately from this, since the integral of a polynomial around any closed rectifiable curve is 0.

Proof. By the Hahn-Banach Theorem, it is enough to prove that if L is a continuous linear functional on $H(G)$ such that $L((w-w_n)^{-1})=0$ for each $w_n\in W$ (recall our convention about multiplicities) then $L=0$. This reduces to showing that if $F(w_n)=0$ for each $w_n\in W$, where $[F]\in H_0(G')$, then $F=0$ since

$$F^{(k)}(w) = k!\langle(z-w)^{-(k+1)}, [F]\rangle, \qquad k=0,1,2,\cdots$$
$$F^{(k)}(\infty) = -k!\langle z^{k-1}, [F]\rangle, \qquad k=1,2,3,\cdots.$$

But this follows easily, since a function holomorphic on a connected set, that vanishes on a sequence with a limit point in that set, must vanish identically there.

For the purposes of the next theorem, we place on $H_0(G')$ the weak topology as the dual of $H(G)$. The strong topology was studied in [**5**], and the next lemma

is equivalent to Theorem 13 on page 39 of that paper, since the weakly convergent sequences and strongly convergent sequences on the dual of a Montel space are the same.

LEMMA 1. *A sequence* $\{[F_n]\}$ *of elements of* $H_0(G')$ *is convergent to* $[F]$ *in the weak topology of* $H_0(G')$ *as the dual of* $H(G)$ *if and only if there exists a single open set* $W \supseteq G'$, *representatives* F_n' *of* $[F_n]$, $n=1, 2, 3, \cdots$, *and a representative* F' *of* $[F]$, *such that each* F_n' *and* F' *are holomorphic on* W *and such that* F_n' *converges uniformly to* F' *on* W.

Proof. We use the uniform boundedness principle in the form that if $\{T_n\}$ is a sequence of continuous linear functionals on the Frechet space $H(G)$ such that $T(f) = \lim T_n(f)$ exists for each $f \in H(G)$, then $\lim T_n(f) = 0$ as $f \to 0$ uniformly for $n = 1, 2, 3, \cdots$. Now suppose F_n converges weakly to F in $H_0(G')$ and choose a point $z_0 \in \partial G$. (If $z_0 = \infty$, a slight modification is necessary.) Let $T_n(f) = \langle f, F_n \rangle$. Then

$$\frac{F_n^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{F_n(z)}{(z - z_0)^{k+1}}\, dz = T_n\left(\frac{1}{(z - z_0)^{k+1}}\right).$$

By the uniform boundedness principle, we have $|T_n(f)| \leqq \sigma \|f\|_K$ for some constant σ and some compact subset K of G, so that

$$\left|\frac{F_n^{(k)}(z_0)}{k!}\right| \leqq \sigma\left(\frac{1}{\rho(z_0, K)}\right)^{k+1},$$

where $\rho(z_0, K)$ is the distance from z_0 to K, and thus the power series for F_n around z_0 has a positive radius of convergence that is independent of n. This implies that all the F_n are analytic in some one open set $A_1 \supseteq G'$. An easy extra argument shows that the F_n are uniformly bounded on some slightly smaller open set $A_2 \supseteq G'$ and then a simple argument with normal families shows that the F_n converge uniformly to F on a still smaller open set $A \supseteq G'$.

INTERPOLATION THEOREM. *Let* G *be an open subset of the complex plane, and let* $\{z_n\}$, $n=1, 2, 3, \cdots$, *be a sequence of points of* G *with no limit point in* G. *Let* $p_1, p_2, p_3, \cdots$ *be a sequence of positive integers. Then given any family* $\{a_{n,k}\}$ *of complex numbers, where* $k=0, 1, \cdots, p_n-1$ *and* $n=1, 2, 3, \cdots$, *there is a function* f *holomorphic on* G *such that* $f^{(k)}(z_n) = a_{n,k}$ *for* $k=0, 1, \cdots, p_n-1$ *and* $n=1, 2, 3, \cdots$.

REMARK. This result easily implies the Mittag-Leffler Theorem. To obtain a meromorphic function with prescribed principal part at a given discrete sequence of points, we need only take $f(z) = g(z)/h(z)$, where g and h interpolate the obvious sequence of values at the required points. For example, near $z=0$, we want, say,

$$g(z)/h(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_l}{z^l} + \phi(z)$$

where $\phi(z)$ is analytic. We choose $h(z)=z^l(1+k(z))$ near $z=0$, and want

$$\frac{g(z)}{1+k(z)} = a_1 z^{l-1} + a_2 z^{l-2} + \cdots + a_l + z^l\phi(z).$$

Choosing $k(z)=z^l m(z)$, we want only

$$g(z) = a_1 z^{l-1} + a_2 z^{l-2} + \cdots + a_l + z^l\psi(z)$$

near $z=0$, and this can be done by the Interpolation Theorem. Then, as in [7, p. 305], one can use the Mittag-Leffler Theorem to prove the Weierstrass Theorem that there is a function on G whose zero set is any given set, subject to the obvious discreteness condition. A slight modification of our proof, using the Laplace transform instead of the Cauchy transform, proves the corresponding result for entire functions of several complex variables.

Proof of the interpolation theorem. Denote by S the linear space in $H_0(G')$ generated by $[(z_n-w)^{-(k+1)}]$ for k and n as indicated, and define the linear functional L on S by putting

$$L([(z_n - w)^{-(k+1)}]) = - a_{n,k}/k!. \tag{4}$$

We shall prove that S is a closed subspace of $H_0(G')$ and that L is continuous for the topology induced on S by the weak topology on $H_0(G')$. Then by the Hahn-Banach Theorem, L may be extended to be continuous on all of $H_0(G')$. Since the dual space of $H_0(G')$ is $H(G)$, there exists a function $f\in H(G)$ such that $L([F])=\langle f, [F]\rangle$. Therefore, in view of (3) and (4), we have

$$f^{(k)}(z_n)/k! = - \langle f, [z_n - w]^{-(k+1)}\rangle = - L([(z_n - w)^{-(k+1)}]) = a_{n,k}/k!,$$

so that f is an appropriate interpolating function.

We prove explicitly that L is continuous—the same proof actually proves also that S is closed. It is enough to show that $L^{-1}(0)=\{[F]\in S: L([F])=0\}$ is a closed subspace of S, and it is therefore enough to prove that $L^{-1}(0)$ is a closed subspace of $H_0(G')$. But by a corollary of the Banach-Dieudonné Theorem ([6], p. 275), it is enough to prove that $L^{-1}(0)$ is sequentially closed. To this end, suppose that $[F_n]\in L^{-1}(0)$, $[F]\in H_0(G')$, and that $\lim [F_n]=[F]$. Then in view of the lemma, there is an open set A containing G', and there exist representatives that we designate by F_n and F, of $[F_n]$ and $[F]$ respectively, such that F_n and F are holomorphic on A, with F_n converging uniformly to F there. Without loss of generality, we suppose that each component of A intersects G'. Hence F_n must be of the form

$$F_n(w) = \sum_{j=0}^{k_n} \sum_{s=0}^{p_j-1} A_{j,s}^{(n)} \frac{1}{(z_j - w)^{s+1}} \tag{5}$$

for some family $\{A_{j,s}^{(n)}\}$ of complex numbers. However, only a finite number of the z_j lie exterior to A. Hence, only a finite number of the terms $(z_j-w)^{-(s+1)}$ can actually appear (i.e., have $A_{j,s}^{(n)}\neq 0$) in all the F_n together. We may thus suppose

that $k_n = N$, a constant, for $n=1, 2, 3, \cdots$ in (5). Then, since $F_n \to F$ uniformly on compact subsets of A, it follows that $A_{s,j} = \lim_{n\to\infty} A_{s,j}^{(n)}$ exists, and thus

$$F(w) = \sum_{j=0}^{N} \sum_{s=0}^{p_j-1} A_{j,s} \frac{1}{(z_j - w)^{(s+1)}} \cdot$$

Hence $[F] \in S$ and

$$0 = \lim_{n\to\infty} L([F_n]) = -\lim_{n\to\infty} \sum_{j=0}^{N} \sum_{s=0}^{p_j-1} A_{j,s}^{(n)} (a_{j,s}/s!)$$

$$= -\sum_{j=0}^{N} \sum_{s=0}^{p_j-1} A_{j,s}(a_{j,s}/s!) = L([F]).$$

Hence $[F] \in L^{-1}(0)$ and $L^{-1}(0)$ is closed. This completes the proof of the theorem.

IDEAL THEOREM. *Every closed ideal in $H(G)$ is principal.*

Proof. Let I be a closed ideal in $H(G)$. For $f \in H(G)$ denote by $Z(f)$ the zero set of f with multiplicities counted, and by $Z(I)$ the intersection of $Z(f)$ for those f in I. Given a discrete sequence Z of complex numbers (with multiplicity), let $I(Z)$ be the ideal consisting of all those $f \in H(G)$ that vanish at least on Z. Since, by the Weierstrass Theorem, there is a function f whose zero set is precisely Z, each ideal $I(Z)$ is principal, and to prove our theorem, we need show only that $I = I(Z(I))$. To prove this, we use the Hahn-Banach Theorem, and need only prove that if $L \in H(G)'$ and L annihilates I, then L annihilates each $g \in H(G)$ such that $Z(g) \supseteq Z(I)$. To prove this, let $[\Phi]$ be the element of $H_0(G')$ that corresponds to L. We will show that Φ is a rational function with poles only on $Z(I)$, of the correct multiplicities, and the rest follows easily. To prove this assertion about Φ we shall show that for each $f \in I$, $f\Phi$ has an analytic continuation. Let Γ be the cycle in G, mentioned earlier, that winds once, in the negative direction, around a compact set K outside of which Φ is analytic and let $\Psi = f\Phi$, so that Ψ is analytic in G except possibly on K. Also choose K so that each component of the complement of K intersects G'. Since for each $g \in H(G)$, $fg \in I$, we have

$$0 = L(fg) = \frac{1}{2\pi i} \int_\Gamma g(\zeta) f(\zeta) \Phi(\zeta) d\zeta = \frac{1}{2\pi i} \int_\Gamma g(\zeta) \Psi(\zeta) d\zeta.$$

From this, we conclude that Ψ has an analytic extension to $H(G)$. For let, for any such cycle Γ, $\tilde{\Psi}_\Gamma(z) = (1/2\pi i)\int_\Gamma (\Psi(\zeta)/(\zeta - z)) d\zeta$ for those z that Γ winds around once. It is easy to see, by the Cauchy Integral Theorem, that $\tilde{\Psi}_\Gamma$ is independent of Γ, and we shall show that $\tilde{\Psi}_\Gamma(z) = \Psi(z)$ for all suitable z. Given z_0 in $G \backslash K$, let Γ_1 be a cycle that winds once around K and has winding number 0 around z_0, let Γ_2 be a small circle that winds once around z_0, and let $\Gamma = \Gamma_1 + \Gamma_2$. Then

$$\tilde{\Psi}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{\Psi(\zeta)}{\zeta - z_0}\,d\zeta = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\Psi(\zeta)}{\zeta - z_0}\,d\zeta + \Psi(z_0).$$

Letting

$$A(w) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\Psi(\zeta)}{\zeta - w}\,d\zeta$$

for w outside Γ_1, we see that $A(w)=0$ for $w\in G'$ since $(\zeta-w)^{-1}$ then belongs to $H(G)$, and the same holds for all the derivatives of $A(w)$. By the uniqueness theorem for analytic functions, $A(z_0)=0$ so that $\tilde{\Psi}_\Gamma(z_0)=\Psi(z_0)$ and the result is proved.

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SOME PROPERTIES OF SPACES OF UNIFORMLY QUASI-CONTINUOUS FUNCTIONS

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1. In [3], Hildebrandt has extended the Ascoli-Arzela Theorem to the space of quasi-continuous functions on a number interval $[a, b]$. In this note we generalize the concept of quasi-continuous function to a function whose domain is a particular type of order space X and whose range is a subset of a metric space Y and we obtain a form of the Ascoli-Arzela Theorem for a subset of the collection of quasi-continuous functions on X into Y. Using this theorem we obtain a characterization of the compact continuous linear transformations from a

normed linear space into the Banach space of complex valued uniformly quasi-continuous functions on a generalized interval, and show that each such transformation is uniformly approximatable by transformations of finite range.

2. DEFINITION 1. *Suppose* $(X, <)$ *is a linearly ordered set. The statement that* $(X, <)$ *is a generalized interval means that* $<$ *is a gap free and antisymmetric order for* X *and that* X *has a smallest element* x_0 *and a largest element* x_f *relative to* $<$.

Whenever in this paper a generalized interval $(X, <)$ is considered as a topological space it is assumed that $(X, <)$ has been topologized with the order topology.

If x is an element of a generalized interval $(X, <)$ and x is not x_0 then $D_-(x)$ will denote the set $\{y: (y\in X)\wedge(y<x)\}$, directed by $\geqq$. Similarly if x is not x_f then $D_+(x)$ will denote the set $\{y: (y\in X)\wedge(y>x)\}$, directed by $\leqq$. There is a rather natural generalization of the concept of a quasi-continuous function to functions on a generalized interval into a metric space in terms of these directed sets.

DEFINITION 2. *The statement that a function* f *on a generalized interval* $(X, <)$ *into a metric space* Y *is quasi-continuous means that each of* $\{f(y): y\in D_-(x)\}$, $\{f(y): y\in D_+(x)\}$, $x\in(x_0, x_f)$; $\{f(y): y\in D_+(x_0)\}$; *and* $\{f(y): y\in D_-(x_f)\}$ *is a Cauchy net in* Y.

In the real case every quasi-continuous function on a number interval is uniformly approximatable by step functions. This need not be true in general, but it is with those functions which are uniformly approximatable by step functions that we shall be concerned in this paper. We shall call such a function uniformly quasi-continuous. As a formal definition we take:

DEFINITION 3. *The statement that a function* f *on a generalized interval* $(X, <)$ *into a metric space* (Y, ρ) *is uniformly quasi-continuous means that if* $\epsilon>0$, *there exists a subdivision* D *of* X *such that if* x_i *and* x_{i-1} *belong to* D *and* p *and* q *belong to* (x_{i-1}, x_i) *then* $\rho(f(p), f(q))<\epsilon$.

The subdivision D of Definition 3 will be said to be an ϵ-subdivision for f, in symbols, $D(\epsilon, f)$. It is clear that if $D(\epsilon, f)$ and E refines D then $E(\epsilon, f)$.

Throughout the remainder of this paper it will be assumed that a generalized interval $(X, <)$ and a metric space (Y, ρ) are given and that all functions introduced have domain X and range in Y.

It is easily shown that if f is a uniformly quasi-continuous function then f is quasi-continuous. If X is compact, it is a straightforward compactness argument to show that every quasi-continuous function is uniformly quasi-continuous.

It follows from this observation that Definition 5.1 of [1] is equivalent to the modified definition given in [2]. Hence Theorems 5.1 and 5.2 of [1] follow from either of these definitions.

It is an immediate consequence of Definition 2 that every continuous function on X into Y is quasi-continuous. Since it follows from Theorem 8.11 of [4]

that a generalized interval is uniformizable, one is led to ask whether a continuous function on X into Y is uniformly quasi-continuous if and only if it is uniformly continuous. If $(X, <)$ is order complete, and hence compact, the answer is clear from the preceding remarks. If X is not compact it is order isomorphic to a dense subspace of the order complete generalized interval X^* which consists of all initial segments of X which have no largest element, together with $\{x_0\}$; the order on X^* being proper set inclusion. Therefore if f is a uniformly continuous function on X into Y it may be extended to a uniformly continuous, and hence uniformly quasi-continuous, function $\bar{f}$ on X^* into the completion of Y. Then if $\epsilon > 0$ and D is a subdivision of X^* and $D(\epsilon/2, \bar{f})$ it follows from the fact that X^* is a generalized interval and that X is dense in X^* that there exists a subdivision E of X^*, each of whose points is in X, such that $E(\epsilon, \bar{f})$. Hence f is uniformly quasi-continuous. Conversely, if f is a continuous function on X into Y which is uniformly quasi-continuous then it may be extended to a continuous function $\bar{f}$ on X^* into the completion of Y in the natural way. Since X is dense in X^*, the order topology on X is equivalent to the relativized X^* topology which in turn is equivalent to the topology on X generated by the uniformity on X^* relativized to X. This implies that f is uniformly continuous. These results give a further justification for the use of the name uniform quasi-continuity in Definition 3.

3. It is easily shown from Definition 3 that the range of a uniformly quasi-continuous function is totally bounded. Therefore the collection of all uniformly quasi-continuous functions on X into Y is a metric space with the metric P defined by

$$\mathrm{P}(f, g) = \sup_{x \in X} \rho(f(x), g(x)), \qquad \forall (f, g) \in Y^X \times Y^X.$$

This metric space will be denoted by Q_X^Y. It is easily shown that Q_X^Y is a complete space if Y is.

With these preliminaries completed we may proceed to the main result, a characterization of the subsets of Q_X^Y which have compact closure.

THEOREM 1. *A subset C of Q_X^Y has compact closure if and only if $\overline{C[x]}$ is compact in Y for each x in X and C is equi-uniformly quasi-continuous.*

Proof. Suppose $\bar{C}$ is compact. It is an immediate consequence of the definition of P that $\overline{C[x]}$ is compact for each x in X. If $\epsilon > 0$ there exists a finite sequence, $\{f_i\}_{i=1}^p$ in C such that if f is in C there exists an integer $k(f)$, $1 \leqq k(f) \leqq p$, such that $\mathrm{P}(f, f_{k(f)}) < \epsilon/3$. For each integer i, $1 \leqq i \leqq p$ let D_i be a subdivision of X such that $D_i(\epsilon/3, f_i)$, and let E denote the subdivision $\bigcup_{i=1}^p D_i$. If x_{j-1} and x_j are elements of E and p and q are elements of (x_{j-1}, x_j) then for any f in C,

$$\rho(f(p), f(q)) \leqq 2\mathrm{P}(f, f_{k(f)}) + \rho(f_{k(f)}(p), f_{k(f)}(q)) < \epsilon$$

Hence C is equi-uniformly quasi-continuous.

Conversely, suppose that C satisfies the conditions of the theorem and that $\{f_{1i}\}_{i=1}^{\infty}$ is a sequence in C. Let D_1 denote a subdivision of X such that $D_1(1, f)$ for all f in C and let E_1 denote a refinement of D_1 which contains at least one point between each pair of points of D_1. Since for each x of X, $\overline{C[x]}$ is compact, there exists a subsequence $\{f_{2i}\}_{i=1}^{\infty}$ of $\{f_{1i}\}_{i=1}^{\infty}$ which is uniformly Cauchy on E_1 and such that

$$\rho(f_{2i}(x_k^{(1)}), f_{2j}(x_k^{(1)})) < 1, \qquad \forall x_k^{(1)} \in E_1, \;\; i, j = 1, 2, 3, \cdots .$$

Let D_2 denote a subdivision of X such that $D_2(\frac{1}{2}, f)$ for all f in C and let E_2 be a refinement of D_2 which contains at least one point between each pair of points of D_2. The construction now proceeds by induction. Given the sequence $\{f_{n-1i}\}_{i=1}^{\infty}$ $n=3, 4, \cdots$, and the subdivisions D_{n-1} and E_{n-1}, there exists a subsequence $\{f_{ni}\}_{i=1}^{\infty}$ of $\{f_{n-1i}\}_{i=1}^{\infty}$ which is uniformly Cauchy on E_{n-1} and such that

$$\rho(f_{ni}(x_k^{(n-1)}), f_{nj}(x_k^{(n-1)})) < 1/(n-1), \qquad \forall x_k^{(n-1)} \in E_{n-1}, \;\; i, j = 1, 2, 3. \cdots .$$

Let D_n be a subdivision of X such that $D_n(1/n, f)$ for all f in C and let E_n be a refinement of D_n which contains at least one point between each pair of points of D_n. Consider the sequence $\{f_i\}_{i=1}^{\infty}$ where $f_i = f_{ii}$. Suppose $\epsilon > 0$, and let N be a positive integer greater than $3/\epsilon$. If each of p and q is an integer greater than N then f_p and f_q are in $\{f_{N+1i}\}_{i=1}^{\infty}$. Therefore if x belongs to E_N, $\rho(f_p(x), f_q(x)) < \epsilon$. If x is in X and x does not belong to E_N then there exist points x_{j-1} and x_j of D_N such that x is in (x_{j-1}, x_j) and there exists a point $\bar{x}$ of E_N such that $\bar{x}$ is in (x_{j-1}, x_j). Then

$$\rho(f_p(x), f_q(x)) \leqq \rho(f_p(x), f_p(\bar{x})) + \rho(f_p(\bar{x}), f_q(\bar{x})) + \rho(f_q(x), f_q(\bar{x})) < \epsilon,$$

hence $\{f_i\}_{i=1}^{\infty}$ is a Cauchy sequence in Q_X^Y. For each x in X, $\{f_i(x)\}_{i=1}^{\infty}$ is then a Cauchy sequence in Y and therefore since $\overline{C[x]}$ is compact there exists a function f on X into Y to which $\{f_i\}_{i=1}^{\infty}$ converges pointwise. It is then an immediate consequence of Definition 3 and the definition of P that f is in Q_X^Y and $\{f_i\}_{i=1}^{\infty}$ P-converges to f. Since $\{f_i\}_{i=1}^{\infty}$ is a subsequence of $\{f_{1i}\}_{i=1}^{\infty}$, $\overline{C}$ is compact. This completes the proof.

4. Suppose that Y is C, the complex plane. Then Q_X^C with addition and scalar multiplication defined in the usual way is a Banach space and each element of the collection of linear functionals defined by

$$\phi_x f = f(x), \qquad x \in X, \qquad f \in Q_X^C$$

is continuous. Suppose further that V is a normed linear space and $\mathcal{K}$ is a continuous linear transformation on V into Q_X^C, and let $M\mathcal{K}$ be the function on X into V^* defined by

$$M\mathcal{K}(x) = \phi_x \circ \mathcal{K}, \qquad x \in X.$$

If $\mathcal{K}$ is compact then it follows from Theorem 1 that for each $\epsilon>0$ there exists a subdivision D of X such that if x_{j-1} and x_j are elements of D and p and q belong to (x_{j-1}, x_j) then

$$|\mathcal{K}y(p) - \mathcal{K}y(q)| = |(\phi_p \circ \mathcal{K} - \phi_q \circ \mathcal{K})y| < \epsilon,$$

for all y in the closed unit ball in V. Hence

$$\|\phi_p \circ \mathcal{K} - \phi_q \circ \mathcal{K}\| = \|M_{\mathcal{K}}(p) - M_{\mathcal{K}}(q)\| < \epsilon,$$

and $M_{\mathcal{K}}$ is uniformly quasi-continuous.

If on the other hand $M_{\mathcal{K}}$ is uniformly quasi-continuous and S is a bounded subset of V, then if $\epsilon>0$ there exists a subdivision D of X such that, if x_{j-1} and x_j are elements of D and p and q belong to (x_{j-1}, x_j), then

$$\|M_{\mathcal{K}}(p) - M_{\mathcal{K}}(q)\| < \epsilon/M,$$

where M is a bound for S, and consequently

$$|\mathcal{K}y(p) - \mathcal{K}y(q)| < \epsilon, \qquad \forall y \in S.$$

It then follows immediately from Theorem 1 that $\mathcal{K}[S]$ has compact closure. Hence we have shown:

THEOREM 2. *A continuous linear transformation $\mathcal{K}$ from a normed linear space V into Q_X^C is compact if and only if $M_{\mathcal{K}}$ is uniformly quasi-continuous.*

If $M_{\mathcal{K}}$ is uniformly quasi-continuous, then for each positive number ϵ there exists a step function J on X into V^* such that

$$\sup_{x\in X} \|M_{\mathcal{K}}(x) - J(x)\| < \epsilon.$$

If one defines the linear transformation $\mathcal{J}$ on V into Q_X^C by

$$\mathcal{J}y(x) = J(x)y, \qquad \forall y \in V, \ \forall x \in X$$

then it is clear from the definition of a step function that $\mathcal{J}$ is of finite range. It also follows that

$$\|(\mathcal{K} - \mathcal{J})y\| = \sup_{x\in X} |M_{\mathcal{K}}(x)y - J(x)y| < \epsilon\|y\|, \qquad \forall y \in V.$$

Hence we obtain:

THEOREM 3. *Every compact continuous linear transformation from a normed linear space into Q_X^C is uniformly approximatable by linear transformations of finite range.*

Suppose finally that a function on the real line with values in a metric space is defined to be uniformly quasi-continuous if and only if it is uniformly approximatable by step functions; a step function still being allowed to have only a finite number of jumps. It is obvious that every such function into C is bounded and it follows that the collection of all such functions into C is a Banach space

with the sup norm. This space will be denoted by Q_R^C. It is easily seen that Q_R^C is isometrically isomorphic to that closed subspace of $Q_{R^\#}^C$ whose elements are continuous at plus and minus infinity. Since $R^\#$ is a generalized interval one may readily show:

THEOREM 4. *A continuous linear transformation $\mathcal{K}$ on a normed linear space V into Q_R^C is compact if and only if $M_{\mathcal{K}}$ is uniformly quasi-continuous. Furthermore every such transformation is uniformly approximatable by transformations of finite range.*

5. With obvious and quite small modifications the results of Sections 3 and 4 may be applied to those subspaces of the uniformly quasi-continuous functions whose elements are left continuous, right continuous or continuous, respectively.

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MEASURES IN DENUMERABLE SPACES

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1. Introduction. The purpose of this paper is essentially didactic: to call attention to and make explicit certain properties of measures on denumerable spaces (i.e., spaces with a denumerable number of points) that justify the conventional treatment of such spaces in probability texts. After our results were obtained we discovered that the main one could be derived easily from some theorems of Boolean algebra. Since our theorems have not, to our knowledge, been previously pointed out, and since our proofs are in a measure-theoretic setting that requires no knowledge of Boolean algebra, we have been motivated to contribute this note.

By a *ring* $\mathfrak{R}$ of subsets of a space Ω is meant a nonempty class of subsets that is closed under finite union and difference. If $\mathfrak{R}$ is closed under countable union, it is called a *σ-ring*. The terms *algebra* and *σ-algebra* refer, respectively, to rings and σ-rings containing the space Ω. The sets in a ring $\mathfrak{R}$ are called *$\mathfrak{R}$-measurable*. We use the term *measure* to mean an extended real-valued, nonnegative, countably-additive set function defined on a ring.

The *power set*, $\mathcal{P}(\Omega)$, is the σ-algebra of all subsets of Ω. A measure μ is called *continuous* if each singleton set A is measurable, and $\mu(A)=0$ for all singletons. The *trivial measure* is the measure on $\mathcal{P}(\Omega)$ that vanishes identically. The symbol "$\varnothing$" denotes the empty set.

Our results fill a small gap in the literature on measure theory concerning the possibility of extending a measure from a given σ-ring to the power set.

Let $\mathfrak{M}$ denote the family of all measures on all σ-rings of an arbitrary space Ω, and let $\mathfrak{N}$ denote the family of all measures on $\mathcal{P}(\Omega)$. It may happen that each measure in $\mathfrak{M}$ is the restriction of at least one measure in $\mathfrak{N}$, or loosely speaking, that the class of all measures on all σ-rings is no richer than the class of all measures on the power set. In this case we shall say that Ω, or more precisely the cardinal number of Ω, has the *full extension property*; it is clear that either all sets of the same cardinality have the full extension property or none do. It is evident also that if a cardinal $\aleph$ fails to have the full extension property, so does every cardinal $\aleph^*$ with $\aleph^*>\aleph$. (This is also true if "σ-ring" is replaced by "σ-algebra" in the formulation of the notion of full extension property.)

Ulam [7] has shown that the only real-valued continuous measure on the power set of a space whose cardinality is less than the first weakly inaccessible cardinal is the trivial one. (A cardinal number $\aleph_\alpha$ is called weakly inaccessible if (a) $\aleph_\alpha>\aleph_0$, (b) α is a limit ordinal, (c) $\aleph_\alpha$ is not the sum of fewer than $\aleph_\alpha$ numbers each of which is less than $\aleph_\alpha$.) Hence, in particular, if the cardinality of Ω is $\aleph_1$, $\mathcal{P}(\Omega)$ cannot carry a nontrivial continuous measure. On the other hand, an example in Sect. 2 demonstrates the existence of a space Ω of cardinality $\aleph_1$ which has a σ-algebra $\mathfrak{F}$ of subsets containing all singletons and a nontrivial continuous probability measure μ on $\mathfrak{F}$. By Ulam's Theorem μ cannot be extended to $\mathcal{P}(\Omega)$, so that $\aleph_1$ and hence all greater cardinals fail to have the full extension property. Intuitively it seems clear that if card $\Omega\leqq\aleph_0$, all measures on sub-σ-rings of $\mathcal{P}(\Omega)$ should be compatible with the assignments of masses to points, and thus that $\aleph_0$ should have the full extension property. We shall show that this is so in Section 3. The connection between our results and Boolean algebra is described in Section 4. The implications for probability theory are discussed in Section 5.

2. An example of a nonextendible measure. Let α be any ordinal greater than zero, and let Ω_α denote the set of all ordinal numbers less than α. Let α^* denote the smallest ordinal α such that Ω_α is uncountable. Then the cardinal number of Ω_{α^*} is $\aleph_1$.

We shall call a set $A\subset\Omega_{\alpha^*}$ a cosection if there exists a number $\beta\in\Omega_{\alpha^*}$ such that $\alpha>\beta$ implies $\alpha\in A$. It is not difficult to verify from well-known properties of Ω_{α^*} ([2], p. 29) that the class of sets $\mathfrak{F}$ consisting of all cosections and their complements is a σ-algebra. Moreover, the function

$$\mu\colon \mathfrak{F}\to[0,1]$$

defined by setting

$$\mu(A) = \begin{cases} 1, & \text{if } A \text{ is a cosection} \\ 0, & \text{if } A^c \text{ is a cosection} \end{cases}$$

is a probability measure which, by Ulam's Theorem, cannot be extended to $\mathcal{P}(\Omega_{\alpha^*})$. We therefore have the following result:

THEOREM 2.1. *If card* $\Omega \geqq \aleph_1$, Ω *does not have the full extension property.*

3. Structure of σ-rings and measures in a countable space. As is shown in Theorem 3.1, every σ-ring of subsets of a countable space has a particularly simple structure, an important feature of which is implied by the following lemma:

LEMMA 3.1. *Let* Ω *be a space containing a countable number of points. If* $\mathfrak{F}$ *is a nonempty class of subsets of* Ω *closed under countable union or countable intersection, then* $\mathfrak{F}$ *is closed, respectively, under arbitrary union or intersection.*

Proof. We give the proof only for the case of intersections; the proof for unions is similar.

Suppose that $\mathfrak{F}$ is closed under countable intersection. Let $\mathcal{C} \subset \mathfrak{F}$ be an arbitrary class of sets and put

$$D = \bigcap_{A \in \mathcal{C}} A.$$

Consider the complementary set D^c. If D^c is empty, then $\mathcal{C}$ is the class whose only member is Ω, so that $D \in \mathfrak{F}$.

If D^c is not empty, to each $\omega \in D^c$ there corresponds a set A_ω^c such that $\omega \in A_\omega^c$ and $A_\omega \in \mathcal{C}$. Let

$$E = \bigcap_{\omega \in D^c} A_\omega;$$

plainly $E \supset D$. Since D^c is countable and $A_\omega \in \mathcal{C}$, we have $E \in \mathfrak{F}$. Noting that

$$\omega \in D^c \Rightarrow \omega \in A_\omega^c \Rightarrow \omega \in E^c,$$

that is, $D \supset E$, we conclude that $D = E \in \mathfrak{F}$.

Now let $\mathcal{R}$ denote a σ-ring of subsets of a countable space Ω. We define a binary relation "$\sim$" on Ω by setting

$$\omega \sim \omega'$$

if and only if every set in $\mathcal{R}$ that contains ω' also contains ω. It is clear from the definition that this relation is reflexive and transitive; it follows from the properties of $\mathcal{R}$ that the relation is also symmetric. For suppose that $\omega \sim \omega'$ but

$$\omega' \nsim \omega.$$

Then there exists a set $A \in \mathcal{R}$ such that $\omega \in A$ and $\omega' \notin A$. Setting $\Omega_{\mathcal{R}} = \bigcup_{E \in \mathcal{R}} E$, it follows from Lemma 3.1 that $\Omega_{\mathcal{R}} \in \mathcal{R}$. Hence

$$\Omega_{\mathfrak{R}} - A \in \mathfrak{R},$$
$$\omega' \in \Omega_{\mathfrak{R}} - A,$$

and $\omega \notin \Omega_{\mathfrak{R}} - A$, which contradicts the relation $\omega \sim \omega'$.

Thus "$\sim$" is an equivalence relation on Ω. Plainly $\omega \sim \omega'$ if and only if $\omega \in \bigcap_{\omega' \in A \in \mathfrak{R}} A$. Hence we have proved:

LEMMA 3.2. *The equivalence class,* $[\omega']$, *of* ω' *is given by* $[\omega'] = \bigcap_{\omega' \in A \in \mathfrak{R}} A$.

It follows from Lemmas 3.1 and 3.2 that the class of sets $\{[\omega'] : \omega' \in \Omega_{\mathfrak{R}}\}$ is a measurable partition of $\Omega_{\mathfrak{R}}$, i.e., one in which each set is in $\mathfrak{R}$.

An *atom* of a ring $\mathfrak{R}$ of sets is a nonempty, $\mathfrak{R}$-measurable set A whose only $\mathfrak{R}$-measurable subsets are A and $\emptyset$. The theorem below shows that a σ-ring in a countable space is generated by its atoms.

THEOREM 3.1. *If* Ω *is countable and* $\mathfrak{R}$ *is an arbitrary* σ*-ring of subsets of* Ω, *there exists a countable, measurable partition of* $\Omega_{\mathfrak{R}}$ *into atoms. The atoms are just the equivalence classes* $[\omega]$, $\omega \in \Omega_{\mathfrak{R}}$, *and each* $\mathfrak{R}$*-measurable set* A *is the union of the atoms contained in* A.

Proof. Suppose there is a point $\omega \in \Omega_{\mathfrak{R}}$ and a set $B \in \mathfrak{R}$ such that B is a nonempty, proper subset of $[\omega]$. Then there are equivalent points $\omega' \in B$ and $\omega'' \in [\omega] - B$, which contradicts the definition of equivalence.

Thus the sets $[\omega]$, $\omega \in \Omega_{\mathfrak{R}}$, are atoms which form a measurable partition of $\Omega_{\mathfrak{R}}$; plainly there are no other atoms. To complete the proof we note that $B \in \mathfrak{R}$ implies

$$B = B \cap \Omega_{\mathfrak{R}} = B \cap \bigcup_{\omega \in \Omega_{\mathfrak{R}}} [\omega] = \bigcup_{\omega \in \Omega_{\mathfrak{R}}} B \cap [\omega] = \bigcup_{[\omega] \subset B} [\omega],$$

since $B \cap [\omega] = [\omega]$ if $[\omega] \subset B$ and is empty otherwise.

Let $\mathfrak{F}$ and $\mathfrak{F}^*$ be σ-algebras of subsets of spaces Ω and Ω^*, respectively. We shall say that $\mathfrak{F}$ and $\mathfrak{F}^*$ are isomorphic if there exists a 1:1 mapping ϕ of $\mathfrak{F}$ onto $\mathfrak{F}^*$ that preserves countable unions and complements, i.e.,

$$\text{(a)} \quad \phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} \phi(A_i),$$
$$\text{(b)} \quad \phi(A^c) = (\phi(A))^c.$$

It follows easily from (a) and (b) that ϕ also preserves countable intersections, differences, and set inclusion. We can now state the following corollary to Theorem 3.1:

COROLLARY. *If* Ω *is countable, any* σ*-algebra* $\mathfrak{F}$ *of subsets of* Ω *is isomorphic to the* σ*-algebra* $\mathfrak{F}^*$ *of all subsets of the set* Ω^* *of all atoms of* $\mathfrak{F}$.

Proof. We should like to emphasize that each point $\omega^* \in \Omega^*$ is a subset of Ω. Thus if A is a subset of Ω and A^* is a subset of Ω^*, ω^* can bear the relation of

inclusion to A and the relation of membership to A^*. With this in mind, we define a mapping

$$\phi: \mathfrak{F} \to \mathfrak{F}^*$$

by setting, for $A \in \mathfrak{F}$, $\phi(A) = \{\omega^* \in \Omega^*: \omega^* \subset A\}$.

The mapping ϕ is "onto" since, if $A^* \in \mathfrak{F}^*$, then the set A defined by

$$A = \left(\bigcup_{\omega^* \in A^*} \omega^* \right) \in \mathfrak{F}$$

has the property

$$\phi(A) = A^*.$$

Moreover, ϕ is 1:1; for if $A \in \mathfrak{F}$, $B \in \mathfrak{F}$, $A \neq B$, there exists a point ω in one of the sets that is not in the other, say $\omega \in A$, $\omega \notin B$. Then $[\omega] \subset A$ and $[\omega] \subset B^c$, so that $\phi(A) \neq \phi(B)$.

To show that ϕ preserves countable unions, it suffices to note that for any sequence $A_1, A_2, \cdots$ of $\mathfrak{F}$-measurable sets,

$$\phi(A_i) = \{[\omega]: \omega \in A_i\}$$

and

$$\phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \left\{[\omega]: \omega \in \bigcup_{i=1}^{\infty} A_i\right\} = \bigcup_{i=1}^{\infty} \{[\omega]: \omega \in A_i\} = \bigcup_{i=1}^{\infty} \phi(A_i).$$

Similarly the relation

$$\begin{aligned} \phi(A^c) &= \{[\omega]: \omega \in A^c\} = \{[\omega]: \omega \in \Omega\} - \{[\omega]: \omega \in A\} \\ &= \phi(\Omega) - \phi(A) = \Omega^* - \phi(A) = (\phi(A))^c \end{aligned}$$

shows that ϕ preserves complements. Thus $\mathfrak{F}$ and $\mathfrak{F}^*$ are isomorphic.

It follows easily from Theorem 3.1 that $\aleph_0$ has the full extension property. For completeness the argument is given below.

THEOREM 3.2. *Let Ω be a countable set, $\mathfrak{R}$ a σ-ring of subsets, and μ an arbitrary measure on $\mathfrak{R}$. There exists a measure μ^* on the class of all subsets of Ω whose restriction to $\mathfrak{R}$ is μ.*

Proof. For each equivalence class $[\omega]$, $\omega \in \Omega_{\mathfrak{R}}$, let $p_{[\omega]}: [\omega] \to \overline{R}_+$ be any function such that

$$\sum_{\alpha \in [\omega]} p_{[\omega]}(\alpha) = \mu([\omega]).$$

Assume that $\Omega_{\mathfrak{R}} = \Omega$. Let $p: \Omega \to \overline{R}_+$ be the mapping whose restriction to $[\omega]$ is $p_{[\omega]}$ for all $\omega \in \Omega$. For each set $A \subset \Omega$ define

$$\mu^*(A) = \sum_{\omega \in A} p(\omega).$$

Clearly $\mu^*\colon \mathcal{P}(\Omega)\to\overline{R}_+$ is a measure and $\mu^*([\omega])=\mu([\omega])$, $\omega\in\Omega$. More generally, for $B\in\mathcal{R}$ we have from Theorem 3.1,

$$\mu^*(B) = \mu^*\left(\bigcup_{[\omega]\subset B}[\omega]\right) = \sum_{[\omega]\subset B}\mu^*([\omega]) = \sum_{[\omega]\subset B}\mu([\omega]) = \mu(B).$$

If $\Omega_{\mathcal{R}}\neq\Omega$, we define p on $\Omega_{\mathcal{R}}$ as before and define it arbitrarily on $\Omega-\Omega_{\mathcal{R}}$.

4. Related theorems of Boolean algebra. Let $\mathcal{A}$ be a Boolean algebra and let m be an infinite cardinal. $\mathcal{A}$ is said to be *m-complete* if for every indexed family $\{A_t\}_{t\in T}$, where card $T=m$ and $A_t\in\mathcal{A}$, the join $\bigcup_{t\in T}A_t$ exists in $\mathcal{A}$. This is equivalent to the condition that for every m-indexed family $\{A_t\}_{t\in T}$, the meet $\bigcap_{t\in T}A_t$ exists in $\mathcal{A}$. If $\mathcal{A}$ is an m-complete Boolean algebra for every m, then $\mathcal{A}$ is said to be a *complete* Boolean algebra. (For the definitions of Boolean algebra, infinite joins, and infinite meets, see [4].) A Boolean algebra $\mathcal{A}$ is said to satisfy the *m-chain condition* provided that every set of disjoint elements in $\mathcal{A}$ has cardinality $\leqq m$. (Since an algebra of sets may be viewed as a Boolean algebra, the terminology introduced in this section may also be applied to algebras of sets.)

In this terminology, Lemma 3.1 of the previous section has an obvious corollary.

COROLLARY TO LEMMA 3.1. *A σ-algebra of subsets of a countable space is a complete Boolean algebra.*

It is interesting to note that this corollary is a special case of a much more general theorem of Boolean algebra due to Tarski.

THEOREM (Tarski). *Every m-complete Boolean algebra $\mathcal{A}$ satisfying the m-chain condition is a complete Boolean algebra.*

For a detailed proof of this theorem the reader is referred to [4, 5]. The proof rests on the inductively proved fact that if the join $\bigcup_{t\in T}A_t$ exists for any m-indexed set $\{A_t\}_{t\in T}$ of disjoint elements of a Boolean algebra $\mathcal{A}$, then $\mathcal{A}$ is m-complete. Under the hypothesis of Tarski's Theorem it is evident that the join of any indexed set of disjoint elements exists.

Since a σ-algebra of subsets of a countable space is not only $\aleph_0$-complete but also satisfies the $\aleph_0$-chain condition, the corollary to Lemma 3.1 is a special case of Tarski's Theorem.

Our Theorem 3.1 also has a more general Boolean algebra counterpart. An element $A\neq 0$ of a Boolean algebra $\mathcal{A}$ is said to be an *atom* of $\mathcal{A}$ if for every $B\in\mathcal{A}$ the inclusion

$$B \subset A$$

implies that either $B=0$ or $B=A$. A Boolean algebra is called *atomic* if for every element $A\neq 0$ there exists an atom $B\subset A$. The following theorem, due to

Lindenbaum and Tarski [4, 6], relates completeness to atomicity:

THEOREM (Lindenbaum and Tarski). *A complete Boolean algebra $\mathfrak{A}$ is isomorphic to a complete algebra of sets if and only if $\mathfrak{A}$ is atomic. In this case $\mathfrak{A}$ is isomorphic to the algebra of all subsets of the set of all atoms of $\mathfrak{A}$.*

Two Boolean algebras are called *isomorphic* if there exists a 1:1 mapping of one onto the other that preserves binary join and complement.

Since, as we have seen, a σ-algebra of subsets of a countable space is complete, it follows from the Theorem of Lindenbaum and Tarski that it is atomic. It is evident that the atoms of a Boolean algebra are disjoint. Hence, there are only countably many atoms in a σ-algebra of subsets of a countable space Ω. Clearly the union of these atoms is Ω, so that they form a countable, measurable partition of the space.

5. Implications for probability theory. In the Kolmogorov formulation of the axioms of probability, the mathematical description of a random experiment $\mathcal{E}$ consists of a triple $(\Omega, \mathfrak{F}, P)$, where Ω is a set, $\mathfrak{F}$ is a σ-algebra of subsets of Ω, and $P: \mathfrak{F} \to [0, 1]$ is a probability measure. In applications of the Kolmogorov model each point in Ω is interpreted as a possible "primary" or "indecomposable" outcome of $\mathcal{E}$, and each set $A \in \mathfrak{F}$ is interpreted as the event that "the outcome of $\mathcal{E}$ is one of the points in A." Thus the intuitive notion of event is formalized as a set and, moreover, it is postulated that the only events of $\mathcal{E}$ to which probabilities are associated are the sets in $\mathfrak{F}$. These probabilities are given, of course, by the measure P.

In this approach any σ-algebra of subsets of Ω is admissible as an event class; in particular, for example, it is not required that the singleton sets themselves be events, despite the fact that each point is intuitively thought of as a possible outcome. Keeping this intuitive interpretation of the points of Ω in mind, it is natural for the student of probability to wonder why an arbitrary subset of Ω should not be regarded as a possible event to which a probability is attached. Indeed, from the point of view of the scientist or engineer interested in applications, the definition of an event as a member of a distinguished σ-algebra of subsets may seem decidedly unnatural. Certainly the problem of justifying it is faced by every teacher of probability.

One argument, which we shall not elaborate here, rests on a distinction between events and observable events [3]. This argument gives an intuitive interpretation of the fact that probability is not defined for all events, only for observable events. However, a purely mathematical argument can also be given, based on the theorems proved in the previous section. Although in every application of probability the governing probability measure must satisfy the axioms of probability, i.e., be $\geqq 0$, countably additive, and have $P(\Omega) = 1$, each application typically involves its own additional set of conditions, conditions dictated by a combination of empirical and theoretical considerations peculiar to the

particular application; these additional restrictions may, in a certain sense, clash with the general probability axioms.

Consider, for example, the idealized experiment of making an infinite sequence of tosses of a coin. Suppose that each toss is made independently of the others and that the tosses are made under identical conditions. A possible outcome of this experiment is an infinite sequence of heads and tails such as

$$H, T, T, T, H, T, H, H, \cdots,$$

where H stands for head and T for tail. The sample space Ω for this experiment may be taken to be the collection of all infinite sequences of H's and T's. Let p, $0<p<1$, denote the probability of head in a single toss. The set of all sequences having exactly k H's among the first n coordinates corresponds to the event that exactly k heads occurred among the first n tosses, and elementary probabilistic considerations lead to the conclusion that to this set of sequences we should associate the probability $\binom{n}{k}p^k(1-p)^{n-k}$. It can be shown that this assignment of probabilities leads to a unique probability measure on the ring of all subsets of Ω determined by conditions on a finite number of coordinates. (A set A is said to be determined by conditions on a finite number of coordinates if there exists an integer n such that for each point ω in Ω the first n coordinates of ω determine whether or not ω is in A.) It is evident that the singleton sets do not belong to the ring of sets to which we have thus far attached probabilities. Suppose now that we try to extend the probability measure we have defined to some σ-algebra of subsets of Ω containing all the singletons. Consider the singleton set whose only member is the point $(H, H, H, \cdots)$. For every integer n, this event is contained in the event "n heads in the first n tosses." Since probability is monotonic, it follows that the probability of the singleton is, for every n, less than or equal to p^n. This implies that the probability of the singleton is zero. A similar argument leads to the conclusion that each singleton set must have probability zero. Thus, if we could extend the probability measure we originally defined to some σ-algebra containing all of the singleton sets, the extended probability measure would have to be continuous. Since the cardinality of the space under consideration is that of the continuum, it follows from Ulam's Theorem [7], under the continuum hypothesis, that $\mathcal{P}(\Omega)$ cannot carry a nontrivial continuous measure. (This was also shown independently by Banach and Kuratowski [1].) Yet physical considerations dictate that each singleton in Ω must have probability zero. Hence, if the probability model is to be faithful to the physical situation, it is impossible in the present example to define an event to be an arbitrary subset of Ω. (At least this is so if the continuum hypothesis is adopted as an axiom of set theory. Even if it is not, if the question of whether or not $\mathcal{P}(\Omega)$ can carry a nontrivial continuous measure is decidable from the other axioms of set theory, it must be decided in the negative, since this is the conclusion when the continuum hypothesis is used.) It is, of course, possible to define a continuous measure on the σ-algebra generated by the ring with which

we started, which is done in the usual treatment of this problem in probability theory. By restricting the notion of event to such a σ-algebra, we obtain a model that fits the physical situation at the price of seeming artificiality in the definition of event. This example illustrates the fact that the class of probability measures on arbitrary σ-algebras in a space whose cardinality is that of the continuum is more useful than measures obtained by restriction from the power set. Hence, the decisive advantage of Kolmogorov's definition of event is that it leads to a larger and more useful class of probability spaces than would result from defining an event to be an arbitrary subset of a sample space.

The treatment of countable spaces in probability is in striking contrast to that of uncountable spaces. In the former case it is assumed invariably that the event class is the power set of the sample space. There appears to be no justification for this in the literature other than the fact that the procedure of assigning probabilities to all singletons leads in a simple way to a measure on the power set. Yet how can we be sure that we shall not encounter an experiment in which the sample space is countable and the conditions associated with the experiment are incompatible with all measures arising from point masses? The assurance is given by Theorem 3.2 which shows that the class of all measures on all σ-algebras of subsets of a countable space is no richer than the class of measures on the power set, i.e., the class arising from point masses.

Another way of viewing the situation in the countable case is suggested by the corollary to Theorem 4.1, which shows that no theory in which events are defined as elements of an arbitrary σ-algebra can be more general than one in which events are defined as members of the power set; for each σ-algebra $\mathfrak{F}$ of subsets of a countable space is isomorphic to the power set $\mathcal{P}$ of some space, so that a mathematical model involving $\mathfrak{F}$ can be replaced by one involving $\mathcal{P}$. In the countable case, therefore, the definition of an event as an arbitrary subset of the sample space is not only natural but correct.

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INTERSECTIONS OF MAXIMAL IDEALS IN SEMIGROUPS

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The intersections of maximal ideals of a semigroup S happen to have rather remarkable properties; this paper is devoted to some of them. We prove that two different families of maximal ideals have different intersections; that an ideal which contains an intersection of maximal ideals is itself an intersection of maximal ideals. Assume that S has a kernel K. Then K is an intersection of maximal ideals if and only if S/K is a mutually annihilating sum of semigroups which are 0-simple or null of order two; the principal factors of S are then K and the quotients of S by its maximal ideals. As an application, a semisimple semigroup S with a principal series of length $n+1$ has at most n maximal ideals, $n!$ principal series and 2^n nonempty ideals; in each case, the equality occurs if and only if the kernel K of S is an intersection of maximal ideals.

The reader is referred to [1], especially to Section 2.6, for the definitions and notation, except that we allow an ideal to be empty. Then, for any subset A of S, there exists a largest ideal contained in A (namely, the union of all ideals contained in A); we shall denote it by $M(A)$.

1. Maximal ideals and $\mathcal{J}$-classes. 1. The following result is well known:

PROPOSITION 1. *An ideal I of S is maximal if and only if $S-I$ is a $\mathcal{J}$-class.*

Proof. Assume first that $S-I$ is a $\mathcal{J}$-class and let A be an ideal of S such that $I\subset A\subseteq S$, and $a\in A-I$. Then $J_a\subseteq A$, hence $A=S$ and I is maximal. If conversely I is maximal, then, for each $a\in S-I$, $I\cup J(a)=S$, where $J(a)$ is the principal ideal generated by a. Therefore $b\in J(a)$ for all $b\in S-I$. Since similarly $a\in J(b)$, $S-I$ is contained in a $\mathcal{J}$-class. This $\mathcal{J}$-class is not contained in I, thus is disjoint from I and is finally equal to $S-I$.

Before studying the intersections of maximal ideals, we give a first consequence of this result. Call an ideal I of S *prime* if $AB\subseteq I$ implies $A\subseteq I$ or $B\subseteq I$ whenever A and B are ideals of S; it is easy to show that I is prime if and only if $aS^1b\subseteq I$ implies $a\in I$ or $b\in I$ whenever $a, b\in S$, and we can use this second condition to define prime ideals. On the other hand, if $a\in S-S^2$, then $S-a$ is a (maximal) ideal; call such an ideal *trivial.*

PROPOSITION 2. *A maximal ideal of S is either trivial or prime.*

Proof. Assume that M is a nontrivial maximal ideal of S and take $a, b\in S-M$. Since M is nontrivial, $a=uv$ for some $u, v\in S$; in fact, $u, v\in S-M$. By Proposition 1, $u\in J_a$, $v\in J_b$ and $u=sat$, $v=xby$ for some $s, t, x, y\in S^1$. Then $a=uv=satxby\notin M$, whence $atxb\notin M$ and $aS^1b\not\subseteq M$.

COROLLARY 3. *Every maximal ideal of S is prime if and only if $S=S^2$, and then, if I is an intersection of maximal ideals, $aS^1a\subseteq I$ implies $a\in I$ whenever $a\in S$.*

Proof. Observe that a trivial ideal is not prime. The second assertion is trivial.

2. We come now to some deeper properties of the intersections of maximal ideals. For simplicity we shall refer to them as *IMI* ideals. We start with the following

LEMMA 4. *If $a \in S$, $M(S-a)$ is a maximal ideal of S if and only if J_a is a maximal element of $S/\mathcal{J}$.*

Proof. Assume that $M(S-a)$ is maximal and that $J_a \leqq J_b$, i.e. $a \in J(b)$. Since $a \notin M(S-a)$, $M(S-a) \cup J(a) = S$. But $b \notin M(S-a)$, or else $a \in J(b) \subseteq M(S-a)$. Hence $b \in J(a)$ and $J_a = J_b$, which proves that J_a is maximal.

If conversely J_a is maximal, and if A is an ideal of S such that $M(S-a) \subset A$, then $a \in A$, since $A \nsubseteq S-a$. If now x is any element of S, either $J(x) \subseteq M(S-a)$ and $x \in A$; or $J(x) \nsubseteq M(S-a)$, $J(x) \nsubseteq S-a$, $a \in J(x)$, $J_a \leqq J_x$, $J_a = J_x$, and $x \in J(a) \subseteq A$. Hence $A = S$. This completes the proof.

Denote now by K_1 the intersection of all maximal ideals of S; it is an ideal of S, possibly empty, equal to S if S has no maximal ideal.

THEOREM 5. *An element x of S is not in K_1 if and only if J_x is maximal in $S/\mathcal{J}$.*

Proof. If $x \notin K_1$, there exists a maximal ideal M of S such that $x \notin M$. Then $M = M(S-x)$ and J_x is maximal by the lemma. If conversely J_x is maximal, then $M(S-x)$ is a maximal ideal of S which does not contain x, hence $x \notin K_1$.

It follows that S has maximal ideals if and only if $S/\mathcal{J}$ has maximal elements.

THEOREM 6. *An ideal A of S is an IMI ideal if and only if it contains K_1. In particular, an ideal which contains an IMI ideal is an IMI ideal.*

Proof. Clearly every *IMI* ideal contains K_1. Let conversely A be an ideal containing K_1. If $A = S$, then A is intersection of the empty family of maximal ideals. If $A \neq S$, then, for each $x \in S-A$, J_x is maximal by Theorem 5, and $M(S-x)$ is by Lemma 4 a maximal ideal of S which contains A but not x. Then A is the intersection of all $M(S-x)$ when $x \in S-A$. Hence A is an *IMI* ideal. The last assertion is now obvious.

3. For convenience we now assume that S has maximal ideals (e.g. S satisfies the *ACC* on ideals or the multiplication on S is continuous for some compact topology). Denote by $\mathfrak{M}$ the set of all maximal ideals of S. If A is an ideal of S, denote by $\mathfrak{M}(A)$ the set of all maximal ideals which contain A. If $\mathcal{E}$ is a subset of $\mathfrak{M}$, denote by $I(\mathcal{E})$ the intersection of all the ideals of $\mathcal{E}$. Theorem 6 can be restated as: $A = I(\mathfrak{M}(A))$ if and only if A is an ideal containing K_1.

THEOREM 7. *$I(\mathcal{E}) \subseteq I(\mathcal{F})$ is equivalent to $\mathcal{F} \subseteq \mathcal{E}$. In particular, $I(\mathcal{E}) = I(\mathcal{F})$ if and only if $\mathcal{E} = \mathcal{F}$.*

Proof. Assume that $I(\mathcal{E}) \subseteq I(\mathcal{F})$ but that there is $M_0 \in \mathcal{F} - \mathcal{E}$. Take any $x \in S$. If $x \in I(\mathcal{E})$, then $x \in M_0$. If $x \notin I(\mathcal{E})$, then $x \notin M$ for some $M \in \mathcal{E}$. But $M \neq M_0$, hence $M \cup M_0 = S$; and again $x \in M_0$. Therefore $M_0 = S$, which is absurd. This proves the first assertion; the second is then obvious.

Since $I(\mathfrak{M}(I(\mathcal{E})))=I(\mathcal{E})$ by Theorem 6, it follows that $\mathfrak{M}(I(\mathcal{E}))=\mathcal{E}$ for all $\mathcal{E}\subseteq\mathfrak{M}$.

PROPOSITION 8. *If A and B are IMI ideals, then*

$$\mathfrak{M}(A\cap B)=\mathfrak{M}(A)\cup\mathfrak{M}(B),\qquad \mathfrak{M}(A\cup B)=\mathfrak{M}(A)\cap\mathfrak{M}(B).$$

Proof. Observe that $A\cap B$ is trivially *IMI*, and that $A\cup B$ is *IMI* by Theorem 6. Then

$$I(\mathfrak{M}(A)\cup\mathfrak{M}(B))=I(\mathfrak{M}(A))\cap I(\mathfrak{M}(B))=A\cap B=I(\mathfrak{M}(A\cap B))$$

by Theorem 6, and the first formula follows from Theorem 7. Furthermore $I(\mathfrak{M}(A)\cap\mathfrak{M}(B))$ is, by Theorem 7, the smallest ideal of the form $I(\mathcal{E})$ which contains both A and B; hence it is $A\cup B$, and the formula follows.

4. If A is an *IMI* ideal, we may call codimension of A the cardinal number codim A = Card $\mathfrak{M}(A)$ and dimension of A the cardinal number dim A = Card $(\mathfrak{M}-\mathfrak{M}(A))$. Then dim S = Card $\mathfrak{M}$, while dim A + codim A = dim S for every *IMI* ideal A.

From Proposition 8 then follows that

$$\dim A+\dim B=\dim A\cap B+\dim A\cup B$$

for all *IMI* ideals A, B. The same formula holds for the codimension. Finally:

PROPOSITION 9. *If A is an IMI ideal of S, then* $\dim S=\dim A+\dim S/A$.

Proof. Since $I\to I/A$ is a one-to-one, order-preserving mapping of the set of all ideals I of S containing A onto the set of all nonempty ideals of S/A, it follows that the maximal ideals of S/A are in one-to-one correspondence with the maximal ideals of S containing A; hence dim S/A = Card $\mathfrak{M}(A)$ = codim A. The formula follows.

2. Intersective semigroups.

An intersective semigroup is a semigroup S having a minimal nonempty ideal K such that $K=K_1$—i.e., the intersection of all maximal ideals of S. This rules out the case of a simple semigroup, for if S is simple then S has a unique maximal ideal $\emptyset\neq S$. It also rules out the case when S has no maximal ideal, since then $K_1=S$ cannot be a minimal ideal unless S is simple. Hence, if S is intersective, $K\neq S$ and $\mathfrak{M}\neq\emptyset$. A 0-simple semigroup is a trivial example of an intersective semigroup.

THEOREM 10. *The principal factors of an intersective semigroup S are isomorphic to K and to the semigroups S/M, where $M\in\mathfrak{M}$.*

Proof. Let $x\in S-K$, $I(x)$ be the set of all elements of $J(x)$ which do not generate $J(x)$; then $I(x)$ is maximal among the ideals of S contained in $J(x)$ and $P_x=J(x)/I(x)$ is the principal factor of S associated with x. Since $x\in S-K$, $K\subset J(x)$; also $K\subseteq I(x)$. By Theorem 6, $J(x)=I(\mathcal{E})$, $I(x)=I(\mathcal{F})$, where $\mathcal{E}=\mathfrak{M}(J(x))$, $\mathcal{F}=\mathfrak{M}(I(x))$. By Theorem 7, $\mathcal{E}\subset\mathcal{F}$; and $\mathcal{E}\subset\mathcal{G}\subset\mathcal{F}$ would imply

$I(x) \subset I(\mathcal{G}) \subset J(x)$, which is not possible. Hence $\mathfrak{F}-\mathcal{E}$ contains exactly one element, say M; in other words, $I(x)=J(x)\cap M$. Then

$$P_x = J(x)/I(x) = J(x)/J(x) \cap M \cong J(x) \cup M/M = S/M$$

since M is maximal and $x \in J(x)-M$.

Let conversely M be a maximal ideal of S; take $x \in S-M$. Then M is the unique maximal ideal of S to which x does not belong, for, if $M' \in \mathfrak{M}$, $M' \neq M$, then $M \cup M'=S$ and $x \in M'$. By the above, $P_x \cong S/M_1$ for some $M_1 \in \mathfrak{M}$ such that $x \notin M_1$; since M_1 must there be equal to M, S/M is a principal factor of S. Finally, if $x \in K$, $P_x=K$, which completes the proof.

THEOREM 11. *A semigroup S having a minimal nonempty ideal K is intersective if and only if S/K is a mutually annihilating sum of semigroups which are* 0-*simple or null of order two.*

(More precisely, there exists a family $(S_\alpha)_{\alpha \in I}$ of semigroups which are 0-simple or null of order two, with a common zero, such that $S_\alpha \cap S_\beta=\{0\}$ whenever $\alpha \neq \beta$ and that $S/K=\bigcup_{\alpha \in I} S_\alpha$, with the multiplication such that each S_α is a subsemigroup of S/K and that $S_\alpha S_\beta=0$ when $\alpha \neq \beta$.)

Proof. First observe that S is intersective if and only if S/K is intersective. Hence we may assume for the proof that $K=0$, i.e. that S has a zero.

Assume that S is an intersective semigroup with zero. Then, by Theorem 5, J_x is a maximal element of $S/\mathcal{J}$ for every $x \neq 0$; in other words, $J(x)$ is maximal in the set of all principal ideals of S, partially ordered by inclusion. From this follows that $J(x)$ is also a 0-minimal ideal of S. Indeed, let A be an ideal of S such that $0 \subset A \subseteq J(x)$ and $a \in A$, $a \neq 0$. Then $0 \subset J(a) \subseteq A \subseteq J(x)$, and, since $J(a)$ is a maximal principal ideal by the above, $J(a)=J(x)$ and $A=J(x)$.

It follows that, if $J(x) \neq J(y)$, then $J(x) \cap J(y)=0$ and also $J(x)J(y)=0$ (since contained in $J(x) \cap J(y)$). Therefore S is a mutually annihilating sum of the $J(x)$ for $x \neq 0$. Since each $J(x)$ is a 0-minimal ideal, S is a mutually annihilating sum of semigroups which are 0-simple or null. Now a null semigroup is a mutually annihilating sum of null semigroups of order two, whence S is a mutually annihilating sum of semigroups which are 0-simple or null of order two.

Assume conversely that S is a mutually annihilating sum of semigroup S_α which are 0-simple or null of order two, i.e. which have no proper ideals. Then each S_α is an ideal of S. Also each $M_\alpha=\bigcup(S_\beta; \beta \neq \alpha)$ is an ideal of S. In fact M_α is a maximal ideal of S; for, if A is an ideal of S such that $M_\alpha \subset A$, then $0 \subset A \cap S_\alpha \subseteq S_\alpha$, and $A \cap S_\alpha=S_\alpha$ whence $A=S$. Finally the intersection of all M_α is 0, hence S is intersective. This completes the proof.

COROLLARY 12. *An intersective semigroup S is semisimple if and only if $S=S^2$.*

Proof. Except for K, the principal factors of S and S/K are the same, so once again we are reduced to the case when S has a zero. Keeping the above notation, it follows from Theorem 7 that the M_α are the maximal ideals of S,

since $\bigcap_{\alpha\in I} M_\alpha=0$. Hence the semigroups S/M_α are all the principal factors of S; since

$$S/M_\alpha = M_\alpha \cup S_\alpha/M_\alpha \cong S_\alpha/M_\alpha \cap S_\alpha = S_\alpha,$$

the principal factors of S are isomorphic to the S_α. They are all 0-simple if and only if no S_α is null, if and only if $S=S^2$. (One could also proceed directly.)

We shall now characterize the semisimple and intersective semigroups among the semisimple semigroups with a principal series. Note that the full transformation semigroup $\mathfrak{I}_X$ on a finite set X such that Card $X\geqq 3$ has a unique maximal ideal which is different from the kernel; such a semigroup is semisimple with a principal series, but not intersective.

If S is a semisimple semigroup, recall that an ideal of an ideal of S is also an ideal of S, and there is no distinction in S between the principal series and the composition series. If S has a principal series, all the principal series $S=S_1\supset S_2 \cdots S_n\supset\emptyset$ have the same length n; call this length the principal dimension of S (pr. dim S). Observe that S contains then maximal ideals (e.g. S_2) and that every chain of (pairwise different) ideals can, by Zorn's Lemma, be embedded into a maximal chain, i.e. into a principal series. Observe also that S has a kernel K; S_n is always equal to K. Finally S has n $\mathcal{J}$-classes, namely S_n and the S_i-S_{i+1} $(i=1,\cdots,n-1)$.

PROPOSITION 13. *If S is a semisimple semigroup with a principal series,* dim $S\leqq$ pr. dim $S-1$.

Proof. Let $M_1,\cdots,M_p$ be pairwise different maximal ideals of S. Then

$$S\supset M_1\supset M_1\cap M_2\supset M_1\cap M_2\cap M_3\supset\cdots\supset\bigcap_{j=1}^{p} M_j$$

is a chain of ideals which are pairwise different by Theorem 7, and has length $p+1$. Since it can be embedded into a principal series, $p+1\leqq$ pr. dim S. It follows that dim $S\leqq$ pr. dim $S-1$.

THEOREM 14. *A semisimple semigroup S with a principal series is intersective if and only if* dim $S=$ pr. dim $S-1$.

Proof. Set pr. dim $S=n$. If S is intersective of kernel K, there is a chain

$$S\supset M_1\supset M_1\cap M_2\supset\cdots\supset\bigcap_{j=1}^{p} M_j = K\supset\emptyset,$$

where the M_j $(j=1,\cdots,p)$ are all the maximal ideals of S. This chain is in fact a principal series. Indeed, for any $k=1,\cdots,p$, $M_k\cup\bigcap_{j=1}^{k-1} M_j=S$ since $\bigcap_{j=1}^{k-1} M_j\not\subseteq M_k$ by Theorem 7. Hence, setting $A=\bigcap_{j=1}^{k-1} M_j$,

$$A/A\cap M_k\cong A\cup M_k/M_k = S/M_k$$

is a principal factor of S, thus is 0-simple, whence there is no ideal between A

and $A\cap M_k$. Therefore the length $p+1$ of the chain is equal to n, i.e. $\dim S = \text{pr. dim } S-1$.

If conversely S has $n-1$ maximal ideals, then the chain constructed from them has length n and is made of pairwise different ideals; therefore it is a principal series, and its last term, the kernel of S, is an IMI ideal, i.e. S is intersective. This completes the proof.

The construction above suggests that the maximal ideals of S could be intersected in any order to make the series. This leads to our next theorem.

PROPOSITION 15. *Let S be a semisimple semigroup with a principal series. Set this time* pr. dim $S=n+1$. *Then S has at most $n!$ principal series.*

Proof. By Proposition 13, S has at most n maximal ideals. Hence in the construction of the principal series

$$S = S_1 \supset S_2 \supset S_3 \supset \cdots \supset S_n \supset S_{n+1} \supset \emptyset$$

we have at most n possible choices for S_2. Having chosen S_2, S_3 must be a maximal ideal of S_2; since pr. dim $S_2=$ pr. dim $S-1$, we have at most $n-1$ possible choices for S_3. This goes to S_{n+1}, for which, since pr. dim $S_n=2$, we have at most 1 possible choice (which in fact is clear since S_{n+1} must be the kernel of S). Finally we have at most $n(n-1)\cdots 1=n!$ fashions to construct a principal series of S.

THEOREM 16. *Let S be a semisimple semigroup with a principal series; set* pr. dim $S=n+1$. *Then S is intersective if and only if S has exactly $n!$ principal series.*

Proof. If S is intersective, then S has, by Theorem 14, n maximal ideals. There are $n!$ fashions of indexing them from 1 to n, and two different indexings lead, by Theorem 7, to principal series which differ by at least one term. Therefore S has at least, thus exactly by Proposition 15, $n!$ principal series.

If conversely S is not intersective, then S has, by Theorem 14, $p<n$ maximal ideals. The reasoning of Proposition 15 gives then at most $p\cdot(n-1)!<n!$ principal series, since there are at most p possible choices for S_2. Hence S has less than $n!$ principal series, which completes the proof.

Finally we obtain a characterization in terms of ideals.

PROPOSITION 17. *A semisimple semigroup S with a principal series of length $n+1$, has at most 2^n nonempty ideals.*

Proof. Then S has exactly n $\mathcal{J}$-classes different from K. Associate to each nonempty ideal A of S the set $J_A=\{J_x;\, x\in A-K\}$. Observe that $A=K\cup\bigcup(J_x;\, x\in A-K)$; hence $J_A=J_B$ implies $A=B$, in other words the mapping $A\to J_A$ is one-to-one. It maps the set of all nonempty ideals of S into the set of all subsets of J_S. Since J_S has n elements, S has at most 2^n nonempty ideals.

THEOREM 18. *A semisimple semigroup S with a principal series of length $n+1$ is intersective if and only if it has 2^n nonempty ideals.*

Proof. Assume that S is intersective. Then $\mathcal{E}\to I(\mathcal{E})$ is a one-to-one mapping from the set of all subsets of $\mathfrak{M}$ onto the set of all nonempty ideals of S. Hence S has exactly 2^n nonempty ideals. If conversely S has 2^n nonempty ideals, the mapping $A\to J_A$ above is onto, hence there exist n nonempty ideals A_i of S such that J_{A_i} has $n-1$ elements; in other words, such that $S-A_i$ is a $\mathcal{J}$-class, different from K. Hence the A_i are maximal ideals by Proposition 1 and S is intersective by Theorem 14.

The reader is referred to the bibliography of [1] for further references to the original papers about composition series, principal series and semisimple semigroups.

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THE RING OF REAL POLYNOMIALS

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We will discuss some properties of the ring $R[x]$ of polynomials over the real field R which depend on a special property of R: the only automorphism on R is the identity automorphism. Our results are probably widely known, but except for one brief reference described in the Notes at the end of this paper, I have not seen them in any textbooks or other literature on algebra.

One can easily see that if q is a fixed polynomial, then the correspondence $p\to p(q)$ as p ranges over $R[x]$ is an endomorphism on $R[x]$. Our main result for real polynomials in one variable is that (Theorem 2) if ϕ is a nonzero ring endomorphism on $R[x]$, then $\phi p=p(\phi x)$ for all polynomials p in $R[x]$, i.e., any nonzero endomorphism can be generated by the substitution of a suitable fixed polynomial, namely the endomorph of the polynomial x. If ϕ is an automorphism, then ϕx is of the first degree; and it follows that (Theorem 6) the correspondence $\phi\to\phi x$ is an anti-isomorphism from the group of all ring automorphisms on $R[x]$ onto the group of all nonsingular affine transformations on R. We also discuss an analogy between these properties of $R[x]$ and certain properties of the ring $C(R)$ of all real continuous functions on R.

For polynomials in two variables we obtain $\phi p=p(\phi x, \phi y)$ for any nonzero endomorphism ϕ and any polynomial p, but even if ϕ is an automorphism, the polynomials ϕx and ϕy need not be of the first degree. The group of all automorphisms on $R[x, y]$ is anti-isomorphic with the group of all polynomial transformations of R^2 onto R^2 having polynomial inverses. This latter group contains the affine group on R^2 as a proper subgroup.

The members of $R[x]$ will be looked upon as real-valued polynomial functions on R. Bold face letters denote constant valued functions. For example, $\mathbf{r}$ is

the function having the constant value r. By $\mathbf{R}$ we mean the subring of $R[x]$ consisting of all real constants. The letter x always denotes a function, the identity mapping on R, rather than an arbitrary member of R. Hence if p is a polynomial function, then $p(x)$, i.e., the composition of x by p, is the same function.

1. The ring of real polynomials in one variable.

THEOREM 1. *Let $F(U)$ be any ring of real functions on a nonempty set U such that $F(U)$ contains $\mathbf{R}(U)$, the ring of all real constants on U. If ϕ is a ring endomorphism on $F(U)$, then $\phi\mathbf{r}=r\cdot\phi\mathbf{1}$ for all real constant functions $\mathbf{r}$ on U.*

Proof. For each element u of U, the mapping $h_u\colon R\to R$ given by $h_u(r)=(\phi\mathbf{r})(u)$ is an endomorphism on R. But an endomorphism on R is either the zero or the identity endomorphism; i.e., either $h_u(r)=0$ for all r in R or $h_u(r)=r$ for all r in R. Hence for a particular u, either $(\phi\mathbf{r})(r)=0$ for all r in R or $(\phi\mathbf{r})(u)=r$ for all r in R. In either case $(\phi\mathbf{r})(u)=r\cdot(\phi\mathbf{1})(u)$.

The endomorph of any idempotent is an idempotent and the only idempotents in $R[x]$ are $\mathbf{0}$ and $\mathbf{1}$. Hence:

COROLLARY. *If ϕ is a nonzero ring endomorphism on $R[x]$, then $\phi\mathbf{r}=\mathbf{r}$ for all real constants $\mathbf{r}$.*

One interpretation of the corollary to Theorem 1 is that if we also consider $R[x]$ as an algebra over the real field, then the algebra structure is completely determined by the ring structure. More precisely, if ϕ is a nonzero ring endomorphism on $R[x]$, then

$$\phi(ap) = \phi(\mathbf{a}p) = \phi(\mathbf{a})\phi(p) = a\phi(p).$$

Hence ϕ is also an algebra endomorphism. Since $R[x]$, considered as an algebra over R, is generated by x, we would expect a ring endomorphism ϕ on $R[x]$ to be completely determined by ϕx. In fact we obtain:

THEOREM 2. *Let ϕ be any nonzero ring endomorphism on $R[x]$. Then $\phi p=p(\phi x)$ for all polynomials p in $R[x]$.*

Proof. Let $p(x)=\sum_{k=0}^{n} a_k x^k$. Then

$$\phi p = \phi\left(\sum \mathbf{a}_k x^k\right) = \sum (\phi\mathbf{a}_k)(\phi x)^k = \sum \mathbf{a}_k(\phi x)^k = p(\phi x).$$

It follows from Theorem 2 that an endomorphism on $R[x]$ is order preserving. That is, if $p(t)\geqq 0$ for all t, then so is $(\phi p)(t)$.

If q is any fixed real polynomial, then the correspondence $p\to p(q)$ as p ranges over $R[x]$ is a nonzero endomorphism. Theorem 2 tells us that every nonzero endomorphism is induced by the substitution of some fixed polynomial $q=\phi x$. Since q is the image of x under the endomorphism $p\to p(q)$, changing q changes the endomorphism. Hence there is a natural one-to-one correspondence between the set $R[x]$ of all real polynomials in x and the set of all nonzero endomorphisms on $R[x]$. The zero endomorphism is the only endomorphism which cannot be

induced by the substitution of a suitable polynomial.

The nonzero endomorphisms may be divided into two classes according to whether or not ϕx is a constant.

Case 1. *The polynomial ϕx is a constant.* Then $\phi(R[x]) = \boldsymbol{R}$. Since there is a unique isomorphism from $\boldsymbol{R}$ onto R, namely: $\boldsymbol{r} \leftrightarrow r$, we may interpret ϕ as a homomorphism from $R[x]$ onto R and we observe that ϕ is a *point* or *fixed* homomorphism, i.e., there is a point $t_0 = \phi x$ in R such that $\phi p = p(t_0)$ for all p in $R[x]$. Every nonzero homomorphism from $R[x]$ into R is of this type.

In this case the kernel $\phi^{-1}(0)$ of ϕ consists of all polynomials p such that $p(t_0) = 0$. The residue class ring modulo this kernel is isomorphic to the real field.

Case 2. *The polynomial ϕx is not a constant.* We describe these endomorphisms by the next three theorems:

THEOREM 3. *If ϕ is an endomorphism on $R[x]$ and ϕx is not a constant, then ϕ is a monomorphism.*

Proof. Suppose $\phi p = p(\phi x) = \mathbf{0}$. If ϕx is not a constant, then its range is an infinite set. Hence this equation holds only if p is the constant $\mathbf{0}$.

If p is not a constant, then $\deg(\phi p) = \deg p(\phi x) = (\deg p)(\deg \phi x) \geqq \deg \phi x$. Hence if $\deg(\phi x) > 1$, then $\phi(R[x])$ does not contain any first degree polynomials. Therefore:

THEOREM 4. *If ϕ is an endomorphism on $R[x]$ and* $\deg \phi x > 1$, *then ϕ is not an epimorphism.*

THEOREM 5. *Let ϕ be a nonzero endomorphism on $R[x]$. Then ϕ is an automorphism if and only if* $\deg \phi x = 1$.

Proof. After Theorems 3 and 4, we have only to prove: if $\deg \phi x = 1$, then ϕx is an epimorphism. To prove this let p be any polynomial in $R[x]$ and let $\phi x = ax + \boldsymbol{b}$, $a \neq 0$. Then $p((x - \boldsymbol{b})/a)$ is a polynomial and

$$\phi p((x - \boldsymbol{b})/a) = p((\phi x - \boldsymbol{b})/a) = p((ax + \boldsymbol{b} - \boldsymbol{b})/a) = p(x).$$

Since each p is the image of some polynomial, ϕ is an epimorphism.

We may restate Theorem 5 as:

THEOREM 5'. *Let ϕ be a nonzero endomorphism on $R[x]$. Then ϕ is an automorphism if and only if there is a nonsingular affine transformation $ax + \boldsymbol{b}$ ($a \neq 0$) on R such that $\phi p = p(ax + b)$ for all p in $R[x]$.*

According to Theorem 5, to each ring automorphism on $R[x]$ there corresponds a unique, nonsingular affine transformation, $\phi x = ax + b$, on R. It follows from our remarks above that this correspondence is a one-to-one mapping from the set of all ring automorphisms on $R[x]$ onto the set of all nonsingular affine transformations on R. Now consider two isomorphisms, ϕ and ψ, on $R[x]$ and let $\phi x = ax + \boldsymbol{b}$ and $\psi x = cx + \boldsymbol{d}$. Then for any polynomial p,

$$\begin{aligned}(\phi\psi)(p) &= p((\phi\psi)x)\\ &= p(\phi(\psi x))\\ &= p(\phi(cx+d))\\ &= p(c(\phi x)+d)\\ &= p(c(ax+b)+d).\end{aligned}$$

Hence the automorphism $\phi\psi$ corresponds to the affine transformation $c(ax+b)+d$ which is the affine transformation resulting from first applying the affine transformation $\phi x = ax+b$ and then $\psi x = cx+d$. That is:

THEOREM 6. *The correspondence $\phi \to \phi x$ is an anti-isomorphism from the group of all ring automorphisms on $R[x]$ onto the group of all nonsingular affine transformations on R.*

In the theorems above we could replace the field R by any other field F having the property: the only automorphism on F is the identity. If it is also a subfield of R—say the field Q of real rationals—then the group of affine transformations of the resulting version of Theorem 6 could be interpreted as the group of all nonsingular affine transformations on R with coefficients in F.

There are other examples of the correspondence exhibited by Theorem 6 between a group of transformations on the real line and a ring of real functions on the real line. If ϕ is an automorphism on the ring $C(R)$ of all real continuous functions on the real line R, then there is a homeomorphism h of R onto itself such that $\phi f = f(h)$ for all f in $C(R)$. (Note: This result also holds if R is replaced by any member U of a wide class of topological spaces. For $U = R$, $h = \phi x$ where x is the identity map on R.) The classical method of constructing h involves "fixed ideals." An ideal I in $C(R)$ is fixed if there is a point r_0 in R such that $f(r_0) = 0$ for all f in I. If M is a fixed, maximal ideal, then $C(R)/M$ is isomorphic to R; but if M is a free (nonfixed), maximal ideal, then $C(R)/M$ is a non-Archimedean ordered field containing R as a proper subfield. Hence if $M(r)$ is a maximal ideal, fixed at r, then its automorphic image $\phi(M(r))$ is fixed at some point $h(r)$. The correspondence

$$r \to M(r) \to \phi(M(r)) = M(h(r)) \to h(r)$$

gives the desired homeomorphism h.

In a similar way, we could have constructed an affine transformation α on R from an automorphism ϕ on $R[x]$ such that $\phi p = p(\alpha)$ for all polynomials p by showing that the fixed, maximal ideals in $R[x]$ can be distinguished algebraically from the free, maximal ideals. An ideal in $R[x]$ is fixed and maximal if and only if it is a principal ideal generated by a first degree polynomial; while it is free and maximal if and only if it is a principal ideal generated by an irreducible, second degree polynomial. The residue class ring $R[x]/I$ of an ideal I generated by a first degree polynomial is isomorphic to R; while $R[x]/I$ is isomorphic to the complex field K if I is generated by an irreducible polynomial. Ob-

serve here that, as in the case for $C(R)$, K contains R as a proper subfield; but, unlike the case for $C(R)$, K is not ordered.

2. Rings of real polynomials in two or more variables. Not all of the results of the previous section can be extended to the ring $R[x, y]$ of polynomials in two variables. If we interpret x as the mapping $x: R^2 \to R$ such that $x(s, t) = s$ for all (s, t) and y as the mapping $y: R^2 \to R$ such that $y(s, t) = t$ for all (s, t), then $R[x, y]$ may be substituted for $F(U)$ in Theorem 1. Hence a ring endomorphism is *ipso facto* an algebra isomorphism. Corresponding to Theorem 2, we can show:

THEOREM 7. *Let ϕ be any nonzero ring endomorphism on $R[x, y]$. Then $\phi p = p(\phi x, \phi y)$ for all polynomials p $(=p(x,y))$ in $R[x,y]$.*

Hence, it follows that every nonzero endomorphism on $R[x,y]$ can be induced by the substitution of a suitable pair of polynomials in (x,y). Hence there is a natural correspondence between nonzero endomorphisms on $R[x,y]$ and polynomial mappings from R^2 into R^2. However, unlike the one variable case, *the polynomial mapping corresponding to an automorphism need not constitute an affine transformation,* as we will demonstrate below.

If one defines a *nonsingular bi-polynomial transformation* on R^2 as a mapping from R^2 onto R^2 given by polynomial functions with an inverse which is also given by polynomial functions, then one can show:

THEOREM 8. *Let ϕ be a nonzero endomorphism on $R[x, y]$. Then ϕ is an automorphism if and only if the pair of functions $(\phi x, \phi y)$ constitute a nonsingular bi-polynomial transformation on R^2. The correspondence $\phi \to (\phi x, \phi y)$ is an anti-isomorphism from the group of all ring automorphisms on $R[x,y]$ onto the group of all nonsingular bi-polynomial transformations on R^2.*

A nonsingular bi-polynomial transformation in *two or more* variables need not be an affine transformation as is shown by the example:

$$\begin{aligned} s' &= s & \qquad s &= s' \\ t' &= t + s^2 & \qquad t &= t' - (s')^2. \end{aligned}$$

However, we observe:

THEOREM 9. *The Jacobian $|J(T)|$ of a nonsingular bi-polynomial transformation T is a nonzero constant.*

Proof. Since T and T^{-1} are given by polynomial functions, $|J(T)|$ and $|J(T^{-1})|$ are polynomials. But $1 = |J(T)| \cdot |J(T^{-1})|$. Since the polynomial $|J(T)|$ has a multiplicative inverse, it must be a constant.

NOTES. The proof of Theorem 1 is adapted from the proof, given in [**3**, p. 23 problem 11], of a similar theorem. The corollary to Theorem 1 can also be proved directly by observing that a nonzero element of $R[x]$ is in R if and only if it has a reciprocal in $R[x]$ and then using the fact that the only nonzero endomorphism on R is the identity.

Theorem 6 is a special case of problem 8, p. 143 of [6]:

Show that, if F is a field, the group of all automorphisms of $F[x]$ [which are the identity on F] is isomorphic with the substitutions $x \to ax+b$, $a \neq 0$ and b in F.

Apparently MacLane and Birkhoff obtain an isomorphism rather than an anti-isomorphism because they write $p\phi$ rather than ϕp.

The properties of $C(R)$ described in this paper are developed in [2] and [4]. Also see [3].

That $R[x]/(x^2+1)$ is isomorphic to the complex field K is a standard example given in several introductory works, see for example [5, pp. 66–67]. The general case follows from the special case by virtue of the fact that any irreducible second degree polynomial ax^2+bx+c can be transformed into x^2+1 by a suitable affine transformation.

The example of a nonsingular bi-polynomial transformation in two variables which is not an affine transformation is based on an example in problem 6, p. 363 of [1] (p. 359 of the third edition).

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A GENERAL CHAIN RULE FOR DERIVATIVES AND THE CHANGE OF VARIABLES FORMULA FOR THE LEBESGUE INTEGRAL

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As the title suggests, our purpose is to give very general statements of two classical theorems and to present their proofs in what we believe to be the clearest and most elegant form. In doing so, we follow the broad outlines of the original proofs of de la Vallée Poussin [7]. Our path is by way of an important result (Theorem 1) on critical values. It is believed to be new, though it was conjectured in [8] and a weak form occurs as early as the paper of de la Vallée Poussin.

The reader may wish to compare our proof of the change of variables theorem with weaker versions in Carathéodory [1], Graves [2], Hewitt and Stromberg [3], and Riesz and Nagy [4].

All functions considered in this paper are real valued functions of a single real variable.

1. A theorem on critical values. To place the conclusions of this section in their proper context, we first recall a result from Saks' treatise [5, p. 227]:

If g has a finite derivative on a set E, with $g'=0$ almost everywhere there, then the Lebesgue measure of the image of E is zero, that is $mg(E)=0$.

The proof is simple. If $E_k = \{t\in E: |g'(t)| \leqq k\}$, then

$$m^*g(E) \leqq \sum_{k=0}^{\infty} m^*g(E_k) \leqq \sum_{k=0}^{\infty} km^*(E_k) = 0$$

with m^* denoting outer Lebesgue measure (the only nontrivial step here is the second one, and this follows from the well-known inequality $m^*g(E_k)\leqq km^*(E_k)$, (cf. Saks [5] p. 226). Saks' result is, of course, just a one dimensional version of a later very general multidimensional theorem due to Sard [6] though the latter requires stronger differentiability assumptions.

We now state the main conclusion of this section, an exact converse of the preceding result.

THEOREM 1. *If g has a derivative (finite or infinite) on a set E with $mg(E)=0$, then $g'=0$ almost everywhere on E.*

We note that in combination with the foregoing result, Theorem 1 can be given the following succinct formulation: Let g have a finite derivative on a set E. Then $mg(E)=0$ if and only if $g'=0$ almost everywhere on E.

Proof of Theorem 1. Let B be the subset of E where $|g'(t)|>0$, and define

$$B_n = \{t \in B: |g(s) - g(t)| \geqq |s - t|/n \quad \text{for } |s - t| < 1/n\}.$$

Noting that $B=\bigcup B_n$, we fix n and let $A=I\cap B_n$ where I is any interval of length less than $1/n$. The problem of showing $m(B)=0$ thus reduces to showing $m(A)=0$.

To do this, let $\{I_k\}$ be a sequence of intervals such that $g(A)\subset\bigcup I_k$ but $\sum m(I_k)<\epsilon$ (recall that $mg(A)=0$). Let $A_k=g^{-1}(I_k)\cap A$. Then noting that $\bigcup A_k$ certainly covers A, we have

$$m^*(A) \leqq \sum m^*(A_k) \leqq \sum \sup_{s,t\in A_k} |s-t| \leqq \sum n \sup_{s,t\in A_k} |g(s)-g(t)|,$$

the latter from the fact that $A_k\subset I\cap B_n$. Now $\sup_{s,t\in A_k}|g(s)-g(t)|\leqq m(I_k)$ since $g(A_k)\subset I_k$. Hence

$$m^*(A) \leqq \sum nm(I_k) \leqq n\epsilon.$$

But n is fixed, while ϵ may be chosen arbitrarily small. It follows that $m(A)=0$, and this completes the proof.

An obvious but interesting consequence of Theorem 1 is the following:

COROLLARY 1. *If g has a derivative on a set E, then $g'=0$ almost everywhere on any subset of E where g is constant.*

2. The chain rule for derivatives. We use the symbol $F \circ g$ for the composite function with values $F(g(t))$. Use of this symbol will imply that the range of g is contained in the domain of F; in particular, $g([a, b]) \subset [c, d]$ in the following theorems. The chain rule for the derivative of $F \circ g$ is an elementary matter when F possesses a finite derivative *everywhere*. Our results are directed toward the situation where F does not satisfy this strong condition.

We recall that a function is said to satisfy Lusin's condition N if it maps null sets into null sets. Absolutely continuous functions, for example, satisfy this condition ([5] p. 225).

THEOREM 2. *Let F, g, and $F \circ g$ have finite derivatives almost everywhere on their domains $[c, d]$ and $[a, b]$. If F satisfies condition N, then the chain rule*

$$(F \circ g)' = (f \circ g) \cdot g'$$

holds almost everywhere on $[a, b]$, where f is any function equivalent to F'.

REMARK. We can eliminate the assumption that the derivatives in question be finite, since by the Denjoy-Young-Saks Theorem [4, p. 18] this is a consequence of the fact that the derivatives exist almost everywhere. Also, inspection of the proof shows that we need only assume that F satisfies condition N on the set Z where F' does not exist or $F' = \infty$ or $F' \neq f$.

Proof of Theorem 2. Let Z be as above, $S=g^{-1}(Z)$, and $T=[a, b]-S$. For $t \in T$ we have

$$\Delta(F \circ g) = [f \circ g + \epsilon(\Delta g)] \cdot \Delta g \qquad (\Delta g = g(t + \Delta t) - g(t)),$$

where $\epsilon(\Delta g)$ goes to 0 with Δg. Dividing by Δt and letting Δt tend to zero gives the chain rule at every point of T where g' exists, that is, almost everywhere on T.

On the other hand, it is easy to see that $mg(S)=0$. In view of condition N it therefore follows that $mF(g(S))=0$. Hence by Theorem 1, g' and $(F \circ g)'$ are both zero almost everywhere on S, completing the proof.

The necessity of condition N is revealed in the following example. Let g be strictly increasing with domain and range $[0, 1]$, and such that $g'=0$ almost everywhere (cf. [4] pp. 48, 49). Let $F=g^{-1}$. Then $(F \circ g)'=1$, which together with $g'=0$ shows that the chain rule does not hold.

COROLLARY 2. *Let F and g have finite derivatives almost everywhere on $[c, d]$ and $[a, b]$ respectively, and suppose that g' is zero at most on a null set. Then $F \circ g$ has a finite derivative almost everywhere on $[a, b]$ and the chain rule holds.*

Proof. Let S and T be as in the proof of the theorem. On T we proceed exactly as before. Moreover, since $mg(S)=0$ we have $g'=0$ almost everywhere on S. But then, according to the assumption that g' is zero at most on a null set, S itself must be a null set. Hence the chain rule holds almost everywhere on $[a, b]$, and the proof is complete.

Our next result is an obvious consequence of Theorem 2 and well-known facts about absolutely continuous functions, but is precisely what we need in the next section for the change of variables formula.

COROLLARY 3. *If g and $F \circ g$ have finite derivatives on $[a, b]$ and if F is absolutely continuous on $[c, d]$, then the chain rule holds.*

A less trivial consequence of Theorem 2 is the following:

COROLLARY 4. *If g is monotone on $[a, b]$ and if F is absolutely continuous on $[c, d]$, then $F \circ g$ has a finite derivative almost everywhere on $[a, b]$ and the chain rule holds.*

To see this we note that the composite function $F \circ g$ must have bounded variation, whence the hypotheses of Theorem 2 are all evidently satisfied.

We conclude the section with a result which follows immediately from the fact that a Lipschitz function of a bounded variation function has bounded variation, and is therefore differentiable almost everywhere.

COROLLARY 5. *If F is Lipschitz on $[c, d]$ and g is of bounded variation on $[a, b]$, then $F \circ g$ has a finite derivative almost everywhere on $[a, b]$ and the chain rule holds.*

3. The change of variables formula. In this section we prove a general change of variables theorem for the Lebesgue integral, and use it to give simple proofs of three standard results on change of variables.

THEOREM 3. *Suppose that g has a finite derivative almost everywhere on $[a, b]$ and that f is integrable on $[c, d]$. Then $(f \circ g)\cdot g'$ is integrable, and the change of variables formula*

$$(*) \qquad \int_{g(\alpha)}^{g(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(g(s))g'(s)ds$$

holds for all α, β in the domain of g, if and only if the composite function $F \circ g$ is absolutely continuous, where $F(x) = \int_c^x f(u)du$.

Proof. The "only if" statement is obvious; we turn to the "if" part. The hypotheses on F and g permit us to employ Corollary 3, so that the chain rule $(F \circ g)' = (f \circ g)\cdot g'$ holds. Since $F \circ g$ is absolutely continuous, its derivative $(f \circ g)\cdot g'$ is integrable and

$$F(g(\beta)) - F(g(\alpha)) = \int_{\alpha}^{\beta} f(g(s))g'(s)ds.$$

as desired.

Theorem 3 is a quite general result, more general than the standard theorems which we state below as corollaries. Note that we do not require that g be absolutely continuous. On the other hand, the requirement that $F \circ g$ be absolutely continuous means roughly that $f \circ g$ must be 0 where g is not absolutely continuous. To give an example, the change of variables formula holds for $f(x) = x$, $g(t) = t \sin(1/t)$ on $[-1, 1]$ even though g is not absolutely continuous there.

COROLLARY 6. *Suppose g is monotone and absolutely continuous and f is integrable. Then $(f \circ g) \cdot g'$ is integrable and the change of variables formula holds.*

To prove this, we need only note that the monotonicity and absolute continuity of g imply the absolute continuity of $F \circ g$, where F is as in the theorem.

In a similar vein, we note that g absolutely continuous and f bounded imply that $F \circ g$ is absolutely continuous. This yields

COROLLARY 7. *Suppose g is absolutely continuous and f is bounded and measurable. Then $(f \circ g) \cdot g'$ is integrable and the change of variables formula holds.*

COROLLARY 8. *Suppose that g is absolutely continuous and f and $(f \circ g) \cdot g'$ are integrable. Then the change of variables formula holds.*

Proof. Let $\{f_n\}$ be any sequence of bounded measurable functions which are dominated by an integrable function and converge to f almost everywhere on the domain $[c, d]$ of f (for example, f_n could be chosen as f truncated at n). Applying dominated convergence and Corollary 7, we obtain

$$\int_{g(\alpha)}^{g(\beta)} f(x)dx = \lim_{n\to\infty} \int_{g(\alpha)}^{g(\beta)} f_n(x)dx = \lim_{n\to\infty} \int_{\alpha}^{\beta} f_n(g(s))g'(s)ds.$$

At this point, without using the integrability of $(f \circ g) \cdot g'$, we have obtained a kind of limiting change of variables formula (see [1] pp. 561–562; Graves [2] p. 223 gives an example where the complete change of variables formula fails). Now specializing f_n as suggested above and using the integrability of $(f \circ g) \cdot g'$, we may pass the limit inside the integral thus obtaining the claimed conclusion.

4. Change of variables for open intervals. In Section 3, the domain of g was always the finite closed interval $[a, b]$. It is of interest to ask what happens when the domain is $(-\infty, \infty)$ or in fact any open or half-open subinterval I of $(-\infty, \infty)$. That is, under the assumption that g has limiting values (finite or infinite) at the open endpoints of I, does the change of variables formula (*) also hold when α, β are the endpoints of I?

To be precise, consider functions g which are defined on I (as above) and which have been assigned values at the open endpoints of I by means of the associated limits at these points. Then, assuming that f is defined on some interval J containing $g(I)$, we shall say that *the change of variables formula* (*) *holds in the extended sense* provided that (*) is valid not only for all $\alpha, \beta \in I$ but also when α, β are allowed to take endpoint values. With this understood, straight-

forward limit arguments show that *Corollaries* 6 *and* 8 *hold as stated for the change of variables formula in the extended sense; moreover we can even replace absolute continuity by local absolute continuity.*

The corresponding generalization of Corollary 7, and indeed of Theorem 3, is not valid; this may be seen by considering, for example, the pair of functions $g(s)=(\sin s)/s$ on $I=[1, \infty)$ and $f(x)=1$ on $J=[-1, 1]$. Note that

$$f(g(s))g'(s) = g'(s) = (s \cos s - \sin s)/s^2$$

which is not integrable on $[1, \infty)$. Nevertheless, we do have the following

THEOREM 4. *Suppose that g has a finite derivative almost everywhere on I and that f is integrable on J. Then $(f \circ g) \cdot g'$ is integrable on I and the change of variables formula* (*) *holds in the extended sense if and only if the composite function $F \circ g$ is locally absolutely continuous and of bounded variation on I, where $F(x) = \int_c^x f(u)du$.*

Proof. The "only if" statement is easily verified. For the "if" part, we note first that local absolute continuity implies, by Theorem 3, that the change of variables formula holds for any closed subinterval of I. Now by an extension of the standard argument for closed intervals (e.g., [**3**] p. 284), the bounded variation of $F \circ g$ on I implies that its derivative $(f \circ g) \cdot g'$ is integrable there. Hence by taking limits as α, β approach the endpoints of I, we conclude that the change of variables formula holds in the extended sense.

It is now straightforward to demonstrate the following analogue of Corollary 7.

COROLLARY 9. *Suppose that g is locally absolutely continuous and of bounded variation on I, and that f is bounded and measurable on J. Then $(f \circ g) \cdot g'$ is integrable and the change of variables formula holds in the extended sense.*

If we are willing to interpret $\int f(g(s))g'(s)ds$ as an *improper Lebesgue integral*, that is, as the limit of integrations over closed subintervals of I, then we have the further:

COROLLARY 10. *Suppose that g is locally absolutely continuous on I and that f is a bounded integrable function on J. Then the change of variables formula again holds in the extended sense.*

Acknowledgments. After publication of [**8**], A. Zygmund communicated his conviction that Theorem 1 was true and F. Cater supplied a lengthy demonstration of this fact as part of a general study of Dini derivates.

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MATHEMATICAL NOTES

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AN ELEMENTARY PROOF OF BOCHNER'S FINITELY ADDITIVE RADON-NIKODYM THEOREM

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Bochner in [1] showed that if a finitely additive measure ν defined on a Boolean algebra $\mathcal{B}$ of subsets of a set Ω is absolutely continuous with respect to a nonnegative finite measure μ defined on $\mathcal{B}$, then for each $\epsilon>0$, there is a simple function f such that for all $A\in\mathcal{B}$, $\int_A f d\mu$ differs from $\nu(A)$ by not more than ϵ. In other words, he showed that $\mathfrak{D}$, the set of measures of the form $fd\mu$, is dense in the Banach space $\mathcal{A}$ of all measures ν that are absolutely continuous with respect to μ; or, equivalently, $\nu\in\mathcal{A}$ implies that ν is in the closure of $\mathfrak{D}$.

The map $f\to fd\mu$ of $L_1(\mu)$ into $\mathcal{A}$ is, of course, an isometry. As is well known, if μ is countably additive, $L_1(\mu)$ is complete, so the set of measures of the form $fd\mu$ for f in $L_1(\mu)$ is complete, and hence is a closed subspace of $\mathcal{A}$. Hence, from Bochner's Theorem, it follows that every ν which is absolutely continuous with respect to μ is of the form $fd\mu$. Thus his theorem can be viewed as a generalization of the usual Radon-Nikodym Theorem which applies to not necessarily countably additive μ.

His proof used the countably additive Radon-Nikodym Theorem, and was based on Lebesgue's Theorem that a monotone function has a derivative almost everywhere. In [2], he and Phillips presented a proof that did not depend upon the countably additive theory, but was based on the theory of vector lattices. In another interesting paper [3], de Finetti rediscovered Bochner's Theorem, with still another proof, but his version seems slightly less general than Bochner's.

This note offers a self-contained and elementary proof of Bochner's Theorem. Though no use is made of the countably additive theory and the theory of vector lattices, I could not have found this elementary proof had I not studied [1], [2], and [3] and received valuable suggestions from David Freedman and

David Gilat. A number of improvements in style were suggested by Leonard J. Savage.

1. Definitions and notations. $\mathcal{B}$ is a Boolean algebra of subsets of a set Ω. A *measure* is a finitely additive real-valued function μ defined on $\mathcal{B}$. For $A \in \mathcal{B}$, $\mu^+(A) = \sup \mu(E \cap A)$ over all $E \in \mathcal{B}$, $\mu^- = (-\mu)^+$, and $|\mu| = \mu^+ + \mu^-$. If $|\mu|(\Omega) < \infty$, μ is of *bounded variation*, and $\|\mu\| = |\mu|(\Omega)$. If, for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $A \in \mathcal{B}$, $|\mu|(A) < \delta$ implies $|\nu|(A) < \epsilon$, then ν is *absolutely continuous with respect to* μ. If, for all $A \in \mathcal{B}$, $\nu(A) \leqq \mu(A)$, write $\nu \leqq \mu$. For each $A \in \mathcal{B}$, the function that is 1 on A and 0 off A is an *indicator function*. A finite linear combination of indicator functions is a *simple* function. For any simple function f, $fd\mu$ is the measure that assigns to each $E \in \mathcal{B}$, the value $\int_E f d\mu$.

2. The Theorem.

THEOREM 1. *For measures μ and ν of bounded variation with $\mu \geqq 0$, these three conditions are equivalent*:

(a) *ν is absolutely continuous with respect to μ.*

(b) *For each $\epsilon > 0$, there is a measure ω and a positive number k such that $-k\mu \leqq \omega \leqq k\mu$ and $\|\nu - \omega\| < \epsilon$.*

(c) *For each $\epsilon > 0$, there is a simple function f such that $\|\nu - fd\mu\| < \epsilon$.*

The proof that (a) implies (b) uses the following known fact:

LEMMA 1. *Every two measures μ and ν have a greatest lower bound. This measure in fact assigns to each $A \in \mathcal{B}$ the infimum over all $E \in \mathcal{B}$ of $\mu(A \cap E) + \nu(A - E)$.*

Proof. Plainly, if δ' is a measure that is less than μ and less than ν, then $\delta'(A)$ is at most the infimum above, call it $\delta(A)$. Moreover, $\delta(A) \leqq \nu(A)$, as is easily seen by letting E be the empty set. Similarly, by letting E be A, it becomes plain that $\delta(A) \leqq \mu(A)$. Moreover, as is not difficult to verify, δ is finitely additive. Thus δ is the largest measure which is bounded from above by both μ and ν.

Proof that (a) *implies* (b): Since ν^+ and ν^- can be handled separately, there is no real loss in supposing that $\nu \geqq 0$. There is then a $\delta(\epsilon) > 0$ such that $\mu(E) < \delta(\epsilon)$ implies $\nu(E) < \epsilon$. Let $k = \nu(\Omega)/\delta(\epsilon)$, and let $\omega = \nu \wedge k\mu$ be the infimum of ν and $k\mu$. That k and ω satisfy (b) is easily verified thus: By considering separately the cases $\mu(E) \leqq \delta(\epsilon)$ and $\mu(E) > \delta(\epsilon)$, verify:

$$\nu(E) - k\mu(E) < \epsilon \qquad \text{for all } E. \tag{1}$$

Verify next that

$$(\nu \wedge \beta)(\Omega) = \inf_E[\nu(E^c) + \beta(E)] \tag{2}$$

for every finite nonnegative measure β. Hence, since $\nu \wedge \beta \leqq \nu$,

$$\begin{aligned} \|\nu - \nu \wedge \beta\| &= \nu(\Omega) - (\nu \wedge \beta)(\Omega) \\ &= \sup_E[\nu(E) - \beta(E)]. \end{aligned} \tag{3}$$

For $\beta = k\mu$, (1) and (3) together imply

$$\|\nu - \nu \wedge k\mu\| \leqq \epsilon. \tag{4}$$

The proof that (b) implies (c) uses a lemma in which it is convenient to use the notation $\omega \leqq \mu + \epsilon$ if, for all $A \in \mathcal{B}$, $\omega(A) \leqq \mu(A) + \epsilon$. Note that ω and μ are measures, but ϵ is a positive number.

LEMMA 2. *Let* $\epsilon \geqq 0$, $k > 0$, μ *and* ω *finite measures,* $\mu \geqq 0$, *which satisfy*

$$-k\mu - \epsilon < \omega < k\mu + \epsilon. \tag{5}$$

Then for each $\epsilon' > \epsilon$, *there is a two-valued* f *such that*

$$-\frac{k}{2}\mu - \epsilon' < \omega - fd\mu < \frac{k}{2}\mu + \epsilon', \tag{6}$$

and a simple f *such that*

$$-\epsilon' < \omega - fd\mu < \epsilon'. \tag{7}$$

Proof. Choose $A \in \mathcal{B}$ so that for all $E \in \mathcal{B}$, $\omega(A) > \omega(E) - (\epsilon' - \epsilon)$. Let f equal $k/2$ on A and $-k/2$ off A. Verify

$$\omega(E \cap A) - \int_{E \cap A} fd\mu \leqq \frac{k}{2}\mu(E \cap A) + \epsilon, \tag{8}$$

and, since $\omega(E \cap A^c) < \epsilon' - \epsilon$,

$$\omega(E \cap A^c) - \int_{E \cap A^c} fd\mu < \epsilon' - \epsilon + \frac{k}{2}\mu(E \cap A^c). \tag{9}$$

In view of (8) and (9),

$$\omega(E) - \int_E fd\mu < \frac{k}{2}\mu(E) + \epsilon', \tag{10}$$

which establishes the second inequality in (6). The first inequality in (6) has, of course, an analogous proof because $\omega(A^c)$ is nearly minimal.

Next, verify, by induction, that for each $\epsilon^* > \epsilon$ and each positive integer n, there is a simple f for which

$$-\frac{k}{2^n}\mu - \epsilon^* < \omega - fd\mu < \frac{k}{2^n}\mu + \epsilon^*. \tag{11}$$

As is easily seen, this implies the existence of a simple f which satisfies (7). This proves the lemma.

Proof that (b) *implies* (c): For $\epsilon > 0$, choose ω so as to satisfy (b). For that ω, choose f so as to satisfy (7), so $\|\omega - fd\mu\| < 2\epsilon'$. Plainly then

$$\|\nu - fd\mu\| \leqq \|\nu - \omega\| + \|\omega - fd\mu\| < \epsilon + 2\epsilon',$$

which proves (c).

Proof that (c) *implies* (a): Plainly, the set of measures of bounded variation that are absolutely continuous with respect to μ is closed with respect to the norm. And (c) simply states that ν is in the closure of the set of measures of the form $fd\mu$ which are evidently absolutely continuous with respect to μ.

For me, the above proof of Bochner's Theorem is an attractive approach to its corollary, the usual Radon-Nikodym Theorem. But some will prefer to derive Bochner's Theorem as a corollary to the countably additive theorem by transferring μ and ν to a field of sets $\mathcal{B}'$ which, as a Boolean algebra, is isomorphic to $\mathcal{B}$, and on which every finitely additive measure is countably additive. The existence of such a $\mathcal{B}'$ is assured by a famous theorem of M. H. Stone, which implies that each Boolean algebra $\mathcal{B}$ is isomorphic to a field of compact subsets of some compact space.

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ON SOBCZYK'S PROJECTION THEOREM

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To determine whether or not there exists a projection (a bounded linear operator P such that $P^2=P$) from a given Banach space onto a given infinite dimensional closed subspace is usually extremely difficult. One of the first significant results in this direction was given by Sobczyk [5] who proved the following: Let c_0 be the Banach space consisting of sequences of complex numbers which converge to zero with norm given by $\|\{\alpha_i\}\| = \sup|\alpha_i|$. If X is a separable Banach space containing a subspace N which is linearly isometric to c_0, then there exists a projection from X onto N of norm at most 2. Köthe [4] gave a simpler proof of this result and showed, moreover, that if A is an isomorphism, i.e., a linear homeomorphism, from c_0 into a separable Banach space X, then there exists a left inverse B of A with $\|B\| \leqq 2\|A^{-1}\|$ and a projection onto the range of A with norm at most $2\|A\|\,\|A^{-1}\|$.

The purpose of this paper is to present a simpler proof of Köthe's result.

For V a subset of X and M a subset of the conjugate space X',

$$V^{\perp} = \{x' \in X' \mid x'V = 0\}, \qquad {}^{\perp}M = \{x \in X \mid x'x = 0 \text{ for all } x' \in M\}.$$

For A a linear operator, $R(A)$ denotes the range of A and $N(A)$ denotes the kernel of A. For U, V subspaces of a vector space Y, $Y=U\oplus V$ means that $Y=U+V$ and $U\cap V=(0)$.

An r-ball in a normed linear space is the set of elements which have norm at most r.

THEOREM. *Let A be an isomorphism mapping c_0 into a separable Banach space X. There exists a w^* closed subspace M of X' such that*

$$X' = M \oplus R(A)^{\perp}; \qquad X = {}^{\perp}M \oplus R(A).$$

Furthermore, the projection P from X onto $R(A)$ with kernel ${}^{\perp}M$ has norm at most $2\|A\|\,\|A^{-1}\|$ and $B=A^{-1}P$ is a left inverse of A with norm at most $2\|A^{-1}\|$.

The following lemma is essentially due to Köthe [3, p. 335] which he proved for a class of spaces now called Köthe spaces. The proof we give is only a slight modification of the one appearing in [3].

LEMMA. *Let V be a subspace of a separable normed linear space X. Suppose $\{x_n'\}$ is a sequence in an r-ball $K\subset X'$ such that $\{x_n'\}$ converges pointwise to 0 on V. Then there exists a sequence $\{z_n'\}$ in $V^{\perp}\cap K$ such that $\{x_n'+z_n'\}$ converges pointwise to 0 on X.*

Proof. Let $y_1, y_2, \cdots$ be dense in X. Given $\{y_1, y_2, \cdots, y_k\}$ and $\epsilon>0$, there exists an integer N such that $n\geqq N$ implies the existence of a $w_n'\in V^{\perp}\cap K$ such that

$$(1) \qquad |(x_n' + w_n')y_i| \leqq \epsilon, \qquad 1 \leqq i \leqq k.$$

For suppose this is not the case. Then there exists a subsequence $\{y_n'\}$ of $\{x_n'\}$ such that

$$(2) \qquad |(y_n' + w')y_{i(n)}| \geqq \epsilon$$

for all $w'\in V^{\perp}\cap K$ and some $y_{i(n)}$, $1\leqq i(n)\leqq k$. Since an r-ball in the conjugate space of a separable normed linear space is compact and metrizable with respect to the w^* topology (Dunford and Schwartz [1], pp. 424, 426), there exists a subsequence $\{u_n'\}$ of $\{y_n'\}$ which converges in the w^* topology to some $x'\in K$. In particular, $x'(v)=\lim u_n'(v)=0$ for each $v\in V$. Thus $x'\in V^{\perp}\cap K$ and $\{u_n'-x'\}$ converges pointwise to 0 on X. But this contradicts (2). Therefore (1) holds. Let $N_1<N_2<\cdots$ be positive integers such that $n\geqq N_k$ implies the existence of a $w_n'\in V^{\perp}\cap K$ such that $|(x_n'+w_n')y_i|\leqq 1/k$, $1\leqq i\leqq k$. Define $\{z_n'\}$ by $z_n'=0$, $1\leqq n<N_1$ and $z_n'=w_n'$ for $n\geqq N_1$. Then $\lim_{n\to\infty}(x_n'+z_n')y_i=0$ for each i. Since $\{x_n'+z_n'\}$ is bounded and $\{y_i\}$ is dense in X, it is easy to see that $\{x_n'+z_n'\}$ converges pointwise to 0 on X.

COROLLARY. *Let $\{x_n'\}$ satisfy the hypotheses in the lemma. There exists a subsequence $\{y_n'\}$ of $\{x_n'\}$ and an element $z'\in V^{\perp}\cap K$ such that $\{y_n'+z'\}$ converges pointwise to 0 on X.*

Proof. $V^\perp$ is w^* closed and therefore $V^\perp \cap K$ is w^* compact. Since K is compact and metrizable with respect to the w^* topology, there exists a subsequence $\{z'_{n_j}\}$ of $\{z'_n\}$ ($\{z'_n\}$ is the sequence in the lemma) which converges in the w^* topology to some $z' \in V^\perp \cap K$. Since $\{x'_{n_j} + z'_{n_j}\}$ converges pointwise to 0 on X, it follows that $\{x'_{n_j} + z'\}$ also converges pointwise to 0 on X.

Proof of the theorem. Since A has a bounded inverse, its conjugate A' maps X' onto l_1 (Goldberg [2], II.3.11). Let e_i be the element in l_1 consisting of 1 in the ith place and 0 elsewhere. Suppose $A'w'_i = e_i$. Then for $\gamma = \{\gamma_i\} \in c_0$,

$$|w'_i A\gamma| = |e_i(\gamma)| = |\gamma_i| \leqq \|\gamma\| \leqq \|A^{-1}\| \|A\gamma\|.$$

By the Hahn-Banach Theorem, there exists an $x'_i \in X'$ such that $x'_i = w'_i$ on $R(A)$ and $\|x'_i\| \leqq \|A^{-1}\|$. Now $\lim_{i\to\infty} x_i A\gamma = \lim_{i\to\infty} \gamma_i = 0$ and $\{x'_i\}$ is in the $\|A^{-1}\|$ ball of X'. Therefore, by the lemma, there exists a sequence $\{z'_i\} \subset R(A)^\perp = N(A')$ such that $\|z'_i\| \leqq \|A^{-1}\|$ and $\{(x'_i + z'_i)x\}$ is in c_0 for each $x \in X$. Let $y'_i = x'_i + z'_i$ and let

$$M = \left\{ \sum_1^\infty \alpha_i y'_i \,\middle|\, \{\alpha_i\} \in l_1 \right\}.$$

It is easy to see that $X' = M \oplus N(A') = M \oplus R(A)^\perp$. We now show that $X = {}^\perp M \oplus R(A)$. Suppose $A\gamma \in {}^\perp M$, $\gamma = \{\gamma_i\} \in c_0$. Then, in particular,

$$0 = y'_i A\gamma = A'y'_i(\gamma) = e_i(\gamma) = \gamma_i, \qquad 1 \leqq i,$$

whence $0 = \gamma = A\gamma$. Given $x \in X$, we know that $\beta = \{y'_i x\}$ is in c_0. Moreover, $x - A\beta \in {}^\perp M$ since

$$\left(\sum_1^\infty \alpha_i y'_i\right) x = \left(\sum_1^\infty \alpha_i e_i\right)\beta = \left(\sum_1^\infty \alpha_i A' y'_i\right)\beta = \left(\sum_1^\infty \alpha_i y'_i\right) A\beta$$

for $\{\alpha_i\} \in l_1$. Thus $X = {}^\perp M \oplus R(A)$ and therefore

$$X' = ({}^\perp M)^\perp \oplus R(A)^\perp = M \oplus R(A)^\perp.$$

Since $M \subset ({}^\perp M)^\perp$, it follows that $M = ({}^\perp M)^\perp$ or, equivalently, M is w^* closed.

Now $\|y'_i\| \leqq \|x'_i\| + \|z'_i\| \leqq 2\|A^{-1}\|$. Therefore for $x \in {}^\perp M$ and $\gamma \in c_0$,

$$(3) \quad 2\|A^{-1}\| \|x + A\gamma\| \geqq \sup_i |y'_i(x + A\gamma)| = \sup_i |A'y'_i \gamma| = \sup_i |\gamma_i| = \|\gamma\|.$$

For P the projection onto $R(A)$ with kernel ${}^\perp M$, (3) implies

$$\|P(x + A\gamma)\| = \|A\gamma\| \leqq \|A\| \|\gamma\| \leqq 2\|A\| \|A^{-1}\| \|x + A\gamma\|, \qquad x \in {}^\perp M, \gamma \in c_0.$$

Thus $\|P\| \leqq 2\|A\| \|A^{-1}\|$. Finally, $B = A^{-1}P$ is a left inverse of A and $\|B(x + A\gamma)\| = \|\gamma\| \leqq 2\|A^{-1}\| \|x + A\gamma\|$, $x \in {}^\perp M$, $\gamma \in c_0$. Hence $\|B\| \leqq 2\|A^{-1}\|$.

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FROBENIUS GROUPS AND WEDDERBURN'S THEOREM

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The purpose of this note is to show how some standard results from the theory of Frobenius groups (which are reviewed below) can be used together with the geometry of a finite Desarguesian plane to give a new arrangement for the proof of Wedderburn's Theorem that every finite division ring is commutative.

Our interest in this particular proof of Wedderburn's Theorem comes from a consideration of the program of Artin in *Geometric Algebra* [1]. Artin's title is a take-off on "algebraic geometry" a subject in which algebraic structures are used to study geometric objects. Artin, in *Geometric Algebra*, uses geometry as a tool to accomplish a deep investigation of the algebraic structure of certain of the classical groups. Now from the geometry of a Desarguesian plane it is possible to construct a division ring k, which is commutative if and only if the Theorem of Pappus holds in the geometry. (See Chapter II of [1].) Hence Wedderburn's Theorem implies that in each finite Desarguesian plane the Theorem of Pappus holds. Now our proof of Wedderburn's Theorem will reverse the roles of algebra and geometry in that we shall use properties of the Desarguesian plane for an essential step in the argument. Thus we shall use geometry as a tool to prove this purely algebraic theorem about the multiplicative group of a finite division ring.

A *permutation representation* (G, X) of a group G consists of a nonempty set X on which the group acts transitively. We call (G, X) a *Frobenius representation* if each $\sigma \in G$, $\sigma \neq 1$, has at most one fixed point. Each group has two trivial Frobenius representations: (1) (G, X) is the regular representation of G, and (2) X is a set with one element. By a *Frobenius group* we mean a finite group G which has a non-trivial Frobenius representation.

We shall now review some well-known facts about Frobenius groups. The proofs of these can be found in either [2] or [5]. Let G be a Frobenius group with a nontrivial Frobenius representation (G, X). The set

$$N = \{\sigma \mid \sigma \in G \quad \text{and} \quad \sigma \text{ has no fixed points}\} \cup \{1\}$$

is a normal subgroup of G. For any $x \in X$, we denote by H_x the subgroup of G consisting of all elements of G leaving x fixed. The various subgroups H_x are conjugate to each other. We choose a particular H_x and denote it by H. The three groups (N, G, H) form an exact sequence:

$$1 \to N \to G \to H \to 1.$$

Z_p with the nonsingular orthogonal geometry whose metric is given by the quadratic form x^2+y^2, we know from p. 143 of [1] that each element of Z_p^* must appear as the square of some vector in V. Thus we have $x, y \in Z_p$ such that $x^2+y^2=-1$. We can check that $(x\sigma+y\tau)^2=1$. But $x\sigma+y\tau$ cannot equal either 1 or -1, for 1 and -1 commute with every element of k while it is impossible for $x\sigma+y\tau$ to commute with both σ and τ. This contradiction establishes the proposition.

Now we have shown that k^* is a group in which every Sylow subgroup, including the even Sylow subgroup if it exists, is cyclic. Such a group is called a *metacyclic* group and its structure is simple and well known. A good description of metacyclic groups is found in Chapter 9 of [3]. Now we know from p. 322 of [4] that if $\sigma \notin$ Center k^*, then there exists an inner automorphism of k^* mapping σ onto some $\sigma^i \neq \sigma$. We now prove:

PROPOSITION 2. *If H is a metacyclic group and if for each $\sigma \notin$ Center (H), there exists an inner automorphism of H mapping σ onto σ^i $(\neq \sigma)$, then H is cyclic.*

Proof: We can easily see by examining the proof of Theorem 9.4.3 of [3] that the group H has the following properties:

(i) H' (the commutator subgroup of H) is cyclic, as is H/H', and

(ii) if $\tau H'$ is a generator of H/H', then the order of $\tau H'$ in H/H' equals the order of τ in H.

Now suppose H is not cyclic and let $\tau \in H$ be chosen so that $\tau H'$ generates H/H'. Then there exists a $\sigma \in H$ such that $\sigma\tau\sigma^{-1}=\tau^i$ where $1<i<$order of τ in H. But H' contains $\sigma\tau\sigma^{-1}\tau^{-1}=\tau^{i-1}$ so that the order of $\tau H'$ in H/H' is less than the order of τ in H. This contradiction establishes our proposition. Wedderburn's Theorem follows from the above proposition by taking $H=k^*$.

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ON FUNCTIONAL VALUES OF FAMILIES OF ANALYTIC FUNCTIONS

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I. Introduction. Erdös [1] proved the following result: Let F denote an arbitrary nondenumerable family of functions analytic on a domain D. The continuum hypothesis is false if and only if for every F there exists some complex z_0, such that the set of functional values $T(z_0, F)=\{g(z_0); g\in F\}$ is nondenumerable.

In this note we show that:

THEOREM 1. *For each F there is some complex z_1 such that the set $T(z_1, F)$ is infinite and has a finite limit point.*

THEOREM 2. *If F is replaced by any nondenumerable family of polynomials P, then $T(z, P)$ is nondenumerable for all but at most finitely many z. If all the polynomials in P are of the same degree, say n, then the above is true for all z except at most n.*

II. Proofs of Theorems 1 and 2.

Proof of Theorem 1. We may assume that D is bounded. Let S_n be a disc with radius n and center at the origin. All elements of some nondenumerable subfamily F^* of F map D inside S_n for some n. There must exist a point z_1 such that $T(z_1, F^*)$ is infinite. For assume that for each z such a set is finite. Then for some integer m there exists a nondenumerable number of z for which $T(z, F^*)$ has only m values, say $a_1(z), a_2(z), \cdots, a_m(z)$. Thus, out of any $m+1$ functions $g \in F^*$, two must be equal for a set of z with a limit point and hence must be identical. This completes the proof of Theorem 1.

Proof of Theorem 2. There exists an integer n such that nondenumerably many of the elements of F are of degree n. It therefore suffices to show that the assertion is valid for any nondenumerable family of polynomials whose members are of degree n. We prove this by mathematical induction.

For the case $n=1$ we consider a nondenumerable family of linear functions $F_1 = \{A_\alpha(z-z_0)+B_\beta\}_{\alpha \in A, \beta \in B}$ where A and B are nondenumerable index sets and A_α and B_β depend on z_0. If $\{B_\beta\}_{\beta \in B}$ is nondenumerable for every z_0, then there is nothing more to prove in this case. If, on the other hand, $\{B_\beta\}_{\beta \in B}$ is denumerable for some z_0, then there is a fixed $B_{\beta'}$ to which there corresponds a nondenumerable set $\{A_\alpha\}_{\alpha \in A'}$, $A' \subset A$, with the property that $\{A_\alpha(z-z_0)+B_{\beta'}\}_{\alpha \in A'}$ is contained in F_1. Thus, for any $z_1 \neq z_0$ the functional values $\{A_\alpha(z_1-z_0)+B_{\beta'}\}_{\alpha \in A'}$ form a nondenumerable set.

Assume that the theorem is valid for polynomials of degree $n-1$. Let F_n denote a nondenumerable family of polynomials of degree n, say

$$F_n = \{A_{\alpha n}(z - z_0)^n + A_{\alpha(n-1)}(z - z_0)^{n-1} + \cdots + A_{\alpha 0}\}_{\alpha i \in A_i}, \quad i = 0, 1, \cdots, n.$$

We note again that $A_{\alpha i}$ depend on z_0.

If $\{A_{\alpha 0}\}$ is nondenumerable for every z_0, then our proof is complete. Otherwise for some fixed z_0 and a corresponding fixed $A_{\alpha 0'}$ one can find a nondenumerable family

$$F_n' = \{A_{\alpha n}(z - z_0)^n + \cdots + A_{\alpha 1}(z - z_0) + A_{\alpha 0'}\}_{\alpha i \in A_i}, \quad i = 1, \cdots, n,$$

which is contained in F_n. Each member of this family may be written in the form $(z-z_0)P(z)+A_{\alpha 0'}$, where $P(z)$ is a polynomial of degree $n-1$. We denote the corresponding nondenumerable family of the P's by F_n''. Since the assertion of the theorem is valid for F_n'' it follows that it is also valid for F_n'. This completes our proof.

In a similar manner one can prove:

THEOREM 3. *Let g_i be a given sequence of functions analytic on a domain D. Let F be any nondenumerable family of functions of the form*

$$\sum_{i=1}^{k} \lambda_i g_i, \qquad k = 1, 2, 3, \cdots; \ \lambda_i \text{ complex numbers.}$$

Then $T(z, F)$ is nondenumerable for all z with the possible exception of denumerably many.

We conclude with the following:

CONJECTURE. If in Theorem 3, λ_i are replaced by entire functions of zero order, then the theorem is still valid.

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SUBPERMANENTS

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In the paper [2], the authors suggest the following generalization of the van der Waerden conjecture.

CONJECTURE. *If A is an n-square doubly stochastic matrix and $P_k(A)$ denotes the sum of all the kth order subpermanents of A, then*

$$P_k(A) \geqq P_k(J_n), \tag{1}$$

where J_n is the n-square matrix all of whose entries are $1/n$.

They prove that the only local minimum of $P_k(A)$ (i.e., a minimum for which $a_{ij} > 0$) is at J_n. The techniques are closely related to those in [1]. The purpose of the present note is to establish an inequality for the function $P_k(A)$ which will establish the conjecture in the event A is semidefinite symmetric. The van der Waerden conjecture was proved for this case in [1]. Our main result is the following:

THEOREM 1. *If A and B are any two n-square complex matrices, then*

$$\left| P_k(AB) \right| \leqq P_k(AA^*)^{\frac{1}{2}} P_k(B^*B)^{\frac{1}{2}}. \tag{2}$$

We also confirm the fact that the unique positive semidefinite symmetric square root of a doubly stochastic matrix has every row and column sum equal to 1. (The entries are not necessarily nonnegative.) If we use this fact, together with Theorem 1, we are able to prove

THEOREM 2. *If A is doubly stochastic and positive semidefinite symmetric, then $P_k(A) \geqq P_k(J_n)$.*

To see that Theorem 2 follows from Theorem 1, suppose A is positive semidefinite symmetric doubly stochastic, and let S be its unique positive semidefinite symmetric square root, with all row and column sums equal to 1, i.e.,

$$A = S^2 = SS^* = S^*S, \tag{3}$$

and $J_nS = SJ_n = J_n$. Hence

$$\begin{aligned} P_k(J_n) &= \left| P_k(J_nS) \right| \\ &\leqq P_k(J_n^2)^{\frac{1}{2}} P_k(S^*S)^{\frac{1}{2}} \\ &= P_k(J_n)^{\frac{1}{2}} P_k(A)^{\frac{1}{2}}. \end{aligned}$$

Cancelling $P_k(J_n)^{\frac{1}{2}}$ from both sides and squaring, we obtain the desired result.

To prove Theorem 1, let $K(A)$ denote the kth induced power matrix of A [3]. (The usual notation for $K(A)$ is $P_k(A)$, which for obvious reasons we do not use here.) The entries of $K(A)$ are the following

$$\binom{n+k-1}{k}^2$$

numbers:

$$\frac{\operatorname{per}(A[\alpha \mid \beta])}{\sqrt{\nu(\alpha)\nu(\beta)}}, \qquad \alpha, \beta \in G_{k,n}. \tag{4}$$

The notation in (4) means the following: the sequence set $G_{k,n}$ is the totality of

$$\binom{n+k-1}{k}$$

nondecreasing sequences of length k chosen from the integers $1, \cdots, n$; the number $\nu(\alpha)$ is the product of the factorials of the multiplicities of the distinct integers appearing in α; and $A[\alpha|\beta]$ is the k-square matrix whose (i, j) entry is

$$a_{\alpha_i\beta_j}.$$

The entries (4) of $K(A)$ are arranged in doubly lexicographic order with respect to α and β. It is well known that the induced power matrix is a representation of the set of n-square matrices, i.e.,

$$K(AB) = K(A)K(B).$$

Now, let x and y be arbitrary

$$\binom{n+k-1}{k}\text{-tuples}$$

of complex numbers. Then, by the Cauchy-Schwarz inequality applied to the standard inner product in Euclidean space, we have

$$
\begin{aligned}
|(K(AB)x, y)| &= |(K(A)K(B)x, y)| \\
&= |(K(B)x, K(A)^*y)| \\
&= |(K(B)x, K(A^*)y)| \qquad (5)\\
&\leqq (K(B)x, K(B)x)^{\frac{1}{2}}(K(A^*)y, K(A^*)y)^{\frac{1}{2}} \\
&= (K(B^*B)x, x)^{\frac{1}{2}}(K(AA^*)y, y)^{\frac{1}{2}}.
\end{aligned}
$$

In (5), let x and y both be the following

$$\binom{n+k-1}{k}\text{-tuple:}$$

in the positions corresponding to the strictly increasing sequences α in the lexicographic ordering of $G_{k,n}$, let the components of x and y be one, and let all other components be zero. Call this vector e. From the definition of $\nu(\alpha)$, it is clear that for all such α, $\nu(\alpha)=1$. Now, it is obvious that for any matrix A,

$$(K(A)e, e) = \sum_{\alpha,\beta} \operatorname{per} A[\alpha|\beta], \tag{6}$$

in which the summation is over all strictly increasing sequences of length k from $1, \cdots, n$. In other words, (6) is the statement $(K(A)e, e) = P_k(A)$. From (5), with $x=y=e$, we have $|P_k(AB)| \leqq P_k(AA^*)^{\frac{1}{2}}P_k(B^*B)^{\frac{1}{2}}$.

Our proof of Theorem 2 will be complete as soon as we establish that A possesses a positive semidefinite symmetric square root S satisfying $SJ_n=J_nS=J_n$. Since A is symmetric, let U be an orthogonal matrix such that

$$U^TAU = D = \operatorname{diag}(\alpha_1, \cdots, \alpha_n),$$

in which $\alpha_1=1$. Then $S=U\operatorname{diag}(\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_n})U^T$. Now,

$$s_{ij} = \sum_{\nu=1}^{n} u_{i\nu}\sqrt{\alpha_\nu}\, u_{j\nu},$$

so that

$$\sum_{j=1}^{n} s_{ij} = \sum_{j=1}^{n}\sum_{\nu=1}^{n} u_{i\nu}\sqrt{\alpha_\nu}\, u_{j\nu} = \sum_{\nu=1}^{n} u_{i\nu}\sqrt{\alpha_\nu}\sum_{j=1}^{n} u_{j\nu}. \tag{7}$$

The first column of U is the unit characteristic vector corresponding to the characteristic root 1 of A, and hence may be chosen so that $u_{j1} = (1/n)^{\frac{1}{2}}$, $j=1, \cdots, n$. Since the columns of U are orthogonal, it follows that for $\nu\geqq 2$, $\sum_{j=1}^{n} u_{j\nu}=0$.

Furthermore, $\sum_{j=1}^{n} u_{j1}=n^{\frac{1}{2}}$. Thus (7) becomes $u_{i1}\sqrt{\alpha_1}n^{\frac{1}{2}}=1$. Similarly, $\sum_{i=1}^{n} s_{ij}=1$, completing the proof.

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A SEQUENCE OF FAMILIES CONVERGING IN AN EQUICONVERGENT MANNER

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The theorem of Arzela-Ascoli states that a NASC for a family B of real-valued continuous functions defined on a compact metric space X to be compact in $C(X)$ is that the family B be uniformly bounded and equicontinuous. In showing that B is compact in $C(X)$ the difficult part is generally to show that B is equicontinuous. The following theorem may be of some value in this regard. We first, however, state one definition.

Definition. *If the pair (X, d) is a metric space, we say that a sequence of families of real-valued functions $F_n = \{f_\alpha^n\}$, α belonging to some index set A, each f_α^n defined on X, converges to the family $\{f_\alpha\}$ in an equiconvergent manner as $n \to \infty$ if, for each $\epsilon > 0$, there exists an integer N, independent of $\alpha \in A$, such that $n \geqq N$ implies*

$$|f_\alpha^n(x) - f_\alpha(x)| < \epsilon;$$

for all $x \in X$ and all $\alpha \in A$.

Theorem. *If for each $n = 1, 2, \cdots$, $F_n = \{f_\alpha^n\}$ is a family of equicontinuous real-valued functions defined on a compact metric space (X, d) and if $\{f_\alpha^n\}$ converges to $\{f_\alpha\}$ in an equiconvergent manner as $n \to \infty$, then the family $\{f_\alpha\}$ is equicontinuous.*

Proof. Since each family F_n is equicontinuous, we know that for every $\epsilon > 0$ there exists a $\delta_n > 0$, independent of α, such that $d(x, y) < \delta_n$ implies $|f_\alpha^n(x) - f_\alpha^n(y)| < \epsilon/3$ for all $\alpha \in A$, by use of the definition. Also since $\{f_\alpha^n\}$ converges to $\{f_\alpha\}$ *in an equiconvergent manner* we know that if $\epsilon > 0$ then there exists an integer N independent of α such that for all $x \in X$, $\alpha \in A$,

$$|f_\alpha^n(x) - f_\alpha(x)| < \epsilon/3$$

for all $n \geqq N$. Now pick $\epsilon > 0$. We conclude that there exists a $\delta_n > 0$, independent of $\alpha \in A$, such that if $d(x, y) < \delta_n$, then for all $\alpha \in A$ we have

$$\begin{aligned} |f_\alpha(x) - f_\alpha(y)| &\leqq |f_\alpha(x) - f_\alpha^n(x)| + |f_\alpha^n(x) - f_\alpha^n(y)| + |f_\alpha^n(y) - f_\alpha(y)| \\ &\leqq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

AN EXTENSION OF A THEOREM ON MONOMIAL GROUPS

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A theorem of Taketa says that if the complex irreducible representations of a finite group are monomial, that is, induced from one-dimensional representations of subgroups, then the group is solvable ([2]; an exposition is given in [1], Section 52). The question of which solvable groups have all their irreducible representations monomial is only partly answered (see [1] and the references there).

The purpose of this note is to extend the result of Taketa. For this, let S be a family of simple groups. Call a finite group G an *S-group* if all the factors in a composition series for G are members of S, and call a representation over a field K of any finite group G an *S-representation* if the image of G in the corresponding linear group over K is an S-group. Then the extension is this:

THEOREM. *Let K be a field and let S be a family of simple groups that includes the cyclic group of order p if the characteristic of K is the prime p. Suppose G is a finite group such that every irreducible representation of G over K is either an S-representation or is induced from an S-representation of a subgroup. Then G is an S-group.*

Proof. The Theorem of Taketa comes on taking S to be the family of cyclic simple groups, and the proof of the present theorem is patterned after Taketa's. Representations will be understood as modules over group algebras, and if L is a representation of a subgroup of G, L^G is the induced representation (again, see [1]). All representations are to be over K.

Suppose G satisfies the hypotheses and N is a normal subgroup of G. Let L be an irreducible representation of G/N regarded as one of G. If $L=M^G$ where M is a representation of a subgroup H of G, then $L|N$ (L restricted to N) is a direct sum of modules induced from subgroups of N of the form $gHg^{-1}\cap N$, g in G, by the Subgroup Theorem of Mackey ([1], Section 44). But $L|N$ is a direct sum of copies of the identity representation. Since direct examination of the definition of an induced representation shows that a representation induced from one of a proper subgroup cannot be a direct sum of copies of the identity representation, it can only be that the groups $gHg^{-1}\cap N$ are N itself. Then $N\subseteq H$. Since M is an H-submodule of M^G, N must be the identity on M. So M is a representation of G/N, and in G/N, L is induced from M. Therefore the hypotheses hold for the quotient G/N.

By way of induction, then, let G be a group of minimal order satisfying the hypotheses but not the conclusion. Then G has a unique minimal normal subgroup N (possibly G itself). For if there were two minimal normal subgroups N_1 and N_2, then the normal series $G\supseteq N_1N_2\supseteq N_1\supseteq 1$, the isomorphism of N_1N_2/N_2 and N_1 (from $N_1\cap N_2=1$), and the induction would show G to be an S-group.

Then a representation of G either is faithful or has N in its kernel. But if all irreducible representations of G have N in their kernels, the set of elements $g-1$,

g in N, is in the radical of the group algebra of G and these elements are nilpotent. Since then N would be a p-group (p the characteristic of K, necessarily not 0 in this case), G would be an S-group.

Then, let M be a faithful irreducible representation of least degree. M could not be an S-representation, so $M=L^G$, L an S-representation of a (proper) subgroup H. Letting I_A stand for the identity representation of a group A, form $(I_H)^G$. Since this is reducible (I_G is a submodule) and its degree is at most that of M, all its irreducible constituents have lower degree than that of M. So they all contain N in their kernels. Now $(I_H)^G$ cannot be faithful; for if it were, since the matrices representing N can be simultaneously triangularized with 1's on the diagonal, N would be faithfully represented as a nilpotent linear group. The minimality of N would then imply N was elementary Abelian. But as it cannot be a p-group (where p is the characteristic of K), $(I_H)^G | N$ would be completely reducible and therefore be the identity on N.

Thus N is in the kernel of $(I_H)^G$. By the same argument as above with the Subgroup Theorem it follows that $H \supseteq N$. From the minimality of N it follows that if N_0 is a proper normal subgroup of N the intersection of the conjugates of N_0 by the members of G is 1. So if N/N_0 were an S-group, a step-by-step argument on these conjugates, like the argument with N_1 and N_2 above, would imply that N was an S-group.

Suppose then that N is not in ker L, the kernel of L. Then as $H/\ker L$ and therefore $N \ker L/\ker L$ are S-groups, $N/N \cap \ker L$ is an S-group and N is an S-group as above. Thus $N \subseteq \ker L$. But that would imply that M was not faithful, since N is normal in G.

Thus the assumption on G is untenable, and the proof complete. A corollary is this:

COROLLARY. *It cannot be that the irreducible representations other than the identity of a simple group, over the field K, are all induced from proper subgroups.*

For, take G to be a minimal counterexample and then let S be the collection of simple groups of lower orders than that of G (along with the cyclic group of order p if the characteristic of K is the prime p).

Since this note was submitted (March 1967) a paper of L. Dornhoff (Math. Z., 100(1967) 226–256) has appeared containing the special case of the theorem given here in which K is the complex field. The paper deals in some depth with the question at the end of the first paragraph of the present note.

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SOME PICTORIAL COMPACTIFICATIONS OF THE REAL LINE

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1. Introduction. The general theory of compactifications of a completely regular space, X, either from a classical Tychonoff cube or from the modern Gelfand point of view is well known (see, e.g., [1] pp. 223–227). It turns out that in case the original space is not compact, there are many different compactifications; in fact, there is one for every algebra of bounded continuous real-valued functions on X which is closed in the uniform norm and which contains enough functions to separate points from closed sets. Even in the case where X is the real line, R, one rarely talks about anything but the one-point, the two-point and the Stone-Čech compactifications. The first two are quite tame while the last is impossible to picture. The purpose of this paper is to present a certain class of compactifications of R which are quite easy to picture.

2. The main result. First, we state the basic definition:

If X is a topological space, a compactification *of X is a compact Hausdorff space, Y, together with a map $f: X \to Y$ such that:*

(i) *f is a homeomorphism of X and* $\operatorname{im} f$ *(the image of f) where* $\operatorname{im} f$ *has the relative topology which it inherits as a subset of Y.*

(ii) $\operatorname{Im} f$ *is dense in Y.*

Two compactifications $f: X \to Y$ and $g: X \to Z$ are said to be equivalent if there is a homeomorphism $h: Z \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z \arrow[ur, "h"'] & \end{tikzcd}$$

commutes, i.e., $f = h \circ g$.

One normally associates X with its image in the compactification, in which case the commutative diagram is replaced with the statement that h leaves X pointwise fixed.

We will be concerned with compactifications of $[0, \infty)$. If $f: [0, \infty) \to Y$ is a compactification, we will say that Y-$\operatorname{im} f$ has been added to make the compactification. The main result is the theorem:

Let X be a compact Hausdorff space and let $g: [0, \infty) \to X$ be a continuous map with the property that for each $a > 0$, $g([a, \infty))$ is dense in X. Then $[0, \infty)$ has a compactification in which X has been added to make the compactification.

We note that g was not required to be either injective, or if injective, a homeomorphism onto im g. We also emphasize that X is not itself the compact extension.

Proof. Let $I=[0,1]$ and define f: $[0,\infty)\to X\times I$ by $f(a)=(g(a),h(a))$ where $h(a)=a/(1+a)$ is a homeomorphism of $[0,\infty)$ and $[0,1)$. For convenience set $G=g\circ h^{-1}$.

We first show that f is a homeomorphism of $[0,\infty)$ and $\operatorname{im} f$. It is obvious that f is continuous since its coordinates are continuous and 1-1 since h is 1-1. Moreover, f is open, for if $A\subset[0,\infty)$ is open, $h[A]\subset[0,1)$ is open and thus $f[A]=(X\times h[A])\cap\operatorname{im} f$ is relatively open in $\operatorname{im} f$.

Next, we show that $\overline{\operatorname{im} f}=(X\times\{1\})\cup\operatorname{im} f$. For let $(x,r)\in X\times I$ with $r\neq 1$ and $x\neq G(r)$. We show that $(x,r)\notin\overline{\operatorname{im} f}$; for let B and C be disjoint open sets in X about x and $G(r)$ respectively. Then $(B\times G^{-1}[C])$ is a neighborhood of (x,r) which does not intersect $\operatorname{im} f$. On the other hand any $(x,1)\in\overline{\operatorname{im} f}$; for let $U\times(b,1]$ be a rectangular neighborhood of $(x,1)$, and let $a=h^{-1}(b)$. Then, by the density assumption, $U\cap g((a,\infty))\neq\emptyset$; say $g(c)\in U$. Then $f(c)=(g(c),h(c))\in U\times(b,1]$. Thus $(x,1)\in\overline{\operatorname{im} f}$.

Thus our result is proven; for f: $[0,\infty)\to\overline{\operatorname{im} f}$ is a compactification and $\overline{\operatorname{im} f}-\operatorname{im} f=X$.

3. Some examples. Since $(0,\infty)$ is homeomorphic to the real line, given g as in the main theorem (and given a *specific* homeomorphism of $(-\infty,\infty)$ and $(0,\infty)$), we can regard f: $(0,\infty)\to\overline{\operatorname{im} f}$ as a compactification of R. We get this compactification "by putting a point at one end of R and X at the other end"; thus we will call it the point-X compactification (actually a point-X compactification since the way R lies in $\overline{\operatorname{im} f}$ depends not only on X but on the exact map g and on the homeomorphism of $(-\infty,\infty)$ and $(0,\infty)$). Given two compactifications of $[0,\infty)$ following the theorem, say by adding X and Y respectively, we can view one as a compactification of $(-\infty,0)$ and join the two together at 0 and so get an $X-Y$ compactification or if $X=Y$ a two$-X$ compactification. Finally given a map g satisfying the hypothesis of the theorem, one can consider the two$-X$ compactification and "glue" the two copies of X together; a "one$-X$" compactification results. This terminology agrees with the usual one-point, two-point terminology in the case that X is a single point.

The prime example of an X and a g obeying the conditions of the theorem, in fact, the example that motivated the theorem is the winding line on the torus $S^1\times S^1$. If S^1 is represented by real coordinates mod 2π, and g: $[0,\infty)\to S^1\times S^1$ is defined by $g(a)=(a,ta)$ with t a fixed irrational number, then g meets the hypothesis of the theorem. In this way, one can construct one- and two-torus compactifications.

To obtain a geometric picture of a torus-point compactification, we imbed $(S^1\times S^1)\times I$ in R^3 as a toroidal shell. In fact, without changing the construction of the main result, we can shrink $(S^1\times S^1)\times\{0\}$ into a circle and so view our compactification as being embedded in a solid torus. Then we take a copy of the real line, start it at the center of a cross-section and let it spiral out towards the surface, winding around longitudinally as we spiral outward; only the surface need be added to give us a compact set.

Of course, we need not stop with two dimensions or with the torus. We can get a winding line on an n-dimensional torus or we can go to a countable number of dimensions or even an uncountable number of dimensions since the reals have uncountable dimension over the rationals. Or one can wind about a two-dimensional sphere as if one were winding a ball of yarn and thereby find sphere-torus, point-sphere and assorted other compactifications. Again, one is not restricted to two-dimensional spheres. More exotic spaces (like $S^n \times S^m$ or a nest of circles tangent at one point) can be used.

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LEFT ARTINIAN RINGS THAT ARE DIVISION RINGS

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Zariski and Samuel point out that if R is a commutative ring with identity and no proper divisors of 0, and R satisfies the descending chain condition, then it is a field [**1**, p. 203]. Surprisingly, we can omit the assumptions of commutativity and an identity and prove the following theorem:

If R is a left Artinian ring with no proper divisors of 0, *then R is a division ring.*

Proof. Recall that a left Artinian ring is one in which every properly descending chain of left ideals is finite. A semisimple ring is a left Artinian ring with zero radical [**2**, Chapter 2]. A semisimple ring has a multiplicative identity [**2**, p. 29]. Now if R is left Artinian with no proper divisors of 0, then it is semisimple and hence has an identity 1. Consider R as a left R-module. All the submodules are left ideals, so R is an Artinian left R-module. Let $b \in R$, $b \neq 0$, and define a function f on R:

$$f(x) = xb.$$

Then f is an endomorphism of R as a left R-module and f is one-to-one. Now a one-to-one endomorphism of an Artinian module is an automorphism [**3**, p. 23]. Hence for all b in R different from 0, there exists x in R such that $xb = 1$. Thus R is a division ring.

The author is a National Science Foundation fellow.

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CORRECTION TO "THE SUPERSET TOPOLOGY"

NORMAN LEVINE, Ohio State University

Paragraph 11 of "The Superset Topology," this MONTHLY, 75(1968) 745–746, should read:

11. In this section we will characterize some of the well-known topologies for a set X in terms of S-topologies. Let $\mathfrak{I}$ be a topology for the set X. Then

(a) $\mathfrak{I}$ is the cofinite topology for X iff $(X, \mathfrak{I})$ is compact, $\mathfrak{I}$ is an S-topology for X and finite sets are closed.

(b) $\mathfrak{I}$ is the cocountable topology for X iff $(X, \mathfrak{I})$ is Lindelöf, $\mathfrak{I}$ is an S-topology for X and countable sets are closed.

(c) If X has two or more elements, then $(X, \mathfrak{I})$ is discrete iff $(X, \mathfrak{I})$ is disconnected and $\mathfrak{I}$ is an S-topology for X.

(d) $\mathfrak{I} = \{A : A = \emptyset \text{ or } A^* \subseteq A \subseteq X\}$ for some fixed nonempty set A^* iff $\mathfrak{I}$ is an S-topology for X and $\emptyset \neq \bigcap\{O : \emptyset \neq O \in \mathfrak{I}\} \in \mathfrak{I}$.

RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

IS A BODY SPHERICAL IF ITS HA-MEASUREMENTS ARE CONSTANT?

VICTOR KLEE, University of Washington

Two problems. Let B be a body and P a plane in Euclidean 3-space E^3. Imagine an instrument that measures, for each plane P' parallel to P, the area of the intersection $P' \cap B$, and then records the supremum of these areas. Denote the supremum by $\eta_P B$ and call it the *HA-measurement of B* relative to P. The next section explains the HA prefix and tells why such measurements are of interest. Here we state and discuss two unsolved problems concerning HA-measurements:

Is a body B spherical if its HA-measurements are constant — that is if $\eta_P B$ has the same values for all planes P?

Is a body determined by its full set of HA-measurements?

In order to make these questions precise, the terms *body*, *spherical*, and *determine* must be defined. At a minimum, each body B is to be nonempty, bounded, and the closure of its interior. And any of the following conditions may be incorporated into the definition of *body*: B is connected; the boundary S of B is connected; B is contractible; B is starshaped; B is convex. A body is said to be *spherical* provided that it is the union of a family of concentric spheres. Under any of the additional conditions just listed, a spherical body is actually a spherical ball or (in one case only) a region bounded by two concentric spheres.

Note that $\eta_P B$ is merely a number associated with the pair (P, B). It does not tell which translates P' of P produce sections $P' \cap B$ of large area. Indeed, it is plain that two bodies have the same HA-measurements if one body can be obtained from the other body by translation or by reflection in a point. Thus we say that two bodies are *equivalent* provided that one can be obtained from the other in this way, and that a body B is *determined* by certain conditions provided that any body satisfying these conditions is equivalent to B.

If too weak a notion of *body* is used, bodies are not determined by their HA-measurements. For example, a spherical ball of radius 1 has the same HA-measurements as the region bounded by two concentric spheres of radii $r > 1$ and $(r^3 - 1)^{1/3}$. However, for any of the definitions of body suggested above it is not known whether a body with constant HA-measurements must be spherical. It is known (Lifshitz and Pogorelov [12]) that two bodies having the same HA-measurements are equivalent if they are starshaped, centrally symmetric, and their sections of maximum area parallel to any given plane pass through the center. The last condition is always satisfied in the convex centrally symmetric case treated by Funk [9]. However, without symmetry assumptions it is unknown whether two bodies having the same HA-measurements are equivalent when both have connected boundaries and even when both are convex.

Physical background: Fermi surfaces of metals. The above problems arise in the study of the *Fermi surface* S of a metal. S is the boundary, in velocity space, between the velocity states that are occupied and those that are not occupied at absolute zero in the "electron gas" formed by the valence electrons of the metal. We refer to the set of all occupied states as the *Fermi body* B of the metal, so that S is the boundary of B. If the electron gas in question were a free gas, S would be merely a sphere centered at the origin. However, the electron velocities in a metal are strongly influenced by the positively charged ions that form the lattice structure of the metal, and in conjunction with less important effects this accounts for the asphericity of the Fermi surface S. The symmetries of S are those of the underlying crystal lattice, and as might be expected S is especially simple in the case of monovalent metals.

As the temperature of the substance is raised above absolute zero, some of the velocity states outside B are occupied and some of those inside B are vacated. Nevertheless, knowledge about B or S is useful in predicting such important properties of the substance as its thermal, acoustic, and electrical con-

ductivity, its superconductivity at low temperatures, etc. Experiments have been devised for measuring various properties of the Fermi body B or Fermi surface S. By means of the so-called *deHaas-vanAlphen effect*, magnetism induced in the substance by a strong magnetic field at low temperatures leads to what has here been called the *HA-measurement of B* relative to a plane P orthogonal to the field. HA-measurements have been used to determine the Fermi surfaces of copper, silver, and gold, to suggest a model for the Fermi surface of lead, and to yield useful information about other Fermi surfaces. The problems of the first section are natural and basic ones in the theory of the deHaas-vanAlphen effect.

The reader is referred to Mackintosh [13] for a fine expository account of Fermi surfaces and attempts to determine them, to Lifshitz and Kosevich [12], Shoenberg [14], and some of their references for more detailed discussions of the deHaas-vanAlphen effect.

Mathematical background: outer and inner M-measures. Actually, the problems of the first section antedate the notion of a Fermi surface. They belong to a class of problems that will now be described. Let B be a body and M an m-dimensional flat in Euclidean d-space E^d. The *outer M-measure* of B, here denoted $\xi_M B$, is the m-measure (length when $m=1$, area when $m=2$, etc.) of the orthogonal projection of B on M. The *inner M-measure* of B, here denoted $\eta_M B$, is the supremum of the m-measures of the intersections $M'\cap B$ as M' ranges over all translates of M. In the German literature, the numbers $\xi_M B$ and $\eta_M B$ are called the *aussere Quermass* and the *innere Quermass* of B with respect to M. Plainly $\xi_M B \geqq \eta_M B$. The body B is said to be of *constant outer* [resp. *inner*] *m-measure* provided that $\xi_M B$ [resp. $\eta_M B$] has the same value for all m-flats M in E^d.

It is known that the convex bodies of constant outer 1-measure are identical with those of constant inner 1-measure. In fact, these are the famous convex bodies B of *constant width*, usually defined by the condition that the distance between two parallel supporting hyperplanes of B is the same for all such pairs of hyperplanes. For each $d>1$, E^d contains nonspherical convex bodies of constant width. For $d=2$ they were first studied by Euler [7]. For $d=3$ they are of interest in connection with a measurement of Fermi surfaces proposed by Kohn [10]. For any plane P, his experiment measures the lengths and the directions of the segment(s) xy such that x and y are points of S at which there are tangent planes parallel to P. When B is of constant width w, xy is always of length w and orthogonal to P, so that any convex body of constant width has the same "Kohn measurements" as a spherical ball. (I am indebted to Professor Walter Kohn for introducing me to Fermi surfaces and supplying some of the references given here.)

Convex bodies in E^3 of constant outer 2-measure are said to be of *constant brightness*. A nonspherical body of this sort was constructed by Blaschke [1] and other constructions were given by Firey [8]. It appears to be unknown whether convex bodies in E^3 are determined by their outer and inner 2-measures

(see p. 80 of [2] and pp. 133–134 of [3]) or even whether a convex body B in E^3 must be spherical if it is of constant brightness and has constant HA-measurements. Apart from that, sphericity of B is implied by any two of the following four conditions: constant width; constant brightness; constant HA-measurements; central symmetry. The following questions appear to be open for all $1<m<d$, except that (as noted above) the answer to $Q_\xi(3, 2)$ is affirmative:

$Q_\xi(d, m)$: *Does E^d contain nonspherical convex bodies of constant outer m-measure?*(*)

$Q_\eta(d, m)$: *Does E^d contain nonspherical convex bodies of constant inner m-measure?*

For the original sources of results stated above without reference, see the classical survey of Bonnesen & Fenchel [4]. For extensions of some of the above notions, results and problems to arbitrary finite-dimensional normed spaces, see Eggleston [6] and Chakerian [5].

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(*) Added in proof: This question has recently been answered affirmatively, for all $1<m<d$, by W. J. Firey.

CLASSROOM NOTES

EDITED BY DAVID DRASIN

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ELEMENTARY PROOFS OF BASIC INEQUALITIES

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1. Introduction. Most mathematics courses include a discussion of inequalities, and usually each inequality is proved by means of its own particular trick. In this note we draw attention to the fact that many inequalities can be proved simply by differentiating a suitable function. To prove an inequality $C \leqq D$ say, one looks for a differentiable real function $f(x)$ such that (i) $f(0)=C$, (ii) $f(1)=D$, and (iii) $f(x)$ increases for $0 \leqq x \leqq 1$. Once such an $f(x)$ has been found the conditions under which we have $C=D$ are determined by requiring $f(x)$ to be constant. If a dash denotes differentiation, then condition (iii) is satisfied if $f'(x) \geqq 0$ for $0 \leqq x \leqq 1$. It is satisfied more elegantly if $f'(0)=0$ and $f(x)$ is convex for all x. We recall that $f(x)$ is convex for all x if $f''(x) \geqq 0$ for all x, and that then

$$f(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2) \leqq \tfrac{1}{2}f(x_1) + \tfrac{1}{2}f(x_2) \quad \text{for all } x_1, x_2.$$

2. Notation. Let N denote a positive integer. The variable x may represent any real number, but $\alpha, \beta, \gamma, \cdots$ denote positive reals. We write

$$\sum \alpha, \quad \sum \alpha_i, \quad \text{or} \quad \sum_i \alpha_i \quad \text{for} \quad \sum_{i=1}^{N} \alpha_i,$$

with a similar notation for multiple sums. Also we set

$$H = N/\sum \alpha^{-1}, \quad G = (\alpha_1 \cdots \alpha_N)^{1/N}, \text{ and } A = N^{-1}\sum \alpha$$

so that H, G, and A are the harmonic, geometric and arithmetic means respectively of $\alpha_1, \alpha_2, \cdots, \alpha_N$.

3. Cauchy's Inequality. This inequality is $c(0) \leqq c(1)$, where $c(x)$ is the function

$$c(x) = \left(\sum \alpha_i^{1+x}\beta_i^{1-x}\right)\left(\sum \alpha_j^{1-x}\beta_j^{1+x}\right)$$
$$= \sum_{i,j} \alpha_i\beta_i\alpha_j\beta_j(\alpha_i\beta_j/\alpha_j\beta_i)^x.$$

A sum of convex functions is convex, and γ^x is convex, so $c(x)$ is convex. It is easy to see that $c'(0)=0$, and therefore $c(0) \leqq c(1)$. We have $c(0)=c(1)$ iff $\alpha_i\beta_j/\alpha_j\beta_i = 1$ for all i, j.

4. The arithmetic-geometric mean inequality. This inequality is $G \leqq A$. To prove it we choose the convex function

$$a(x) = N^{-1}G\sum (\alpha_i/G)^x.$$

Again it is easy to see that $a'(0)=0$, and so $a(0)=G\leqq a(1)=A$ as required, with equality iff all α_i are equal.

5. Slight variation of technique. It is obvious now how to prove

$$\alpha_1\alpha_2\cdots\alpha_N \leqq \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\cdots\alpha_N^{\alpha_N} \leqq \alpha_1^{\alpha_1^2}\alpha_2^{\alpha_2^2}\cdots\alpha_N^{\alpha_N^2}. \tag{1}$$

However to prove that

$$\beta_1^{\beta_1}\beta_2^{\beta_2}\cdots\beta_N^{\beta_N} \geqq B^{\Sigma\beta}, \tag{2}$$

where $B=N^{-1}\Sigma\beta$, we approach the problem from the inside, so to speak. In the range $-\beta_i<x<\beta_j$ the positive function

$$b(x) = (\beta_i + x)^{\beta_i+x}(\beta_j - x)^{\beta_j-x}$$

is convex, and symmetric about $x=\frac{1}{2}(\beta_j-\beta_i)$. However we only need the fact that $b'(x)=b(x)\log[(\beta_i+x)/(\beta_j-x)]<0$ for $0<\beta_i+x<\beta_j-x$, to deduce that

$$\beta_i^{\beta_i}\beta_j^{\beta_j} > B^B(\beta_i+\beta_j-B)^{\beta_i+\beta_j-B} \quad \text{if } \beta_i < B < \beta_j. \tag{3}$$

We can convert the left hand side of (2) to the right hand side by at most $N-1$ applications of (3), so (2) is proved.

Another variation of the technique is needed to prove that if $v(x)$ is a convex function with minimum at $x=0$, then so is

$$u(x) = \sum \alpha_i^{v(x)},$$

provided $v(0)=0$ and $G\geqq 1$. Here we must first show that

$$w(x) = \sum \alpha_i^{v(x)} \log \alpha_i \geqq 0 \quad \text{for all } x.$$

6. The exponential mean function. This function is

$$e(x) = \left(\frac{1}{N}\sum \alpha_i^x\right)^{1/x} \qquad \text{for } x \neq 0.$$

Using l'Hospital's Rule that under certain conditions

$$\lim f/g = \lim f'/g', \tag{4}$$

we obtain

$$\lim_{x\to 0} \log e(x) = \lim_{x\to 0} \frac{(1/N)\sum \alpha^x \log \alpha}{(1/N)\sum \alpha^x} = \log G,$$

so we put $e(0)=G$ to make $e(x)$ continuous for all x, with $e(-1)=H$, $e(0)=G$ and $e(1)=A$. It is difficult to discuss the convexity of $e(x)$, but a simple calculation shows that

$$\left(\sum \alpha_i^x\right)\frac{e'(x)}{e(x)} = x^{-2}\log\left[\beta_1^{\beta_1}\beta_2^{\beta_2}\cdots\beta_N^{\beta_N}/B^{\Sigma\beta}\right], \tag{5}$$

where $\beta_i = \alpha_i^x > 0$, and B is defined in (2). It follows from (2) that $e'(x) > 0$ for $x \neq 0$. In fact $e'(x)$ is continuous for all x, and applying (4) twice to the right hand side of (5) shows that

$$e'(0) = \tfrac{1}{2} G[(N^{-1} \sum \log^2 \alpha_i) - \log^2 G].$$

Thus we have shown that $H \leqq G \leqq A$ with equality iff all α_i are equal.

7. Minkowski's inequality. Given any real number r and functions $g_1(x), \cdots, g_N(x)$ which are positive in some interval containing 0, we set $f(x) = [\sum g_i^r(x)]^{1/r} = [\sum g^r]^{1/r}$. Then

$$\begin{aligned} f''(x) = {} & (r-1)[\sum g^r]^{(1/r)-2}[(\sum g^r)(\sum g^{r-2}g'^2) - (\sum g^{r-1}g')^2] \\ & + [\sum g^r]^{(1/r)-1}[\sum g^{r-1}g''] \end{aligned}$$

and $(\sum g^r)(\sum g^{r-2}g'^2) - (\sum g^{r-1}g')^2 \geqq 0$, by Cauchy's inequality. Hence if the $g_i(x)$ are all convex and $r > 1$ then f is convex. Further if the $g_i(x)$ are all linear, say $g_i(x) = \delta_i + \epsilon_i x$, then all $g_i''(x) = 0$, and so f is convex or concave according as $r > 1$ or $r < 1$.

Now Minkowski's inequality (for three sums) is

$$[\sum (\alpha_i + \beta_i + \gamma_i)^r]^{1/r} \lesseqgtr [\sum \alpha_i^r]^{1/r} + [\sum \beta_i^r]^{1/r} + [\sum \gamma_i^r]^{1/r}$$

according as $1 \gtrless r$. To prove it we set $\delta_i = (\alpha_i + \beta_i + \gamma_i)/3$ and, for $j = 1, 2, 3$, set $\delta_i + \epsilon_{i,j}$ equal to $\alpha_i, \beta_i, \gamma_i$ respectively. Thus Minkowski's inequality is $m(0) \gtrless m(1)$, where $m(x)$ is the convex or concave function

$$m(x) = \left[\sum_i g_{i,1}^r\right]^{1/r} + \left[\sum_i g_{i,2}^r\right]^{1/r} + \left[\sum_i g_{i,3}^r\right]^{1/r},$$

with $g_{i,j} = \delta_i + \epsilon_{i,j}x$. It is easy to show that $m'(0) = 0$, because $\epsilon_{i,1} + \epsilon_{1,2} + \epsilon_{i,3} = 0$.

8. Hölder's inequality. We have $(1/p) + (1/q) = 1$, where $p, q > 1$, and set $h(x) = s(x)^{1/p} t(x)^{1/q}$ where

$$s(x) = \sum \mu \left(\frac{\alpha^p}{\mu}\right)^x \quad \text{and} \quad t(x) = \sum \mu \left(\frac{\beta^q}{\mu}\right)^x,$$

and $\mu = \mu_i = \alpha_i \beta_i$. Then Hölder's inequality is $h(0) \leqq h(1)$. Straightforward, but tedious, calculation shows that

$$h'(0) = s(0)^{(1/p)-1} t(0)^{(1/q)-1} \sum_{i,j} \mu_i \mu_j \log\left(\frac{\alpha_i \beta_i}{\mu_i^{1/p} \mu_j^{1/q}}\right),$$

which is zero because the log terms cancel in pairs. Further

$$0 \leqq 2h''(x) = s^{(1/p)-2} t^{(1/q)-2} u,$$

where

$$u = \sum_{i,j,k,l} \mu_i\mu_j\mu_k\mu_l \left(\frac{\alpha_i^p \alpha_j^p \beta_k^q \beta_l^q}{\mu_i\mu_j\mu_k\mu_l}\right)^x v,$$

and the factor v in this sum is

$$v = \frac{1}{pq}\log^2\left(\frac{\alpha_i^p \mu_j}{\alpha_j^p \mu_i}\right) + \log^2\left(\frac{\alpha_j\beta_k}{\mu_j^{1/p}\mu_k^{1/q}}\right) + \frac{1}{pq}\log^2\left(\frac{\beta_k^q \mu_l}{\beta_l^q \mu_k}\right)$$
$$+ \log^2\left(\frac{\beta_k\alpha_i}{\mu_k^{1/q}\mu_i^{1/q}}\right).$$

This proves Hölder's inequality.

9. Remarks. The function $c(x)$ was selected by D. K. Callebaut [1]. It was generalized and used to obtain inequalities for gamma functions in [4]. The function $a(x)$ was generalized beyond recognition in [2]. The inequality (1) generalizes an unpublished one of Lim Voon Ka. We believe the treatment here of $e(x)$ to be new. The above discussion of $m(x)$ and $h(x)$ are special cases of work done in [3].

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THE EVALUATION OF $\int_a^b x^k dx$

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The evaluation of $\int_a^b x^k dx$ $(0 \leqq a \leqq b)$, prior to the introduction of the Fundamental Theorem of Calculus, may be obtained in a manner exhibited in [2] or [3]. Another approach will be presented in which the restrictions on the partitions such as occur in [2] and [3] will be unnecessary. The reader may be interested in comparing the present treatment with that in [1] where the method of telescoping series is also used.

Let $0 \leqq a = x_1 < x_2 < \cdots < x_{n+1} = b$ be a partition of $[a, b]$.

Since $x_i^k \leqq x_i^{k-j} x_{i+1}^j \leqq x_{i+1}^k$ $(j = 0, 1, \cdots, k)$, each subinterval $[x_i, x_{i+1}]$ will contain the point $(x_i^{k-j} x_{i+1}^j)^{1/k}$; and therefore, if $x_{i+1} - x_i$ is denoted by Δx_i,

$$\int_a^b x^k dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^n x_i^{k-j} x_{i+1}^j \Delta x_i \quad \text{for any } j \in \{0, 1, \cdots, k\}.$$

Thus

$$
\begin{aligned}
(k+1)\int_a^b x^k\,dx &= \lim_{\max \Delta x_i\to 0} \sum_{j=0}^{k}\left(\sum_{i=1}^{n} x_i^{k-j} x_{i+1}^{j}\Delta x_i\right)\\
&= \lim_{\max \Delta x_i\to 0} \sum_{i=1}^{n}\left(\sum_{j=0}^{k} x_i^{k-j} x_{i+1}^{j}\Delta x_i\right)\\
&= \lim_{\max \Delta x_i\to 0} \sum_{i=1}^{n}\left(\sum_{j=0}^{k} x_i^{k-j} x_{i+1}^{j+1} - \sum_{j=0}^{k} x_i^{k-j+1} x_{i+1}^{j}\right)\\
&= \lim_{\max \Delta x_i\to 0} \sum_{i=1}^{n}\left(\sum_{j=1}^{k+1} x_i^{k-j+1} x_{i+1}^{j} - \sum_{j=0}^{k} x_i^{k-j+1} x_{i+1}^{j}\right)\\
&= \lim_{\max \Delta x_i\to 0} \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{k} x_i^{k-j+1} x_{i+1}^{j} + x_{i+1}^{k+1}\right)\right.\\
&\qquad\qquad \left. - \left(\sum_{j=1}^{k} x_i^{k-j+1} x_{i+1}^{j} + x_i^{k+1}\right)\right]\\
&= \lim_{\max \Delta x_i\to 0} \sum_{i=1}^{n} \left(x_{i+1}^{k+1} - x_i^{k+1}\right).
\end{aligned}
$$

This last sum telescopes and we have $\int_a^b x^k dx = [1/(k+1)](b^{k+1} - a^{k+1})$.

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CONDITIONS FOR A LOOP TO BE A GROUP AND FOR A GROUPOID TO BE A SEMIGROUP

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D. Watson has found (this Monthly, 74 (1967) 843–844) an "improved rule" to determine whether a loop is a group. It says, "In the multiplication table of the loop choose any four places forming the vertices of a rectangle. Suppose that the entries in these places are

$$
\begin{matrix} q & r \\ p & s. \end{matrix}
$$

If the loop is a group, then all other rectangles having p, q, r as entries at successive vertices, with p and q sharing a column, will have s as the entry at the fourth vertex. The converse is also true."

Let us elaborate a bit on the rectangles in question:

	y_1	y_2	y_3	y_4
x_1		q		r
x_2	q		r	
x_3		p		s
x_4	p		s	

So the "improved rule" states:

If $x_1y_2=x_2y_1(=q)$ *and* $x_1y_4=x_2y_3(=r)$ *and* $x_3y_2=x_4y_1(=p)$, *then* $x_3y_4=x_4y_3(=s)$.

But this is the well-known Reidemeister condition which is necessary and sufficient for the associativity of loops (see [1], also, e.g., [2, 3, 5, 6, 7]).

Even the proofs are essentially the same as in the above-mentioned paper.

The proof of the statement, that the Reidemeister condition is necessary and sufficient for a loop to be a group, is in one of its simplest forms the following:

1. *Necessary.* In a group $(xy)^{-1}=y^{-1}x^{-1}$, and $x_3y_2=x_4y_1$, $x_1y_2=x_2y_1$, $x_1y_4=x_2y_3$ imply $x_3y_4 = x_3y_2(x_1y_2)^{-1}x_1y_4 = x_4y_1(x_2y_1)^{-1}x_2y_3 = x_4y_3$.

2. *Sufficient.* A loop is a group, if it is associative. So put into the Reidemeister condition $x_1=y_1=u$, the unit element of the loop, in order to get the condition

$$(R') \qquad \text{if } y_2 = x_2 \quad \text{and} \quad y_4 = x_2y_3 \quad \text{and} \quad x_3y_2 = x_4, \quad \text{then } x_3y_4 = x_4y_3.$$

The condition (R') gives associativity immediately:

$$x_3(x_2y_3) = x_3y_4 = x_4y_3 = (x_3y_2)y_3 = (x_3x_2)y_3,$$

and this finishes the proof.

In loops R' is equivalent to the Reidemeister condition (see, e.g., [1] p. 413). On the other hand, (R') *is evidently equivalent to the associativity in any groupoid* (not only in loops). We have just seen that (R') implies associativity, and the proof did not make any use either of quasigroup- or of loop-properties. Similarly does associativity imply (R') in any groupoid: if $y_2=x_2$, $y_4=x_2y_3$ and $x_3y_2=x_4$, then $x_3y_4 = x_3(x_2y_3) = (x_3x_2)y_3 = (x_3y_2)y_3 = x_4y_3$.

F. W. Light has stated (see [4]) the following associativity test for groupoids, when the operation is given by a Cayley multiplication (CM)-table:

"We form a new table with the x_2 row of the CM-table as top index line and the $y_2(=x_2)$ column as left-hand index column of the new table. Each entry x_2y in the x_2 row of the CM-table tells us what column of the CM-table to copy down as the y column of the new table, and each entry xy_2 in the $y_2(=x_2)$ column of the CM-table tells us which row of the CM-table should be compared with the x row of the new table. If they coincide for all x_2 and x, y in the groupoid, then the groupoid is associative, that is, a semigroup." (We have slightly changed the quotation from [4] p. 7, so that it should conform to our purpose.)

In this light, *Light's test is identical with* (R') (which in turn is a special case of the Reidemeister condition). We have $y_2=x_2$, and the products x_2y stand in the top index line, the products xy_2 in the left-hand index column of the new

table. Specify in particular $y=y_3$, $x=x_3$, and denote $y_4=x_2y_3$ and $x_4=x_3y_2$. As the columns (lines) of the new table(s) are copied unchanged from the original CM-table (only their placement is changed), the element of the column with the top index y_4 $(=x_2y_3)$ of the new table which is in the line with x_3y_2 as left-hand index will be the same as the element of the original CM-table in the column with the same top index y_4, but in the line with the left-hand index x_3, that is, the element x_3y_4. This has to be compared with the element of the original CM-table in the line with the left-hand index x_4 $(=x_3y_2)$ and in the column with the top index y_3, that is, with the element x_4y_3. If they are equal, then $x_3y_4=x_4y_3$, that is, Light's test states

the equalities $y_2=x_2$ *and* $y_4=x_2y_3$ *and* $x_4=x_3y_2$ *imply* $x_3y_4=x_4y_3$

as associativity condition, and this is exactly (R').

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A DEFINITE INTEGRAL FOR BESSEL'S FUNCTION

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The integral representation for Bessel's function

$$J_n(x) = \frac{1}{\pi}\,\frac{x^n}{1\cdot 3\cdot 5\cdot \;\cdots\; \cdot(2n-1)}\int_0^{\pi} \cos(x\cos\theta)\sin^{2n}\theta d\theta$$

is well known [1, p. 24]. Here is a simple method of deriving this formula from the differential equation for $J_n(x)$ by a slight modification of the method given in [2, Art. 136]. Bessel's equation

$$\frac{d^2z}{dx^2}+\frac{1}{x}\,\frac{dz}{dx}+\left(1-\frac{n^2}{x^2}\right)z = 0 \tag{1}$$

reduces to

$$\frac{d^2v}{dx^2}+\frac{2n+1}{x}\,\frac{dv}{dx}+v = 0, \tag{2}$$

where $z=x^nv$. Assume

$$v = \int_a^b T \cos(xt)dt, \tag{3}$$

where x and t are independent, T is an unknown function of t, and the limits of integration a and b (supposed independent of x) are to be determined by substituting this proposed value of v in the differential equation.

Since $dv/dx = -\int_a^b tT \sin(xt)dt$ and $d^2v/dx^2 = -\int_a^b t^2 T \cos(xt)dt$, the result of the substitution may be expressed in the form

$$\int_a^b (1 - t^2)Tx \cos(xt)dt - \int_a^b (2n + 1)tT \sin(xt)dt = 0, \tag{4}$$

which must be identically satisfied. Upon integration by parts, (4) becomes

$$\left[(1 - t^2)T \sin(xt)\right]_a^b - \int_a^b \left[(1 - t^2)\frac{dT}{dt} + (2n - 1)tT\right] \sin(xt)dt = 0, \tag{5}$$

and this will be identically satisfied provided that

$$(1 - t^2)\frac{dT}{dt} + (2n - 1)tT = 0 \tag{6}$$

and that a and b are such that

$$\left[(1 - t^2)T \sin(xt)\right]_a^b = 0. \tag{7}$$

From (6) we obtain

$$T = A(1 - t^2)^{n-\frac{1}{2}}, \tag{8}$$

and (7) will obviously be satisfied if $a = -1$ and $b = 1$. It thus follows that $v = A \int_{-1}^1 (1 - t^2)^{n-\frac{1}{2}} \cos(xt)dt$ is a solution of (2), and

$$z = Ax^n \int_{-1}^1 (1 - t^2)^{n-\frac{1}{2}} \cos(xt)dt \tag{9}$$

is a solution of Bessel's equation.

If we put $t = \cos\theta$ in (9) we get $z = Ax^n \int_0^\pi \sin^{2n}\theta \cos(x\cos\theta)d\theta$. Using

$$\cos(x\cos\theta) = 1 - \frac{x^2\cos^2\theta}{2!} + \frac{x^4\cos^4\theta}{4!} - \cdots,$$

and integrating term-by-term with the aid of the formula

$$\int_0^{\pi/2} \sin^n x \cos^m x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n}{2} + 1\right)},$$

and comparing with

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4\cdot 2!(n+1)(n+2)} - \cdots\right],$$

we get

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^\pi \cos(x\cos\theta)\sin^{2n}\theta d\theta. \tag{10}$$

If n is a positive integer, (10) reduces to

$$J_n(x) = \frac{1}{\pi}\,\frac{x^n}{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}\int_0^\pi \cos(x\cos\theta)\sin^{2n}\theta d\theta. \tag{11}$$

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ON NONMEASURABLE SETS

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When measure theory is developed by the double-barrelled approach, utilizing both outer and inner measures, the student has no difficulty in visualizing the shortcomings of a nonmeasurable set. More usually outer measure alone is developed, and although Carathéodory's definition that a set E is measurable if, for every set T in the space X,

$$\mu(T) = \mu(T\cap E) + \mu(T\cap E')$$

makes it evident that measurable sets are well-behaved, it does not make clear to what extent a nonmeasurable set misbehaves. Indeed, since the negation of "for every set T" reads "for some set S," it would appear that the nonmeasurability of E might be due to its lamentable interaction with one particularly intractable set S, in which case it is hardly fair to put all the blame on E. That this is not the case is made clear by the following

THEOREM. *Let $(X, \mathfrak{A}, \mu)$ be an arbitrary measure space, with E a non-μ-measurable set. There is a positive number ϵ, depending solely on E, such that if A and B are measurable sets with $A\supset E$ and $B\supset E'$, then $\mu(A\cap B)\geqq\epsilon$.*

Proof: Suppose the theorem false. Then for each natural number n one can find measurable sets $C_n\supset E$, $D_n\supset E'$, with $\mu(C_n\cap D_n)<1/n$. Then, forming the measurable sets $C=\bigcap_{n=1}^\infty C_n$ and $D=\bigcap_{n=1}^\infty D_n$, we have $C\supset E$, $D\supset E'$, and $\mu(C\cap D)=0$. The measurability of C implies, for any $T\subset X$, $\mu(T)=\mu(T\cap C)+\mu(T\cap C')$. Since $C'\cup(C\cap D)\supset D$, we have

$$(T\cap C')\cup(T\cap C\cap D)\supset T\cap D,$$

so

$$\mu(T \cap C') + \mu(T \cap C \cap D) \geqq \mu(T \cap D).$$

But $\mu(T\cap C\cap D) \leqq \mu(C\cap D) = 0$, which yields $\mu(T) \geqq \mu(T\cap C) + \mu(T\cap D)$. Since $C \supset E$ and $D \supset E'$, this gives us $\mu(T) \geqq \mu(T\cap E) + \mu(T\cap E')$, which proves E to be measurable, contrary to the hypothesis. The number ϵ can be realized as

$$\epsilon = \inf \{\mu(A \cap B) : A, B \text{ measurable}, A \supset E, B \supset E'\}.$$

ANOTHER PROOF OF THE FORMULA $\sum 1/k^2 = \pi^2/6$

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The result contained here is well known; a somewhat more difficult proof is given in [2] and that source also lists several of the standard proofs.

For any positive integer n we have by means of Fejér's kernel [1, p. 222]

$$F_n(x) = \frac{1}{2(n+1)} \frac{\sin^2 (n+1)(x/2)}{\sin^2 (x/2)} = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kx \geqq 0.$$

On the one hand

$$\begin{aligned} M_n = \int_0^\pi x F_n(x) dx &= \frac{\pi^2}{4} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \\ &\quad + \frac{1}{n+1} \left\{ \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \right\} \\ &= \left\{ \frac{\pi^2}{12} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2} \right\} + \left\{ \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right\} + O\left(\frac{\log n}{n+1}\right), \qquad n \to \infty, \end{aligned}$$

this partition of $\pi^2/4$ being suggested by

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + 2\frac{\pi^2}{12}, \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

which is valid in view of the identity

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{2} \sum_{k=1}^{[n/2]} \frac{1}{k^2}.$$

On the other hand, as $x/\pi \leqq \sin (x/2)$ on $[0, \pi]$, it follows (cf. [1] p. 163) that

$$\begin{aligned} M_n &\leqq \frac{\pi^2}{2(n+1)} \int_0^{(n+1)\pi/2} \frac{\sin^2 t}{t} dt \\ &\leqq \frac{\pi^2}{2(n+1)} \left\{ \int_0^{\pi/2} dt + \int_{\pi/2}^{(n+1)\pi/2} \frac{dt}{t} \right\} = O\left(\frac{\log (n+1)}{n+1}\right), \qquad n \to \infty. \end{aligned}$$

Finally, for $n \to \infty$ we obtain at the same time

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}, \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

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PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N. J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before September 30, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2172. *Proposed by S. I. Drobnies, San Diego State College*

Prove or disprove: The complex number z belongs to the set $\{\omega : |\omega| - \operatorname{Re}\omega \leq \frac{1}{2}\}$ if and only if z is a product ac such that $|\bar{c} - a| \leq 1$.

E 2173. *Proposed by Emanuel Vegh, Naval Research Laboratory*

If p is a prime ($p \neq 2$, 3, 5, 11, or 17) there are three distinct quadratic nonresidues of p, whose sum is divisible by p.

E 2174. *Proposed by Klaus Steffen, Johannes Gutenberg University, Mainz, Germany*

Consider the set M of arrays in a row of the $2n$ symbols $a_1, \cdots, a_n, b_1, \cdots, b_n$ such that a_i precedes a_{i+1} and b_i precedes b_{i+1} for $1 \leq i \leq n-1$. An inversion is a pair (a_i, b_i) such that a_i does not precede b_i. For $0 \leq k \leq n$ let v_k be the number of elements of M with k inversions. Prove or disprove that $v_0 = v_1 = \cdots = v_n$.

Clearly $v_k = v_{n-k}$. (Cf. E 2083 [1969, 419].)

E 2175. *Proposed by Harry Pollard, Purdue University*

Find the maximum value of the function

$$\exp\left(-\frac{1}{x}e^{-x}\right) + \exp(-xe^{-1/x}), \qquad 0 < x < \infty.$$

E 2176. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

Find all continuous real functions f such that

$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x)+f(y)}{f(x)-f(y)}.$$

E 2177. *Proposed by P. M. Gibson, University of Alabama, Huntsville*

Let r be a nonzero complex number, and m an integer. Suppose that $A = (a_{ij})$, $B = (b_{ij})$ are $n \times n$ complex matrices with $b_{ij} = r^{m+i-j}a_{ij}$ for $i, j = 1, \cdots, n$. Show that if λ is a characteristic value of A of multiplicity k then $r^m\lambda$ is a characteristic value of B of multiplicity k. Use this to show that if A is a tridiagonal matrix of odd order with zero diagonal then A is singular.

SOLUTIONS OF ELEMENTARY PROBLEMS

Generalized Fermat Numbers

E 2087 [1968, 542]. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

For every nonzero integer k show that $(2k)^{2^n}+1$, $n = 1, 2, \cdots$, are relatively prime integers.

Solution by Neal Felsinger, Yale University. The following is a simple generalization of Polya's idea to demonstrate the infinitude of primes by constructive means (see Hardy and Wright, *Introduction to the Theory of Numbers*). Let $F_n = (2k)^{2^n}+1$ and suppose $m < n$. If p is a positive integer we have

$$x^{2p} - 1 = (x+1)(x^{2p-1} - x^{2p-2} + x^{2p-3} - \cdots - x^2 + x - 1).$$

Putting $x = (2k)^{2^m}$ and $p = 2^{n-m-1}$, it follows that $F_m | (F_n - 2)$, whence $(F_m, F_n) \leq 2$. Since F_n is odd, this implies $(F_m, F_n) = 1$.

Also solved by fifty-two other readers. This problem appears also as an exercise in a number of texts—Niven & Zuckermann, C. Long, and Sierpinski were cited by solvers.

Imbedding the Platonic Solids in a Lattice

E 2088 [1968, 542]. *Proposed by H. E. Chrestenson, Reed College*

In problem 5014 [1963, 447] it was shown that a regular n-gon can be imbedded in a cubic lattice (of arbitrary dimension) only if $n = 3$, 4 or 6, and in these cases dimension 3 suffices. Answer the analogous question for the five Platonic solids.

Solution by Michael Goldberg, Washington, D.C. The regular tetrahedron, cube, and regular octahedron can be imbedded in a cubic lattice of three dimensions. The four points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ form a regular tetrahedron. The eight points $(\pm 1, \pm 1, \pm 1)$ form a cube. The six points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ form a regular octahedron. Since the regular icosahedron and the regular dodecahedron both contain regular pentagons, they cannot (by problem 5014) be imbedded in a cubic lattice of any dimension.

Also solved by Simeon Reich (Israel) and the proposer.

The question was also raised by Hadwiger, Debrunner, and Klee in *Combinatorial Geometry in the Plane*, p. 43, sec. 2.

A Roller Point Path

E 2089 [1968, 542]. *Proposed by J. F. Randolph, University of Rochester*

A roller is constructed from three circular arcs each having its center at one vertex of an equilateral triangle and joining the other two vertices. In the ordinary cycloid problem replace the circle by this roller and find the locus of a point on its boundary as it rolls without slipping along a straight line.

Solution by Norman Miller, Queen's University. The curve, like the cycloid, has a periodic sequence of arches.

Case I. If, in the tricord "roller," the boundary point that traces the locus is a vertex, then each arch has symmetry about a vertical line. Denote the line on which the roller rolls by L and a side of the triangle by s. Suppose the tracing point P begins at A with an arc of the roller tangent to L at A. When a second vertex reaches L at B, where $AB=\pi s/3$ and angle $ABP=\pi/6$, then P has traced an arc of a cycloid with cusp at A. While the roller is pivoted at B, P describes an arc of a circle of radius s and angle $\pi/3$. At the conclusion of this second step PB is perpendicular to L. Since P is now the center of a circle rolling on L, it describes a line segment parallel to L and of length $\pi s/3$. A vertex is now at C on L where $BC=\pi s/3$ and PC is perpendicular to L. The remainder of the arch consists of two arcs, one circular and one cycloidal, which are mirror images of the first two described.

Case II. If the point tracing the locus is an interior point of one of the circular arcs, the resulting arch has symmetry only if this point is the midpoint of its arc. In any case the arch consists of seven curvilinear arcs. These, in order, are: a cycloidal arc, a circular arc, a prolate cycloidal arc, a circular arc, then, reversing the original order, a prolate cycloidal, a circular, and a cycloidal arc.

Also solved by Michael Goldberg, and by Bohuslav Mišek (Czechoslovakia).

A Vector Solution to a Triangle Equality

E 2090 [1968, 542]. *Proposed by J. C. Brooks, Georgia Institute of Technology*

In a triangle with sides a, b, c prove that the distances from the centroid G to the incenter I and the excenters I', I'', I''' satisfy the relation

$$(s-a)GI'^2 + (s-b)GI''^2 + (s-c)GI'''^2 - (s)GI^2 = 2abc.$$

Solution by M. G. Greening, University of New South Wales, Australia. We have

$$(1) \quad s = \sum (s-a), \quad (2)\ \mathbf{AG} = \tfrac{1}{3}(\mathbf{AB} + \mathbf{AC}), \quad (3)\ \mathbf{AI}\cdot\mathbf{AB} = c(s-a),$$
$$\mathbf{AI}\cdot\mathbf{AC} = b(s-a), \quad (4)\ \mathbf{AI'} = (s/(s-a))\mathbf{AI}.$$

$$\begin{aligned}
\sum (s-a)GI'^2 - sGI^2 &= \sum (s-a)\{GI'^2 - GI^2\} \qquad \text{by (1)}\\
&= \sum (s-a)\{\,|\mathbf{AI'} - \mathbf{AG}|^2 - |\mathbf{AI} - \mathbf{AG}|^2\}\\
&= \sum (s-a)\{AI'^2 - AI^2 - 2\mathbf{AG}\cdot(\mathbf{AI'} - \mathbf{AI})\}\\
&= \sum (s-a)\left\{a(2s-a)\sec^2\frac{A}{2} - \frac{2}{3}\,\frac{a}{s-a}\,(b+c)(s-a)\right\}\\
&= \sum \frac{abc(b+c)}{s} - \sum \frac{abc(b+c)\cos A}{s}\\
&= 4abc - 2abc = 2abc.
\end{aligned}$$

Also solved by Ragnar Dybvik (Norway), Norman Miller, J. M. Quoniam (France), V. V. Rao (India), Simeon Reich (Israel), C. V. Subbarama Iyer (India), C. S. Venkataraman (India), Gregory Wulczyn, and the proposer.

Roots of Monic Polynomials and Integral Domains

E 2091 [1968, 542]. *Proposed by J. O. Kiltinen and T. J. Grilliot, Duke University*

Consider the following two properties for a commutative ring with identity:

(i) If P is a monic polynomial over A and deg $P = n$, then P has at most n roots in A.

(ii) A is not an integral domain.

Are there any commutative rings with identity possessing both of these properties?

Solution by Klaus Steffen, University of Mainz, Germany. There are exactly two rings (commutative and with identity) possessing properties (i) and (ii), namely $Z_4 = Z/4Z$, where Z is the set of integers, and $Z_2[\epsilon]$, where $\epsilon^2 = 0$. In fact, let A be a commutative ring with identity satisfying (i) and (ii). Then if $s, t \in A$:

$$(*) \qquad s \neq 0 \neq t, \quad s\cdot t = 0 \Rightarrow s = t,$$

for otherwise $(x-s)(x-t)$ would be a monic polynomial of degree 2 with three roots s, t, 0. Since A is not an integral domain there exists, by (*), a nonzero element ϵ of A with $\epsilon^2 = 0$. Now, if $r \in A$ we have $(r\epsilon)\epsilon = r\epsilon^2 = 0$ and by (*) either $r\epsilon = 0$ (which implies $r = 0$ or $r = \epsilon$ again by (*)) or $r\epsilon = \epsilon$ (which implies $(r-1)\epsilon = 0$ and $r - 1 = 0$ or $r - 1 = \epsilon$). Therefore A contains only four elements 0, 1, ϵ, $\epsilon + 1$ and $A = Z_4$ or $A = Z_2[\epsilon]$ according as $1 + 1 = \epsilon$ or $1 + 1 = 0$.

To prove that Z_4 and $Z_2[\epsilon]$ actually have property (i) observe that a poly-

nomial of degree $n \geqq 4$ has at most card $Z_4 = \text{card } Z_2[\epsilon] = 4$ roots and that a monic polynomial of degree 1 has exactly 1 root. If P is a monic polynomial of degree 2 with root a, then we can write $P(x) = (x-a)(x-b)$ and $P(x) = 0$ iff $x = a$, $x = b$, or $x - a = x - b = \epsilon$. In either case P has at most two roots. If Q is a monic polynomial of degree 3 with root a, we can write $Q(x) = (x-a)P(x)$ where P is monic and of degree 2. Now $Q(x) = 0$ iff $x = a$, x is a root of P, or $x - a = \epsilon = P(x)$. If P has no root then Q has at most 2 roots. If, however, b is a root of P we have $P(x) = (x-b)(x-c)$ and $(x-b)(x-c) = \epsilon$ only if $x - b = \epsilon$ or $x - c = \epsilon$. Therefore $x - a = \epsilon = P(x)$ only if $a = b$ or $a = c$ and Q has at most three roots.

Also solved by Einar Andresen (Norway), Anders Bager (Denmark), L. Carlitz, F. D. Cheek II, P. R. Chernoff, Thomas Elsner, N. J. Fine, G. J. Ford, Robert Gilmer & Robert Loraine, Marvin Gruber, D. A. Hejhal, J. M. Katz, D. C. B. Marsh, C. P. Milies (Uruguay), J. C. Nichols, D. P. Sumner, and the proposers. (Many of the solutions merely indicated that Z_4 satisfies the conditions.)

Magic Number Star

E 2092 [1968, 542]. *Proposed by A. Domergue, Paris, France*

Let a regular star polygon be constructed by dividing a circle into $n (n \geqq 5)$ equal parts and drawing the chords joining alternate points of division. Each of the n chords will carry four points of intersection. If distinct positive integers are assigned to each intersection in such a way that the sum of the 4 numbers on each chord is a constant, we have a magic number star. Such a number star may be characterized by Δ, the difference between the smallest and largest numbers employed.

(a) Construct a magic star pentagon in which the constant is 1968 and such that Δ is smallest possible. (b) Do the same for a magic star hexagon. (c) Show that there is no magic star pentagon with constant sum 1967.

Solution by D. C. B. Marsh, Colorado School of Mines. We take the three parts in reverse order. (c) By summing the numbers on every chord of a magic star pentagon we find that twice the sum of the ten distinct positive integers equals five times the star's constant. Thus, for integral values, this constant must be even. No magic star pentagon with constant sum 1967 exists.

(b) For a magic star hexagon there are 12 intersections, whence $\Delta \geqq 11$. The value 11 could occur iff a magic star hexagon exists with 12 consecutive integral values assigned to the intersections: $a, a+1, \cdots, a+11$; but, summing over all chords, one then has the equation $12a + 66 = 3(1968)$ which has no integral solution. The next larger value ($\Delta = 12$) would require that the intersections be assigned integral values v_i, $486 \leqq v_i \leqq 496$, $v_i \neq 492$. This configuration is possible (Fig. 1).

(c) For a magic star pentagon with constant 1968, arguments as given above show that Δ cannot be 9. The argument showing that $\Delta \neq 10$ is difficult to formulate (essentially, the least and greatest numerical entries are "played off" against one another). Here, finally, $\Delta = 11$, and one may exhibit Fig. 2.

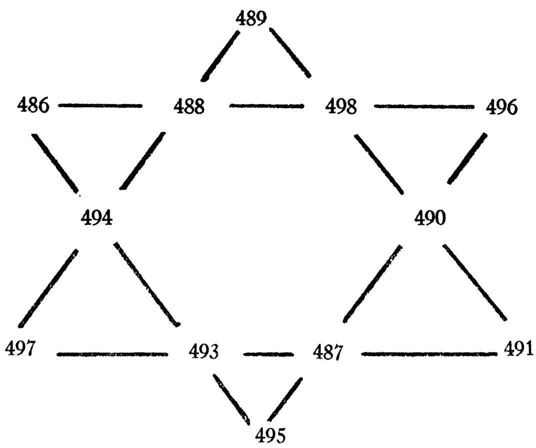


FIG. 1

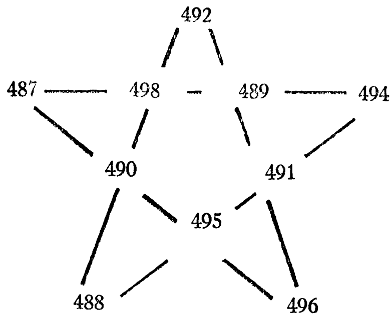


FIG. 2

Also solved by Walter Bluger, Michael Goodman, H. L. Nelson, and the proposer.

Magic Pentagrams for 1962 and 1964 are given in the *Mathematics Magazine*, v. 35(1962), p. 228; v. 37(1964), pp. 49–50. The latter reference treats the general magic pentagram.

sin cos $c=c$

E 2093 [1968; 543, 779]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York*

Prove that there exist unique numbers c and d in the interval $[0, \frac{1}{2}\pi]$ such that $c<d$ and

(A) $\sin\cos c=c$, (B) $\cos\sin d=d$.

Solution by D. A. Hejhal, University of Chicago. We put $\theta(x)=\sin(\cos x)-x$, $0\leqq x\leqq\frac{1}{2}\pi$. Then $\theta(0)=\sin 1>0$, $\theta(\frac{1}{2}\pi)=-\frac{1}{2}\pi<0$, also $\theta'(x)<0$, $0\leqq x\leqq\frac{1}{2}\pi$. Hence there is exactly one $c\in(0,\frac{1}{2}\pi)$ such that $\sin(\cos c)=c$.

If $\cos(\sin d)=d$, then $\sin\{\cos(\sin d)\}=\sin d$ and $\sin d\in(0,\frac{1}{2}\pi)$, but $\sin\{\cos(c)\}=c$ and c is unique. Therefore $c=\sin d$. Similarly $d=\cos c$ and d is unique. Further, $c=\sin d$ and $\sin x<x$ for $x>0$ imply $c<d$.

Also solved by Anders Bager (Denmark), Walter Bluger, M. J. Brown, F. D. Cheek II, M. S. Demos, William Fox, Michael Goldberg, C. G. Khatri & A. M. Vaidya (India), Lew Kowarski, Lauwerens Kuipers, Beatriz Margolis (Argentina), D. E. Meyers, Norman Miller, Edward Moylan, G. B. Parrish, Simeon Reich (Israel), F. E. Sullivan, Gregory Wulczyn, and the proposers.

Several solvers made the computations: $c=.69482$, $d=.76817$, approximately.

A Case of Nonintersecting Ranges

E 2094 [1968, 543]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York*

Define $\sin_n x=\sin(\sin_{n-1}x)$ and $\cos_n x=\cos(\cos_{n-1}x)$ with $\sin_1 x=\sin x$ and $\cos_1 x=\cos x$. For which values of n are there solutions to the equation $\cos_n x=\sin_n x$, $0\leqq x\leqq\frac{1}{2}\pi$?

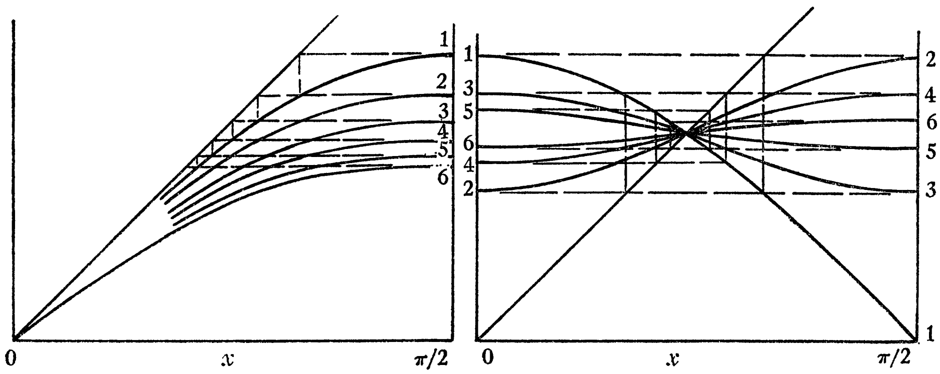
Solution by Michael Goldberg, Washington, D. C. The functions $\sin_n x$ and $\cos_n x$ are continuous and monotonic in the interval zero to $\frac{1}{2}\pi$. Their values range over the intervals given below:

x	$\sin_1 x$	$\sin_2 x$	$\sin_3 x$	$\sin_4 x$	$\sin_5 x$
0	0	0	0	0	0
$\frac{1}{2}\pi$	1.00	.842	.745	.678	.627
x	$\cos_1 x$	$\cos_2 x$	$\cos_3 x$	$\cos_4 x$	$\cos_5 x$
0	1.00	.540	.858	.654	.793
$\frac{1}{2}\pi$	0	1.00	.540	.858	.654

The graphs of the functions are shown in the figures. If they are superimposed, the curve for $\sin_n x$ intersects the curve for $\cos_n x$ when $n=1$ and $n=3$. For $n \geqq 5$, the range of values of the two functions are mutually exclusive. Hence the equation $\cos_n x = \sin_n x$ is satisfied only for $n=1$ and $n=3$.

Also solved by D. S. Bassan & C. G. Khatri & A. M. Vaidya (India), Walter Bluger, David Gootkind, D. C. B. Marsh, Norman Miller, H. C. Nelson, G. B. Parrish, Simeon Reich (Israel), J. G. Rosenstein, Arpád Varecza & László Varecza (Hungary), and the proposer.

Several solvers pointed out that $\sin_2 x = \cos_2 x$ iff $\sin x + \cos x = \frac{1}{2}\pi$, an impossibility.



On Subsets of a Finite Group Commuting with a Subgroup

E 2095 [1968, 669]. *Proposed by C. C. Lindner, Coker College, Hartsville, S. C.*

Let G be a finite group written multiplicatively, H a subgroup of G, and k a positive integer. Show that if $KH \neq HK$ for every subset K of G containing k elements, then $k < [G:H]$.

Solution by Michael Merritt, Baylor University. Let n denote $[G:H]$ and let m be any positive integer such that $n \leqq m \leqq o(G)$. Now $G = Hk_1 \cup Hk_2 \cup \cdots \cup Hk_n$, where $k_i \in G (i=1, 2, \cdots, n)$ and $Hk_i \cap Hk_j = \varnothing (i \neq j)$. Let $K' = \{k_1, k_2, \cdots, k_n\}$. Then $HK' = G$. Moreover $o(K'H) = o(G)$, for $k_i h_1' = k_j h_2'$ implies $h_1' h_2'^{-1} = k_j k_i^{-1}$. But then $H = H(h_1' h_2'^{-1}) = H(k_j k_i^{-1})$ so that $Hk_i = Hk_j$. Thus

$i=j$ and we must have $o(K'H)=n\cdot o(H)=o(G)$. Hence $K'H=G=HK'$. If K is any subset of G containing k elements and $K\supseteq K'$, then $HK=HK'=G=K'H=KH$. Therefore, if $HK\neq KH$ for every subset K of G containing k elements, then $k<[G:H]$.

Also solved by Anders Bager (Denmark), S. M. Gagola, Jr., M. G. Greening (Australia), Eleanor G. Jones, Erwin Just, J. F. Leetch, Bob Prielipp, Hans Schwerdtfeger, Klaus Steffen (Germany), D. P. Sumner, and the proposer.

Editorial Note. Gagola proves that for each integer k with $k\leqq[G:1]$ one can find a subset K of k elements with the property that $KH=HK$. Thus the hypothesis of the problem is vacuously true.

A Variance System

E 2096 [1968, 669]. *Proposed by Seymour Geisser and Paul Schillo, State University of New York at Buffalo*

Let m and n be positive integers such that $m<n$; and, for each positive integer $t\leqq n$, let x_t be a real number. Can the equations

$$m^2\left(\sum_{t=1}^{m} x_t^2 - 1\right) = m\left(\sum_{t=1}^{m} x_t\right)^2 + 1, \tag{1}$$

$$n^2\left(\sum_{t=1}^{n} x_t^2 - 1\right) = n\left(\sum_{t=1}^{n} x_t\right)^2 + 1 \tag{2}$$

both be true?

I. *Solution by W. W. Leutert, Stamford, Connecticut.* Let

$$f(k) = k\sum_{t=1}^{k} x_t^2 - \left(\sum_{t=1}^{k} x_t\right)^2 - k - 1/k.$$

By regrouping and expanding the sums involved it follows that

$$f(k+1) - (1+1/k)f(k) = k\left(x_{k+1} - (1/k)\sum_{t=1}^{k} x_t\right)^2 + 1/(k^2+k) + 1/k^2.$$

Thus, regardless of the choice of x_{k+1} we have $f(k+1)>f(k)$. Relation (1) is equivalent to $f(m)=0$. By mathematical induction, (2) cannot be true, and vice versa.

II. *Solution by the proposers.* For each positive integer

$$k \leqq n, \qquad \text{let } \bar{x}_k = \sum_{t=1}^{k} x_t/k \quad \text{and}$$

$$y_k = \sum_{t=1}^{k} (x_t - \bar{x}_k)^2 = \sum_{t=1}^{k} x_t^2 - k\bar{x}_k^2.$$

Since $m<n$, $y_m\leqq y_n$. Equations (1) and (2) are equivalent to $y_m=1+1/m^2$ and $y_n=1+1/n^2$; they cannot both be true because $1+1/m^2>1+1/n^2$.

It can be shown by a less concise argument that the problem would still have a negative answer if each exponent 2 were to be replaced by an even integer greater than 2.

Also solved by M. A. Bershad, N. L. Johnson, J. C. Koop, D. C. B. Marsh, R. E. Meyer, Norman Miller, Judith Richman, P. N. Somerville, and Julius Vogel.

Vogel demonstrates the inconsistency of the system:

$$m^s\left(\sum_{t=1}^{m} x_t^2 - 1\right) = m^{s-1}\left(\sum_{t=1}^{m} x_t\right)^2 + 1,$$

$$n^s\left(\sum_{t=1}^{n} x_t^2 - 1\right) = n^{s-1}\left(\sum_{t=1}^{n} x_t\right)^2 + 1$$

for $n>m$ and $s>0$.

A Root of Unity as a Zero of a Polynomial

E 2097 [1968, 670]. *Proposed by Harley Flanders, Purdue University*

Let $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, where $1\geqq a_1\geqq a_2\geqq\cdots\geqq a_n\geqq 0$. Let λ be a complex root of f such that $|\lambda|\geqq 1$. Prove λ is a root of unity.

Solution by the proposer. λ is a root of $(x-1)f(x)$, hence

$$\lambda^{n+1} = (1-a_1)\lambda^n + (a_1-a_2)\lambda^{n-1} + \cdots + (a_{n-1}-a_n)\lambda + a_n.$$

By the triangle inequality,

$$\begin{aligned} |\lambda|^{n+1} &\leqq (1-a_1)|\lambda|^n + (a_1-a_2)|\lambda|^{n-1} + \cdots + (a_{n-1}-a_n)|\lambda| + a_n \\ &\leqq (1-a_1)|\lambda|^n + (a_1-a_2)|\lambda|^n + \cdots + (a_{n-1}-a_n)|\lambda|^n + a_n|\lambda|^n \\ &= |\lambda|^n, \end{aligned}$$

hence $|\lambda|\leqq 1$, $|\lambda|=1$, and we have the case of equality in the triangle inequality. Hence the numbers

$$(1-a_1)\lambda^n, \qquad (a_1-a_2)\lambda^{n-1},\cdots,(a_{n-1}-a_n)\lambda,\ a_n$$

are nonnegative multiples of a single complex number. These numbers cannot all vanish or $1=a_1=a_2=\cdots=a_n=0$.

If all vanish but one, then $1=a_1=\cdots=a_r$, $a_{r+1}=\cdots=a_n=0$, $\lambda^n+\lambda^{n-1}+\cdots+\lambda^{n-r}=0$, $\lambda^r+\lambda^{r-1}+\cdots+1=0$, $\lambda^{r+1}=1$. If two terms do not vanish, then we divide them to deduce that for some $s\geqq 1$, $\lambda^s>0$, hence $\lambda^s=1$.

Also solved by Leonard Carlitz, F. W. Carroll, Michael Goldberg, M. G. Greening (Australia), Marvin Gruber, R. K. Meany, Simeon Reich (Israel), Bernd Schmidt & Klaus Steffen (Germany), Peter Ungar, and J. P. Williams.

Reich notes that $|\lambda|=1$ follows from Q. G. Mohammad, *Location of the zeros of polynomials*, this MONTHLY, 74(1967) Theorem B, p. 292.

Rings over which Polynomials have a Finite Number of Roots

E 2098 [1968, 670]. *Proposed by J. O. Kiltinen and T. J. Grilliot, Duke University*

Suppose that A is a commutative ring with identity which has the property

that every nonzero polynomial with coefficients in A has only finitely many roots in A. Prove that A is either an integral domain or is finite.

Solution by Herbert Elliott, Jr., Nazareth College of Rochester, N. Y. The hypothesis that A be commutative and have an identity is superfluous. Suppose that A is an infinite ring with nonzero elements a, b such that $ab=0$. Then the map $x \to bx$ is an endomorphism of the additive group of A. So the image bA and the kernel $b^{-1}(0)$ of that map cannot both be finite because A is infinite. If bA (resp. $b^{-1}(0)$) is infinite, then the polynomial aX (resp. bX) has infinitely many roots in A.

Also solved by Einar Andresen (Norway), P. R. Chernoff, G. J. Ford, W. F. Fox, Robert Gilmer, D. L. Grant, Erwin Just, Gesing Leung (Hong Kong), W. G. McArthur, Brian Parshall & Mutt Parshall, Niel Shilkret, Klaus Steffen (Germany), D. P. Sumner, and the proposers.

A Problem in Extended Analytic Geometry

E 2099 [1968, 670]. *Proposed by R. S. Underwood, Texas Technological College*

Let all numbers involved be real. Given

$$\frac{x^2}{a^2} - \sum_{i=1}^{n} \frac{y_i^2}{b_i^2} = 1, \tag{1}$$

let $\sum_{i=1}^{n} A_i^2 b_i^2 = k^2 a^2$, with the A_i arbitrary except that not all are zero. Find the sole common real solution (x, y_i) of (1) and

$$2kx - \sqrt{3} \sum_{i=1}^{n} A_i y_i = ka \qquad (k > 0,\ a > 0). \tag{2}$$

Show that the solution is unique and that there is no solution if the right side of (2) is replaced by K, with $|K| < ka$.

I. *Solution by the proposer.* By Case 1, Article 4 of *New Results in Extended Analytic Geometry* (this MONTHLY, March 1965), the locus of (1) by the plotting rule there given is the silhouette of a hyperboloid of two sheets with the bounding locus $X^2/a^2 - Y^2/(k^2a^2) = 1$. The locus of (2) is the line tangent to (1) at the point $(2a, ka\sqrt{3})$. This yields the single solution: $x = 2a$, $y_i = b_i^2 A_i \sqrt{3}/(ka)$. To get y_i we use (5.20) page 72 of reference [5] in the cited article, for which the proof of uniqueness follows. When K is as stated, the line does not touch the locus of (1).

II. *Solution by L. E. Clarke, University of East Anglia, England.* Suppose that (1) and (2) (with the right-hand side replaced by K) have a common solution. Then by Cauchy's inequality

$$k^2(x^2 - a^2) = \left(\sum A_i^2 b_i^2\right)\left(\sum y_i^2/b_i^2\right) \geqq \left(\sum A_i y_i\right)^2 = \tfrac{1}{3}(2kx - K)^2. \tag{a}$$

Therefore

$$\text{(b)} \qquad 0 \geqq k^2x^2 - 4kKx + K^2 + 3k^2a^2 = (kx - 2K)^2 + 3(k^2a^2 - K^2),$$

which cannot hold if $|K| < ka$.

If $K = ka$, then $x = 2a$ (for (b) to hold) and, for some λ,

$$y_i/b_i = \lambda A_i b_i$$

for all i, as (a) must now hold with equality, and the A_i are not all zero. Substitution in (2) yields $\lambda = \sqrt{3}/(ka)$, and the required unique solution is $x = 2a$, $y_i = A_i b_i^2 \sqrt{3}/(ka)$.

Also solved by M. A. Bershad, Ted Cullen, R. J. Driscoll, D. C. B. Marsh, Norman Miller, Simeon Reich (Israel), and Judith Richman.

Six Relations

E 2100 [1968, 670]. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

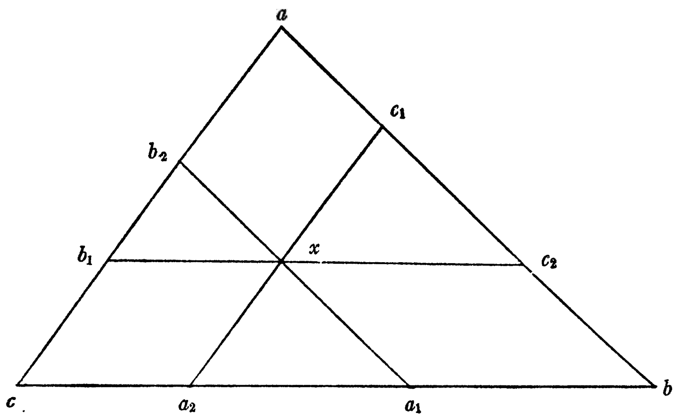
Show that any five of the relations

$$(1)\ \frac{x - a_1}{a_1 - a_2} = \frac{a - b}{b - c}, \qquad (2)\ \frac{x - b_1}{b_1 - b_2} = \frac{b - c}{c - a}, \qquad (3)\ \frac{x - c_1}{c_1 - c_2} = \frac{c - a}{a - b},$$

$$(4)\ x + a = b_2 + c_1, \qquad (5)\ x + b = c_2 + a_1, \qquad (6)\ x + c = a_2 + b_1$$

imply the sixth. Interpret this set of consistent relations geometrically letting a, b, c be the affixes, in the complex plane, of a triangle of reference ABC and other numbers be those of other points.

Solution by Michael Goldberg, Washington, D. C. Take any triangle, represented by the vertices a, b, c. Take any point x in the plane of the triangle. Draw parallels to the sides of the triangle through the point x. Let the intersections of these parallels with the sides be a_1, a_2, b_1, b_2, c_1, c_2, as shown in the figure. Then the six given relations hold. If one of the six points, say c_2, is omitted, then x



can be found as the intersection of a_2c_1 with b_2a_1, and then c_2 is determined as the intersection of b_1x with ab.

Also solved by A. F. Gentzel, Jr., Simeon Reich (Israel), and the proposer.

Three Parabolas and a Triangle

E 2101 [1968, 670]. *Proposed by H. Demir, Middle East Technological University, Ankara, Turkey*

ABC is a triangle. Let P_a denote the parabola tangent to the sides AB, AC at B, C respectively. The parabolas P_b and P_c are similarly defined. Let these parabolas intersect at the points A', B', C' inside ABC. Denote the areas of the (curvilinear) triangular regions ABC, $A'B'C'$, $AB'C'$, $BC'A'$, $CA'B'$, $A'BC$, $B'CA$, $C'AB$ by Δ, Δ_0, Δ_a', Δ_b', Δ_c', Δ_a'', Δ_b'', Δ_c''. Then prove

$$(1) \qquad \Delta_a' = \Delta_b' = \Delta_c'(\equiv \Delta_1), \qquad \Delta_a'' = \Delta_b'' = \Delta_c''(\equiv \Delta_2),$$

$$(2) \qquad \Delta_0 : \Delta_1 : \Delta_2 : \Delta = 15:17:5:81.$$

Solution by the proposer. Under parallel projections the nature of conics, the tangency and ratios of segments and areas are invariant, and any triangle can be transformed into an equilateral triangle. Hence it will suffice to prove the assertion for an equilateral triangle. So, part (1) is already proved.

To prove (2), let ABC be an equilateral triangle located in the coordinate plane such that $A=(1, \sqrt{3})$, $B=(0, 0)$, $C=(2, 0)$. The equations of parabolas P_a and P_c are found to be

$$(1) \qquad P_a\colon\ y = (x - \tfrac{1}{2}x^2)\sqrt{3},$$

$$(2) \qquad P_c\colon\ \sqrt{3}\,x^2 + 6xy + 3\sqrt{3}\,y^2 - 16y = 0.$$

From (1) and (2) we obtain

$$(3) \qquad \Delta_0 + 2\Delta_1 + \Delta_2 = \sqrt{3}\int_0^2 (x - \tfrac{1}{2}x^2)dx = \tfrac{2}{3}\sqrt{3} = \tfrac{2}{3}\Delta,$$

$$(4) \qquad y = \frac{8\sqrt{3}}{9} - \frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{9}\sqrt{4-3x} \qquad (0 \leqq x \leqq \tfrac{4}{3}).$$

We find, therefore,

$$(5) \qquad \Delta_2 = 2\int_0^1 y\,dx = \frac{5\sqrt{3}}{81} = \frac{5}{81}\Delta,$$

with

$$(6) \qquad \Delta_0 + 3\Delta_1 + 3\Delta_2 = \Delta.$$

Solving the system (3), (5), (6) for Δ_0, Δ_1, Δ_2, we get the required result.

Also solved by Anders Bager (Denmark), Jordi Dou (Spain), Michael Goldberg, M. G. Greening (Australia), Norman Miller, J. M. Quoniam (France), and A. Zujus.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (with double spacing) and should be mailed before October 31, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5646 [1969, 94]. **Correction.** The third integral should read

$$\int_{-\pi/4}^{\pi/4} f(\theta)\left[\frac{1}{\cos 2\theta} - 1\right]^{1/2} d\theta = 0.$$

Several copies of the MONTHLY appeared without the exponent $\frac{1}{2}$.

5671. *Proposed by Gerard Letac, University of Montreal.*

In a group G call S a splitting set if $S\subset S\,S$. Show that for each n there exists a finite splitting set in the group of integers with $0\notin S+S+\cdots+S$ (n terms). However, if G is a topological group and S is a compact splitting set, the identity e must belong to the closure of the semigroup generated by S.

5672. *Proposed by Stephen Weingram, Purdue University*

A continuous map $f:R^n\to R^n$ is a contraction outside the set K if there is a constant c $(0<c<1)$ such that for any two points x, y outside K, $|f(x)-f(y)| \leqq c|x-y|$. Prove that if $f:R^n\to R^n$ is a contraction outside a compact set K, it has a fixed point.

5673. *Proposed by Stanley E. Payne, Miami University, Ohio*

Let G be a group with subgroups A and B such that $(A:A\cap B)=(B:A\cap B) =k\geqq 3$, and such that there is an $n<\infty$ and an $x\in G$ with Ax^i, Bx^i, $1\leqq i\leqq n$, being all the distinct left cosets of A and B in G. Then prove that $ABA\cap BAB \neq A+B$.

5674. *Proposed by L. Carlitz, Duke University*

It is proved in problem 5542 [1968, 1021] that the following statement is incorrect: If a, b are algebraic over F of degree m and n respectively and if m and n are relatively prime, then ab is algebraic over F of degree mn.

Show that the statement is correct when $F=GF(q)$, $q=p^n$, p prime, $n\geqq 1$.

5675. *Proposed by M. Slater, University of Bristol, England*

Let R be an associative ring, A an ideal of R, and U a nonzero ideal of A. Under suitable conditions U contains a nonzero ideal of R (cf. Jacobson, *Structure of Rings*, p. 65). Construct an example in which the conclusion is false, and in addition (1) A is as well-behaved as possible; (2) $U^2\neq(0)$; (3) $U^2\neq(0)$ and $A^2=A$.

5676. *Proposed by Albert Wilansky, Lehigh University*

Let $X=\{0\}\cup(1, \infty)$ with the ordinary topology of the real line. Suppose that a complete metric is given for this topology. Must $(1, \infty)$ contain a closest point to $\{0\}$?

SOLUTIONS OF ADVANCED PROBLEMS

Constrained Inequalities

5182 [1964, 326]. *Proposed by D. Ž. Djoković, University of Waterloo, Canada*

Prove or disprove the following inequality: $x^3\{y^2(z+1)^2+t(y+z+yz)^2\} \leqq 1$, where x, y, z, t are nonnegative real numbers such that $3x+2y+2z+t=5$.

5183 [1964, 326]. *Proposed by D. Ž. Djoković, University of Waterloo, Canada*

Prove or disprove the following inequality:

$$x^2\{v[z(y+1)(u+1)+yu]^2+u^2(y+z+yz)^2\} \leqq 1,$$

where x, y, z, u, v are nonnegative real numbers such that $2(x+y+z+u)+v=5$.

Solution by the proposer. Using methods in my paper in the Proceedings of the Glasgow Mathematical Association, 6 (1963), 1–10, the inequality of H. S. Shapiro in problem 4603 [1956, 191] for $n=10$, is shown to be equivalent to

$$\begin{aligned} x_1^2\{ & x_3x_5x_7[1+(x_2+x_4+x_6+x_8)+(x_2x_4+x_2x_6+x_2x_8+x_4x_6+x_4x_8+x_6x_8) \\ & +(x_2x_4x_6+x_2x_4x_8+x_2x_6x_8+x_4x_6x_8)+x_2x_4x_6x_8] \\ & +x_3x_5x_6x_8(1+x_2+x_4+x_2x_4)+x_3x_7x_4x_6(1+x_2+x_8+x_2x_8) \\ & +x_5x_7x_2x_4(1+x_6+x_8+x_6x_8)+x_3x_4x_6x_8(1+x_2)+x_5x_2x_4x_6x_8 \\ & +x_7x_2x_4x_6(1+x_8)+x_2x_4x_6x_8\} \leqq 1, \end{aligned} \tag{1}$$

where $2x_1+x_2+x_3+\cdots+x_8=5, (x_i \geqq 0)$. Our problems are special cases of (1) for

(i) $x_1=x_3=x, \quad x_2=0, \quad x_4=x_8=z, \quad x_5=x_7=y, \quad x_6=t:$

(ii) $x_1=x, \quad x_2=x_8=y, \quad x_3=x_7=z, \quad x_4=x_6=u, \quad x_5=v.$

Shapiro's inequality

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geqq \frac{n}{2}, \qquad \text{(replace } x_{n+j} \text{ by } x_j) \tag{2}$$

has been proved for $n=10$ in P. P. Nowosad, *Isoperimetric Problems in Algebras*, Comm. Pure Appl. Math., 21 (1968) 401–465.

Editorial Note. For comments on Shapiro's inequality, see this MONTHLY, 67 (1960) 87, and 66 (1959) 489–491. Other references are R. A. Rankin, *An inequality*, Math. Gazette, 42(1958)39–42. A. Zulauf, *Note on some inequalities*, Math. Gazette, 42(1958)42. A. Zulauf, *On a conjecture of L. J. Mordell*, Hamburg Abhandlungen, 22(1958)24. A. Zulauf, *On a conjecture of L. J. Mordell*, II, Math. Gazette, 43(1959)182–184.

The present state of knowledge regarding (2) seems to be: true for $n \leqq 8$ and $n=10$; undecided for $n=9$, 11, 12, 13, 15, 17, 19, 21, 23, 25; false for other values.

Euler's Constant, a Ramanujan Identity

5600 [1968, 685]. *Proposed by L. Carlitz, Duke University*

Ramanujan (Collected Papers, p. 325) has stated the following formula for Euler's constant:

$$C = \log 2 - \frac{2}{3^3 - 3} - 2\left(\frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}\right)$$
$$- 3\left(\frac{2}{15^3 - 15} + \frac{2}{18^3 - 18} + \cdots + \frac{2}{39^3 - 39}\right) - \cdots.$$

Provide a proof.

Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands. We remark that

$$\frac{2}{(3m)^3 - 3m} = \frac{1}{3m - 1} + \frac{1}{3m} + \frac{1}{3m + 1} - \frac{1}{m}.$$

Therefore, the partial sums of the series to be summed are

$$S_n = \log 2 + \sum_{k=1}^{\frac{1}{2}(3^n - 1)} k^{-1} - n \sum_{k=\frac{1}{2}(3^n+1)}^{\frac{1}{2}(3^{n+1}-1)} k^{-1}.$$

Using $\sum_{k=1}^{m} k^{-1} = C + \log m + o(1)$, we find for S_n

$$S_n = C + \log(3^n - 1) - n\left\{\log \frac{3^{n+1} - 1}{3^n - 1}\right\} + o(1)$$

and since the second and third terms on the right are $n \log 3 + o(1)$ and $-n \log 3 + o(1)$ respectively, we have $S_n = C + o(1)$.

Also solved by M. G. Beumer (Netherlands), Robert Breusch, M. G. Greening (Australia), Heiko Harborth (Germany), Robert Heller, and the proposer.

Beumer remarks that a solution may be found in Ramanujan's paper in the Messenger of Mathematics, 46 (1917) 73–80. See also the note following problem 5601 below.

Euler's Constant, a Variable Series

5601 [1968, 685]. *Proposed by L. Carlitz, Duke University*

Let k be an integer, $k > 1$. Show that Euler's constant satisfies

$$C = \sum_{n=1}^{\infty} \frac{\epsilon_n}{n}\left(\frac{\log n}{\log k}\right), \qquad \epsilon_n = \begin{cases} k - 1 & (k \mid n) \\ -1 & \text{otherwise.} \end{cases}$$

For $k = 2$ this result reduces to problem 4353 [1951, 116].

I. *Solution by Robert Breusch, Amherst College.* The problem should read

$$C = \tfrac{1}{2} \log k + \sum_{n=1}^{\infty} \frac{\epsilon_n}{n}\left(\frac{\log n}{\log k}\right).$$

Proof: The series clearly converges. Let

$$S_{m,k}=\sum_{n=1}^{mk}\frac{\epsilon_n}{n}\left(\frac{\log n}{\log k}\right)=\sum_{r=1}^{m}\frac{k}{kr}\,\frac{\log(kr)}{\log k}-\sum_{n=1}^{mk}\frac{\log n}{n\cdot\log k}$$

$$=\sum_{r=1}^{m}\frac{1}{r}\,\frac{\log k+\log r}{\log k}-\frac{1}{\log k}\sum_{n=1}^{mk}\frac{\log n}{n}$$

$$=\sum_{r=1}^{m}\frac{1}{r}-\frac{1}{\log k}\sum_{n=m+1}^{mk}\frac{\log n}{n}$$

$$=\log m+C+O\left(\frac{1}{m}\right)-\frac{1}{\log k}\cdot\frac{1}{2}\left[\log^2(mk)-\log^2 m\right]+O\left(\frac{\log m}{m}\right).$$

Thus

$$C=\tfrac{1}{2}\log k+\lim_{m\to\infty}S_{m,k}=\tfrac{1}{2}\log k+\sum_{n=1}^{\infty}\frac{\epsilon_n}{n}\left(\frac{\log n}{\log k}\right).$$

II. *Solution by Heiko Harborth, Braunschweig, Germany.* To obtain the result of the problem we need [ ]'s instead of parentheses for the factor $\log n/\log k$. Then for fixed r, we have

$$\sum_{n=1}^{k^r-1}\frac{\epsilon_n}{n}\left[\frac{\log n}{\log k}\right]=\sum_{i=0}^{r-1}\sum_{n=k^i}^{k^{i+1}-1}\frac{\epsilon_n}{n}\left[\frac{\log n}{\log k}\right]=\sum_{i=0}^{r-1}i\sum_{n=k^i}^{k^{i+1}-1}\frac{\epsilon_n}{n}$$

$$=\sum_{i=1}^{r-1}i\left\{-\sum_{n=k^i}^{k^{i+1}-1}\frac{1}{n}+\sum_{n=0}^{k^i-k^{i-1}-1}\frac{1}{n+k^{i-1}}\right\}$$

$$=-\sum_{i=1}^{r-1}i\sum_{n=k^i}^{k^{i+1}-1}\frac{1}{n}+\sum_{i=1}^{r-1}i\sum_{n=k^{i-1}}^{k^i-1}\frac{1}{n}=\sum_{n=1}^{k^r-1}\frac{1}{n}-r\sum_{n=k^{r-1}}^{k^r-1}\frac{1}{n}.$$

Then from

$$\log k=\int_{k^{r-1}}^{k^r}\frac{dt}{t}\leqq\sum_{n=k^{r-1}}^{k^r-1}\frac{1}{n}\leqq\int_{k^{r-1}-1}^{k^r-1}\frac{dt}{t}=\log k+\log\frac{1-k^{-r}}{1-k^{1-r}},$$

it follows, with $r\to\infty$, that

$$\sum_{n=1}^{\infty}\frac{\epsilon_n}{n}\left[\frac{\log n}{\log k}\right]=C.$$

Also solved by M. G. Beumer (Netherlands), Bengt Fornberg (Sweden), J. H. van Lint (Netherlands), and the proposer (to whom go apologies for the incorrect parentheses).

Beumer informs us that the identities in 5600 and 5601 are contained, along with some others, in his paper, *On Vacca-like expansions of Euler's constant*, due to appear in a future issue of the MONTHLY.

Multipoint Completion of the Reals

5602 [1968, 685]. *Proposed by A. Wilansky, Lehigh University*

It is well known that R (the reals) has no 3-point compactification. Does it have a 3-point completion? (This means: is there a complete metric space such that the removal of 3 nonisolated points leaves a subspace homeomorphic with R?)

Solution by M. D. Mavinkurve, Siddharth College, Bombay, India. The following gives a positive answer. In the Euclidean plane which is a complete metric space, the graph G of the function $y=\sin(1/x)$, $0<x<1$, being homeomorphic with R admits a metric in which it is complete. Hence G is a G_δ set. If p, q, r are any three points on the closed segment $[-1, 1]$ of the y axis, the set $S=G \cup \{p, q, r\}$ is also a G_δ set, for it can be realized as $\bigcap_{n=1}^{\infty}(O_n \cup A_n)$ where $\bigcap_n O_n$ is a realization for G and A_n is the union of $(1/n)$-spherical neighborhoods of p, q, r. Therefore the set S is a G_δ in a complete metric space and admits a metric which makes it into a complete metric space (see Kelley, *General Topology*, p. 207) such that the removal of three nonisolated points from it leaves a subspace homeomorphic with R.

The example also shows that for each $n \geqq 1$ there is a completion of R

Also solved by P. R. Chernoff, J. A. Isbell, W. G. McArthur, L. E. Ward, Jr., and the proposer.

Chernoff notes how R admits a completion using $\aleph_0$ points and Isbell and Ward give solutions for a c-point completion. These are possible as modifications of the solution given above.

An Integral Transform

5604 [1968; 686, 910]. *Proposed by R. A. Struble, North Carolina State University at Raleigh*

If the function f is bounded and continuous on $[0, \infty)$ and for some $a>0$ satisfies

$$0 \leqq f(t) \leqq \frac{a}{3}\int_0^t e^{-a(t-s)}f(s)ds + \frac{a}{3}\int_t^\infty e^{a(t-s)}f(s)ds + e^{-at/2}$$

for $t \geqq 0$, then $f(t) \to 0$ as $t \to \infty$.

Solution by R. K. Meany, Iowa State University. Let

$$K(t, s) = \begin{cases} \frac{1}{3}ae^{a(s-t)}, & 0 \leqq s \leqq t, \\ \frac{1}{3}ae^{-a(s-t)}, & t < s < \infty, \end{cases}$$

and let $g(t)=e^{-at/2}$. The given inequality is then

$$0 \leqq f(t) \leqq \int_0^\infty K(t, s)f(s)ds + g(t).$$

The relevant facts needed for the kernel K are:

(i) $$0 \leqq K(t, s) \leqq M,$$

(ii) $$K(t, s) \to 0 \quad \text{when } t \to \infty \text{ with } s \text{ fixed},$$

(iii) $$\int_0^\infty K(t, s)ds \leqq \delta < 1.$$

Write $Tf(t) = \int_0^\infty K(t, s) f(s)ds$, and for $n = 2, 3, \cdots$, let $T^n f(t) = T(T^{n-1}f(t))$. The hypothesis becomes

$$0 \leqq f \leqq Tf + g,$$

and iteration yields

$$f \leqq T^n f + T^{n-1}g + \cdots + Tg + g.$$

Suppose that $f(t) \leqq A$. It then follows from (i) and (iii) that for $n = 1, 2, \cdots$,

$$0 \leqq T^n f(t) \leqq \delta^n A.$$

Furthermore it follows from (i), (ii) and (iii) that for all n, $T^n g(t) \to 0$ when $t \to \infty$.

Given ϵ, choose n so large that $\delta^n A < \epsilon$. Then choose τ so large that for $t > \tau$, $T^{n-1}g(t) + \cdots + Tg(t) + g(t) < \epsilon$. It then follows that for $t > \tau$, $f(t) \leqq 2\epsilon$.

Also solved by P. R. Chernoff, D. A. Hejhal, Thomas Hughes, C. V. L. Smith, Alberto Torchinsky, and the proposer.

Density Values of a Periodic Function

5605 [1968, 686]. *Proposed by Neal Felsinger, State University of New York at Buffalo*

Let f be a continuous, periodic function with least positive period p from R onto $[-1, 1]$. Then the sequence $\{f(n)\}$ is known to be dense in $[-1, 1]$ if and only if p is irrational. Does the result still hold if f is not defined on $\{\alpha + kr | \alpha$ and r real constants, k an integer$\}$, but continuous on its domain and having range $(-\infty, \infty)$?

Solution by the proposer. If p is rational, $p = a/b$ and $f(n + kb) = f(n)$ for all k. So $\{f(n)\}$ is finite and cannot be dense in $[-1, 1]$. If p is irrational we need the following theorem: *If α and β are real constants, α irrational, then for any positive integer N and any $\epsilon > 0$ there exist integers $n > N$ and k such that* $|n\alpha - \beta - k| < \epsilon$.

Given x, there is a sequence $\{x_i\}$ such that $f(x_1) = x$, $x_1 > 0$ and $x_{i+1} = x_i + p$. Hence $f(x_i) = x$ for all i. By periodicity and continuity for any $\epsilon > 0$, there exists $\delta > 0$ such that $|x_i - y| < \delta$ implies $|f(x_i) - f(y)| < \epsilon$ (since the δ that works for x_1 works for all x_i). If N and ϵ are given, there exist integers n and k, $n > N$, such that $|np + x_1 - k| < \delta$ or $|x_{n+1} - k| < \delta$ which implies $|f(x_{n+1}) - f(k)| < \epsilon$ or $|x - f(k)| < \epsilon$. Therefore x is a limit point.

An affirmative answer follows after noting that f may be undefined for some integer k, but this can happen at most once and may be ignored in a limiting process.

Editorial Note. The statement of the problem, as originally printed, allowed f to be undefined on the set $\{\alpha+kr \mid \alpha, r$ fixed real numbers, k an integer$\}$. The fact that f is periodic shows that the set on which f is undefined should be in reality of the form $\{\alpha+kp \mid \alpha$ fixed, k an integer, p the designated period$\}$.

Convex Sums of Orthogonal Projectors

5607 [1968, 686]. *Proposed by E. A. Power, University College, London, England*

If $M_i (i=1, 2, \cdots, n; n>1)$ are distinct orthogonal projectors in Hilbert space show that for no set of positive numbers p_i with $\sum_{i=1}^n p_i=1$ is $M=\sum_{i=1}^n p_i M_i$ a projector. Show that the result is false if M_i are nonorthogonal projectors.

Solution by Bernd Schmidt and Klaus Steffen, Johannes Gutenberg University, Germany. A projector is orthogonal if and only if it is selfadjoint. Therefore M, the indicated linear combination of orthogonal projectors, is an orthogonal projector if and only if $M^2=M$; we shall show that this is equivalent to $M_1 = \cdots = M_n = M$.

Let $M^2=M$. By the triangle and the Schwarz inequalities we have

$$\begin{aligned} \|Mx\|^2 &\leqq \left(\sum p_i \|M_i x\|\right)^2 = \left\{\sum \sqrt{p_i}\left(\sqrt{p_i}\|M_i x\|\right)\right\}^2 \\ &\leqq \left(\sum p_i\right)\left(\sum p_i \|M_i x\|^2\right) = \sum p_i \cdot (M_i x, x) = (Mx, x) = \|Mx\|^2. \end{aligned}$$

Therefore $(\sqrt{p_1}, \cdots, \sqrt{p_n})$ and $(\sqrt{p_1}\|M_1 x\|, \cdots, \sqrt{p_n}\|M_n x\|)$ are linearly dependent, i.e. $\|M_1 x\| = \cdots = \|M_n x\| (=\|Mx\|)$. The assertion follows now from

$$(M_i x, x) = \|M_i x\|^2 = \|Mx\|^2 = (Mx, x)$$

for all x in the space.

If M_i are nonorthogonal we get a counterexample from the identity

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}.$$

Also solved by P. R. Chernoff, M. A. Ettrick, D. A. Hejhal, "Young Archimedes", and the proposer & H. Kestelman.

Cardinality and the Axiom of Choice

5609 [1968, 686]. *Proposed by R. E. Maas, University of Santa Clara, California*

For any set S and any mapping $T: S \to \mathrm{Re}_{+0}$ (the additive semigroup of nonnegative real numbers) define

$$\mathrm{SUM}(T \mid S) = \operatorname*{lub}_{\substack{S' \subset S \\ S' \text{ finite}}} \left(\sum_{n \in S'} T(n)\right).$$

Prove that if SUM $(T \mid S) < \infty$, then card $\{s \in S \mid T(s) > 0\} \leqq \aleph_0$ without assuming the Axiom of Choice.

Partial Solution by D. T. Adams, University of Maryland. Let $\mathrm{SUM}(T|S) = a < \infty$. There can be in S at most one point x such that $T(x) = a$, for if there were two or more such points, we would have $\mathrm{SUM}(T|S) > a$, a contradiction. Similarly, for any positive integer n there can be at most n points in S whose image under T is $\geqq a/n$, for if not, we have the same contradiction.

Using the Axiom of Replacement, construct a map f from the natural numbers (excluding zero, of course) to the power set of S as follows: for n a natural number,

$$f(n) = \left\{x \in S \mid T(x) \in \left[\frac{a}{n+1}, \frac{a}{n}\right)\right\}.$$

Now the range of f contains at most a countable number of sets, each of which must be finite. Hence we have

$$\mathrm{card}\{s \in S \mid T(s) > 0\} \leqq \aleph_0$$

since clearly $\{s \in S \mid T(s) > 0\}$ = the range of f.

Also solved (partially) by R. L. Enison, M. L. Laplaza, Necdet Üçoluk, and the proposer.

Editorial Note. The finite sets $f(n)$ may be established on the basis of the weaker axiom of separation. Although inductive procedures may be offered, the intrinsic question remains to prove that "the union of a countable number of pairwise disjoint finite sets is countable" without the Axiom of Choice. See P. Suppes, *Axiomatic Set Theory*, Chapter 8, specifically p. 243, exercise 9.

The problem actually constitutes a part of the proof of the theorem that the discontinuities of a monotonic function form a denumerable set.

REVIEWS

EDITED BY KENNETH O. MAY

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.*

Elements of Modern Topology. By Ronald Brown. McGraw-Hill, New York, 1968. xvi+351 pp. $10.95. (Telegraphic Review, Jan. 1969.)

The view of topology presented by this book is unusual; one might call it "a category theorist's view of topology." The major attraction of this view is the early introduction of category theory and groupoids as a background from

which to view homotopy theory. Much space—perhaps more than will please most readers—is devoted to groupoids; one has the feeling that the author's enthusiasm for this subject has so captured his fancy that he cannot relinquish it. This emphasis has also caused the author to slight that aspect of topology that looks toward analysis. For example, the Tychonoff Theorem is stated but not proved, as is also the Tietze Extension Theorem. No mention is made of completely regular spaces, paracompactness, or of complete metric spaces. Also algebraic topology is treated only in part; for example homology theory is not touched upon.

The first three chapters treat general topology with emphasis as indicated above; chapters 4 and 5 deal with identification spaces, cells, cell complexes and simplicial complexes. Chapter 6 introduces categories and groupoids; and it along with chapters 7 and 8 deal with homotopy theory in general, emphasis being on the notion of groupoid. Chapter 9 deals with covering spaces, and finally there is an appendix on functions, cardinality, and the Axiom of Choice.

The book has a nice format; typographical errors are few. The description of the Cantor set on page 11 as a set of ternary decimals is ambiguous; what is intended is that the Cantor set is that set of ternary decimals in the closed unit interval which *can* be written without using the digit "1." Finally the reviewer feels that though the book has merit and will serve as a useful reference, it is not adequate as a text for either general or algebraic topology, since too many topics in both areas have been left untreated or treated only cursorily.

J. D. BAUM, Oberlin College

C *Integrals and Operators.* By Irving Segal and Ray Kunze. McGraw-Hill, New York, 1968. xi+308 pp. $10.50. (Telegraphic Review, May 1968.)

Intended as a first graduate course in contemporary real analysis, the book covers integration theory, real variable theory, and elementary functional analysis with the aim of exposing the student to modern analytical thinking rather than equipping him with encyclopedic knowledge or professional skill. It is intended to serve as an introduction for students specializing in analysis and as a possible terminal course for others.

The first six chapters form a systematic (but not standard) introduction to real analysis. Integration theory is presented by Daniell extension, applied to an integration lattice. Measurability is defined with respect to the σ-ring generated by a ring $\mathcal{R}$; σ-algebra measurability is called local measurability. This attitude shades the treatment of many topics; e.g., decomposition and Radon-Nikodym Theorems, and L_p spaces. Baire measurability plays the leading role in topological measure and integration. Topologies for various modes of convergence are emphasized and treated precisely throughout the book. Chapters 7 through 10 present special topics (invariant integration, Banach algebras and Hilbert space, spectral analysis in Hilbert space, representations of locally compact groups, unbounded operators) and can be studied independently by anyone who understands integration and linear duality. It is regrettable, how-

ever, that the authors did not achieve a higher degree of independence for these sections by including at least an adequate index and perhaps a glossary for the deluge of technical terms (many of them nonstandard) which are used.

The authors are distinguished mathematicians and the book is of the high quality expected. High points are many lucid heuristic discussions and the treatment of spectral theory and diagonalization. There are ample exercises (349) amplifying nearly everything in the text proper, and of all degrees of difficulty. There is no shortage of excellent competition for the course intended, and selection of the book should be based on a preference for the authors' approach (unique among current texts) to the basic material, and the selection of special topics. The book can be highly recommended for the special topics, with a slight reservation because of the inadequate index. The announced prerequisites of some topology, set theory and linear algebra are accurate. In particular, the topology prerequisite should not be taken lightly; a concurrent course is minimally sufficient, even for the first part of the book.

KEITH PHILLIPS, New Mexico State University, Las Cruces

TELEGRAPHIC REVIEWS

The following abbreviations indicate possible uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level)—18 (second graduate year). A boldface star (★) marks a notable book of general interest.

Algebra

Elements of Number Theory. By. I. A. Barnett (Ohio Univ.). Prindle, Weber & Schmidt, Boston, Mass., 1969. x+213 pp. $8.50. This is intended primarily for future junior and senior high school teachers of mathematics. "Elegance has been sacrificed in favor of a basic approach that requires no sophisticated background . . ." There is plenty of material for a substantial half-year course. T (14), TT.

Algebra. By Roger Godement (Faculty of Sciences, Paris). Hermann, Paris, and Houghton Mifflin, Boston, 1968. 638 pp. $15.00. A hefty introduction in seven parts: set theory (reasoning, equality, function, etc.); groups, rings, fields (including complex numbers); modules over a ring (including vector spaces, matrices) finite dimensional vector spaces (including linear equations); determinants (multilinear functions, alternating bilinear mappings, alternating multilinear mappings, determinants, affine spaces), polynomials and algebraic equations (including derivations), and reduction of matrices (eigenvalues, canonical forms, Hermitian forms). T (15–16).

Lie Groups. Lie Algebra. By Melvin Hausner and J. T. Schwartz (both of New York Univ.). Gordon and Breach, New York, 1968. x+229 pp. $9.50 (cloth) $4.50 (paper). This volume in the series *Notes on Mathematics and its Applications* is a fairly comprehensive exposition of its subject along the lines developed by Lie, Killing, Cartan, Weyl, Dynkyn, Harish-Chandra, Gantmacher, and Bourbaki. P.

Analysis

Theory of Ordinary Differential Equations. By Randal H. Cole (Univ. of Western Ontario). Appleton-Century-Crofts, New York, 1968. xi+273 pp. $8.50. Topics include existence and uniqueness of solutions, the two point boundary problem, self-adjoint eigenvalue problems (in which the Lebesgue integral is introduced), Sturmian theory,

and non-self-adjoint eigenvalue problems. T (16–17).

Lectures on the Numerical Solution of Linear, Singular, and Nonlinear Differential Equations. By D. Greenspan (Univ. of Wisconsin). Prentice-Hall, Englewood Cliffs, N.J., 1968. 185 pp. $6.95. This is a survey growing out of lectures delivered by the author at summer conferences at the University of Michigan. Solutions are suitable for use by digital computers. There is an extensive bibliography and references to it in the text. P, L.

Linear Analysis. By Ralph Henstock (Univ. of Lancaster). Plenum Press, New York, 1968. 448 pp. $19.50. The author describes his book as "an outline of the general summability theory of series and integrals, using functional analysis . . .", and considers it a possible introduction to such treatises as Dunford and Schwartz. It is based in part on the author's own approach to integration and summability. T (17), P.

Classical Harmonic Analysis and Locally Compact Groups. By Hans Reiter (Univ. of Utrecht). Clarendon Press, Oxford, 1968. xi+200 pp. $12.00 (paper). This is intended as a rather thorough introduction to some of the methods and ideas of Wiener, Carleman and Weil. Chapter headings are classical harmonic analysis and Wiener's theorem, function algebras and the generalization of Wiener's theorem, locally compact groups and Haar measure, locally compact abelian groups and the foundations of harmonic analysis, functions on locally compact abelian groups, Wiener's theorem and locally compact abelian groups, the spectrum and its applications, and functions on general locally compact groups. There are a selective bibliography and historical notes in the text. T (17), S, P, L.

Applications

Filtering for Stochastic Processes with Applications to Guidance. By Richard S. Bucy (Univ. of Southern Calif.,) and Peter D. Joseph (TRW Systems Group, Redondo Beach, Calif.). Interscience, New York, 1968. xviii+195 pp. $12.95. Here "filtering" refers to the problem of estimating a random signal process from observations corrupted by noise. The purpose of these lectures is to provide a detailed derivation of the Kalman-Bucy filter, describe its asymptotic property and deal partially with the problem of nonlinear filtering. P,L.

Variational Calculus in Science and Engineering. By Marvin J. Forray. McGraw-Hill, New York, 1968. xi+221 pp. $14.95. After being slighted for decades, the calculus of variations is assuming once more considerable practical and theoretical importance. Topics in this book include the Euler-Lagrange development, Hamilton's principle and Lagrange's equation, the Rayleigh-Ritz method, and the methods of Galerkin, Kantorovich, and Euler. There are many worked examples, but no exercises. T (15), S, P, L.

Nonlinear Programming. Sequential Unconstrained Minimization Techniques. By Anthony V. Fiacco and Garth P. McCormick (Research Analysis Corp, McLean, VA). Wiley, New York, 1968. xiv+210 pp. $9.95. The authors describe their book as "a comprehensive reference for the evolution, theory, and computational implementation of auxiliary function sequential unconstrained methods and a concise reference for mathematical programming theory." There are a historical section and a bibliography. T, P, S, L.

Information Theory and Reliable Communication. By Robert G. Gallager (MIT). Wiley, New York, 1968. xiv+588 pp. $16.95. Although designed for both engineers and mathematicians, this book emphasizes precise statements of results and careful proof. Chapter titles are Communication systems and information theory, Measure

of information, Coding for discrete sources, Discrete memoryless channels and capacity, The noisy-channel coding theorem, Techniques for coding and decoding, Memoryless channels with discrete time, Waveform channels, and Source coding with a fidelity criterion. There are historical notes and references at the end of each chapter, a bibliography and a glossary of symbols. T (15–17), L.

Readings in Mathematical Economics. Edited by Peter Newman. Vol. I: Value Theory. Vol. II: Capital and Growth. Johns Hopkins Press, Baltimore, 1968. Vol. I: xii +390 pp. $10.00 (cloth) $3.50 (paper). Vol. II: ix+358 pp. $10.00 (cloth) $3.50 (paper). Forty-five papers from the mid thirties to 1967. The first volume stresses mathematical techniques that have made equilibrium analysis into a rigorous doctrine. The second volume covers many lively issues. S, P, L.

Application of Finite Difference Equations to Shell Analysis. By Mircea Soare. Pergamon, New York, and Publishing House of the Academy of the Socialist Republic of Romania, Bucharest, 1967. xxi+438 pp. $23.00 This is a revised and augmented translation of the Romanian original published in 1960. P.

Mathematical Ideas in Biology. By J. Maynard Smith (Univ. of Sussex). Cambridge, New York, 1968. vii+152 pp. $5.00 (cloth) $1.95 (paper). A useful book for biologists who have had a little calculus, an excellent supplementary text for a mathematics course for biologists and a good source of problems for the mathematics professor. S, P.

Calculus

Prelude to Calculus and Linear Algebra. By John Olmsted (Southern Illinois Univ.). xix +332 pp. $6.00. *Basic Concepts of Calculus.* By John M. H. Olmsted. xiv+405 pp. $6.50. *A Second Course in Calculus.* By John M. Olmsted. xv+291 pp. $6.50. Appleton-Century-Crofts, New York, 1968. This three volume set is a condensation and rearrangement of Olmsted's *Calculus with Analytic Geometry.* T (13–14).

Two-Dimensional Calculus. By Robert Osserman (Stanford Univ.). Harcourt, Brace & World, New York, 1968. xvii+456 pp. $11.95. The intention is to present the fundamental ideas that distinguish several-variable from one-variable calculus by limiting the context to functions of two real variables, which can be done without the apparatus of linear algebra. The treatment is classical but not incompatible with a possible following course given in the abstract modern manner. T (14).

Computers

University Education in Computing Science. Proceedings of a conference on graduate academic and related research programs in computing science, held at the State University of New York at Stony Brook, June 1967. Edited by Aaron Finerman (SUNY at Stony Brook). Academic, New York, 1968. xvi+237 pp. $12.00. The papers in this report tell us a great deal about computers as well as about education in computing science. The book is therefore of quite broad interest. P, L.

★ *Calculus and the Computer Revolution.* By Richard W. Hamming (Bell Telephone Lab.,). Houghton Mifflin, Boston, Mass., 1968. x+72 pp. $1.75 (paper). This booklet is a rewritten and expanded version of the pamphlet with the same title published by CUPM in 1966. Though addressed to teachers and students of calculus, it goes beyond the original purpose of showing the importance of calculus concepts in modern computing. The author touches the general relationship between computing and mathematics, the possibilities and limitations of computers, the symbol manipulating powers of computers, and various common misconceptions and unanswered questions. These matters deserve more extended discussions than they get here. P, L.

Education

Computer-Assisted Instruction: A Survey of the Literature. 3rd ed. Edited by Albert E. Hickey. Entelek Inc., Newburyport, Mass., 1968. viii+150 pp. $8.00. P, L.

Algebra for Elementary Teachers. By Philip L. Hosford (New Mexico State Univ.). Harcourt, Brace & World, New York, 1968. xiv+246 pp. $8.75. After three pages on "The Philosophy of Modern Mathematics," there are six chapters on real numbers, linear equations, quadratic equations, systems of equations and inequalities, complex numbers, and other topics. Solutions to exercises are scattered in the book, and the student is directed to the solution and then back to his original place. This is called "programming" but it appears to differ from the usual problems set and answers only in that the answers are scattered instead of being collected at the back of the book. TT.

Pi Mu Epsilon Journal. Subscription $2.00 for two years, 1000 Asp Avenue, Room 215, Univ. of Oklahoma, Norman, Okla. 73069. Volume 4, No. 9, published in the fall of 1968, contains several short papers, problems, book reviews, and lists of initiates. The journal is a good one and should be in any undergraduate mathematics library, but it is too bad that the various journals, primarily for undergraduates, do not consolidate to create a really substantial and frequently appearing journal for and by undergraduates.

★ *Contests in Higher Mathematics.* Hungary 1949–1961. In memoriam Miklos Schweitzer. Edited by G. Szasz, L. Geher, I. Kovacs and L. Pinter. Akademiai Kiado, Budapest, 1968. 260 pp. $10.00. The contests are those for university students and named for Schweitzer, who died in 1945 at the age of 22 in the battle for the liberation of Budapest from the Nazis. After a description of the contest, statement of the problems, and a list of winners, the bulk of the volume gives solutions classified under 9 topics. The book ends with a brief biographical note on Schweitzer. P, L.

Soviet Secondary Schools for the Mathematically Talented. By Bruce Ramon Vogeli. National Council of Teachers of Mathematics, Washington, D.C., 1968. ix+100 pp. $2.00 (paper). Food for thought by university mathematicians, since such a program depends on their initiative. P.

General

Mathematics The Art of Reason. By William P. Berlinghoff (College of Saint Rose). Heath, Boston, Mass., 1968. x+260 pp. $7.95. An introduction intended for the general education of college freshmen, this book covers a number of topics from the point of view of the "new math", including some attention to algebraic structures, the number system, form, analytic geometry, probability, infinity and a twenty-four page survey of the history of mathematics. T (13).

Excavation and other verse. By Katharine O'Brien (University of Maine). Anthoensen Press, Portland, Maine, 1967. 67 pp. $3.95. Among the poems are fourteen on mathematics. I liked best the poem entitled "Moment of truth." Its first stanza begins "If you start at the bottom, the going is rough—five measly axioms barely enough." Its second stanza begins "Up at the top the going is rougher—the paradoxes will make you suffer." The conclusion: "It's safer to fiddle around in the middle". S, P, L.

★*Mathematical Snapshots.* Third American Edition, Revised and Enlarged. By H. Steinhaus. Oxford Univ. Press, New York, 1969. 311 pp. $7.50. This great classic whose first edition appeared in 1950, presents interesting mathematical ideas with a minimum of words and with many fine pictures. S, P, L.

History

Courbes Géométriques Remarquables. By H. Brocard and T. Lemoyne. Two volumes. Blanchard, Paris, 1967. Vol. I. viii+451 pp. 40.F. Vol. II. viii+198 pp. 30. F. Contains a great deal of historical and bibliographic information on special curves. The original of vol. 1 was published in 1919; vol. 2 was published in 1967 for the first time. L.

Nicole Oresme and the Medieval Geometry of Qualities and Motions. A treatise on the uniformity and difformity of intensities known as *Tractatus de configurationibus qualitatum et motuum*. Edited with an introduction, English translation, and commentary by Marshall Clagett. Univ. of Wisconsin Press, Madison, 1968. xiii+713 pp. $15.00. The introduction covers 157 pages, and there are also a bibliography and indexes of Latin terms and of manuscripts. A very important source for an extraordinary mathematician. P, L.

Hypocycloides et Épicycloides. By J. Lemaire, with a preface by Maurice D'Ocagne. Blanchard, Paris, 1967. vii+287 pp. 28 F. Reprint of a book first published in 1929.

The Annus Mirabilis of Sir Isaac Newton. Tricentennial Celebration. Special issue of the *Texas Quarterly*, Autumn 1967. Vol. 10, No. 3. 287 pp. $1.50. Sixteen papers on the wonderful year of 1666 during which Newton developed integral calculus, verified the composite nature of sunlight, and satisfied himself that the earth's gravitation holds the moon in its orbit. P, L.

Einsteins Vision. Wie steht es Heute mit Einsteins Vision, alles als Geometrie aufzufassen? By John Archibald Wheeler. Springer-Verlag, New York, 1968. vii+108 pp. $4.95. A fascinating looking little book with informative bibliographic notes and a portrait. It ought to be available in English also. P, L.

★*World Who's Who in Science. A Biographical Dictionary of Notable Scientists from Antiquity to the Present*. First edition. Edited by Allen G. Debus. Marquis-Who's Who, Chicago, 1968. xvi+1855 pp. $60.00. In spite of a heavy bias for living scientists and some caprice in their selection, this is a very useful reference work for mathematicians. Bourbaki is included! L.

NOTABLE PAPERS

The Development of Wittgenstein's Philosophy, by D. F. Pears in the *New York Review of Books*, January 16, 1969. This is an extensive review of two books by Wittgenstein and five books about him.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D. C. 20036. Items must be submited at least two months before publication can take place.

PERSONAL ITEMS

East Texas State University: Dr. Archie Brock, Virginia Polytechnic Institute, has been appointed Associate Professor; Dr. John Lamb, University of Texas at Austin, has been appointed Assistant Professor; Dr. Howard Lambert, Texas Technological College, has been appointed Associate Professor.

Sacramento State College: Messrs. Cecil Crawford, Wallace Etterbeek, Roger Leezer and Albert Wehrly, University of California at Davis, have been appointed Assistant Professors; Assistant Professors Nancy J. Poxon and C. H. Tjoelker have been promoted to Associate Professors.

Southern Illinois University: Dr. H. J. Biesterfeldt, Jr., Drexel Institute of Technology, has been appointed Associate Professor; Dr. R. B. Kirk, California Institute of Technology, has been appointed Assistant Professor.

St. Cloud State College: Assistant Professor Allen Brink has been promoted to Associate Professor; Mr. David Lahren has been promoted to Assistant Professor; Dr. Vinnie Miller, Montana State University, has been appointed Associate Professor; Mr. Everett Van Akin has been appointed Assistant Professor.

University of Texas at El Paso: Associate Professor W. J. Leahey, University of Hawaii, has been appointed Professor and Head of the Mathematics Department; Assistant Professor F. B. Strauss, University of Hawaii, has been appointed Associate Professor.

Valdosta State College: Dr. P. S. Chiang, Western Michigan University, and Dr. R. C. Moore, Florida State University, have been appointed Associate Professors; Assistant Professor R. C. Hicks has been promoted to Associate Professor.

Western Washington State College: Dr. Francis Hildebrand, Stanford University, has been appointed Assistant Professor; Associate Professors J. R. Reay and P. T. Rygg have been promoted to Professors.

Westfield State College: Assistant Professors John Bolduc, Helen Peters, J. B. Sbrega, have been promoted to Associate Professors; Associate Professor A. J. Jackowski has been appointed Chairman of the Department of Mathematics.

Youngstown State University: Dr. Howard Banilower, Case Western Reserve University, has been appointed Associate Professor; Assistant Professor R. W. Hurd has been promoted to Associate Professor; Mrs. Nell G. Whipkey has been promoted to Assistant Professor.

Dr. L. R. Amunrud, Montana State University, has been appointed Associate Professor at Eastern Montana College.

Assistant Professor A. F. Baylock, St. Francis College, has been promoted to Associate Professor and appointed Chairman of the Mathematics Department.

Assistant Professor D. G. Beane, College of Wooster, has been promoted to Associate Professor.

Assistant Professor L. W. Beineke, Purdue University of Fort Wayne, has been promoted to Associate Professor.

Dr. V. C. Cateforis, University of Wisconsin, has been appointed Assistant Professor at the University of Kentucky.

Mr. G. C. Fenneman, Wartburg College, has been promoted to Assistant Professor.

Assistant Professor I. I. Glick, George Washington University, has been promoted to Associate Professor.

Dr. Irving Hollingshead, University College, Nairobi, has been appointed Associate Professor at Kutztown State College.

Associate Professor James Householder, Humboldt State College, has been promoted to Professor.

Professor M. A. Hyman of the Mathematics Research Center, University of Wisconsin, has been appointed a Consultant to the Office of Emergency Planning, Executive Office of the President, Washington, D.C. Professor Hyman is on leave of absence from the IBM Corporation.

Associate Professor Hyman Kamel, PMC Colleges, has been promoted to Professor.

Assistant Professor L. R. King, Davidson College, has been promoted to Associate Professor.

Associate Professor D. M. Merriell, University of California at Santa Barbara, has been appointed Professor at Vassar College.

Dr. A. R. Brodsky, UCLA, died on July 10, 1968. He was a member of the Association for eight years.

Professor Emeritus H. H. Downing, University of Kentucky, died on October 20, 1968. He was a member of the Association for fifty-one years.

Mr. W. R. Greaney, West Farms, Bronx, died on November 20, 1968. He was a member of the Association for eighteen years.

INTERNATIONAL COMMISSION ON MATHEMATICAL INSTRUCTION (I.C.M.I.) FIRST INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION

The First International Congress on Mathematical Education will be held in Lyons, France, August 24–30, 1969. The scientific work will take place on three levels: 1. about 15 invited lectures; 2. free communications; 3. panel discussions of the invited lecturers and other invited persons. Sections will be formed for the free communications. The following is a tentative division into sections: (a) Secondary education in mathematics: Geometry; Logic; Algebra; Calculus; Probability and statistics; Numerical methods. (b) Primary education in mathematics. (c) Mathematics for pupils who do not aim at higher studies. (d) Teacher training. (e) Research on methods and contents of mathematics teaching. For further particulars, please write to the secretary of the Congress, M. Glaymann, 43, Boulevard du 11 Novembre 1918, 69 Villeurbanne, France.

FOREIGN SCHOLARS AVAILABLE FOR APPOINTMENT IN U.S. UNIVERSITIES AND COLLEGES UNDER PROVISION OF THE FULBRIGHT-HAYS ACT 1969–1970

The Committee on International Exchange of Persons, Conference Board of Associated Research Councils, has issued a list of foreign scholars available under the provisions of the Fulbright-Hays Act for appointments in American colleges and universities during the academic year 1969–1970. This list, compiled annually, includes information about scholars nominated by the binational Educational Commissions and Foundations abroad for Fulbright-Hays travel grants covering costs of round-trip transportation from the home country to the United States, provided arrangements can be completed for a lecturing or a research appointment with appropriate stipend at an American institution of higher learning. Information regarding the procedures for extending invitations and the conditions of appointment is also provided.

A copy of the list and additional information about individual scholars may be obtained from: Miss Grace E. L. Haskins, Program Officer, Committee on International Exchange of Persons, 2101 Constitution Avenue, N. W., Washington, D. C. 20418.

PRELIMINARY ANNOUNCEMENT

UNITED STATES NAVAL ACADEMY

ANNAPOLIS, MARYLAND

Dedication of CHAUVENET HALL

Friday and Saturday, October 17 and 18, 1969

Symposium

"The State of Mathematics Today"

Six lectures by former Mathematical Association of America CHAUVENET PRIZE awardees, including Paul R. Halmos, Mark Kac, Saunders MacLane, Guido L. Weiss, and Gordon T. Whyburn.

Detailed information on reservations will be published in a later issue.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

PROPOSED REVISION OF THE BY-LAWS OF THE MAA

At the meeting of the Board of Governors held on January 24, 1969, in New Orleans, Louisiana, the Secretary was instructed to submit to a vote of the membership a revision of the By-Laws of the Association in the form indicated below. In accordance with these instructions, a motion will be made at the business meeting of the Association to be held at the University of Oregon on Tuesday, August 26, 1969, to approve these revised By-Laws, which—except for some minor changes made by the Board—were prepared by a Committee on the Revision of the By-Laws of the Association under the chairmanship of Professor W. L. Duren, Jr.

For the guidance of the membership, all changes of substantive nature proposed in the new By-Laws are listed below:

a. The office of Associate Secretary is abolished and his duties and responsibilities assigned to the Executive Director.

b. The Editor of the *Mathematics Magazine* becomes a member of the Board of Governors.

c. The quorum for meetings of the Board, presently being defined as consisting of not less than five members, is redefined as consisting of not less than 25% of the membership of the Board.

d. The provision for "informal action based on a ballot" by the Board of Governors is eliminated.

e. Governors having served less than a year and a half are eligible for reelection for a term of three years (at present, they are eligible only after an interim of three years).

f. Provision is made so that the Association may have more than one official journal.

g. The initiation fee is raised to $4 (it has remained unchanged at $2 since the founding of the Association.)

In addition, certain obsolete provisions and statements which have never been used or have no chance of ever being applied are omitted.

HENRY L. ALDER, *Secretary*

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(To be submitted for a vote of the membership on August 26, 1969)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of the mathematical sciences in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by cooperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

Article II—Membership

1. There shall be two classes of members, ordinary and institutional.

2. Any person interested in the field of collegiate mathematics shall be eligible for election to ordinary membership in the Association.

3. Any institution, academic or corporate, interested in the support of collegiate mathematics shall be eligible for election to institutional membership in the Association.

4. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission. In the case of individuals, the application shall be endorsed by two ordinary members of the Association.

Article III—Board of Governors and Officers

1. The Officers of the Association shall be a President, a President-Elect (only during a year immediately prior to the expiration of a President's term), a Past-President (only during a year immediately following the expiration of a President's term), a First Vice-President, a Second Vice-President, an Editor of its publication entitled "*The American Mathematical Monthly*", a Secretary, and a Treasurer.

2. There shall be a Board of Governors (herein called "the Board") to consist of the officers, the ex-presidents for terms of six years after the expiration of their respective presidential terms, the Editor of its publication entitled *Mathematics Magazine*, the members of the Finance Committee, and additional elected members (herein called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association.

3. There shall be an Executive Committee of the Board consisting of the President, the President-Elect (only during a year immediately preceding the expiration of a President's term), the Past-President (only during a year immediately following the expiration of a President's term), the two Vice-Presidents, the Editor of the *American Mathematical Monthly*, the Secretary, and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or interest to the Association. This Committee shall prepare the agenda for meetings of the Board and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent or as may in any way facilitate the Board's work.

4. At all meetings of the Board of Governors a quorum shall consist of not less than 25 per cent of the membership of the Board, and no business may be validly transacted at a meeting at which less than a quorum is present.

5. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This Committee shall consist of five members including the President, the Secretary, and the Treasurer.

6. The Board shall hold a meeting each year immediately preceding the annual business meeting of the Association. Other meetings of the Board may be held from time to time at the call of the President or of any six (6) members of the Board.

7. Notice of all meetings of the Board shall be given by the Secretary to each member of the Board at least fifteen (15) days prior to the date set therefor.

8. A member of the Board may waive notice with the same effect as if due notice had been given him.

ARTICLE IV—ELECTIONS, APPOINTMENTS, AND TERMS OF OFFICERS AND MEMBERS OF THE BOARD

1. (a) The membership at large shall elect biennially a President-Elect for a term of one year and a First Vice-President for a term of two years and shall elect annually two Governors for terms of three years. The President-Elect shall become President for a two-year term at the expiration of his one-year term as President-Elect and shall become Past-President for a one-year term at the expiration of his term as President.

(b) The membership in each Section shall elect triennially a Governor for a term of three years beginning July 1. For these elections at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section. A Governor who has moved his place of employment from the Section by which he was elected shall be considered to have ended his term of office on the Board.

(c) The Board shall elect at appropriate times by ballot and for terms stated: a Second Vice-President for two years, an Editor of the *American Mathematical Monthly*, an Editor of *Mathematics Magazine*, a Secretary, and a Treasurer, each for five years, and members of the Finance Committee (other than the President, the Secretary, and the Treasurer) for four years.

(d) The beginning and end of the term of every officer and member of the Board (except as provided in Section (b) of this Article) shall occur at the adjournment of the annual business meeting. All officers and members of the Board shall hold over until their respective successors have been duly elected or appointed and qualified.

(e) The President shall be ineligible for reelection as President-Elect or as President. The Vice-Presidents, the Editors, and the Governors shall be eligible for reelection only after an interim equal to their respective terms of office except that Governors having served less than a year and a half shall be eligible for reelection for a term of three years.

(f) The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of business of the Association.

(g) Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and members of the Finance Committee. The Board may make additional nominations.

2. For general elections by the membership of the Association there shall be a Nominating Committee appointed annually by the President with the approval of the Board. The general election shall be conducted in two stages, a primary mail voting concluding approximately five months before the date of the annual meeting and a final voting concluding at the time of the annual meeting. For the primary voting the Nominating Committee shall prepare printed ballots with five or more nominees for President-Elect and three or more for each other office to be filled by the members. Blank spaces on the ballot shall be provided for write-in votes. From the results of the primary voting the Nominating Committee shall prepare a printed ballot for the final voting. This ballot shall be mailed to the membership approximately one month before the annual meeting and the voting shall close on the day of the annual business meeting. The final ballot shall present one nominee for President-Elect, to be selected by the Nominating Committee out of the three persons who received the most votes in the primary voting. For each other office the final ballot shall present two names, one being that of the person who received the highest vote in the primary voting.

3. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual business meeting of the Association. He shall be Chairman of the Executive Committee and of the Finance Committee. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

4. In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

5. The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of

the Board of Governors and of the annual business meeting and special meetings, and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the terms of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

6. The Treasurer shall have the usual duties pertaining to his office including the collection of dues and the supervision and safekeeping of the funds of the Association.

7. (a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, except when they meet in executive session, but he shall not be *ex officio* a member of these bodies. He shall be especially responsible for implementing and coordinating Section activities.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee.

Article V—Business Meetings of the Association

1. A business meeting of the Association shall be held annually, at such time and place as the Board may direct. Other business meetings of the Association may be called from time to time by the Board or by the President of the Association to be held at such time and place as may appear from the call.

2. Notice of any business meeting of the Association shall be given by the Secretary to each member of the Association at least thirty (30) days prior to the date set for each meeting.

3. Any member of the Association may waive notice with the same effect as if due notice had been given him.

4. At all business meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present.

Article VI—Sections

1. In the interest of more effective promotion of the objectives of the Association on a local level, the United States, Canada and their possessions shall be subdivided by the Board of Governors into non-overlapping geographical areas, and a Section of the Association shall be established in each of these areas. The subdivision into non-overlapping areas may be changed by the Board, upon recommendation by the Committee on Sections (see paragraph 7).

2. Each member of the Association residing in the United States, Canada or their possessions shall belong to one and only one Section. He will belong to the Section in whose geographic area he resides, except that a member who resides in one area and is employed in a different area may elect the Section to which he prefers to belong. Any member may petition the Committee on Sections for reassignment of his membership to another Section.

3. Each Section shall adopt a set of By-Laws which, along with any subsequent changes, must be approved by the Board. The geographic area covered by a Section shall be described in the By-Laws for the Section.

4. If there are members of the Association residing in a geographic area in which no Section has been organized, any ten or more members of this Association residing or employed in this area may petition the Board for authority to organize a Section covering that area.

5. A group of not less than twenty-five members of an existing Section may petition the Board to partition the area and the Section into two or more Sections. The Board shall have authority to approve or deny this petition. The Board may specify conditions under which such action may be accomplished. It may conduct a poll of some or all members of the Association in the Section to determine the advisability of allowing the proposed partition of the Section. If separate Sections

are approved then each new Section must prepare its own set of By-Laws to be approved by the Board.

6. A group of not less than twenty-five members residing or employed in that part of the area of an existing Section which they desire to become part of another existing Section may petition the Board to redefine the geographic boundaries of the Sections affected. The Board shall have authority to approve or deny this petition. It may conduct a poll of some or all members of the Sections involved to determine the advisability of permitting such action.

7. There shall be a standing Committee on Sections through which the Board shall maintain general supervision over the activities of all Sections. This Committee, in particular, shall study all matters involving creation of Sections or modification of boundaries of Sections and make appropriate recommendations to the Board.

8. The Association shall not be obligated to pay from its treasury any of the expenses of a Section except as the Board may provide.

Article VII—Official Publications

1. The Association shall publish at least one official journal, of which one shall be sent free to all members of the Association in accordance with Article VIII.

2. The Board shall have full control of the publication and sale of each official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors for each official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other publications.

5. The Board shall fix the price of each official journal and of any other publications of the Association, but in no case shall an official journal be sold to nonmembers for less than the annual dues of ordinary members.

Article VIII—Dues

1. Ordinary members of the Association shall pay an initiation fee of four dollars ($4) at the time of election. The Board of Governors may authorize the admission to ordinary membership of individuals and classes of applicants without payment of the initiation fee.

2. The Board shall establish the annual dues and privileges of membership for ordinary and institutional members. The dues of ordinary members shall include a subscription to one of the official journals.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, that member shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year may have their dues prorated for the balance of the year, except when they desire to receive the full current volume of an official journal.

5. Any ordinary member who because of age is no longer in active service, who is in good standing at the time of his retirement, and who has been a member of the Association for twenty years, may, upon notifying the central office of said retirement, be exempt from the payment of dues, with the privilege of obtaining an official journal at an annual cost of half of the dues of an ordinary member.

Article IX—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual business meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (2/3) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in each official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association or amendments to these By-Laws shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

REPORT OF THE TREASURER FOR THE YEAR 1968

Following is a summary of the report of the Treasurer of the Association for the year 1968. The report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes to have a copy of the full report may obtain one by writing to the Washington office.

Unusual expenses were incurred during 1968 as a result of the move to Washington and the change in administrative staff. In addition, the processing of dues and other accounts payable to the Association was somewhat interrupted during the period of the move and the setting up of the new office. The result was a deficit in the Current Fund for the year.

Nevertheless, the total assets of the Association have increased during the year, due largely to the payment of $29,369 by the NSF for indirect costs for 1966 and 1967, the receipt of a bequest of $15,000 from the estate of the late Lester R. Ford and a substantial appreciation in the value of our securities during 1968.

HOLDINGS OF THE ASSOCIATION	*January 1, 1968*	*December 31, 1968*
American Security and Trust Co., checking acct.	—	41,793
M and T Trust Company	30,157	71
American Security and Trust Co., special acct. (NSF)	—	84,552
M and T Trust Company, special acct. (NSF)	12,831	100
American Security and Trust Co., third acct.	—	150
M and T Trust Company, third acct.	367	—
Securities at Market Values	195,862	207,287
Deposit accounts	1,925	1,925
	241,142	335,878

FUNDS OF THE ASSOCIATION		
Current Fund	9,520	−46,455
MATHEMATICS MAGAZINE	2,236	11,712
Awards Fund	1,753	1,753
General Fund	81,656	115,499
Carus Fund	77,887	86,338
Chace Fund	164	2,426
Houck Fund	9,822	10,784
Dunkel Fund	20,849	19,406
Lester R. Ford Fund	10,327	25,747
Greenwood Fund	1,072	2,362
CEM Subcommittee on TV	1,772	1,772
High School Contests	2,483	10,525
Committee on Institutes	1,477	2,690
Committee on Advisement and Personnel	5,000	4,592
NSF Fund	14,757	86,577
Contributions Fund	367	150
	241,142	335,878

EDWARD A. CAMERON, *Treasurer*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The fourteenth annual meeting of the Northeastern Section of the MAA was held at the University of Bridgeport, Connecticut, on November 30, 1968. The registered attendance was seventy-one, including sixty members of the Association. Chairman Guilford Spencer presided at both the morning and afternoon sessions.

At the business meeting Grace E. Bates, Chairman of the Nominating Committee, proposed the following slate of officers for the coming year: Chairman, W. H. Crawford, Mount Allison University; Vice-Chairman, M. C. Gemignani, Smith College; Secretary-Treasurer, G. W. Best, Phillips Academy. The slate was elected unanimously.

The following program was presented:

1. *A curricular equivalence relation between statistics and calculus*, by R. M. Kozelka, Williams College.

2. *A topological miscellany*, by W. W. Comfort, Wesleyan University.

3. *An elementary extension of the Lebesgue dominated convergence theorem*, by Norton Starr, Amherst College.

4. *Empirical logic*, by D. J. Foulis, University of Massachusetts.

G. W. BEST, *Secretary-Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The forty-third annual meeting of the Philadelphia Section of the MAA was held at the Drexel Institute of Technology, Philadelphia, on November 23, 1968. The Section Chairman, Professor Samuel McNeary of the Drexel Institute of Technology, presided at the meeting. The meeting was attended by 165 persons including 125 members of the Association.

At the business meeting, amendments to the By-Laws were passed which provide for a new position of Vice-Chairman and an increase from three to six in the number of Executive Committee members-at-large. The following officers were elected: Chairman, Professor Willard Baxter, University of Delaware; Vice-Chairman, Professor Hugh Albright, LaSalle College; Members of the Executive Committee, Professor Alexander Beck, Olney High School (Philadelphia); Professor Frederick Cunningham, Jr., Bryn Mawr College; Professor William Jones, Lafayette College; Professor Joseph Mamelak, Community College of Philadelphia.

The two top performers from the Section on the 1967 Putnam Competition were recognized and awarded an annual membership in the MAA. They are J. C. Mather and R. S. Fowler, both of Swarthmore College. Honorable mention citations were also presented to Alan Feldman and Mary Kramer, Swarthmore College, David Brubaker, Lebanon Valley College, and Eugene Hamilton, University of Delaware.

A resolution was passed expressing the Section's deep sense of loss due to the death on March 3, 1968, of Professor Emil Amelotti, Villanova University, who was chairman of the Section from November, 1966, until his death.

The following papers were presented:

1. *Characters of finite groups*, by C. W. Curtis, University of Oregon.

2. *A comparison of two uniqueness theorems for the ordinary differential equation* $y'=f(x, y)$, by J. P. Diaz, Rensselaer Polytechnic Institute.

3. *Two year colleges—CUPM panel*, by Malcolm Pownall, Colgate University and James Mettler, Pennsylvania State University (Schuylkill Haven).

4. *SMSG—A second round of curriculum development*, by Donald Richmond, Williams College.

A. E. FILANO, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

Fifty-third Annual Meeting, Miami, Florida, January 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
FLORIDA
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MISSOURI
NEBRASKA
NEW JERSEY
NORTH CENTRAL
NORTHEASTERN, Williams College, Williamstown, June 21, 1969.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA-ARKANSAS
PACIFIC NORTHWEST, University of Oregon, Eugene, August 25–27, 1969.
PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.
AMERICAN MATHEMATICAL SOCIETY, University of Oregon, Eugene, Oregon, August 26–29, 1969.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Pennsylvania State University, June 23–26, 1969.
ASSOCIATION FOR COMPUTING MACHINERY
ASSOCIATION FOR SYMBOLIC LOGIC
CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Milwaukee, Wisconsin, November 27–29, 1969.
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS, New York City, August 19–22, 1969.
MU ALPHA THETA, University of Oregon, Eugene, Oregon, August 27, 1969.
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Brown Palace Hotel, Denver, Colorado, June 17–20, 1969.
PI MU EPSILON, University of Oregon, Eugene, Oregon, August 26–27, 1969.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham Hotel, Washington, D. C., June 10–12, 1969.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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VOLUME 76 NUMBER 6

CONTENTS

(*continued on inside cover*)

JUNE-JULY 1969

NOTICE TO AUTHORS

Articles should be typewritten on good quality paper, triple spaced with wide margins. Submit the original (and a duplicate if convenient) keeping a complete copy for your protection against loss. Please follow the format used in current issues of the MONTHLY.

The manuscript must make the author's intent clear to the printer. A paper which is accepted for publication by the editor may not be acceptable to the printer and will be returned to the author for retyping. Please use the American Mathematical Society *Manual for Authors of Mathematical Papers* as your guide and please conform to correct English usage.

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NOTICE

Since May 1968, about 470 manuscripts have been submitted for main articles. Most of these, perhaps 8 or 9 out of 10, are technical specialized research, unsuitable for the MONTHLY. Of course, not all of the remainder clear the obstacle course of referees and editors.

The backlog of material left over from the previous editorship will be exhausted in a few months. Not very much really good material is submitted voluntarily, and the editor would welcome more. If you are planning an expository or survey article you think suitable for the MONTHLY, please send me an outline as soon in advance as possible.

HARLEY FLANDERS, *Editor*

HOW TO SOLVE LINEAR INEQUALITIES

DAVID GALE, University of California, Berkeley

1. Introduction. Suppose you were asked to solve a system of simultaneous linear inequalities of modest size, say for example, four inequalities in three unknowns. How would you proceed? Or suppose that the size of the problem was immodest so that machine computation was appropriate? How should the machine proceed?

These questions, it seems to me, are natural ones to ask, for linear inequalities come up almost as often as linear equations in all sorts of applications; yet I believe very few mathematicians can give a good answer to them. I suspect, given a little time, a competent mathematician could devise some sort of finite algorithm which for any system of inequalities would either produce a solution or else show that none existed. (This is what we shall mean by the word "solve," as opposed to finding all solutions. We will mention this latter problem in the final section.) It would be surprising, though, if he could on the spur of the moment come up with a procedure that would do the job using only a "reasonable" amount of computation. By a reasonable amount of computation I mean an amount of the same order of magnitude as that involved in solving systems of equations. In fact, this raises a mathematical question. Do there exist such reasonable procedures, or is the inequality problem intrinsically of a higher order of computational complexity then the equation problem? I would like to expound briefly on the present curious state of affairs regarding this question.

David Gale is known for his research in Mathematical Economics, Game Theory, and the Geometry of Convex Sets. After taking his degree at Princeton under A. W. Tucker, he taught at Princeton, Brown, and (his present appointment) Berkeley. He spent a Fulbright year in Copenhagen, a Guggenheim year in Osaka, was a Berkeley Visiting Miller Professor another year, and currently is back in Copenhagen as an NSF Fellow. He is a Fellow of the Econometric Society. *Editor.*

The usual method for solving linear equations is ordinary "elimination"; solve Equation 1 for x_1 and then substitute this expression into Equations 2 through m, etc. In this method the basic step is this elimination, and after each of the n variables have been eliminated, thus after n such steps, the solution emerges (this description is not intended to be precise or rigorous). The algorithm we are about to describe makes use of these same eliminations or, as we shall call them, *replacement* steps. The number of such steps will in general be greater than n, but not much greater, perhaps as much as $2n$. Klee [5] has constructed examples which indicate that one may run into situations which require as many as (roughly) mn steps and has conjectured that this is the maximum possible. On the other hand, the best upper bound on the number of steps which has been rigorously established is not even a polynomial in m and n but of the order of $\binom{n}{m}$. (See argument at the beginning of Section 3.) Thus there is a large and embarrassing gap between what has been observed and what has been proved. This gap has stood as a challenge to workers in the field for twenty years now and remains, in my opinion, the principal open question in the theory of linear computation.

One further introductory word seems in order. There is one group in the mathematical community who do know how to solve inequalities; these are the people who work in linear programming. The situation here is again curious. Linear programming involves maximizing or minimizing a linear function using variables which are required to satisfy a system of linear inequalities. Thus, in order to solve a linear program one must in the process find a solution of these inequalities. It turns out, on the other hand, that the problem of solving inequalities can itself be thought of as a linear programming problem in which one is minimizing a so-called "artificial objective function." While this approach achieves the desired end, it seems to me to be a backward way of going about things. Logically one would first learn to solve the inequalities and then worry about minimizing or maximizing over the set of solutions. This is the approach taken here. The method used is the lexicographic variant of the simplex method of Dantzig, Orden, and Wolfe [1] which was used by those authors to solve linear programs, and later by Dantzig [2] to solve matrix games, but has not up to now to my knowledge, been used to give a direct method (no artificial objective function) for solving inequalities. A different direct method has been given by Debreu [3] but his procedure is more complicated to describe than the one proposed here, though it may be computationally more efficient in some cases.

2. Solving matrix equations. We begin by reviewing the "standard" method for solving linear equations, slightly generalized and using slightly different terminology from the usual one.

PROBLEM I. *Given an $m\times n$ matrix A and an $m\times r$ matrix B, find an $n\times r$ matrix Y such that $AY=B$.*

It will be convenient to rephrase the problem. Instead of thinking of A and B as matrices we will think of them as sets of m-vectors. Thus

$$A = \{a_1, \cdots, a_n\}$$
$$B = \{b_1, \cdots, b_r\}$$

PROBLEM. *Express each vector* b_k *as a linear combination of the vectors* a_j *if possible.*

We are about to describe what we will call a *replacement algorithm* for solving (I). The following is the fundamental notion needed:

DEFINITION. *Let* $S = \{s_1, \cdots, s_m\}$ *be a basis for m-space and let* $B = \{b_1, \cdots, b_r\}$ *be any set of m-vectors. The tableau of B with respect to S is the* $m \times r$ *matrix* $Y = (y_{ij})$ *such that*

$$b_j = \sum_{i=1}^{m} y_{ij} s_i, \qquad j = 1, \cdots, r.$$

In matrix notation, if we think of S and B as matrices with columns s_i and b_j then Y is simply the solution of the equation $SY = B$, or $Y = S^{-1}B$.

We write tableaus in the following manner:

$$\begin{array}{c|cccc|}
 & b_1, & b_2, & \cdots, & b_n \\ \hline
s_1 & y_{11} & y_{12} & \cdots & y_{1n} \\
s_2 & y_{21} & y_{22} & \cdots & y_{2n} \\
\vdots & & & & \\
s_m & y_{m1} & y_{m2} & \cdots & y_{mn} \\ \hline
\end{array}$$

FIG. 1

REPLACEMENT ALGORITHM. We are going to describe a procedure for constructing a finite sequence of bases. The *initial basis* S_0 consists of the unit vectors $\{e_1, e_2, \cdots, e_m\}$, and each basis S_k in the sequence consists of certain unit vectors and certain vectors a_j of A. Reordering for convenience, we may suppose $S_k = \{a_1, \cdots, a_k, e_{k+1}, \cdots, e_m\}$. We write out the tableau of $A \cup B$ with respect to S_k as shown in Figure 2, where we denote the tableau of A and B with respect to S_k by X_k and Y_k respectively. There are two cases.

CASE I. The last $m-k$ rows of X_k are zero. Then:

(A) If the last $m-k$ rows of Y_k are also zero then Y_k is the desired solution of (I) since it expresses all the b_j linearly in $a_1, \cdots, a_k$.

(B) $y_{ij} \neq 0$ for some $i > k$. Then the problem has no solution; in fact, b_j is not a linear combination of the a_j. To see this note that the condition on X_k shows that the set $A_k = \{a_1, \cdots, a_k\}$ is a basis for A, but b_j is not a linear combination of A_k since the term $y_{ij}e_i$ occurs in the expression for b_j in terms of S_k.

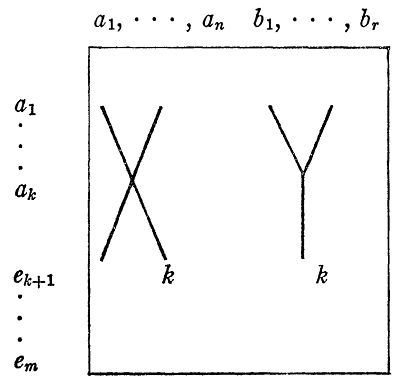

FIG. 2

CASE II. $x_{ij} \neq 0$ for some $i > k$, say $i = k+1$. Then let S_{k+1} be the basis obtained from S_k by *replacing* e_{k+1} by a_j. Thus $S_{k+1} = \{a_1, \cdots, a_k, a_j, e_{k+2}, \cdots, e_m\}$.

The proof that this algorithm solves Problem I is almost immediate. If Case I ever occurs then (A) the solution is either present or (B) it is seen not to exist. If Case I never occurs then after m replacements we will have constructed a basis S_m of vectors a_j from A, and the tableau of B with respect to this basis is the desired solution.

Note that our method always produces a *basic solution*, i.e., a solution Y such that $y_{ij} \neq 0$ only for the basis $s_1, \cdots, s_n$. This proves the following fact which may not be immediately obvious:

THEOREM 1. *If* (I) *has a solution then it has a solution Y in which at least $n-m$ rows of Y are zero.*

We now ask how much computation the replacement algorithm involves. Clearly the only arithmetical step consists in going from a tableau with respect to a basis S to one with respect to S' obtained from S by replacing a single vector. In our present notation, let Y and Y' be the tableaus of B with respect to S and S' and let the ith row of Y and Y' be denoted by y_i and y_i'.

THEOREM 2. *Let* $S = \{s_1, \cdots, s_m\}$ *and suppose* $y_{11} \neq 0$. *Then* $S' = \{b_1, s_2, \cdots, s_m\}$ *is a basis and Y' is given by the rule*

$$y_1' = y_1/y_{11}, \qquad y_i' = y_i - (y_{i1}/y_{11})y_1. \tag{1}$$

Terminology. Operation (1) is known as *pivoting* and the element y_{11} is called the *pivot* element of the operation. It is easiest to remember the operation from Figures 3 and 4.

The pivot element is circled in Y. Pivoting is done by dividing the pivot row by the pivot element, and by adding a suitable multiple of the pivot row to each of the others, the suitable multiple being the one that will give zeros in the pivot column.

$$Y \quad \begin{array}{c|cccc|} & b_1, & b_2, & \cdots, & b_r \\ \hline s_1 & y_{11} & y_{12} & \cdots & y_{1r} \\ s_2 & y_{21} & y_{22} & \cdots & y_{2r} \\ \vdots & & & & \\ s_m & y_{m1} & y_{m2} & \cdots & y_{mr} \\ \hline \end{array}$$

FIG. 3

$$Y' \quad \begin{array}{c|cccc|} & b_1, & b_2, & \cdots, & b_r \\ \hline b_1 & 1 & y'_{12} & \cdots & y'_{1r} \\ s_2 & 0 & y'_{22} & \cdots & y'_{2r} \\ \vdots & & & & \\ s_m & 0 & y'_{m2} & \cdots & y'_{mr} \\ \hline \end{array}$$

FIG. 4

Proof of Theorem. Let Y' satisfy (1). Then $y'_{1j} = y_{1j}/y_{11}$ and $y'_{ij} = y_{ij} - (y_{i1}/y_{11})y_{1j}$ for $i \neq 1$. Thus for $j \neq 1$,

$$y'_{1j}b_1 + \sum_{i=2}^{m} y'_{ij}s_i = (y_{1j}/y_{11})b_1 + \sum_{i=2}^{m} y_{ij}s_i - (y_{1j}/y_{11}) \sum_{i=2}^{m} y_{i1}s_i$$

$$= y_{1j}/y_{11}\left(\sum_{i=1}^{m} y_{i1}s_i - \sum_{i=2}^{m} y_{i1}s_i \right) + \sum_{i=2}^{m} y_{ij}s_i = \sum_{i=1}^{m} y_{ij}s_i = b_j,$$

so Y' is the tableau of B with respect to S'.

From rule (1) we see that each pivot step requires mr multiplications. For the matrix equation problem the number of columns of the tableau is $n+r$ (see Figure 2) and the problem is solved in at most m pivots, so the number of multiplications is at most $m^2(n+r)$. Actually one does somewhat better than this because of the fact that after each pivot one gets columns which are unit vectors, like the first column of Y' in Figure 4.

Of special interest is the case where A and B are square $m \times m$ matrices and particularly where B is the identity matrix so that b_k is the kth unit vector. In this case X, if it exists, is A^{-1} and the pivot method involves exactly m^3 multiplications. Note that this is the number of multiplications used in multiplying a pair of matrices, hence the number involved in checking a proposed solution X of $AX = I$. This suggests that m^3 multiplication is about as few as one could reasonably expect to use in solving the problem.

Finally note that we can follow the steps of the replacement algorithm even if there is no B matrix at all. The final tableau will then yield a column basis for A, and also, if one thinks about it for a moment, a proof that the row and column ranks of A are equal.

3. Solving linear inequalities.

PROBLEM II. *Given an $m \times n$ matrix A and an n-vector a, find an m-vector y such that*

$$y \geqq 0 \quad \textit{and} \quad yA \geqq a.$$

We are treating the case of nonnegative solutions of inequalities. The case in which y is unrestricted in sign can be handled in a similar way but involves some slight technical complication which we prefer to avoid in this exposition. It is convenient to rewrite the problem as follows: Find an m-vector y such that

$$ya_j \geqq \alpha_j \qquad \text{for } j = 1, \cdots, n,$$

$$ye_i \geqq 0 \qquad \text{for } i = 1, \cdots, m,$$

where $\{e_i\}$ are the unit vectors of m-space.

Now there is no difficulty in finding a finite procedure for solving II, for it is easily shown, and will emerge from the procedure to be given here, that if II has a solution then it has a *basic solution*, that is a vector y such that $ya_j = \alpha_j$ and $ye_i = 0$ for some set of m vectors a_j and e_i which form a basis for m-space. One could, therefore, consider all bases among the vectors a_j, e_i and for each such compute the solution y to the corresponding m equations and then substitute this y into II. Eventually one of these vectors would satisfy the system unless there was no solution at all. Of course, this would be an enormously lengthy procedure since it would involve solving possibly

$$\binom{n+m}{m}$$

systems of m equations in m unknowns.

We shall now describe a replacement algorithm for solving II. For this purpose we wish to transform II to a "homogeneous" problem, as follows:

Let $\hat{a}_j$ be the $(m+1)$-vector $(-\alpha_j, a_j)$ and let $\hat{e}_i = (0, e_i)$ for $i = 1, \cdots, m$, and let $e_0 = (1, 0, \cdots, 0)$ so that $e_0, \hat{e}_1, \cdots, \hat{e}_m$ are the unit vectors in $(m+1)$-space. Finally let $\hat{Y}$ be all vectors $(1, y)$ where y is any m-vector.

PROBLEM $\hat{\text{II}}$. *Find y in $\hat{Y}$ such that*

$$y\hat{a}_j \geqq 0 \qquad \textit{for all } j,$$

$$y\hat{e}_k \geqq 0 \qquad \textit{for all } k.$$

It is clear from the definitions that Problems II and $\hat{\text{II}}$ are equivalent.

Now let $S = \{e_0, s_1, \cdots, s_m\}$ be a basis for $(m+1)$-space where s_i is either a vector $\hat{a}_j$ or $\hat{e}_k$, and write the tableau with respect to this basis as shown in Figure 5.

THEOREM 3. *If x_0 and y_0 are nonnegative then $(1, y_0)$ solves $\hat{\text{II}}$ (and y_0 solves* II).

Proof. Let $\hat{y}$ be the $(m+1)$-vector which solves the system

$$ys_i = 0 \qquad i = 1, \cdots, m,$$

$$ye_0 = 1$$

(this vector exists since S is a basis). From the last equation above $\hat{y}$ is in $\hat{Y}$.

Now from the tableau we have

$$\hat{y}\hat{e}_k = y_{0k}(\hat{y}e_0) + \sum_{i=1}^{m} y_{ik}(\hat{y}s_i) = y_{0k}$$

so $\hat{y} = (1, y_0)$ and by assumption $y_0 \geqq 0$. Finally

$$\hat{y}\hat{a}_j = x_{0j}(\hat{y}e_0) + \sum x_{ij}(\hat{y}s_i) = x_{0j} \geqq 0$$

so $y = (1, y_0)$ solves $\hat{\text{I}}\text{I}$, as asserted.

	$\hat{a}_1, \cdots, \hat{a}_n$	$\hat{e}_1, \cdots, \hat{e}_m$
e_0	x_0	y_0
s_1, s_2, $\cdot\cdot\cdot$, s_m	X	Y

FIG. 5

The inequality problem has now become that of finding a basis S so that the tableau of Figure 5 has its first row nonnegative, if such a basis exists. We wish to arrive at this basis by a sequence of replacements starting with the initial basis S_0 consisting of the unit vectors. The initial tableau is given below.

	$\hat{a}_1, \cdots, \hat{a}_n$	$\hat{e}_1, \cdots, \hat{e}_m$
e_0	$-\alpha_1 \cdots -\alpha_n$	$0 \cdots 0$
$\hat{e}_1$, $\cdot\cdot\cdot$, $\hat{e}_m$	A	I

FIG. 6

Now, as in the previous section, we must describe the replacement operation. Suppose then that we have arrived at the tableau of Figure 5, but (x_0, y_0) is not positive so that, say, $x_{0j} < 0$ (or $y_{0k} < 0$). Then by bringing $\hat{a}_j$ (or $\hat{e}_k$) into the next basis S' we can be sure that in the next tableau the entry x'_{0j} (or y'_{0k}) will be zero which would seem to be a step in the right direction. The question which remains to be decided is which vector s_i in S should be replaced by $\hat{a}_j$ (or $\hat{e}_k$) and the success of the method depends on an ingenious criterion for making this decision which we now describe.

DEFINITION. *An m-vector x is called lexicographically positive, or simply l-positive if its first (reading from the left) nonzero coordinate is positive. We write*

$$x \succ 0.$$

A vector x is lexicographically greater than y, written $x \succ y$, if $x - y \succ 0$.

It is clear that for any $x \neq 0$ either $x \succ 0$ or $-x \succ 0$ so that $\succ$ defines a complete ordering of m-space with the further obvious property;

$$\text{if } x, y \succ 0 \qquad \lambda, \mu \geqq 0 \qquad \text{then } \lambda x + \mu y \succ 0.$$

Finally we call a matrix Y *l-positive* if all of its rows are l-positive. The following is the crucial notion for our algorithm:

DEFINITION. *The basis S will be called l-feasible if the matrix Y* (Figure 5) *is l-positive.*

Note that the initial basis of Figure 6 is l-feasible since in this case Y is the identity matrix. We now complete the description of the replacement algorithm. Assume in the tableau of Figure 5 that, say, x_{01} is negative (the argument would be the same for y_{01} negative). There are two cases:

CASE I. The first column of X is nonpositive. Then we have

$$\hat{a}_1 = x_{01} e_0 + \sum_{i=1}^{m} x_{i1} s_i. \tag{2}$$

In this case II has no solution for if $\hat{y}$ solves II then $\hat{y} s_i \geqq 0$ for all i, but then taking scalar product of (2) with $\hat{y}$ gives

$$\hat{y}\hat{a}_1 = x_{01} + \sum x_{i1}(\hat{y} s_i) \leqq x_{01} < 0$$

so $\hat{y}$ cannot solve II.

CASE II. $x_{i1} > 0$ for some i. Then let $I_1 = \{i \mid x_{i1} > 0\}$ and compute y_i / x_{i1} for i in I_1 and choose i_0 in I_1 such that y_i / x_{i1} is *l-minimal*. Then obtain the new basis S' by replacing s_{i_0} by $\hat{a}_1$ (i.e., pivot on $x_{i_0 1}$). The proof that this algorithm terminates depends on the next lemma.

LEMMA 2. *The new basis S' is again l-feasible and the vector y_0' of the new tableau is lexicographically greater than y_0.*

Proof. From (1)

$$y'_{i_0} = y_{i_0} / x_{i_0 1} \quad \text{and since} \quad y_{i_0} \succ 0 \quad \text{and} \quad x_{i_0 1} > 0$$

it follows that $y'_{i_0} \succ 0$. Also for $i \neq i_0$

$$y_i' = y_i - (x_{i1}/x_{i_0 1}) y_{i_0}.$$

If $x_{i1} \leqq 0$ then clearly $y_i' \succ 0$. If $x_{i1} > 0$ then by the choice rule of Case II above,

$$y_i / x_{i1} \succ y_{i_0} / x_{i_0 1}.$$

Equality cannot hold here since this would mean that y_i and y'_{i_0} are proportional which is impossible since Y is nonsingular, so again $y'_i \succ 0$ and hence Y' is l-positive.

Also from (1)

$$y'_0 = y_0 - (x_{01}/x_{i_01})y_{i_0} \quad \text{and since}$$

x_{01} is negative, x_{i_01} is positive and y_{i_0} is l-positive, we have $y'_0 \succ y_0$ as asserted.

THEOREM 4. *The replacement algorithm terminates.*

Proof. Since the vector y_0 depends only on the basis S and since y_0 gets lexicographically larger at each iteration it is clear that no basis can recur. Therefore, one eventually arrives at the situation of Case I in which some column of the tableau is nonpositive, in which case $\hat{\text{I}}$ has no solution, or else eventually (x_0, y_0) becomes nonnegative and y_0 is the desired solution.

4. An Example. Consider the system $y_1, y_2 \geqq 0$

$$\begin{aligned} 2y_1 + y_2 &\geqq 1 \\ y_1 &\geqq 1 \\ -y_2 &\geqq -1. \end{aligned}$$

The initial tableau is then

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{e}_1$	$\hat{e}_2$
e_0	-1	-1	1	0	0
e_1	2	1	0	1	0
e_2	1	0	-1	0	1

Now we will bring $\hat{a}_1$ into the next basis. According to the lexicographic rule, $\hat{a}_1$ must replace e_2. The pivot element has been circled in the tableau above. The next tableau is

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{e}_1$	$\hat{e}_2$
e_0	0	-1	0	0	1
e_1	0	1	2	1	-2
$\hat{a}_1$	1	0	-1	0	1

The only possibility now is to replace $\hat{e}_1$ by $\hat{a}_2$ giving,

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	$\hat{e}_1$	$\hat{a}_2$
e_0	0	0	2	1	-1
$\hat{a}_2$	0	1	2	1	-2
$\hat{a}_1$	1	0	-1	0	1

.

Again there is no choice. We must replace $\hat{a}_1$ by $\hat{e}_2$ and we get,

e_0	1	0	1	1	0
$\hat{a}_2$	2	1	0	1	0
$\hat{e}_2$	1	0	-1	0	1

which gives the solution $y_1=1$, $y_2=0$.

Note the way the row vector y_0 increases lexicographically with each replacement. Note too the interesting fact that the vector $\hat{e}_2$ was replaced on the first pivot step but came back in again in the end. Of course, if we had chosen to bring in $\hat{a}_2$ instead of $\hat{a}_1$ on the first replacement we would have obtained the solution in one step. However, in general there does not seem to be any good way of deciding which vector to bring in order to minimize the number of replacements required to arrive at a solution.

5. Concluding remarks. Having found an (apparently) good way to find at least one solution of a system of inequalities one might now ask for a way of finding *all* solutions, which means in essence finding all basic solutions of II. There do exist procedures for doing this but it is almost impossible to say whether these procedures are reasonable or not because of the fact that the number of basic solutions may increase very rapidly with m and n. The main interest here is theoretical. How many basic solutions can there be for an $m\times n$ system? I should like to conclude by describing very briefly the state of our knowledge (or ignorance) on this matter. For more details see Grünbaum [4].

It is conjectured that the maximum number μ of basic solutions which an $m\times n$ system can have is given by the strange looking formula

$$\mu(m, n) = 2\binom{n + \frac{m-1}{2}}{n} \qquad \text{for } m \text{ odd}$$

$$= \binom{n+\frac{m}{2}}{n} + \binom{n+\frac{m}{2}-1}{n} \qquad \text{for } m \text{ even.}$$

This conjecture has in fact been proved for "most" values of m and n, specifically for all $m \leqq 8$ and for $n \leqq 3$ and $n \geqq (m/2)^2 - 2$. To see what this means, the first unsolved cases are

$$m = 9 \qquad 4 \leqq n \leqq 9.$$

In general for each $m > 8$ there is an interval of values of n for which the conjecture has not been verified.

This strange situation together with the one described in the introduction concerning the number of replacements required to solve an $m \times n$ system are perhaps the most interesting features of what might superficially appear to be a dull and routine problem. To mix metaphors a little, they indicate how close to the surface the so-called frontiers of mathematics sometimes lie.

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SPECTRAL SEQUENCES FOR THE LAYMAN

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There are two methods for obtaining the usual facts relating the iterated homology of a double complex to the total homology. The first is to regard the double complex as a filtered complex and consider the resulting spectral sequence. This requires a mathematical maturity which many mathematicians never attain. The second method, which is accessible to anyone who knows how to take the quotient of two modules, is to chase diagrams. The purpose of this paper is to show that this alternative is not only possible, but furthermore quite simple.

As a student (and since) I had a very difficult time learning spectral sequences, and I felt that certain results in homological algebra and sheaf theory would forever remain inaccessible to me. It came as an immense relief when I found that, in these fields anyway, one does not often need an involved theory of spectral sequences, but rather only an elementary theory of double complexes. The proofs involved in the latter make excellent exercises for someone who is learning how to chase diagrams. At first I thought that this aversion to spectral sequences was a peculiarity of my own. However, over the years I have found that many students have the same problem, without realizing, unfortunately, that in order to handle double complexes one does not need a confusing involved theory. I am by no means claiming of course that there are no honest spectral sequences. But a spectral sequence which arises from a double complex has always struck me as being some sort of a fraud, and the present article is intended to show this.

The list of properties we shall deal with is at least complete enough to obtain all the properties stated in [1, Chapter 15, Section 6]. Since the arguments are always the same, the reader may regard any omissions as exercises in the technique of diagram chasing. I don't think that I can make the paper any simpler, since one man can only confuse another by trying to do his diagram chasing for him.

1. Edge morphisms. A *double complex* is a family of (left) R-modules $\{X^{ij} \mid (i,j) \in Z \times Z\}$, equipped with morphisms $d'_{ij}\colon X^{ij} \to X^{i+1,j}$, $d''_{ij}\colon X^{ij} \to X^{i,j+1}$ satisfying the rules

$$d'_{i+1,j}d'_{ij} = 0, \tag{1}$$

$$d''_{i,j+1}d''_{ij} = 0, \tag{2}$$

$$d''_{i+1,j}d'_{ij} + d'_{i,j+1}d''_{ij} = 0. \tag{3}$$

Because of (1), we can define

$$H_I^{ij} = \text{kernel } d'_{ij}/\text{image } d'_{i-1,j}.$$

Then using (3), we find that the morphisms d'' induce morphisms $\overline{d''_{ij}}\colon H_I^{ij} \to H_I^{i,j+1}$, and because of (2) we have $\overline{d''_{i,j+1}}\,\overline{d''_{ij}} = 0$. Consequently we can define

$$H_{II}H_I^{ij} = \text{kernel } \overline{d''_{ij}}/\text{image } \overline{d''_{i,j-1}}.$$

This module may be defined alternatively as follows. First define Z^{ij} to be the submodule of X^{ij} consisting of all elements x_{ij} such that

$$d'x_{ij} = 0 \quad \text{and} \quad d''x_{ij} = d'x_{i-1,j+1} \tag{4}$$

for some $x_{i-1,j+1} \in X^{i-1,j+1}$. That is, Z^{ij} is the set of all elements of X^{ij} which go into 0 on the right, and which go into something above which comes from something on the left. Also define B^{ij} to be the submodule of X^{ij} consisting of all elements x_{ij} such that

(5) $$x_{ij} = d''x_{i,j-1} + d'x_{i-1,j} \quad \text{where} \quad d'x_{i,j-1} = 0$$

for some $x_{i,j-1} \in X^{i,j-1}$ and $x_{i-1,j} \in X^{i-1,j}$. Thus B^{ij} is the set of all elements of X^{ij} which can be written as the sum of an element from underneath with an element from the left, where the element underneath goes into 0 on its right. Then using (1), (2), and (3), we see that $B^{ij} \subset Z^{ij}$, and so we can define $H^{ij} = Z^{ij}/B^{ij}$. Then it is easily seen that $H_{II}H_I^{ij}$ is isomorphic to H^{ij}.

For each integer n, we define $X^n = \oplus_{i+j=n} X^{ij}$, and we define $d_n\colon X^n \to X^{n+1}$ by the rule

$$d_n x_{ij} = d'x_{ij} + d''x_{ij}.$$

Then using (1), (2), and (3), we find $d_{n+1}d_n = 0$ and so we can define $H^n = Z^n/B^n$, where $Z^n = \text{kernel } d_n$ and $B^n = \text{image } d_{n-1}$. For some fixed pair of integers (p, q) we shall be interested in the relationship between $H^{p,q}$ and H^{p+q}. By a translation, we may assume that $(p, q) = (0, 0)$.

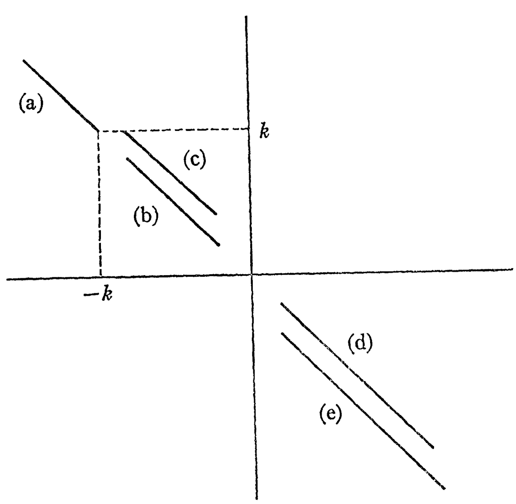
First we define an element of X^0 of the form $x = \sum_{i=0}^{k} x_{-i,i}$ to be a *second quadrant element of* X^0. If x is a second quadrant element and if dx has 0 entries in positions (1, 0) and (0, 1), then $x_{00} \in Z^{00}$. In particular this is true if $x \in Z^0$.

For some fixed integer $k > 0$, we specialize the following properties of the double complex X.

(a) $X^{-i,i} = 0$ for all $i \geqq k$.
(b) $H^{-i,i} = 0$ for $0 < i < k$.
(c) $H^{-i,i+1} = 0$ for $0 < i < k$.
(d) $H^{i,-i} = 0$ for all $i > 0$.
(e) $H^{i,-i-1} = 0$ for all $i > 0$.

The sets of integral points in the plane involved in the above conditions are indicated in the following diagram:

(6)



LEMMA 1.1. (i) *If X satisfies* (a) *and* (c), *then any element of Z^{00} can be extended to a second quadrant element of Z^0.*

(ii) *If X satisfies* (a) *and* (b), *and if z is a second quadrant element of Z^0 such that $z_{00} \in B^{00}$, then $z \in B^0$.*

(iii) *If X satisfies* (e), *and if z is a second quadrant element of B^0, then $z_{00} \in B^{00}$.*

(iv) *If X satisfies* (d), *then any element of Z^0 is congruent* mod B^0 *to a second quadrant element.*

Proof. All points are verified by looking hard at the diagram (6). As examples of the type of diagram chasing involved, we prove parts (i) and (ii). Suppose that X satisfies (a) and (c). Let $z_{00} \in Z^{00}$, so that we can write

$$d'z_{00} = 0, \qquad d''z_{00} + d'x_{-1,1} = 0. \tag{7}$$

Then using (3) and (2) we find $d'd''x_{-1,1} = 0$, and so since also $d''d''x_{-1,1} = 0$, we see that $d''x_{-1,1}$ is an element of $Z^{-1,2}$. By condition (c), the latter is $B^{-1,2}$, and so we can write

$$d''x_{-1,1} + d''y_{-1,1} + d'x_{-2,2} = 0 \quad \text{where} \quad d'y_{-1,1} = 0.$$

Inductively we can produce elements $x_{-i-1,i+1}$ and $y_{-i,i}$ for $0 < i < k$ satisfying

$$d''x_{-i,i} + d''y_{-i,i} + d'x_{-i-1,i+1} = 0, \quad \text{where} \quad d'y_{-i,i} = 0. \tag{8}$$

By condition (a) we have $x_{-k,k} = 0$. Hence adding equations (7) and (8), we find that $z_{00} + \sum_{i=1}^{k-1} (x_{-i,i} + y_{-i,i})$ is an element of Z^0. This proves part (i).

Now assume (a) and (b), and suppose that z is a second quadrant element of Z^0 where $z_{00} \in B^{00}$. Then we can write $z_{00} + d''x_{0,-1} + d'x_{-1,0} = 0$ where $d'x_{0,-1} = 0$. We can add $d(x_{-1,0} + x_{0,-1})$ to z without changing the class of z mod B^0 to obtain an element $z' = z'_{-1,1} + \sum_{i=2}^{k-1} z_{-i,i}$. Using (b) we can repeat this process until we are left with an element of $X^{-k,k}$, and by condition (a) the latter is 0. This proves that $z \in B^0$.

In verifying part (iii) it is useful to keep in mind that $dd = 0$, so that if $dx = z$ where $x \in X^{-1}$, then we may alter x by anything of the form dy without changing z. Thus, starting with the lowest term we can reduce one by one the elements of x in the fourth quadrant to 0. The proof of (iv) is similar.

The various parts of the lemma may now be put together as follows.

PROPOSITION 1.2. *If X satisfies conditions* (a), (b), *and* (c), *then there is a natural morphism $\alpha: H^{00} \to H^0$. If X satisfies* (d), *then α is an epimorphism. If X satisfies* (e), *then α is a monomorphism.*

If X satisfies conditions (d) *and* (e), *then there is a natural morphism $\beta: H^0 \to H^{00}$. If X satisfies* (a) *and* (b), *then β is a monomorphism. If X satisfies* (a) *and* (c), *then β is an epimorphism.*

Proof. The morphism α is defined by assigning to the class of $z_{00} \in Z^{00}$ the class of any second quadrant extension of z_{00} to an element of Z^0. That such an extension exists is guaranteed by (i) of the lemma. That α is well defined follows from (ii).

The morphism β is defined by representing an element of H^0 by a second quadrant element z of Z^0, and assigning to it the class of z_{00}. This is justified by (iv). That β is well defined follows from (iii).

The other assertions follow directly from the various parts of the lemma.

The morphisms α and β are called *edge morphisms*. They are usually defined under stronger conditions. The condition under which they are isomorphism is sometimes referred to as the *maximal term principle* for double complexes.

In view of 1.2, we shall say that "α_{00} is defined" if X satisfies conditions (a), (b), and (c) (all relative to the same positive integer k), and that "β_{00} is defined" if X satisfies conditions (d) and (e). The conditions "α_{ij} is defined" and "β_{ij} is defined" are obtained by translation.

2. Exact sequences. Let $x_{1,0}\in Z^{1,0}$, and let $x_{0,1}$ be as in equation (4) of Section 1. Then it is easily seen that $d''x_{0,1}\in Z^{0,2}$, and that this element is independent mod $B^{0,2}$ of the choice of $x_{0,1}$. Furthermore if $x_{1,0}\in B^{1,0}$, then $d''x_{0,1}\in B^{0,2}$. There results a morphism $\delta\colon H^{1,0}\to H^{0,2}$ (which is known to chess players as the *knight's morphism*).

PROPOSITION 2.1. *If $\alpha_{0,1}$ and $\beta_{1,0}$ are defined, then*

$$H^{0,1}\xrightarrow{\alpha}H^1\xrightarrow{\beta}H^{1,0}\ \textit{is exact.}$$

If $\beta_{1,0}$ is defined, and if for some $k>0$ we have $H^{-i,i+2}=0$ for $0<i<k$ and $X^{-i,i+1}=0$ for $i\geqq k$, then

$$H^1\xrightarrow{\beta}H^{1,0}\xrightarrow{\delta}H^{0,2}\ \textit{is exact.}$$

If $\alpha_{0,2}$ is defined, and if $H^{i,-i+1}=0$ for $i>1$, then

$$H^{1,0}\xrightarrow{\delta}H^{0,2}\xrightarrow{\alpha}H^2\ \textit{is exact.}$$

Proof. It is trivial to show that the three compositions are zero. The relations kernel $\subset$ image are verified using arguments similar to those used in the proof of 1.1. The reader should draw a diagram, and should not write anything down.

REMARK. If one combines the three sets of conditions and adds the condition $H^{1,-1}=0$ (so that since $\beta_{1,0}$ is defined, $H^{i,-i}=0$ for all $i>0$, and consequently by 1.2, $\alpha_{0,1}$ is a monomorphism), one obtains the "exact sequence for terms of low degree" of [**1**, Chap. 15, Section 6].

COROLLARY 2.2. *If $H^{ij}=0$ for all $j\neq 0$, 1, and if there is a positive k such that $X^{ij}=0$ for $j\geqq k$, then for each integer n we have an exact sequence*

$$0\to H^{n-1,1}\to H^n\to H^{n,0}\to 0.$$

Proof. Make a horizontal translation and apply 2.1.

COROLLARY 2.3. *If $H^{ij}=0$ for $i\neq 0$, 1, and if there is a negative k such that $X^{ij}=0$ for $i\leqq k$, then there is an exact sequence*

$$\cdots\to H^{0,n}\to H^n\to H^{1,n-1}\to H^{0,n+1}\to H^{n+1}\to\cdots.$$

Proof. Make a vertical translation and apply 2.1.

3. Morphisms of complexes. Let (X, d', d'') and (Y, e', e'') denote double complexes of R-modules. A *morphism* $f\colon X \to Y$ of double complexes is a family of morphisms $f_{ij}\colon X^{ij} \to Y^{ij}$ satisfying

$$e'_{ij} f_{ij} = f_{i+1,j}\, d'_{ij}, \qquad e''_{ij} f_{ij} = f_{i,j+1}\, d''_{ij}. \tag{1}$$

Using conditions (1), we obtain induced morphisms

$$H^{ij}(f)\colon H^{ij}(X) \to H^{ij}(Y) \tag{2}$$

$$H^{n}(f)\colon H^{n}(X) \to H^{n}(Y). \tag{3}$$

We are interested in finding sufficient conditions on the morphisms (2) in order that the morphisms (3) be monomorphisms (epimorphisms).

PROPOSITION 3.1. (a) *Suppose that $X^{i,-i}=0$ for $i<0$. If $H^{i,-i}(f)$ is a monomorphism and $H^{i,-i-1}(f)$ is an epimorphism for all $i\geqq 0$, then $H^0(f)$ is a monomorphism.*

(b) *Suppose that $X^{i,-i}=0=Y^{i,-i}$ for $i<0$. If $H^{i,-i}(f)$ is an epimorphism and $H^{i,-i+1}(f)$ is a monomorphism for all $i\geqq 0$, then $H^0(f)$ is an epimorphism.*

Proof. The diagram chasing involved here is slightly more complicated than that we have encountered so far owing to the fact that there are two complexes instead of one, and so we shall give a proof of part (a). The reader is again advised, however, that the proof is much easier if one does not try to write anything down.

Suppose $x=\sum_{i=0}^{n} x_{i,-i}\in Z^0(X)$, and that $fx=ey$ where $y=\sum_{i=k}^{m} y_{i,-i-1}$. We may assume $m=n$ by adding 0 terms to x or y if necessary. Now $x_{n,-n}\in Z^{n,-n}(X)$, and $fx_{n,-n}\in B^{n,-n}(Y)$. Consequently since $H^{n,-n}(f)$ is a monomorphism, we can write

$$x_{n,-n} = d''x_{n,-n-1} + d'x_{n-1,-n} \quad \text{where} \quad d'x_{n,-n-1} = 0. \tag{4}$$

But then we find that $y_{n,-n-1}-fx_{n,-n-1}\in Z^{n,-n-1}(Y)$, and so since $H^{n,-n-1}(f)$ is an epimorphism, we can write

$$y_{n,-n-1} - fx_{n,-n-1} - f\bar{x}_{n,-n-1} = e''y_{n,-n-2} + e'y_{n-1,-n-1} \tag{5}$$

where $e'y_{n,-n-2}=0$, and where

$$d''\bar{x}_{n,-n-1} + d'\bar{x}_{n-1,-n} = 0,\ d'\bar{x}_{n,-n-1} = 0. \tag{6}$$

If we alter y by subtracting $e(y_{n,-n-2}+y_{n-1,-n-1})$ from it, we may assume the right side of (5) is 0, and this does not change the fact that $fx=ey$ (remember that $ee=0$). Also if we alter $x_{n,-n-1}$ and $x_{n-1,-n}$ by adding to them $\bar{x}_{n,-n-1}$ and $\bar{x}_{n-1,-n}$ respectively, we see from (6) that this does not change (4). But then equation (5) is reduced to $y_{n,-n-1}=fx_{n,-n-1}$, and so combining this with $fx=ey$, we obtain

$$f(x - d(x_{n,-n-1} + x_{n-1,-n})) = e(y - y_{n,-n-1} - fx_{n-1,-n}). \tag{7}$$

Since the term in outer parenthesis on the left of (7) has 0 at position $(n, -n)$ and below, and the term in parenthesis on the right has 0 at position $(n, -n-1)$ and below, we may now apply induction on n and the fact that $X^{i,-i}=0$ for $i<0$ to deduce that $x \in B^0(X)$.

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THE JORDAN CURVE THEOREM FOR PIECEWISE SMOOTH CURVES

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1. Introduction. It is the purpose of this note to provide an elementary proof of the Jordan Curve Theorem for the class of piecewise smooth curves. The only tools which we require are the notions of compactness, continuity, and the concept of the index of a closed curve relative to a point. Since these topics are included in a standard advanced undergraduate or beginning graduate course in complex analysis, it is our hope that the proof will fit in well with such a course.

We begin with an informal outline of the proof as it would apply to a polygon. In order to prove the Jordan Arc Theorem for a simple polygon, it suffices to demonstrate that its complement is arcwise connected. Suppose this is true for all simple polygons having at most n segments. A simple polygon P_{n+1} having $n+1$ segments is obtained by adjoining a single segment σ to a simple polygon P_n having n-segments. Any two points in the complement of P_{n+1} can be joined by a polygonal arc C in the complement of P_n. If C does not intersect σ, then it clearly lies in the complement of P_{n+1}. If it does intersect σ, then by drawing parallel lines on either side of σ, it is easily seen that C may be replaced by a polygon which does not intersect P_{n+1}. Hence, the complement of P_{n+1} is connected. In order to obtain a valid induction proof, it suffices to note that the complement of a single segment is indeed connected.

Now let P be a simple closed polygon. The Jordan Curve Theorem for P asserts that the complement of P is comprised of two nonempty components E and I. Let Γ be the simple polygon obtained by removing from P a segment σ. Choose ζ to be a point which lies outside of a disk containing P in its interior. Denote by E the set of points in the complement of P which can be joined, in

the complement of Γ to ζ by a polygonal arc which does not cross σ. Let I consist of all other points in the complement of P. Evidently E is connected, since any two of its points can be joined in E to ζ, and hence to each other. In order to show that I is connected, we observe that any point $z\in I$ can be joined in the complement of Γ to ζ by a polygonal arc C_z which necessarily crosses σ. Since $E\cap I=\emptyset$, it follows that C_z must emerge from σ on the opposite side from which it approached σ. Otherwise, by drawing a line parallel to σ, we could show that $z\in E$. By similar reasoning, it can be shown that if $w\in I$, then C_w approaches σ from the same side as C_z. Hence z can be joined to w by an arc which does not cross P. It follows that I is connected. However, it is not immediately evident that it is nonempty. This possibility is excluded by showing that two points on opposite sides of σ have different indices with respect to P, and hence are in different components.

The objective of this paper is accomplished if we can show that the preceding argument remains valid with straight line segments replaced by simple smooth arcs. A moment of reflection shows that the only property of a straight line which was used is that it is not intersected by a line parallel to it. In Lemma 1, it is shown that to each simple smooth arc Γ there can be associated a system of "parallel" (not necessarily simple) arcs which do not intersect it. Lemma 2 is devoted to proving that any point sufficiently close to an interior point of Γ lies on at least one of these parallel arcs. It then follows immediately from Lemmas 1 and 2 that any two points of the complement Γ which are sufficiently close to Γ and on "the same side" of Γ can be joined by an arc not crossing Γ. Lemma 3 provides the step needed to draw the same conclusion concerning points on "opposite sides" of Γ. Lemma 4 provides the tool needed to deduce the Jordan Arc Theorem for arcs consisting of $n+1$ smooth arcs from its truth for those having n smooth arcs.

2. **Definitions and notation.** An arc is said to be *smooth* if it has a C^1 parametrization. A *piecewise smooth arc* is one which is obtained by joining end to end a finite number of smooth arcs. If an arc C is parametrized by $z=\Phi(t)$, $a\leqq t\leqq b$, and $S\subset[a, b]$, we shall denote by $C\{S\}$ the image of S under Φ. At other times, if $z, \zeta\in C$, we shall use $C[z, \zeta]$ to denote a portion of C joining z to ζ. By C' we mean the complement of the arc C with respect to the plane.

3. **Preliminaries.** We begin by introducing, for each smooth arc C, a class of arcs C_ϵ which plays the roll of the lines parallel to a given segment. These arcs will be used to connect points close to C by an arc which does not intersect C.

LEMMA 1. *Let $C: z=\Phi(t)$, $0\leqq t\leqq L$, be a simple smooth arc parametrized by arc-length. Define, for each real ϵ, C_ϵ to be that arc parametrized by $z=\Phi_\epsilon(t)\equiv\Phi(t)+i\epsilon\,\Phi'(t)$, $0\leqq t\leqq L$. There exists a $d>0$ such that $C_\epsilon\cap C=\emptyset$ when $0<|\epsilon|<d$.*

Proof. We begin by showing that portions of C and C_ϵ, corresponding to sufficiently small neighborhoods of the parametric interval $[0, L]$, are disjoint. Let

$t, \tau \in [0, L]$. We then have, after some manipulation,

$$\Phi_\epsilon(t) - \Phi(\tau) = (t - \tau + i\epsilon)\Phi'(t) + \int_\tau^t [\Phi'(s) - \Phi'(t)]ds.$$

By uniform continuity, there exists a $\delta > 0$ such that $|\Phi'(s) - \Phi'(t)| < 1/2$ if $|s-t| < \delta$. Hence, if $|t-\tau| < \delta$, we have

$$|\Phi_\epsilon(t) - \Phi(\tau)| > |t - \tau + i\epsilon||\Phi'(t)| - |t - \tau|/2.$$

Now $|\Phi'(t)| = 1$ since t represents arc-length. Consequently

$$\Phi_\epsilon(t) \neq \Phi(\tau) \quad \text{if } |t - \tau| < \delta \quad \text{and} \quad \epsilon \neq 0. \tag{1}$$

We next prove that for ϵ sufficiently small, each point on C has a neighborhood which is disjoint from C_ϵ. To this end we choose points $0 = t_0 < t_1 < \cdots < t_n = L$ such that $|t_k - t_{k-1}| < \delta/4$. It is then a consequence of (1) that

$$C\{|t - t_k| \leqq \delta/4\} \cap C_\epsilon\{|t - t_k| \leqq \delta/2\} = \varnothing. \tag{2}$$

The point sets $C\{|t-t_k| \leqq \delta/4\}$ and $C\{|t-t_k| \geqq \delta/2\}$ are disjoint and compact since C is simple and the continuous image of a compact set is compact. Hence they have a positive distance d_k. The fact that the portions of C and C_ϵ, corresponding to the set $|t-t_k| \geqq \delta/2$, have a distance at most $|\epsilon|$ then shows that

$$C_\epsilon\{|t - t_k| \geqq \delta/2\} \cap C\{|t - t_k| \leqq \delta/4\} = \varnothing \quad \text{if } |\epsilon| < d_k. \tag{3}$$

By combining (2) and (3) it is easily seen that $C\{|t-t_k| \leqq \delta/4\} \cap C_\epsilon = \varnothing$ if $|\epsilon| < d_k$. It follows that $C \cap C_\epsilon = \varnothing$ if $|\epsilon| < d = \min\{d_k\}$. This completes the proof.

We next use a standard variational argument to show that any point sufficiently close to an interior point of C lies on one of the arcs C_ϵ.

LEMMA 2. *Let C and C_ϵ be as defined in Lemma* 1. *If $z \notin C$ is closer to C than it is to either end point of C, then there exists a $t_0 \in (0, L)$ and an $\epsilon_0 \neq 0$ such that $z = \Phi(t_0) + i\epsilon_0\Phi'(t_0)$, that is $z \in C_{\epsilon_0}$.*

Proof. Since z is closer to C than it is to $\Phi(0)$ or $\Phi(L)$, there exists a $t_0 \in (0, L)$ such that $|z - \Phi(t_0)| = \text{dist}\{z, C\}$. Using the definition of distance and the identity $z - \Phi(t) = z - \Phi(t_0) + \Phi'(t_0)(t - t_0) + o(t - t_0)$, we have

$$|z - \Phi(t_0)|^2 \leqq |z - \Phi(t)|^2 = |z - \Phi(t_0)|^2 + 2 \operatorname{Re}[z - \Phi(t_0)]\overline{\Phi'(t_0)}(t - t_0) + o(t - t_0).$$

It follows from the fact that $t - t_0$ can be either positive or negative that $2 \operatorname{Re}[z - \Phi(t_0)]\overline{\Phi'(t_0)} = 0$. But this is equivalent to $z - \Phi(t_0) = i\epsilon_0\Phi'(t_0)$ for some real $\epsilon_0 \neq 0$. This completes the proof.

The previous two lemmas allow us to say that two points $z \in C_\epsilon$ and $\zeta \in C_\eta$ are on the same or opposite sides of C according to whether ϵ and η have the

same or opposite signs. Note that we have not excluded the possibility of a point being on both sides of C. Fortunately, this is not important for our purpose. Once the Jordan Curve Theorem has been proved, it is an easy exercise to show that if $z\in C_\epsilon$, $0<|\epsilon|<d$, then z is only on one side of C.

The previous two lemmas will be used to show that any two points sufficiently close to and on the same side of C can be joined by an arc in C'. In order to prove the same result for points on opposite sides of C we shall need another lemma.

LEMMA 3. *Let C be as in Lemma 1. There exists a $d>0$ such that the 'half neighborhood' $z=\Phi(L)+\epsilon\,\Phi'(L)e^{i\theta}$, $0<\epsilon<d$, $-\pi/2\leqq\theta\leqq\pi/2$, is disjoint from C.*

Proof. There exists a δ such that $|\Phi'(s)-\Phi'(L)|<1/2$ if $|s-L|<\delta$. We have

$$\Phi(L)+\epsilon\Phi'(L)e^{i\theta}-\Phi(t)=\int_t^L[\Phi'(s)-\Phi'(L)]ds+\Phi'(L)[(L-t)+\epsilon e^{i\theta}].$$

It follows from the triangle inequality and the fact that $|\Phi'(L)|=1$ that the right side of the above equality is greater in absolute value than

$$|L-t+\epsilon\,e^{i\theta}|-(L-t)/2$$

if $L-t<\delta$. The above expression is easily seen to be positive for $\epsilon>0$ and $-\pi/2\leqq\theta\leqq\pi/2$. Hence the 'half neighborhood' is disjoint from $C\{L-\delta\leqq t\leqq L\}$. Let d be the distance from $\Phi(L)$ to $C\{0\leqq t\leqq L-\delta\}$. If $0<\epsilon<d$, then the 'half neighborhood' is disjoint from C. This completes the proof.

LEMMA 4. *Let C be as in Lemma 1, A a compact set and z, ζ two points of $(C\cup A)'$, each of which is closer to an interior point of C than it is to A or to an endpoint of C. If* (i) $C\cap A=\Phi(0)$ *or* (ii) $C\cap A=\Phi(0)\cup\Phi(L)$ *and z, ζ are on the same side of C, then z can be joined to ζ by an arc in $(A\cup C)'$.*

Proof. Let $z_1=\Phi(t_1)$ and $\zeta_1=\Phi(\tau_1)$ be points on C which minimize the respective distances from z and ζ to C. The segments $[z, z_1]$ and $[\zeta, \zeta_1]$ then intersect C only at the points z_1 and ζ_1 respectively. As a consequence of Lemma 2, we have $z=\Phi(t_1)+i\epsilon_0\Phi'(t_1)$ and $\zeta=\Phi(\tau_1)+i\eta_0\Phi'(\tau_1)$ where ϵ_0 and η_0 are nonzero. It follows that z and ζ can be connected in $(A\cup C)'$ to points $z_2=\Phi(t_1)+i\epsilon\Phi'(t_1)$ and $\zeta_2=\Phi(\tau_1)+i\eta\Phi'(\tau_1)$ for all small ϵ, η which have the same signs as ϵ_0 and η_0. Assume that $\eta=\pm\epsilon$ and that $|\epsilon|$ is less than the δ of Lemma 1 and the d of Lemma 3. If case (i) holds we suppose that $t_1<\tau_1$. The arc $C[\Phi(t_1), \Phi(L)]$ is disjoint from A and hence has a positive distance δ from it. Let $|\epsilon|$ be less than δ and the d's of Lemmas 1 and 3. If $\eta=\epsilon$ then, by Lemma 1, the arc $C_\epsilon\{t_1\leqq t\leqq\tau_1\}$ serves to join z_2 and ζ_2 in $(C\cup A)'$. If $\eta=-\epsilon$ we may join z_2 and ζ_2 to the points $\Phi(L)+i\epsilon\Phi'(L)$ and $\Phi(L)-i\epsilon\Phi'(L)$ by the arcs $C_\epsilon\{t_1\leqq t\leqq L\}$ and $C_{-\epsilon}\{\tau_1\leqq\tau\leqq L\}$. It then follows from Lemma 3 that these two points can be joined by an arc in $(C\cup A)'$. The proof of (ii) is similar and will be omitted.

4. The Jordan Arc Theorem. Once one has proved either the Jordan Curve

Theorem or its companion the Jordan Arc Theorem, the proof of the other one is relatively simple. Lemmas 1, 2, and 3 are now used to give a simple proof of the Jordan Arc Theorem for our special class of curves.

THEOREM 1. *The complement of a simple piecewise smooth arc C is an open connected set having C as its boundary.*

Proof. It follows from the fact that C is compact that C' is open and that the boundary of C' is contained in C. Lemma 1 shows that each smooth point of C is a boundary point of C'. Since the boundary of any set is closed, the "corner points" of C are also in the boundary of C'.

It remains to show that C' is connected. We proceed by induction on the number of smooth segments of C to show that C' is arcwise connected. Suppose then that C is a simple smooth arc. If $z, \zeta \in C'$ we may join them by a smooth arc Γ which does not pass through either end point of C. If Γ does intersect C we may, because of the continuity of the parametrization of Γ, join z and ζ in C' to points z_1 and ζ_1 which are arbitrarily close to interior points of C. By Lemma 4, z_1 can be connected to ζ_1 by an arc in C'. Thus any two points in C' can be connected by an arc in C'. Hence C' is arcwise connected.

Suppose now that Theorem 1 is true for arcs having n smooth segments. If C_{n+1} has $n+1$ smooth segments, let C_n denote the first n segments and C the last. If $z, \zeta \in C'_{n+1}$, then by our induction hypothesis, z and ζ can be joined by an arc Γ in C'_n. We may assume that Γ does not pass through an end point of C, since removing a point from an open connected set does not disconnect it. We may then join z and ζ in C'_{n+1} to points z_1 and ζ_1 which are arbitrarily close to interior points of C. It follows from Lemma 4, with $A = C_n$, that z_1 can be connected to ζ_1 by an arc in C'_{n+1}. This completes the proof.

5. The Jordan Curve Theorem. We are now in a position to state and prove the main theorem of this note.

THEOREM 2. *The complement of a simple closed piecewise smooth curve C consists of two components, E and I, each having C as its boundary. Moreover, the index of C is equal to zero in E and, if C is oriented properly, is equal to one in I.*

Proof. We first show that C' consists of at most two components. Since C is compact, there exists a point ζ which lies outside of a disk containing C in its interior. Let E denote the set of points which can be joined to ζ by an arc in C'. E is clearly connected since any two of its points can be connected to ζ by an arc in E. Let $I = C' - E$. If $I \neq \varnothing$, let Γ be the simple piecewise smooth arc obtained by removing an open smooth segment γ from C. By Theorem 1, any point $z_1 \in I$ can be joined to ζ by an arc $\Gamma_{z_1} \subset \Gamma'$. This curve necessarily crosses γ for otherwise z would be in E. As in the proof of Theorem 1, we now choose points z_1', ζ' arbitrarily close to interior points of γ such that $\Gamma_{z_1}[z_1, z_1']$ and $\Gamma_{z_1}[\zeta', \zeta]$ are in C'. We claim that z_1' and ζ' are on opposite sides of γ for otherwise by Lemma 4 z could be joined to ζ by an arc in C'. Let z_2 be another point in I and let z_2' play

the role analogous to z_1'. The point z_2' must be on the same side of γ as z_1' for otherwise z_2 could be connected to ζ by an arc in C'. Since z_1' and z_2' are on the same side of γ, it follows from Lemma 4 that z_1 and z_2 can be connected by an arc in C'. Hence I is connected.

We next consider the difference between the index of C at two points on opposite sides of a smooth portion of C. If $z_0=\Phi(t_0)$ is such a point, it follows from Lemma 1 that, as long as ϵ retains its sign, $z_\epsilon=\Phi(t_0)+i\epsilon\Phi(t_0)$ is in the same component of C'. It follows that

$$\Delta = n(C, z_0 + i\epsilon\Phi'(t_0)) - n(C, z_0 - i\epsilon\Phi'(t_0))$$

is constant for all small $\epsilon>0$. By using the continuity of Φ' at t_0, it is easily shown that

$$\left(\frac{1}{\Phi(t) - z_0 - i\epsilon\Phi'(t_0)} - \frac{1}{\Phi(t) - z_0 + i\epsilon\Phi'(t_0)}\right)\Phi'(t) = \frac{2i\epsilon}{(t-t_0)^2+\epsilon^2} + \mathcal{E},$$

where given an $\eta>0$ there exists a $\delta>0$ such that

$$|\mathcal{E}| < \eta\frac{\epsilon}{[(t-t_0)^2+\epsilon^2]}$$

if $|t-t_0|<\delta$. If $\tilde{C}$ denotes the portion of C corresponding to $|t-t_0|>\delta$, we then have

$$\begin{aligned}\Delta = {}& \frac{1}{2\pi i}\int_{\tilde{C}}\left(\frac{1}{z - z_0 - i\epsilon\Phi'(t_0)} - \frac{1}{z - z_0 + i\epsilon\Phi'(t_0)}\right)dz \\ &+ \frac{1}{2\pi i}\int_{t_0-\delta}^{t_0+\delta}\left(\frac{2i\epsilon}{(t-t_0)^2+\epsilon^2} + \mathcal{E}\right)dt.\end{aligned}$$

The first integral tends to zero as $\epsilon\to 0$ since its integrand is continuous at $\epsilon=0$. In the second integral we substitute $t-t_0=\epsilon s$ and then let $\epsilon\to 0$. We then obtain

$$|\Delta - 1| < \eta.$$

But since Δ is an integer we must have $\Delta=1$. It follows that C' has at least two components. But we already know that C' has at most two components; hence I is not empty. The above argument also shows that each smooth point of C is a boundary point of both E and I. That the 'corners' are boundary points follows from the fact that the boundary is a closed set. Since in E (the unbounded component of C') the index of C is zero, it follows that in I it is ± 1. Hence by reorienting if necessary, we can arrange that it is 1. This completes the proof.

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LEGENDRE'S IDENTITY

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1. Introduction. For natural numbers n and a given prime p, let $e(n)=e_p(n)$ denote the *multiplicity* of p in n; that is, $e(n)$ is the uniquely determined non-negative integer e such that $p^e \mid n$, $p^{e+1}\nmid n$. The Identity of Legendre asserts that

$$e(n!) = \sum_{\alpha>0} [n/p^\alpha], \tag{1}$$

where $[t]$ denotes the largest integer $\leqq t$ for all real t and the summation ranges over all natural numbers α. Evidently, the sum has only a finite number of terms different from zero.

Most texts in the elementary theory of numbers give the classical proof of this identity. The argument is sometimes presented in a rather sketchy manner. It would seem appropriate to point out the two basic features of this proof. The first involves essentially the Fundamental Theorem of Arithmetic in the guise of the additivity property of the function e (Property 1 below). The second involves a transformation of the sum resulting from the application of this property. This transformation can be accomplished by equating row and column sums in a Ferrers diagram. This is essentially the basis of the approach used by Niven and Zuckerman [9], p. 87. (For the idea of a Ferrers diagram we refer to Hardy and Wright [4], Section 19.2.) The transformation can also be effected by interpreting the function e as a "divisor function" (Property 2 below). A precise formulation of this approach is presented in the present note.

We show in Section 3 how the infiniteness of the sequence of prime numbers can be deduced on the basis of the Legendre Identity. An analogous property of the classical Dirichlet divisor function is also proved. It is hoped that this discussion may serve as a background for the treatment of the distribution of prime numbers and of the average order of the divisor function as presented in general texts on the theory of numbers.

Finally, we mention that there exist other proofs of Legendre's formula. An alternate proof by induction is given in Niven and Zuckerman [9], Section 4.1. A logarithmic proof may be found in Landau [7], p. 24.

2. Proof of the identity. First we note two simple properties of $e(n)$.

PROPERTY 1. *The function $e(n)$ is completely additive: For each pair of positive integers, n_1, n_2,*

$$e(n_1n_2) = e(n_1) + e(n_2). \tag{2}$$

REMARK 1. The obvious generalization of this property to k-tuples, $n_1, \cdots n_k$ $(k\geqq 2)$ follows by a simple induction.

PROPERTY 2. *Let $\tau_p(n)$ denote the number of ordered pairs (α, m) of natural numbers α, m, such that $p^\alpha m=n$. Then*

$$e(n) = \tau_p(n). \tag{3}$$

To prove Property 1, let $e_1=e(n_1)$, $e_2=e(n_2)$, $m_1=n_1/p^{e_1}$, $m_2=n_2/p^{e_2}$. Evidently, m_1 and m_2 are integers and $(m_1, p)=(m_2, p)=1$. It follows that $p^{e_1+e_2+1} \nmid n_1n_2$ because otherwise $p \mid m_1m_2$, contradicting the fact that the reduced residue classes (mod p) are closed under multiplication. Property 2 results from the one to one correspondence between the pairs (α, m) and the powers of p dividing n, $p, p^2, \cdots, p^e$, where $e=e(n)$.

In summations, a notation, $i \leqq t$, will mean a sum over the natural numbers $\leqq t$, while (α, m) will mean a summation over the ordered pairs of natural numbers, α, m (subjected to whatever other restrictions are indicated). Vacuous sums will be assumed to have the value 0.

Proof of (1). By Remark 1 and (3),

$$\begin{aligned} e(n!) &= \sum_{k\leqq n} e(k) = \sum_{k\leqq n} \tau_p(k) \\ &= \sum_{k\leqq n} \sum_{\substack{(\alpha,m)\\ p^\alpha m=k}} 1 = \sum_{\substack{(\alpha,m)\\ p^\alpha m\leqq n}} 1 \\ &= \sum_{\alpha>0} \sum_{m\leqq (n/p^\alpha)} 1 = \sum_{\alpha>0} [n/p^\alpha], \end{aligned}$$

since for $t\geqq 0$, $[t]$ represents the number of natural numbers $\leqq t$.

REMARK 2. Note that the convention on vacuous sums is needed in the above proof for the occurring sums of 1.

REMARK 3. Property 1 can also be formulated as follows: e is a homomorphism of the multiplicative semigroup of positive integers into the additive semigroup of the nonnegative integers.

REMARK 4. A divisor function counts the number of decompositions of a number n into a product $d\delta$ of ordered pairs of natural numbers d, δ of a specified type. The ordinary divisor function (denoted τ in [7], d in [4]) counts the totality of unrestricted decompositions. The "unitary" divisor function enumerates those decompositions for which d and δ are relatively prime (cf. [1]). The function τ_p defined above enumerates those for which d is a positive power of a given prime p.

If one applies the above summation process to τ in place of τ_p, it is easily deduced that

$$\sum_{k\leqq n} \tau(k) = \sum_{d\leqq n} [n/d]. \tag{4}$$

The reader is referred to Sierpinski [10] for a discussion of (4) and its refinements. The relation (4) is due to Dirichlet.

The proof of Dirichlet's formula can be formulated in such a way that it is the result of counting in two different ways the lattice points on and under the hyperbola, $xy=n$ (see [4], Section 18.2, p. 262). The above proof of (1) based on τ_p can similarly be put into geometric form in terms of lattice points.

REMARK 5. Legendre's Identity is often cited without reference to the name of Legendre. The priority of Legendre as its discoverer is pointed out by Landau in the note to Section 17 of [6] appearing on p. 884 of the second volume of [6]. The identity of Legendre occurs with a discussion in Legendre's famous "Théorie des Nombres," vol. 1, 3rd ed., 1830 (reprinted 1955 by Blanchard, Paris); in particular, Sections 16–18 of the Introduction. The formula had actually occurred in the previous edition of this work published under a slightly different title in 1808.

3. Asymptotic properties. The Legendre Identity is important as a starting point in investigating the distribution of the prime numbers. Similarly, the identity (4) is a point of departure for investigating the average order of the divisor function $\tau(n)$. For purposes of illustration, in this section we show how from the two identities the simplest nontrivial results for these two problems issue. In particular, we prove that the set of prime numbers is infinite (Euclid's Theorem) and that τ has an infinite mean value. In the next section are listed references to further developments regarding the two problems.

For $n>1$ we have

$$n! = \prod_{p|n!} p^{e(n!)} = \prod_{p\leq n} p^{e(n!)}, \tag{5}$$

the products being restricted to prime numbers p. The first equation in (5) results from the fundamental theorem of arithmetic and the second from the fact that a prime divisor of a product of integers divides at least one factor. Since $[x]\leqq x$, it follows from (1) that

$$e(n!) = \sum_{\alpha=1}^{\infty} [n/p^{\alpha}] \leqq n \sum_{\alpha=1}^{\infty} 1/p^{\alpha} = n/(p-1),$$

on summing a geometric series. Hence by (5)

$$\sqrt[n]{n!} \leqq \prod_{p\leq n} p^{1/(p-1)}. \tag{6}$$

If there were but finitely many prime numbers, the product on the right would remain bounded as $n\to\infty$, in contradiction to the well-known fact,

$$\lim_{n\to\infty} \sqrt[n]{n!} = \infty. \tag{7}$$

A simple proof of this property of the factorial is given in [2].

REMARK 6. The preceding argument suffices to prove somewhat more than Euclid's Theorem, assuming known the properties of logarithms. By (7) the product in (6) must diverge to ∞, and so must its logarithm which is

$$= \sum_{p\leq n} \log p/(p-1) \leqq 2 \sum_{p\leq n} \log p/p$$

since $p\geqq 2$. Therefore

$$\sum_p \frac{\log p}{p} \text{ diverges to } \infty, \tag{8}$$

the summation being over all primes p. This result implies immediately that the primes are infinite, but is short of asserting that the series of the prime reciprocals diverges to ∞ (Euler's Theorem). For any real $t>0$ it is a consequence of L'Hôpital's Rule that

$$\lim_{x\to\infty} \frac{\log x}{x^t} = 0;$$

thus $\log x$ tends to ∞ so slowly that (8) may be viewed as a fairly close approximation of Euler's Theorem. For a simple proof of the latter result the reader is referred to [3]. Further discussion of the above noted asymptotic property of $\log x$ may be found in Hardy and Wright ([4] Sec. 1.7) and Knopp ([5] Sec. 2.1.4).

Let $D(n)$ denote the sum occurring on the left of (4); that is, D is the summatory function of the arithmetical function τ. We wish to show that $D(n)/n$ become infinite as $n\to\infty$. By definition, $[x]>x-1$ for all real x, from which it follows that $[x]>x/2$ if $x\geqq 2$. Since obviously $[x]>x/2$ if $1\leqq x<2$, it follows that $[x]>x/2$ for all $x\geqq 1$ and hence that

$$D(n) = \sum_{d\leqq n} [n/d] > (n/2) \sum_{d\leqq n} 1/d,$$

but the last sum tends to ∞ with n by the divergence of the harmonic series; hence

$$\lim_{n\to\infty} \frac{D(n)}{n} = \infty. \tag{9}$$

On the other hand

$$\lim_{n\to\infty} \frac{D(n)}{n^2} = 0, \tag{10}$$

because, if we note that $x\geqq [x]$ then $0<D(n)=\sum_{d\leqq n} [n/d]\leqq n\sum_{d\leqq n} 1/d$, but by Cauchy's Limit Theorem ([5], Section 2.4) $\lim_{n\to\infty} (1/n)\sum_{d=1}^{n} (1/d)=0$. This suffices to prove (10).

4. Supplementary remarks. Legendre's Identity is an essential feature of the proof of Chebyshev's Theorem concerning the distribution of the primes. There is a simplified version of this proof due to Landau ([7] Theorem 112). Expositions of Landau's proof appear in LeVeque ([8] Sec. 6.7) and in Niven and Zuckerman ([9] Sec. 8.1). We observe that, while consideration of $n!$ suffices for the above proof of Euclid's Theorem, to prove the deeper Chebyshev Theorem requires consideration of the binomial coefficient $(2n)!/(n!)^2$. Further material

on Euclid's Theorem and related topics appears in Hardy and Wright [4] Chap. 2.

Somewhat more than (9) can be proved on the basis of the identity (4). In particular, it can be proved that $D(n)$ is asymptotic to $n \log n$ ([4] Theorem 318). Moreover, a refinement of the method yields the classical theorem of Dirichlet ([4] Theorem 320). For a discussion of this topic from both arithmetical and geometric points of view the reader is referred to LeVeque ([8] pp. 116–119).

If $\pi(x)$ is defined to be the number of primes $\leqq x$, Euclid's Theorem states that $\lim \pi(x)=\infty$. The theorem of Chebyshev asserts that $\pi(x)$ is approximated by $x/\log x$ in the weak sense that $\pi(x)/(x/\log x)$ is contained between positive bounds for all large x. The Prime Number Theorem asserts the approximation in the strong sense that $\lim \pi(x)/(x/\log x)=1$ as $x\to\infty$. For investigating essentially closer approximation to $\pi(x)$ the function $x/\log x$ has to be replaced by the so-called "logarithmic integral," $\mathrm{li}(x)$ or a variant ([6] vol. 1, p. 27). The question as to how closely $\pi(x)$ is approximated by $\mathrm{li}(x)$ is one of the great unsolved problems of the theory of numbers (the "prime number problem"). There is a famous unproved conjecture, the Riemann Hypothesis, whose truth or falsity would settle this problem. For further discussion and references we mention the notes to Chapter 1 of Hardy and Wright [4].

There is a comparably difficult and deep problem relating to the divisor function (the "Dirichlet divisor problem"). Place $\rho(n)=n \log n+(2C-1)n$, where C denotes a certain constant (the Euler constant). The Dirichlet Theorem mentioned above shows that $\rho(n)$ approximates to $D(n)$ in a certain precise sense. In the prime number problem we are interested in the order of magnitude of $\pi(x)-\mathrm{li}(x)$, in the divisor problem with that of $D(n)-\rho(n)$. The reader is referred to the notes of Chapter 18 of [4] for more details concerning the divisor problem.

Added in proof. An interesting interpretation of the function $e_p(n)$ is to be found in Calvin Long's *Number Theory* (§6.8). The latter work also contains a proof of Euler's theorem (§3.2, Theorem 3.5) which may be contrasted with the proof in Landau ([6], vol. 1, §13). For a neat proof of the limit result (7) the reader is referred to G. H. Hardy's *Pure Mathematics* (in the exercises to the section devoted to the limit of x^n).

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NOTES ON TOPOLOGICAL SPACES WITH MINIMUM NEIGHBORHOODS

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The purpose of this paper is to equate the study of a certain class of topological spaces—defined below as "saturated" topological spaces—with that of "preordered" sets. This is done in terms of isomorphism of categories and isomorphism of "complete lattices." Examples selected from various fields are then given. We conclude with a few remarks on connectedness, convergence, and continuity in such spaces.

DEFINITION. *Let a saturated topological space* (STS) *be a space in which any intersection of open sets is itself an open set*; *or, equivalently, every point of which possesses a minimum neighborhood. We shall also refer to such spaces as sets provided with saturated topologies* (*S-topologies*).

One could have thought of calling these spaces "complete" topological spaces, since they are those spaces whose set of open sets constitutes a "complete lattice" (see below, Definition 3, Section 1); but such a term is not appropriate, since it is already used in the literature to designate another class of topological spaces ([4], Chapter 2, Section 3, No. 3).

Trivially, a finite topological space (FTS) is an STS. Indeed, this paper stems from an attempt to study FTS's. But, in the process, the basic fact about an FTS (apart from its finiteness!) finally proved to be that it is saturated. Accordingly, we shall here essentially deal with STS's, and the results on FTS's will be given as corollaries.

The study of STS's is extremely different from that of the more commonly encountered classes of topological spaces. This is of course even more true of FTS's. The only T_1 S-topologies are the discrete topologies; the only compact, or locally compact, S-topologies are thus the discrete topologies (since compact and locally compact topologies—as defined in [4], Chapter 1, Section 9, No. 1 and No. 7—are "separated," i.e., T_2); STS's are only trivially first countable; the only S-topological groups may be shown to be those provided with the topology of the equivalence relation associated with a normal subgroup.

There appears to be very little material concerning either STS's or FTS's in the literature. In particular, the subject is not discussed in the two books of Berge, [1] and [2]. Bourbaki ([4], Chapter 1, Section 1, Exercise 2, p. 135) has a

short exercise in the field of Kolmogorov (i.e., T_0) topological spaces, which is related to this subject, and to which we shall return later.

1. Definitions.

1. *Let a preorder on a set X be any reflexive and transitive relation on X.*

2. *Let a lattice be a partially ordered set X, any two elements of which have an* l.u.b. *and a* g.l.b. *in X; let a semilattice be a partially ordered set X, any two elements of which have a* g.l.b. *in X.*

3. *Let a complete lattice be a partially ordered set X, any nonempty subset of which has an* l.u.b. *and a* g.l.b. *in X; let a complete semilattice be a partially ordered set X, any nonempty subset of which has a* g.l.b. *in X.*

4. *Let an increasing function from a set X provided with a preorder R into a set X' provided with a preorder R' be a function $f: X \to X'$ such that*

$$(\forall a, b \in X)(aRb \text{ implies } f(a)R'f(b)).$$

5. *Let a relation R on a set X be finer than a relation R' on X, if the graph of R is included in the graph of R'; i.e., if the identity function on X, considered as a function of (X, R) into (X, R'), is an increasing function. In such a case we shall also say that R' is coarser than R.*

6. *Let a topology T on a set X be finer than a topology T' on X, if T includes T' (identifying a topology with the set of open sets for that topology); i.e., if the identity function on X, considered as a function of (X, T) into (X, T'), is a continuous function. In such a case we shall also say that T' is coarser than T.*

We shall now define a *category*, for the benefit of those who may not be quite familiar with this concept. (On the theory of categories, see, for example, reference [6].)

7. *A category consists of a class C, the elements of which are called the objects of the category, provided with the following structure.*

(a) *Each pair $(a, b) \in C \times C$ is provided with a set $M(a, b)$, possibly empty, the elements of which are called the morphisms from a to b. The sets of morphisms associated with different pairs must be disjoint. When $\alpha \in M(a, b)$, one often writes: $a \xrightarrow{\alpha} b$, or $\alpha : a \to b$.*

(b) *The sets of morphisms associated with any two pairs (a, b), $(b, c) \in C \times C$ are provided with a product*

$$M(b, c) \times M(a, b) \xrightarrow{\circ} M(a, c).$$

If $\alpha \in M(a, b)$ and $\beta \in M(b, c)$, one then writes $\beta \circ \alpha$ for $\circ(\beta, \alpha)$, and occasionally $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ for $a \xrightarrow{\beta\circ\alpha} c$. This product must be associative, and for any object a of C, there must be a morphism 1_a from a to a which acts as a unit element when multiplied with morphisms from a to any object b, or from b to a.

LEMMA 2. *Given a preorder R on a set X, the set $\{R(x)/x\in X\}$ is actually a basis for the right-topology associated with R.*

This follows from the fact that for all elements x, y of X

$$R(x) \cap R(y) = \bigcup_{z\in R(x)\cap R(y)} R(z).$$

LEMMA 3. *The right- (resp. left-) topology associated with a preorder R on a set X is saturated; moreover the right- (resp. left-) preorder associated with this topology is precisely the original preorder, R.*

1. The equation under Lemma 2 implies that a subset Y of X is open relative to $T_r(R)$ if and only if

$$Y = \bigcup_{y\in Y} R(y).$$

Thus $R(x)$ is a minimum neighborhood of x, relative to $T_r(R)$, so that $T_r(R)$ is saturated.

2. By definition, $P_r(T_r(R))\ (x)=N_x$, $x\in X$. But $N_x=R(x)$. Thus $P_rT_r(R) =R$.

3. Finally, $P_lT_l(R)=P_lT_r(R^{-1})=(P_rT_r(R^{-1}))^{-1}=(R^{-1})^{-1}=R$.

LEMMA 4. *Given an S-topology T on a set X, and given the right- (resp. left-) preorder associated with T, the right- (resp. left-) topology associated with this preorder is precisely the original topology, T.*

1. By definition, $P_r(T)\ (x)=N_x$, $x\in X$. From Lemmas 1 and 2 then it follows that $T_rP_r(T)=T$.

2. Then $T_lP_l(T)=T_l(P_r(T)^{-1})=T_rP_r(T)=T$.

LEMMA 5. *Given two sets X and X', provided with preorders R and R' respectively, a function $f:X\to X'$ is an increasing function if and only if it is continuous relative to the right-topologies (or, equivalently, to the left-topologies) associated with R and R'.*

Because of the definition of left-topologies, here it is sufficient to consider only right-topologies. Then

f is an increasing function

iff

$(\forall x, y \in X)(xRy$ implies $f(x)R'f(y))$

iff

$(\forall x, y \in X)(y \in N_x$ implies $f(y) \subseteq N_{f(x)})$

iff

$(\forall x \in X)(N_x \subseteq f^{-1}(N_{f(x)}))$

iff

f is a continuous function.

Lemmas 3, 4, and 5 show that P_r and P_l are isomorphisms between the category of STS's and the category of preordered sets. It will now be shown that the set of S-topologies on a set X and the set of preorders on X are isomorphic complete lattices, two such isomorphisms being P_r and P_l. It is sufficient to consider only one of those; let us use P_r.

LEMMA 6. *The set of S-topologies on a set X and the set of preorders on X are isomorphic partially ordered sets.*

If the reader reviews the definitions of the "finer than" relations (Definitions 5 and 6) and then applies Lemma 5, he will see that an S-topology T on X is finer than another S-topology T' on X if and only if $P_r(T)$ is finer than $P_r(T')$. The lemma is then seen to follow from the fact that P_r is a one-to-one correspondence.

LEMMA 7. *The set of S-topologies on a set X and the set of preorders on X are complete lattices.*

Relative to the "finer than" relation, any family $(T_k)_{k\in K}$ of S-topologies on X has the l.u.b.

$$\bigcap_{k\in K} T_k$$

in the set of S-topologies on X, and any family $(R_k)_{k\in K}$ of preorders on X has the g.l.b.

$$\bigcap_{k\in K} R_k$$

in the set of preorders on X. The lemma then follows from Lemma 6.

The proof of Theorem 1 is now complete.

SCHOLIUM. If a set X is infinite, the g.l.b. of a set $\mathfrak{J}$ of S-topologies on X, in the lattice of S-topologies on X, is not necessarily equal to the g.l.b. of $\mathfrak{J}$ in the lattice of *all* topologies on X, which is the topology generated by the topologies in $\mathfrak{J}$: the latter topology is not necessarily saturated. For example, let X be the set of real numbers Re, let x be a given element of Re, and, for any natural number n, let T_n be the S-topology on Re with minimum neighborhoods $N_x=(x-(1/n), x+(1/n))$ and $N_y=\{y\}$, for $y\neq x$. The *saturated topology* generated by $\{T_n/n$ a natural number$\}$ is the discrete topology, while the *topology* generated by the same set of topologies is not.

3. The next theorem is similar to Theorem 1, except that it concerns Kolmogorov (i.e., T_0) STS's and partially ordered sets. (Recall that a Kolmogorov topological space is a space such that, given any two points of the space, there is always one, one of whose neighborhoods excludes the other point.) This theorem is substantially the same, although formulated quite differently, as a statement which will be found in Bourbaki (loc. cit.).

THEOREM 2. *The set of Kolmogorov S-topologies on a set X and the set of partial*

order relations on X are canonically isomorphic complete semilattices (with respect to the "finer than" relation). Two such isomorphisms are induced by canonical isomorphisms between the category of Kolmogorov STS's and the category of partially ordered sets. These in turn are induced by the category isomorphisms P_r and P_l.

We already know (Lemma 3) that the right- and left-topologies associated with a partial order on X are saturated. These are also Kolmogorov topologies, because of the antisymmetry property of a partial order relation.

Conversely, given a Kolmogorov S-topology T on X, the right-preorder associated with T—i.e., the relation $P_r(T)$ on X such that $xP_r(T)y$ iff $y \in N_x$—is a partial order relation: it is trivially reflexive and transitive, and it is also antisymmetric, since

$$(x \in N_y \text{ and } y \in N_x) \qquad \text{iff } x = y,$$

by definition of a Kolmogorov space.

P_r and P_l, restricted to the category of Kolmogorov STS's are thus the two required category isomorphisms. The proof of the rest of the theorem is left to the reader.

4. THEOREM 3. *There are two canonical full functors, F of the category $\mathfrak{I}$ of* STS'*s into the category $\bar{\mathfrak{I}}$ of Kolmogorov* STS'*s, and G of the category $\mathcal{P}$ of preordered sets into the category $\bar{\mathcal{P}}$ of partially ordered sets, such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{I} & \xrightarrow{F} & \bar{\mathfrak{I}} \\ T_r \Uparrow\Downarrow P_r & & T_r \Uparrow\Downarrow P_r \\ \mathcal{P} & \xrightarrow{G} & \bar{\mathcal{P}}. \end{array}$$

(The same then also holds with P_l and T_l, of course. Also, recall that, by Lemmas 3 and 4, P_r and T_r are inverses of one another, as are also P_l and T_l.)

The proof again proceeds through a few definitions and lemmas.

DEFINITION. *Given a set X provided with a preorder R, let $E(R)$ be the equivalence relation $R \cap R^{-1}$ on X.*

Notation. Let then $\tilde{R}$ denote the partial order induced on $\tilde{X} = X/E(R)$ by the canonical function $\rho(x)$ of X onto $\tilde{X}$.

Note that R is uniquely determined by $\tilde{R}$, so that $E(R)$ is a very "natural" equivalence relation; indeed, it is the coarsest one having this property of uniquely determining R, given $\tilde{R}$.

LEMMA 8. *Given a set X, provided with a preorder R, the set $\tilde{X}$, provided with the right- (resp. left-) topology associated with $\tilde{R}$, is precisely the quotient space, by the equivalence relation $E(R)$, of X, provided with the right- (resp. left-) topology associated with R.*

A quotient structure may be defined (see [3], Section 2, No. 5 and No. 6) as the finest structure on $\tilde{X}$ such that $\rho(x)$ is a morphism (here, an increasing function, or a continuous function). The lemma then follows from P_r (resp. P_l) being an isomorphism of categories.

LEMMA 9. *Given two sets X and X', provided with preorders R and R' respectively, and given an increasing function $f:X\to X'$, there is a unique increasing function $\tilde{f}:\tilde{X}\to\tilde{X}'$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle\rho(x)}\downarrow & & \downarrow{\scriptstyle\rho(x')} \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X}'. \end{array}$$

This follows from the fact that f maps equivalence classes of $E(R)$ into equivalence classes of $E(R')$.

It is now a simple matter to show that the mapping

$$G\colon (X, R) \to (\tilde{X}, \tilde{R}), \qquad G\colon f \to \tilde{f}$$

is a full functor of the category of preordered sets into the category of partially ordered sets: given three preordered sets X, X', X'', and morphisms $f:X\to X'$, $f':X'\to X''$, then $\widetilde{f'f} = \tilde{f}'\tilde{f}$; and, for every morphism $g:\tilde{X}\to\tilde{X}'$, there is a morphism $h:X\to X'$, such that $g=\tilde{h}$.

The theorem now follows from Lemma 8.

Since $\bar{\mathfrak{I}}$ is a subcategory of $\mathfrak{I}$, the diagram of Lemma 9 shows that ρ is a *natural transformation* ([6], p. 8) between the identity functor of $\mathfrak{I}$ and the functor F; and similarly for $\mathcal{P}$ and G.

5. Corollaries 1, 2, and 3 are immediate.

COROLLARY 1. *The set of topologies on a finite set X and the set of preorders on X are canonically isomorphic (complete) lattices. Two such isomorphisms are induced by canonical isomorphisms between the category of finite topological spaces and the category of finite preordered sets. These in turn are induced by the category isomorphisms P_r and P_l.*

COROLLARY 2. *The set of Kolmogorov topologies on a finite set X and the set of partial order relations on X are isomorphic (complete) semilattices. Two such isomorphisms are induced by canonical isomorphisms between the category of finite Kolmogorov spaces and the category of finite partially ordered sets. These in turn are induced by the category isomorphisms P_r and P_l.*

COROLLARY 3. *There are two canonical full functors, F of the category $\mathfrak{I}$ of* FTS's *into the category $\bar{\mathfrak{I}}$ of Kolmogorov* FTS's, *and G of the category $\mathcal{P}$ of finite preordered sets into the category $\bar{\mathcal{P}}$ of finite partially ordered sets, such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{F} & \bar{\mathfrak{J}} \\ T_r \uparrow\downarrow P_r & & T_r \uparrow\downarrow P_r \\ \mathcal{P} & \xrightarrow{G} & \bar{\mathcal{P}}. \end{array}$$

(The same also holding with P_l and T_l. P_r and T_r are inverses of one another, as are also P_l and T_l.)

Notation. Let N^N be the set of all functions of the set N of natural numbers into itself. Consider N^N to be partially ordered by the relation $f \leqq g$ iff $(\forall n \in N)$ $(f(n) \leqq g(n))$.

COROLLARY 4. *The finite Kolmogorov topological spaces are the finite subspaces of N^N.*

This, in one sense, is trivial, as seen in the following. Given a finite set X, of cardinality n, provided with a partial order R, X is isomorphic to the subset $\{R(x)/x \in X\}$ of the set $\mathcal{P}(X)$ of all subsets of X. But $\mathcal{P}(X)$ may be identified with the subset $\{0, 1\}^n$ of N^N (i.e., the n-dimensional unit cube). So we have the required result.

However, it is possible to embed X into N^N in other ways, so that in some cases X will be identified with a subspace of N^N involving a number of dimensions quite smaller than n. One may, for example, proceed in the following manner.

The partial order R on X is the intersection of the total orders on X which are coarser than R. Let $\{R_i/i \in [1, m]\}$ be the set of these total orders. Then, if $r_i(x)$ is the rank of an element x of X relative to the total order R_i, the function

$$f: x \to (r_i(x))_{i \in [1,m]}$$

of X into $[1, n]^m$ clearly defines an isomorphism of X with $f(X) \subseteq [1, n]^m$, again the result required.

Note that, although m may be as great as $n!$, in general there will be much redundancy in the set $\{R_i\}$, so that the number of dimensions involved in $f(X)$ may often be reduced substantially by (isomorphic) projection into a lower-dimensional subspace of N^N.

6. By definition of an S-topology, the set T^* of all closed sets, relative to a given S-topology T on a set X, is also an S-topology. Because $T^{**} = T$, T and T^* will be called *dual topologies.*

THEOREM 4. *Given a set X, the following two diagrams are commutative:*

$$\begin{array}{ccc} \textit{The S-topologies on } X & \underset{T_r}{\overset{P_r}{\rightleftarrows}} & \textit{The preorders on } X \\ \textit{identity} \updownarrow & & \updownarrow \textit{inversion} \\ \textit{The S-topologies on } X & \underset{T_l}{\overset{P_l}{\rightleftarrows}} & \textit{The preorders on } X \end{array}$$

$$\begin{array}{ccc} \textit{The S-topologies on } X & \underset{T_r}{\overset{P_r}{\rightleftarrows}} & \textit{The preorders on } X \\ *\updownarrow* & & \updownarrow \textit{identity} \\ \textit{The S-topologies on } X & \underset{T_l}{\overset{P_l}{\rightleftarrows}} & \textit{The preorders on } X \end{array}$$

where, as will be remembered (Lemmas 3 and 4), P_r and T_r are inverses of one another, as are also P_l and T_l; moreover all the functions represented in these diagrams are isomorphisms of categories (and of complete lattices).

The commutativity of the first diagram is already known. The commutativity of the second diagram may be seen as follows.

If x is an element of X, the closure C_x of $\{x\}$, relative to the right-topology associated with a preorder R on X, is equal to $R^{-1}(x)$, as the reader may easily show. But C_x is the minimum neighborhood of x, relative to $T_r(R)^*$, so that, by Lemma 1, $\{C_x/x\in X\}$ is a basis of $T_r(R)^*$. Moreover, by definition of T_l and by Lemma 2, $\{R^{-1}(x)/x\in X\}$ is a basis of $T_l(R)\equiv T_r(R^{-1})$. Thus

$$T_l(R) \equiv T_r(R^{-1}) = T_r(R)^*.$$

Similarly,

$$T_r(R) \equiv T_l(R^{-1}) = T_l(R)^*.$$

Hence the theorem.

COROLLARY. *An S-topology T is self-dual—i.e., $T=T^*$—if and only if T is the topology of an equivalence relation.*

Let x be a point of the space. Then, relative to T, $N_x=P_r(T)(x)$; and, relative to T^*, $N_x=P_l(T)(x)$. Thus, if $T=T^*$, then $P_r(T)(x)\equiv(P_r(T))^{-1}(x)$, so that $P_r(T)$ is a symmetric relation, q.e.d.

An important example of self-duality is the (S)-topology generated by $T\cup T^*$, which is the topology of the equivalence relation $E(P_r(T))$ associated with T.

7. The following two propositions are simple applications of the theory developed.

DEFINITION. *Let a topological space X be called quasi-compact if every open cover of X contains a finite open cover of X; or, equivalently, if every ultrafilter in X converges* ([4], Chapter 1, Section 4, No. 1).

A compact space is then a "separated" (i.e., T_2) and quasi-compact space ([4], ibid.).

PROPOSITION 1. *A set X provided with an S-topology T is quasi-compact if and only if there is a finite subset of X which is dense relative to T^*.*

Let R be the right-preorder associated with T, and suppose that T is quasi-compact. Since $\{R(x)/x\in X\}$ covers X, there is a finite subset Y of X, such that $R(Y)=X$. But $R(Y)$ is precisely the closure of Y, relative to T^*.

Conversely, let there be a finite subset Y of X, such that $R(Y)=X$, and let $(0_k)_{k\in K}$ be an open cover of X. Then, because of Lemma 1, for any element y of Y, there is a member $0_{k(y)}$ of the cover, such that $0_{k(y)}\supseteq N_y\equiv R(y)$. Clearly, $(0_{k(y)})_{y\in Y}$ is a finite open cover, such as the one required.

A similar argument leads to the following:

PROPOSITION 2. *A saturated topological space X is separable (i.e., contains a countable dense subset) if and only if every closed cover of X contains a countable closed cover of X.*

8. Examples of saturated topological spaces. (A) It is possible to interpret an S-topology on a set in terms of the associated preorders. For example, let R be the right-preorder associated with an S-topology T on a set X, so that, for $x, y\in X$, xRy iff $N_x\supseteq N_y$, i.e., $N_x=R(x)$. Let us now interpret R as a *domination* relation: namely xRy iff x *dominates* y, so that the set N_x becomes the set of all elements of X dominated by x. Such an interpretation of a preorder relation is sometimes relevant in sociology. We could also interpret X as the set of possible states of a physical or social system—subjected to certain conditions,—and "xRy" as "a transition from state x to state y is possible," or "has nonzero probability," etc. Such interpretations are prevalent, for example, in thermodynamics [5], and, naturally, in Markov chain theory.

Let us now define a subset B of a set A to be *closed relative to a relation* Γ *on* A when $\Gamma(B)\subseteq B$. Then, interpreting R as a "domination" relation, an *open set* relative to the topology T on X is any subset of X which is closed relative to the domination relation, and a *closed set* relative to T is any subset of X which is closed relative to the inverse relation, which might be called, for example, the "dependence" relation.

The *boundary* of a subset of X, relative to T, may also be interpreted in such terms. Consider, for example, the boundary ∂N_x of the minimum neighborhood N_x of a point x of X. Recalling that N_x is the set of the elements of X dominated by x, ∂N_x is then the set of all those elements of X which dominate elements of N_x, but which are not themselves dominated by x (see Figure 1).

Using the fact that

$$xRy \text{ iff } ((x\in\partial N_y \text{ and } y\notin\partial N_x) \text{ or } xE(R)y),$$

it can be shown that the sets ∂N_x completely determine the topology, except for the possible equivalence (by $E(R)$) of certain extremal points of X.

The concept of boundary in a preordered set may perhaps be useful in information retrieval systems where the user is presented successively with sets of titles which are more and more relevant to the subject he is interested in.

(B) A preorder R on X may also be interpreted in terms of its associated topology T: ($\mathfrak{N}$ is the set $\{N_x/x\in X\}$)

—R is an equivalence relation iff $\mathfrak{N}$ is a partition of X;
—R is a partial order iff T is a Kolmogorov S-topology;
—R is a total preorder iff $T=\mathfrak{N}$;
—$\tilde{R}$ (see above, Section 4, "Notation") is the partial order of a semilattice iff $\mathfrak{N}$—relative to T^*, if not relative to T—is closed under the operation of set intersection; the operation $x, y \to$ g.l.b. (x, y) on X is then continuous.

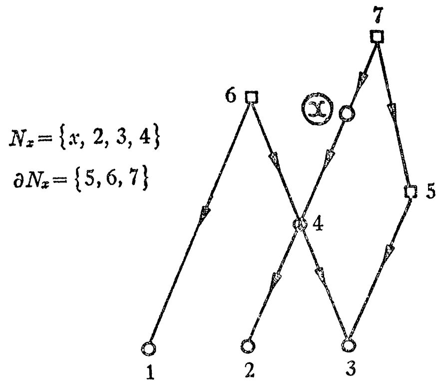

FIG. 1. Illustration of the concept of boundary applied to an ordered set, for one of the two dual topologies associated with the given order on the set.

9. Connectedness, convergence, and continuity. It may perhaps be useful to first note that the topological concept of connectedness is equivalent, in an STS X, to the concept of connectedness in the graphs of the associated preorders on X (see [2]). Thus the component of a point x of X is the minimum open and closed set containing x. The topology which is the intersection of the given topology T with its dual T^* is the topology of the partition of X into its components.

The usual concept of convergence is more or less useful, in STS's. Indeed a filter F converges towards a point x of X if and only if one of the sets of F is included in N_x, and, in finite spaces, the only filters are the so-called "elementary" filters, which are those associated with infinite sequences of points of the space.

Such a definition of convergence is certainly too loose to appeal to the imagination. Take for example the set N of all natural numbers, provided with the right-topology associated with the natural order $\leqq$ on N. Let f be any continuous function of N into X. Then f converges not only to each point of its range, but also to each point of the closure of its range!

The concept of a continuous function between two STS's has much more intuitive meaning, because such a function can be interpreted as an increasing function between preordered sets. We shall call *continuous processes* both continuous functions of N into X like the one just defined, and continuous functions of the set Re of real numbers, provided with the right-topology associated with the natural order $\leqq$ on Re, into X. Such continuous processes have the following

property: if X is finite, X is a Kolmogorov space if and only if every continuous process f is "stationary after a certain k in N (or Re)," i.e.,

$$(\exists k \in N)(\forall n \in N)(n \geqq k \text{ implies } f(n) = f(k)),$$

and similarly for Re.

"Continuous processes" are common in science. For example, consider a thermodynamical system subjected to certain conditions. If we interpret X and the preorder relation on X as above (Section 8, (A)), and if we set the variable $n \in N$ or $t \in \text{Re}$ to be the time, a continuous process is simply a physically possible process under those conditions.

I would like to thank the referee for his numerous and pertinent comments: indeed they led me to rewrite this paper completely.

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ON SIMILARITY IN FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. The concern of this note is the question of necessary and sufficient conditions for continuous functions of two or more variables to be functionally related: the functions f and h are so related if an equation such as

$$f = \alpha \circ h \tag{1.1}$$

is satisfied for some continuous function α, $\alpha \circ h$ designating the composition $\alpha(h)$. Generally speaking, when so related the functions are called similar or almost similar, depending on whether or not α has an inverse.

The underlying space in this paper is the linear space of continuous functions of n variables, $n \geqq 2$, defined on the closed unit cube, E_n, in n-dimensional Euclidean space, the cube being the cartesian product of intervals $[0, 1]$. This space is denoted by $\mathcal{C}_n$; $\mathcal{C}$ stands for the linear space of continuous functions of

a single variable. For points in E_n we use the dual notation $p=(x, y, \cdots)$. The arguments involve a simple examination of the partitions of E_n which are induced by the functions f and h in (1.1). These partitions and their relation to continuity are standard material in topology which falls under the heading of light and monotone mappings (see, e.g. [3]).

2. On the oscillation of functions and similarity. Our initial task in this section is to establish certain properties of the oscillation of the functions of $\mathcal{C}_n$ on certain subsets of E_n. This is prefaced of necessity with the following terminology and notation.

The *restriction* of $h \in \mathcal{C}_n$ to a subset $A \subset E_n$ is designated as $h|A$; the *oscillation* of $h|A$ is marked by $\omega(h|A)$:

$$\omega(h \mid A) = \sup_{p\in A} h(p) - \inf_{p\in A} h(p). \tag{2.1}$$

A *level set* of h, $l_h = l_h(t)$, corresponding to the value t, is defined as

$$l_h(t) = \{p \in E_n \colon h(p) = t\}; \tag{2.2}$$

L_h stands for the family of level sets of h:

$$L_h = \{l_h(t) \colon t \in h(E_n)\}. \tag{2.3}$$

L_h has the simple interpretation as a partition of E_n, and in this context we say that L_f is a refinement of L_h if each level set l_f is a subset of some level set l_h. It must be emphasized that the inclusion $l_h \supset l_f$ referred to here does not imply that $l_f \in L_h$. It merely describes the relation existing between l_f and l_h as point-sets in E_n. In this connection it should also be remembered that the level sets according to (2.2) may be composed of disconnected components.

DEFINITION 2.1. *Consider two functions, f and h, of $\mathcal{C}_n$ (the functions need not be distinct); f is said to be almost similar to h if L_f is a refinement of L_h, the association being designated as*

$$h \succ f. \tag{2.4}$$

The functions are similar, written

$$h \sim f, \tag{2.5}$$

if, and only if, $L_f = L_h$.

According to its definition, similarity is clearly an equivalence relationship; almost similarity, on the other hand, is not: while it is reflexive and transitive, it lacks symmetry. The class of all functions of $\mathcal{C}_n$ which are almost similar to a fixed function contains, therefore, the similarity class of this function as a proper subset, barring the case when it is constant. We could not generate with a suitable equivalence a set from $\mathcal{C}_n$ on which $\omega(f|h)$ would be a norm for fixed h.

The oscillation of f on L_h is defined to be

$$\omega(f \mid h) = \sup_{l_h \in L_h} \omega(f \mid l_h). \tag{2.6}$$

The functionals $\omega(f|h)$ and $\omega(h|f)$ are, in general, distinct when their right entry is fixed. This fact is verified in a nontrivial case.

Example 1. To keep the calculations simple, the example is confined to functions in $\mathcal{C}_2$.

Consider the functions $f=xy$ and $h=x+y$. To compute the number $\omega(f|h)$ we set $x+y=t$ ($0\leqq t\leqq 2$) and substitute for y in $f(x, y)$, therefrom deducing that for each admitted value of t,

$$\sup_{0\leqq x\leqq 1} f(x, t-x) = t^2/4, \tag{2.7}$$

whereas

$$\inf_x f(x, t-x) = \begin{cases} 0 & \text{if } 0\leqq x\leqq t \text{ and } 0\leqq t\leqq 1 \\ t-1 & \text{if } t-1\leqq x\leqq 1 \text{ and } 1\leqq t\leqq 2. \end{cases} \tag{2.8}$$

The conclusion

$$\omega(f\,|\,h) = \tfrac{1}{4} \tag{2.9}$$

is arrived at with a simple calculation.

A similar procedure leads to the result

$$\omega(h\,|\,f) = 1. \tag{2.10}$$

It is easily seen that the relations $\omega(f|h)\geqq 0$, $\omega(af|h)=|a|\omega(f|h)$, $\omega(f+g|h)\leqq\omega(f|h)+\omega(g|h)$, and $\omega(ah+b|h)=0$, a, b, constants, hold for all functions f, g and h, of $\mathcal{C}_n$. These facts are summarized in the statement that, for each fixed $h\in\mathcal{C}_n$, the functional $\omega(f|h)$ determines a seminorm on this space.

Let f and h be given functions of $\mathcal{C}_n$. The main results of this paper are stated as follows:

THEOREM 2.1. *The three statements*, (a) *The equation* $f=X\circ h$ *has a solution* $X\in\mathcal{C}$, (b) $f\succ h$, (c) $\omega(f|h)=0$, *are equivalent.*

THEOREM 2.2. *Let* $f=X\circ h$. *Then the following three statements are equivalent*: (a) *X has an inverse* $X^{-1}\in\mathcal{C}$, (b) $f\sim h$, (c) $\omega(f|h)=\omega(h|f)=0$.

Proof of Theorem 2.1. To begin with, we prove that the requirement (b) is sufficient for (a). For this purpose, designate by l_h^* the collection of all members $l_h\in L_h$ which belong to a single set $l_f\in L_f$; let L_h^* stand for the family of collections l_h^*. Since each of the sets L_f and L_h forms a (closed) covering of E_n, it follows that each l_f intersects some l_h; furthermore, owing to their definition, either $l_f\supset l_h$, or else $l_f\cap l_h=\varnothing$. We are thus led to conclude that $L_f=L_h^*$.

The next part of the scheme is to relate the sets $f(E_n)$ and $h(E_n)$. To accomplish this, we observe that each value $t\in f(E_n)$ specifies a unique member of L_f, namely, $l_f=l_f(t)$; conversely, each level set $l_f\in L_f$ determines one, and only one, point in the range of f. The equivalent association which exists between the values $u\in h(E_n)$ and the members of L_h fails when the latter is replaced by the family L_h^*, and hence a different correspondence is necessitated here.

Realizing that the family L_h^* induces a partition of $h(E_n)$, we mark by u^* the aggregate of those values of u which are determined (in a one-to-one fashion) by the members of l_h^*; let the collection of these be U^*: we assert now that the association in (2.21), which induces a one-to-one correspondence between the sets $f(E_n)$ and U^*, specifies a relationship between the ranges $f(E_n)$ and $h(E_n)$ which suitably defines the function α as demanded in the theorem. To determine this, we argue as follows:

Each point $t \in f(E_n)$ specifies a unique element l_f which, in turn, determines the set l_h^* through the requirement $l_f \supset l_h$, l_h^* giving rise to the unique class u^*, whose entries are obtained from the relation $l_h = l_h(u)$, $l_h \in l_h^*$. The function α is now defined for these values of u as

$$\alpha(u^*) = t:$$

that is, α, being defined pointwise on u^*, is constant on this set.

The same one-to-one correspondence guarantees the pointwise definition of α throughout the interval $h(E_n)$, the resulting function having range $f(E_n)$. It remains, therefore, only to verify the continuity of the function: owing to the continuity of f and h and their demanded relationship this is a matter of routine, and for this reason the verification is omitted here.

This concludes the proof of the claim that (a) follows from (b). To establish the assertion that (b) is, indeed, necessary, we argue as follows:

If L_h is not almost similar to L_f, then it contains at least one element l_h which is not a subset of any l_f: this l_h intersects, therefore, not less than two distinct level sets of f, say, l_f and l_f' : consider two points, $p \in l_f$ and $q \in l_f'$: then $p \neq q$ and it is clear that $h(p) = h(q)$, $\alpha \circ h(p) = \alpha \circ h(q)$, whereas $f(p) \neq f(q)$. Therefore, we can declare the first portion of Theorem 2.1 proved.

We now proceed to demonstrate the relevance of (c) to (b). The argument used in determining the sufficiency of (c) runs like this:

According to its definition, $\omega(f|h) = 0$ if, and only if, $\omega(f|l_h) = 0$ for each $l_h \in L_h$. This, in turn, is valid when, and only when, $f|l_h$ is constant for each member of L_h, the implication being that every member of L_h is contained as a subset in some $l_f \in L_f$. That is, we conclude that $f \succ h$.

The necessity of (c) follows from this fact:

If $\omega(f|h) \neq 0$, then L_h contains at least one member l_h with the property $\omega(f|l_h) \neq 0$. This, however, means that this l_h must intersect (and hence contain) not less than two distinct members of L_f, thereby violating the demanded relation $f \succ h$.

The proof of the theorem is now complete.

Proof of Theorem 2.2. This theorem, which is essentially a symmetric version of the preceding one, follows directly from the latter. Thus, the first claim made in the present theorem will be established once we demonstrate the equivalence of (a) and (b). This, however, is a direct consequence of the fact that $f \sim h$ if, and only if, $f \succ h$ and $h \succ f$. The last assertion made in this theorem follows likewise from Theorem 2.1.

We conclude this section with

COROLLARY 2.1. *Let f and h be given; then $f \succ h$ when, and only when, there is a function $\alpha \in \mathcal{C}$ for which $f \sim \alpha \circ h$.*

3. On the convergence of composite sequences. Prominent among the problems connected with the representation of functions in $\mathcal{C}_n$ in the form (1.1) is that of determining the subclass of $\mathcal{C}_n$ so expressible, when the choice of functions h in (1.1) is suitably restricted. A precise formulation of the problem is this:

Let $\mathcal{C} \circ \mathfrak{D}_n$ stand for the family of all composite functions $\alpha \circ h$, with $\alpha \in \mathcal{C}$ and $h \in \mathfrak{D}_n$, $\mathfrak{D}_n$ being a certain subset of $\mathcal{C}_n$. *We seek necessary and sufficient conditions for any uniformly convergent sequence of functions of $\mathcal{C} \circ \mathfrak{D}_n$ to converge to a function in $\mathcal{C} \circ \mathfrak{D}_n$.*

According to the foregoing, the function $f = \lim_k \alpha_k \circ h_k$ of $\mathcal{C}_n$ belongs to the class $\mathcal{C} \circ \mathfrak{D}_n$ if, and only if, $f \succ h$ for some $h \in \mathfrak{D}_n$; equivalently, necessary and sufficient is the requirement that the infimum

$$\inf_{h \in \mathfrak{D}_n} \lim \omega(\alpha_k \circ h_k \mid h)$$

vanish on $\mathfrak{D}_n$. Clearly, $\lim_k h_k$ belongs to $\mathfrak{D}_n$ only when $\lim_k \omega(\alpha_k \circ h_k \mid h_k) = 0$.

Now consider the function $f = xy$. It was first pointed out by Arnol'd [1] that this function does not admit a representation as

$$f(x, y) = \alpha[a(x) + b(y)] \tag{3.4}$$

with functions of $\mathcal{C}$, despite the fact that it is the uniform limit of the sequence $\{(x+(1/k))(y+(1/k))\}$, each of whose members is expressible in the desired form and is, in addition, strictly monotonic increasing in each variable. Specifically,

$$(x + (1/k))(y + (1/k)) = \exp[\ln(x + (1/k)) + \ln(y + (1/k))].$$

The following is easily verified: Suppose $f = xy$ is representable in the form (3.4); owing to Theorem 2.1, the almost similarity correspondence

$$xy \succ a(x) + b(y)$$

is then required. This, however, is obviously not possible, because the union of the intersection of the x and y coordinate axes with E_2 belongs to L_{xy}, thereby implying the identity $a(x) + b(y) = \text{const.}$ when $x = 0$ or $y = 0$. This last specification requires $a(x) + b(y)$ to be constant throughout E_2, showing thus that (3.4) is not possible. Arnol'd's proof of this impossibility is also simple and can be found in [1].

Vainštein and Kreines established the following sufficient condition [2]: If the uniform limit

$$f = \lim_k \alpha_k[a_k(x) + b_k(y)] \tag{3.5}$$

exists, and if f is strictly monotonic increasing in each variable, then f is of the

form (3.4). (Being zero on the coordinate axes the function $f=xy$ clearly fails to meet the last condition of this proposition.)

It is easily verified that the function $f=\alpha \circ h$ is strictly monotonic in each variable when, and only when, α is strictly monotonic; hence, f and h are similar. In view of Theorem 2.1, therefore, the condition of monotonicity in the result of Vainštein and Kreines is, indeed, not necessary.

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THE DERIVATIVE AS A LINEAR TRANSFORMATION

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1. Introduction. Consider the mapping of E^3 into itself described by

$$F:\quad \begin{aligned} u &= e^x + 2xz - y \\ v &= 4yz - x \\ w &= x^2 + 2y^2 - \sin z. \end{aligned}$$

Following Frechet's lead, the Nevanlinnas [3], Dieudonné [1], and others have defined the derivative of F to be a linear transformation. From this point of view, the derivative of F is determined by the Jacobian matrix

$$F'(p) = \begin{bmatrix} e^x + 2z & -1 & 2x \\ -1 & 4z & 4y \\ 2x & 4y & -\cos z \end{bmatrix}.$$

This matrix is symmetric for every $p=(x, y, z)$. The example, picked from among many which present themselves for the purpose, is intended to demonstrate that symmetry of the matrix $F'(p)$ does not place much of a restriction on the mapping F. It is also intended to create interest in one of the results we shall prove here which says that if the matrix $F'(p)$ is skew symmetric, then F itself must be affine.

Our methods illustrate the use of some structural facts from algebra to prove theorems in analysis. While these theorems can be obtained by the manipulation of subscripts on second and third order partial derivatives, it is from the algebraic point of view that they are most naturally noticed. It is also the point of view from which we gain insight into why the results are true, and from which we are led to raise other interesting questions.

2. Derivatives.

DEFINITION. *Assume that U is an open set in a normed linear space X and that F is a mapping which takes U into a normed linear space Y;*

$$F:\ U \to Y.$$

We say that F is differentiable at a point $x \in U$ if there eixsts a linear transformation $L: X \to Y$, such that for sufficiently small $h \in X$,

$$F(x+h) = F(x) + L(h) + |h|\,\epsilon(x,h),$$

where $\epsilon(x, h)$ is a member of Y which goes to 0 as h goes to 0. When such an L exists, it is called the derivative of F at x, and it is denoted by $F'(x)$.

We are interested in the case where $X = Y = E^n$. In this situation, if F is differentiable throughout U, F' is properly viewed as a mapping

$$F':\ U \to \mathcal{L}(E^n, E^n),$$

where $\mathcal{L}(E^n, E^n)$ is the space of all linear operators on E^n. Regarding $\mathcal{L}(E^n, E^n)$ as a normed linear space in the usual way, we now have a mapping F' of U into a normed linear space. As such, it may itself be differentiable at a point p, in which case the derivative is a linear transformation from E^n into $\mathcal{L}(E^n, E^n)$. This is called the second derivative of F at p and is designated by $F''(p)$.

Note that $F''(p)h \in \mathcal{L}(E^n, E^n)$. It is itself a linear transformation defined for any $k \in E^n$. The expression $[F''(p)h]k$ is usually written in the form $F''(p)(h, k)$ to emphasize that $F''(p)$ is a bilinear transformation on E^n. It would appear that we would have to be careful of the order of h and k in writing $F''(p)(h, k)$, but this is generally not the case. A remarkable theorem [**1**, page 175] tells us that $F''(p)$ is a symmetric bilinear form; that $F''(p)(h, k) = F''(p)(k, h)$. (The usual equality of mixed partial derivatives is a consequence of this theorem.)

Let $\mathfrak{M}^2(E^n, E^n)$ denote the space of all bilinear transformations on E^n. If F is twice differentiable in U, we then have $F'': U \to \mathfrak{M}^2(E^n, E^n)$. It is possible to define a norm on $\mathfrak{M}^2(E^n, E^n)$ so that it too is a normed linear space. Then F'' may be differentiable with derivative $F'''(p)$. And it may be shown that $F'''(p)$ is a multilinear mapping of order 3, and that it is symmetric; i.e., $F'''(p)(h, k, r)$ is invariant under all permutations of h, k, r. In short, the study of the nth derivative corresponds to the study of a symmetric multilinear mapping of order n. If we write $F'''(p)(h, h, h) = F'''(p)h^3$ and similarly for arbitrary n, then it is possible to develop a theorem which parallels Taylor's Theorem.

TAYLOR'S THEOREM. *Suppose that in a spherical neighborhood U of $p_0 \in X$, F has n continuous derivatives and that $F^{(n+1)}$ exists throughout U. Then for any $p \in U$,*

$$\left\| F(p) - \left\{ F(p_0) + F'(p_0)(p-p_0) + \cdots + \frac{1}{n!} F^{(n)}(p_0)(p-p_0)^n \right\} \right\|$$
$$\leqq \frac{1}{(n+1)!} \left\| F^{(n+1)}(p_0 + \lambda(p-p_0))(p-p_0)^{n+1} \right\|,$$

where $\lambda \in (0, 1)$.

Using this notion, the details of a theory of analytic functions in Banach spaces have recently been worked out [5].

We have seen that when $X = E^n$ and $Y = E^n$, it is natural to associate an $n \times n$ matrix with $F'(p)$. Since $F''(p)h$ and $F'''(p)(h, k)$ are also members of $\mathfrak{L}(E^n, E^n)$, they too are naturally associated with $n \times n$ matrices. In general, the kth derivative at p, evaluated at $k-1$ points, corresponds to an $n \times n$ matrix.

We shall be particularly interested in situations in which certain entries of a matrix are equal, symmetric or skew symmetric matrices for example. For this reason, the following remark will be of use to us. It is an easy exercise to verify that it is so.

REMARK. The kth derivative evaluated at $k-1$ points is an $n \times n$ matrix. Suppose that two entries of this matrix are equal or that they differ only in sign. Then the same relationship will hold between corresponding entries of the matrix associated with the next higher derivative.

3. Some results from algebra. We use a computation of Guillemin and Sternberg [2] to obtain some properties of multilinear symmetric operators defined on an inner product space $\mathfrak{H}$. We shall denote the inner product of two elements, say u and v, by $\langle u, v \rangle$. For the case where $\mathfrak{H} = E^n$, we shall think of u and v as column vectors and make use of $\langle u, v \rangle = u^t v$.

THEOREM 1. *Suppose T is a bilinear symmetric operator on $\mathfrak{H}$ with the property that for any three elements u, v, w of $\mathfrak{H}$,*

$$\langle T(u, v), w \rangle + \langle v, T(u, w) \rangle = 0.$$

Then T is identically zero.

Proof. Suppose we write $T(u, v) = (Tu)v$ to emphasize that we are thinking of (Tu) as a member of $\mathfrak{L}(\mathfrak{H}, \mathfrak{H})$. Then using the symmetry of the inner product we may rewrite the hypothesis in the form

$$\langle (Tu)v, w \rangle = -\langle (Tu)w, v \rangle \qquad \text{for any } u, v, w \in \mathfrak{H}, \tag{1}$$

and using the symmetry of T to carry out the following computation,

$$\begin{aligned} \langle (Tu)v, w \rangle &= \langle (Tv)u, w \rangle = -\langle (Tv)w, u \rangle \text{ by (1)} \\ &= -\langle (Tw)v, u \rangle = \langle (Tw)u, v \rangle \text{ by (1)} \\ &= \langle (Tu)w, v \rangle = -\langle (Tu)v, w \rangle \text{ by (1)}. \end{aligned}$$

Then $2\langle T(u, v), w \rangle = 0$ for any $u, v, w \in \mathfrak{H}$ and the assertion is proved.

THEOREM 2. *Suppose T is multilinear of order three and symmetric on $\mathfrak{H}$, and has the property that for any four elements s, t, u, $v \in \mathfrak{H}$,*

$$\langle T(s, t, u), v \rangle + \langle u, T(s, t, v) \rangle = \lambda(s, t) \langle u, v \rangle.$$

where λ is a real valued function defined on $\mathfrak{H}\times\mathfrak{H}$. Then for any two orthogonal unit vectors, x and y, we have $\lambda(x, x) = -\lambda(y, y)$, and if the dimension of $\mathfrak{H}$ is greater than 2, then $\lambda\equiv 0$.

Proof. Writing $T(s, t, u) = T(s, t)u$ and setting $s=t=x$ and $u=v=y$ in the hypothesis, we see that for any x and y in $\mathfrak{H}$,

$$\lambda(x, x)\langle y, y\rangle = 2\langle T(x, x)y, y\rangle. \tag{2}$$

If x and y are chosen to be orthogonal, then by choosing $s=u=x$ and $t=v=y$ in the hypothesis, we get

$$\langle T(x, y)x, y\rangle = -\langle T(x, y)y, x\rangle. \tag{3}$$

Thus, for x and y orthogonal in $\mathfrak{H}$, we have

$$\begin{aligned}
\lambda(x, x)\langle y, y\rangle &= 2\langle T(x, x)y, y\rangle \text{ by (2)}\\
&= 2\langle T(x, y)x, y\rangle \text{ by the symmetry of } T\\
&= -2\langle T(x, y)y, x\rangle \text{ by (3)}\\
&= -2\langle T(y, y)x, x\rangle \text{ by the symmetry of } T\\
&= -\lambda(y, y)\langle x, x\rangle \text{ by (2).}
\end{aligned}$$

The first assertion of the theorem now follows from the fact that when x and y are unit vectors, $\langle y, y\rangle = \langle x, x\rangle = 1$. To see that $\lambda\equiv 0$ when dim $\mathfrak{H}>2$, choose an arbitrary unit vector x in $\mathfrak{H}$. We can then find vectors y and z in $\mathfrak{H}$ such that x, y, and z form an orthonormal set. The theorem immediately gives

$$\begin{aligned}
\lambda(x, x) &= -\lambda(y, y)\\
\lambda(y, y) &= -\lambda(z, z)\\
\lambda(x, x) &= -\lambda(z, z).
\end{aligned}$$

It is easily seen from these equations that $\lambda(x, x) = 0$, and since T is linear and symmetric in s and t, λ is also. We may therefore use the polarization technique on λ to show that $\lambda(s, t) = 0$ for any s, t in $\mathfrak{H}$.

4. Some theorems of analysis. As a first application of these theorems, we use Theorem 1 to obtain a well-known result.

THEOREM 3. *Suppose F: $U\rightarrow E^n$ has two continuous derivatives in U (i.e., $F\in C''(U)$) and that for every $p\in U$, $F'(p)$ is orthogonal. Then F is a rigid motion.*

Proof. Since orthogonal linear transformations preserve angles and distances, we may write $\langle F'(p)v, F'(p)w\rangle = \langle v, w\rangle$.

Differentiation gives

$$\langle F''(p)(u, v), F'(p)w\rangle + \langle F'(p)v, F''(p)(u, w)\rangle = 0.$$

We may rewrite this in the form

$$\langle [F'(p)]^t F''(p)(u, v), w\rangle + \langle v, [F'(p)]^t F''(p)(u, w)\rangle = 0$$

from which it is clear according to Theorem 1 that for any vectors u, v,

$$[F'(p)]^t F''(p)(u, v) = 0.$$

Multiplication on the left by $F'(p)$ gives us $F''=0$ from which we conclude that F' is constant. This means that F is affine, and since $F'(p)$ is a constant orthogonal matrix, the linear part of the affine transformation F is orthogonal. F is a rigid motion.

We turn now to the proof of our previous assertion about $F'(p)$ being skew symmetric.

THEOREM 4. *Let F: $U \rightarrow E^n$ be in $C''(U)$ and suppose that for every $p \in U$, the matrix associated with $F'(p)$ is skew symmetric. Then F is affine.*

Proof. By virtue of the remark above, the matrix associated with $F''(p)u$ is also skew symmetric. Now for a skew symmetric matrix, we know that for arbitrary vectors v, w,

$$v^t[F''(p)u]^t w + v^t[F''(p)u]w = 0.$$

Written in terms of the inner product notation,

$$\langle [F''(p)u]v, w\rangle + \langle v, [F''(p)u]w\rangle = 0.$$

Since $F''(p)$ is known to be a symmetric, bilinear operator on E^n, appeal to Theorem 1 shows us that $F''(p)=0$, from which it follows that F is affine.

THEOREM 5. *Let F: $U \rightarrow E^n$, $n \geqq 3$, be in $C'''(U)$ and suppose that for every p in U, the matrix associated with $F'(p)$ takes the form $\alpha(p)I+S(p)$, where $S(p)$ is skew symmetric. Then if p_0 is in U, we have for p in U,*

$$F(p) = F(p_0) + F'(p_0)(p - p_0) + F''(p_0)(p - p_0, p - p_0).$$

Proof.

$$\begin{aligned} F'(p) &= \alpha(p)I + S(p) \\ F''(p)t &= \alpha'(p)tI + S'(p)t \\ F'''(p)(s, t) &= \alpha''(p)(s, t)I + S''(p)(s, t). \end{aligned}$$

According to the REMARK, we know $S''(p)(s, t)$ is still skew symmetric. We are thus able to write in terms of the inner product notation

$$\langle F'''(p)(s, t)u, v\rangle + \langle u, F'''(p)(s, t)v\rangle = 2\alpha''(p)(s, t)\langle u, v\rangle.$$

Since $n > 2$, appeal to Theorem 2 enables us to conclude that $\alpha''(p) \equiv 0$, hence that $F'''(p)(s, t) = S''(p)(s, t)$. But the skew symmetry of $S''(p)(s, t)$ means

$$\langle S''(p)(s, t)v, u\rangle + \langle v, S''(p)(s, t)u\rangle = 0. \tag{4}$$

The symmetry of $F'''(p)$ in s, t, u guarantees the symmetry of $S''(p)(s, t)u$. Holding s fixed and viewing $S''(p)(s, t)u$ as a bilinear symmetric operator on (t, u) which satisfies (4) means, according to Theorem 1, that $S''(p)=0$. The

conclusion stated now follows from Taylor's Theorem.

One way to define a conformal mapping from a region $U \subset E^n$ into E^n is to require that the matrix corresponding to $F'(p)$ take the form

$$\alpha(p)A(p),$$

where α is a real valued function defined on U and A is a matrix valued function which is orthogonal for every $p \in U$. A theorem of Liouville says that if $n \geqq 3$, then F is a rigid motion or the composite of a rigid motion with an expansion and/or the inversion mapping defined by $F(p) = (p/|p|^2)$. A coordinate free proof of this theorem, cast in the setting of an arbitrary Hilbert space and using the methods described in this paper, has been given by R. Nevanlinna [4].

This problem suggest others. Suppose we know that $F'(p)$ is associated with a matrix that takes the form

$$\alpha(p)A, \tag{5}$$

where A is a constant matrix (not necessarily orthogonal). We have obtained results saying that then α must be constant also.

THEOREM 6. *Let F map an open connected subset U of an n-dimensional space into the same space, and suppose that for each $p \in U$, $F'(p) = \alpha(p)A$, where α: $U \to Re$ and A is an $n \times n$ matrix of rank at least two. Then α is constant; i.e., F is linear.*

The following proof, due to Dr. Robert Ryan of the Office of Naval Research, is much simpler than the one we originally found and is included with his kind permission.

Proof. Pick $p_0 \in U$. Define $R(p) = F(p - p_0)$. For p sufficiently close to p_0, R is defined, and

$$R'(p) = F'(p - p_0) = \alpha(p - p_0)A.$$

Choose p_1, p_2 so that $A^t p_1$ and $A^t p_2$ are linearly independent. Then we may choose a nonsingular matrix B so that $B^t A^t p_i = e_i$ $(i = 1, 2)$ where e_i is the vector whose only nonzero entry is a 1 in the ith position. (Note that if $p = (x_1, \cdots, x_n)$, S: $U \to Re$, then $S'(p)e_i = (\partial/\partial x_i)S(p)$.) Now define by means of the inner product

$$W_i(p) = \langle R(Bp), p_i\rangle, \qquad i = 1, 2.$$

Then $W_i'(p)q = \langle R'(Bp)Bq, p_i\rangle = \alpha(Bp - p_0)\langle q, B^t A^t p_i\rangle$. Hence

$$W_1(p)e_k = \alpha(Bp - p_0)\delta_{1k},$$
$$W_2(p)e_k = \alpha(Bp - p_0)\delta_{2k}.$$

The first of these says that $W_1(p)$, hence $\alpha(Bp - p_0)$, depends only on x_1. The second similarly says $\alpha(Bp - p_0)$ depends only on x_2. Thus, $\alpha(Bp - p_0)$ is constant in a neighborhood of the origin; α is constant in a neighborhood of p_0.

5. Some questions. Viewing the derivative as a linear transformation thus leads to a number of interesting questions. We conclude with two for which we do not know the answers.

The requirement that $F'(p)$ take the form (5) is equivalent to requiring that all the $F'(p)$ be contained in a one dimensional subset of $\mathcal{L}(E^n, E^n)$. What can we learn about F if $F'(p)$ is known to be contained in a two dimensional subset? This means $F'(p)$ must be of the form $\alpha(p)A+\beta(p)B$. (For $n=2$, the holomorphic functions are of this form.)

Turning to a different type of question, suppose it is known that for every $p\in U$, the norm of the linear transformation $F'(p)$ remains constant. Can we say anything more about F? Dale Varberg points out that F need not be linear.

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ON THE CHARACTERISTIC ROOTS OF A MATRIX

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Introduction. If we define

$$P_\kappa = \sum_{\substack{\nu=1\\ \nu\neq\kappa}}^{n} |a_{\kappa\nu}|, \qquad (\kappa = 1(1)n) \tag{1}$$

and

$$\begin{aligned} P_{\kappa\lambda} = {} & |a_{\kappa\lambda}|\, P_\lambda + |a_{\lambda\kappa}|\,(P_\kappa - |a_{\kappa\lambda}|) \\ & + \sum_{\nu=1}^{n} |a_{\kappa\nu}a_{\lambda\nu}| + \sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu} + a_{\kappa\mu}a_{\lambda\nu}|, \end{aligned} \tag{2}$$

where $\kappa\neq\lambda$, $\nu\neq\kappa, \lambda$, $\mu\neq\kappa, \lambda$ and where κ, λ, μ and ν run from 1 to n, then $P_{\kappa\lambda} \leq P_\kappa P_\lambda$ (see [2] p. 553). Using these definitions A. Brauer proved in [2] and [3] the following theorems and results.

THEOREM 1. *Let $A=[a_{\kappa\lambda}]$ be an $n\times n$ matrix with real or complex elements. Then each characteristic root of A lies in the union of the $\binom{n}{2}$ ovals of Cassini*

$$|z - a_{\kappa\kappa}|\,|z - a_{\lambda\lambda}| \leq P_{\kappa\lambda}, \tag{3}$$

where κ, $\lambda = 1(1)n$ and $\kappa\neq\lambda$.

If the matrix A is such that

$$|a_{\kappa\kappa}a_{\lambda\lambda}| > P_{\kappa\lambda}, \quad \kappa, \lambda = 1(1)n; \qquad \kappa \neq \lambda, \tag{4}$$

and ω is a characteristic root, then

$$\begin{aligned} |\omega| &\geqq \tfrac{1}{2} \min_{\substack{\kappa,\lambda=1(1)n \\ \kappa\neq\lambda}} \left\{ |a_{\kappa\kappa}| + |a_{\lambda\lambda}| - ((|a_{\kappa\kappa}| - |a_{\lambda\lambda}|)^2 + 4P_{\kappa\lambda})^{1/2} \right\} \\ &= L > 0, \quad \text{(see [2] p. 24).} \end{aligned} \tag{5}$$

THEOREM 2. *Let* $A = [a_{\kappa\lambda}]$ *be an* $n \times n$ *matrix with real or complex elements. Then if* (4) *holds and all the diagonal elements of* A *are real positive numbers, all the characteristic roots of* A *satisfy* $\mathrm{Re}(\omega) \geqq L > 0$.

In this paper we extend the idea of Theorem 2 to all those matrices which satisfy (4) and whose diagonal elements lie on the imaginary axis. If all the diagonal elements lie on the positive imaginary axis then $\mathrm{Im}(\omega) \geqq L > 0$, while if they all lie on the negative imaginary axis $\mathrm{Im}(\omega) \leqq -L < 0$. By applying these results to skew-hermitian matrices and using the relation $U = \theta(I-S)(I+S)^{-1}$, where U is unitary, S skew-hermitian and θ a root of unity, we obtain a criterion for the stability of U. Finally, we apply the above results to the skew-hermitian matrices $\frac{1}{2}(iA + iA^H)$, $(1/2i)(iA - iA^H)$, where A is an arbitrary matrix, and obtain lower bounds for the real and imaginary parts of the characteristic roots of A.

Main theorems.

THEOREM 3. *If* $A = [a_{\kappa\lambda}] > 0$ *is an* $n \times n$ *positive matrix and* $\rho(A)$ *denotes the spectral radius of* A, *then all the characteristic roots of* A, *except the largest one, lie in the annulus*

$$0 < L \leqq |\omega| \leqq \frac{M - m}{M + m} \rho(A),$$

where $m = \min_{\kappa,\lambda=1(1)n} a_{\kappa\lambda}$, $M = \max_{\kappa,\lambda=1(1)n} a_{\kappa\lambda}$, *provided* (4) *holds.*

Proof. Since A is positive, it has a simple real positive characteristic root which is equal to the spectral radius $\rho(A)$ and all the other characteristic roots satisfy

$$|\omega| \leqq \frac{M - m}{M + m} \rho(A),$$

a result derived from Hopf's inequality (see [1] p. 92). Also since (4) holds $0 < L \leqq |\omega|$. Hence the theorem is proved.

THEOREM 4. *Let* $A = [a_{\kappa\lambda}]$ *be an* $n \times n$ *matrix with real or complex elements. Then if* (4) *holds and all the diagonal elements of* A *lie on the positive imaginary axis, all the characteristic roots of* A *satisfy* $\mathrm{Im}(\omega) \geqq L > 0$.

Proof. If the characteristic root ω of A is one of the elements of the main diagonal, then by hypothesis ω is different from zero and lies on the positive imaginary axis and so the theorem is trivial. We shall assume, therefore, that ω lies in one of the $\binom{n}{2}$ ovals of Cassini

$$|z - ia_{\kappa\kappa}|\,|z - ia_{\lambda\lambda}| \leqq P_{\kappa\lambda}, \qquad \kappa \neq \lambda;\ \kappa, \lambda = 1(1)n,$$

and $P_{\kappa\lambda} > 0$ with $a_{\kappa\kappa} > 0$, for each κ, as stipulated in the hypothesis of the theorem. Writing the equation of the ovals in Cartesian coordinates we get

$$g(x, y) = (x^2 + (y - a_{\kappa\kappa})^2)(x^2 + (y - a_{\lambda\lambda})^2) - P_{\kappa\lambda}^2 = 0.$$

If we put $x^2 + (y - a_{\kappa\kappa})^2 = F$, $x^2 + (y - a_{\lambda\lambda})^2 = H$, then clearly

$$\frac{dy}{dx} = \frac{x(F + H)}{-(y - a_{\kappa\kappa})H - (y - a_{\lambda\lambda})F}.$$

The point with the coordinates

$$x_1 = 0$$
$$y_1 = \tfrac{1}{2}(a_{\kappa\kappa} + a_{\lambda\lambda} - ((a_{\kappa\kappa} - a_{\lambda\lambda})^2 + 4P_{\kappa\lambda})^{1/2})$$

has the smallest distance from the origin. Assume now that $a_{\kappa\kappa} \geqq a_{\lambda\lambda}$. Since $P_{\kappa\lambda} > 0$, $y_1 < \frac{1}{2}(a_{\kappa\kappa} + a_{\lambda\lambda} - ((a_{\kappa\kappa} - a_{\lambda\lambda})^2)^{1/2}) = a_{\lambda\lambda} \leqq a_{\kappa\kappa}$. Hence $y_1 - a_{\kappa\kappa} \leqq y_1 - a_{\lambda\lambda} < 0$.

Now $H > 0$ and $F > 0$ and so the denominator in $(dy/dx)_{(x_1, y_1)}$ is strictly positive, but the numerator is equal to zero. Hence the tangent at (x_1, y_1) is parallel to the x-axis. This tangent does not intersect the oval in any other point, since

$$(y_1 - a_{\kappa\kappa})^2 (y_1 - a_{\lambda\lambda})^2 = P_{\kappa\lambda}^2$$

and so $(x^2 + (y_1 - a_{\kappa\kappa})^2)(x^2 + (y_1 - a_{\lambda\lambda})^2) > P_{\kappa\lambda}^2$, $x \neq 0$. Hence all the points in the interior or on the boundary of the oval have a positive y coordinate greater than or equal to y_1. From this and Theorem 1 we conclude that $\operatorname{Im}(\omega) \geqq y_1 \geqq L > 0$.

COROLLARY 1. *Let* $A = [a_{\kappa\lambda}]$ *be an* $n \times n$ *skew-hermitian matrix. If* (4) *holds and all the diagonal elements of* A *lie on the positive imaginary axis, each characteristic root of* A *lies on the positive imaginary axis and its modulus is greater than or equal to* L.

Proof. Since all the characteristic roots of A are purely imaginary, the corollary follows immediately from Theorem 4.

THEOREM 5. *Let* $A = [a_{\kappa\lambda}]$ *be an* $n \times n$ *matrix with real or complex elements. If* (4) *holds and all the diagonal elements of* A *lie on the negative imaginary axis, all the characteristic roots of* A *satisfy* $\operatorname{Im}(\omega) \leqq -L < 0$.

Proof. The proof is similar to that of Theorem 4.

COROLLARY 2. *Let* $A = [a_{\kappa\lambda}]$ *be an* $n \times n$ *skew-hermitian matrix. If* (4) *holds*

and all the diagonal elements of A lie on the negative imaginary axis, each characteristic root of A lies on the negative imaginary axis and its modulus is greater than or equal to L.

Proof. Since all the characteristic roots of A are purely imaginary, the corollary follows immediately from Theorem 5.

It is known that every unitary matrix can be expressed in the form $U=\theta(I-S)(I+S)^{-1}$, where S is a skew-hermitian matrix and θ a root of unity (a result due to L. Loewy, see [5] p. 79).

THEOREM 6. *Let $U=[u_{\kappa\lambda}]=\theta(I-S)(I+S)^{-1}$ be an $n\times n$ unitary matrix, where S is skew-hermitian and θ a root of unity. Then, if*

$$|s_{\kappa\kappa}s_{\lambda\lambda}| > P_{\kappa\lambda}, \qquad \kappa\neq\lambda;\ \kappa,\lambda = 1(1)n, \tag{6}$$

and all the diagonal elements of S lie on the positive imaginary axis or they all lie on the negative imaginary axis, then provided the quantity 'L' obtained from the matrix S is such that (6a) *is satisfied, each characteristic root of U has a negative real part, namely U is stable.*

Proof. We denote the characteristic roots of S by iy, where y is real, and from the hypothesis of the theorem together with Corollary 1 Theorem 4 and Corollary 2 Theorem 5 we conclude that $|y|\geqq L$. All the characteristic roots of U are of the form

$$\theta\frac{1-iy}{1+iy}$$

and $|\theta(1-iy)/(1+iy)|=1$ as was to be expected. Furthermore, if

$$\begin{aligned}&\frac{\pi}{2}<\arg\theta+\arg\left(\frac{1-iy}{1+iy}\right)\leqq\pi \quad \text{or}\\ &-\pi<\arg\theta+\arg\left(\frac{1-iy}{1+iy}\right)<-\frac{\pi}{2},\end{aligned} \tag{6a}$$

then

$$\operatorname{Re}\left(\theta\frac{1-iy}{1+iy}\right)<0$$

and so U is stable.

Two particular cases are known explicitly. If U has no characteristic root equal to -1, then $U=(I-S)(I+S)^{-1}$ and (6a) is satisfied provided L is greater than 1. If U has no characteristic root equal to 1, then $U=-(I-S)(I+S)^{-1}$ and (6a) is satisfied provided $|y|$ is less than 1.

Let A be a general $n\times n$ matrix and f, F be the least and greatest characteristic roots of the hermitian matrix $\frac{1}{2}(A+A^H)$ and g, G the least and greatest characteristic roots of the hermitian matrix $(1/2i)(A-A^H)$.

If ω is any characteristic root of A, then (see [6] p. 389)

$$f \leqq \mathrm{Re}(\omega) \leqq F, \tag{7}$$

$$g \leqq \mathrm{Im}(\omega) \leqq G. \tag{8}$$

THEOREM 7. *Let $A = [a_{\kappa\lambda}]$ be a general $n \times n$ matrix and ω a characteristic root of A. Suppose*

$$\left| (a_{\kappa\kappa} + \bar{a}_{\kappa\kappa})(a_{\lambda\lambda} + \bar{a}_{\lambda\lambda}) \right| > P_{\kappa\lambda}, \qquad \kappa \neq \lambda;\ \kappa, \lambda = 1(1)n.$$

Then $\mathrm{Re}(\omega) \geqq L > 0$ *if* $\frac{1}{2}(a_{\kappa\kappa} + \bar{a}_{\kappa\kappa}) > 0$ *for each* κ, *and* $\mathrm{Re}(\omega) \leqq -L < 0$ *if* $\frac{1}{2}(a_{\kappa\kappa} + \bar{a}_{\kappa\kappa}) < 0$ *for each* κ, *where*

$$\begin{aligned} P_{\kappa\lambda} = {} & \left| a_{\kappa\lambda} + \bar{a}_{\lambda\kappa} \right| P_\lambda + \left| a_{\lambda\kappa} + \bar{a}_{\kappa\lambda} \right| (P_\kappa - \left| a_{\kappa\lambda} + \bar{a}_{\lambda\kappa} \right|) \\ & + \sum_{\nu=1}^{n} \left| (a_{\kappa\nu} + \bar{a}_{\nu\kappa})(a_{\lambda\nu} + \bar{a}_{\nu\lambda}) \right| + \sum_{\nu<\mu} \left| (a_{\kappa\nu} + \bar{a}_{\nu\kappa})(a_{\lambda\mu} + \bar{a}_{\mu\lambda}) \right. \\ & \left. + (a_{\kappa\mu} + \bar{a}_{\mu\kappa})(a_{\lambda\nu} + \bar{a}_{\nu\lambda}) \right|, \end{aligned}$$

$$P_\kappa = \sum_{\nu=1}^{n} \left| a_{\kappa\nu} + \bar{a}_{\nu\kappa} \right|, \qquad P_\lambda = \sum_{\nu=1}^{n} \left| a_{\lambda\nu} + \bar{a}_{\nu\lambda} \right|$$

and

$$\begin{aligned} L = \tfrac{1}{2} \min_{\substack{\kappa,\lambda=1(1)n \\ \kappa\neq\lambda}} \Big\{ & \left| \tfrac{1}{2}(a_{\kappa\kappa} + \bar{a}_{\kappa\kappa}) \right| + \left| \tfrac{1}{2}(a_{\lambda\lambda} + \bar{a}_{\lambda\lambda}) \right| \\ & - \left(\left(\left| \tfrac{1}{2}(a_{\kappa\kappa} + \bar{a}_{\kappa\kappa}) \right| - \left| \tfrac{1}{2}(a_{\lambda\lambda} + \bar{a}_{\lambda\lambda}) \right| \right)^2 + P_{\kappa\lambda} \right)^{1/2} \Big\}, \end{aligned}$$

$\kappa, \lambda, \mu, \nu = 1(1)n$, $\kappa \neq \lambda$, $\mu \neq \kappa, \lambda$ *and* $\nu \neq \kappa, \lambda$.

Proof. The proof follows directly from Corollary 1 Theorem 4 and Corollary 2 Theorem 5 together with (7).

THEOREM 8. *Let $A = [a_{\kappa\lambda}]$ be a general $n \times n$ matrix and ω a characteristic root of A. Suppose*

$$\left| (a_{\kappa\kappa} - \bar{a}_{\kappa\kappa})(a_{\lambda\lambda} - \bar{a}_{\lambda\lambda}) \right| > P_{\kappa\lambda}, \qquad \kappa \neq \lambda;\ \kappa, \lambda = 1(1)n.$$

Then $\mathrm{Im}(\omega) \geqq L > 0$ *if* $(1/2i)(a_{\kappa\kappa} - \bar{a}_{\kappa\kappa}) > 0$ *for each* κ, *and* $\mathrm{Im}(\omega) \leqq -L < 0$ *if* $(1/2i)(a_{\kappa\kappa} - \bar{a}_{\kappa\kappa}) < 0$ *for each* κ, *where*

$$\begin{aligned} P_{\kappa\lambda} = {} & \left| a_{\kappa\lambda} - \bar{a}_{\lambda\kappa} \right| P_\lambda + \left| a_{\lambda\kappa} - \bar{a}_{\kappa\lambda} \right| (P_\kappa - \left| a_{\kappa\lambda} - \bar{a}_{\lambda\kappa} \right|) \\ & + \sum_{\nu=1}^{n} \left| (a_{\kappa\nu} - \bar{a}_{\nu\kappa})(a_{\lambda\nu} - \bar{a}_{\nu\lambda}) \right| + \sum_{\nu<\mu} \left| (a_{\kappa\nu} - \bar{a}_{\nu\kappa})(a_{\lambda\mu} - \bar{a}_{u\lambda}) \right. \\ & \left. + (a_{\kappa\mu} - \bar{a}_{\mu\kappa})(a_{\lambda\nu} - \bar{a}_{\nu\lambda}) \right|, \end{aligned}$$

$$P_\kappa = \sum_{\nu=1}^{n} \left| a_{\kappa\nu} - \bar{a}_{\nu\kappa} \right|, \qquad P_\lambda = \sum_{\nu=1}^{n} \left| a_{\lambda\nu} - \bar{a}_{\nu\lambda} \right|$$

and

$$L = \frac{1}{2} \min_{\substack{\kappa,\lambda=1(1)n \\ \kappa\neq\lambda}} \left\{ \left| \frac{1}{2i}(a_{\kappa\kappa} - \bar{a}_{\kappa\kappa}) \right| + \left| \frac{1}{2i}(a_{\lambda\lambda} - \bar{a}_{\lambda\lambda}) \right| - \left(\left(\left| \frac{1}{2i}(a_{\kappa\kappa} - \bar{a}_{\kappa\kappa}) \right| - \left| \frac{1}{2i}(a_{\lambda\lambda} - \bar{a}_{\lambda\lambda}) \right| \right)^2 + P_{\kappa\lambda} \right)^{1/2} \right\},$$

$\kappa, \lambda, \mu, \nu = 1(1)n$, $\kappa \neq \lambda$, $\mu \neq \kappa, \lambda$ *and* $\nu \neq \kappa, \lambda$.

Proof. The proof follows directly from Corollary 1 Theorem 4 and Corollary 2 Theorem 5 together with (8).

Example: Consider the matrix

$$A = \begin{bmatrix} 1+i & 3-i.5 & 5+i2 \\ 3-i.5 & .5+i & 4-i.25 \\ 6-i2 & 4+i.25 & -1+i \end{bmatrix}.$$

From Theorem 8 $\mathrm{Im}(\omega) \geqq L \doteqdot .29289$. In fact

$$\omega_1 = -1.85806 + i1.50968,$$
$$\omega_2 = -6.37195 + i0.70887,$$
$$\omega_3 = 8.73001 + i0.78144,$$

correct to five decimal places.

REMARK. The results of this paper can be rewritten to take account of those in [4] which are an improvement of the results in [2].

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COMPACTNESS AND CERTAIN SUBCLASSES OF SCHLICHT FUNCTIONS

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This note will be concerned with single-valued analytic functions which are normalized schlicht functions on the open unit disc centered at the origin of the complex plane. This class of functions will be denoted as S where each function satisfies the conditions of being univalent on $|z| < 1$, $f(0) = 0$, and $f'(0) = 1$. In particular, the subclasses of schlicht functions which will be considered are

those whose members map $|z|<1$ onto domains which are starlike with respect to the origin, convex, and close-to-convex. These subclasses will be denoted as S^*, S^c, and $₵$ respectively. Considerable literature exists concerning the properties of these subclasses of functions with emphasis on the coefficient bounds of their respective power series expansions [1, 2]. It was shown by W. Kaplan [3] that these subclasses satisfy

$$S^c \subset S^* \subset ₵ \subset S. \tag{1}$$

In this note, the term compact will be used in the same sense as used by L. M. Graves [4, p. 357] when concerned with metric spaces. Specifically, a subclass K of a function class M is *compact* in case every infinite sequence of K has a subsequence uniformly convergent on $|z|\leqq r<1$ for every $r<1$ to a limit function in K. Also, a subclass K of a function class M is *compact in* M in case an infinite sequence of K has a subsequence uniformly convergent on $|z|\leqq r<1$ for every $r<1$ to a limit function in M but not in K.

It is well known that S is compact [5, p. 146]. The objectives of this note are to show that each of the subclasses S^*, S^c, and $₵$ is also compact.

Reference will be made to functions which are normalized by $f(0)=1$, single valued, analytic and defined on $|z|<1$ with positive real part. This class of functions will be denoted as R^+. The fact that R^+ is compact will be used to achieve the objectives of this note. Although simple proofs showing that R^+ is compact exist [5, p. 143], the following proof is included because of the interesting concepts involved.

LEMMA 1. *The class R^+ is compact.*

Proof. Let $\{f_k\}$ be an infinite sequence of analytic functions each of which is in class R^+. From the F. Riesz and Herglotz Stieltjes integral representation of functions in R^+ [6, p. 275], there exists for each function $f_k(z)$ a bounded nondecreasing real function $\phi_k(t)$ such that $\phi_k(0)=0$, $\phi_k(2\pi)>0$, and

$$f_k(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\phi_k(t), \qquad |z| < 1, \tag{2}$$

where $\phi_k(t)$ is determined uniquely up to an additive constant at all points of continuity determined by $f_k(z)$; furthermore,

$$f_k(0) = \int_0^{2\pi} d\phi_k(t) = 1, \qquad k = 1, 2, 3, \cdots. \tag{3}$$

From (3), the infinite sequence of total variations $\{V(\phi_k)\}$ is uniformly bounded which implies that the sequence of functions $\{\phi_k\}$ is also uniformly bounded. Since, in addition, each function $\phi_k(t)$ is nondecreasing on $[0, 2\pi]$, there exists a bounded nondecreasing function $\phi(t)$ and a subsequence $\{\phi_{kj}\}$ which converges to $\phi(t)$ for each value of t in $[0, 2\pi]$ [7, p. 221]. Then there exists a corresponding subsequence $\{f_{kj}\}$ such that

$$f_{kj}(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\phi_{kj}(t), \qquad j = 1, 2, 3, \cdots. \tag{4}$$

Let the function $f(z)$ be determined by

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\phi(t). \tag{5}$$

Since the sequence $\{V(\phi_{kj})\}$ is uniformly bounded and the sequence $\{\phi_{kj}\}$ converges to $\phi(t)$ for each t in $[0, 2\pi]$, Kelly's Second Theorem [7, p. 233] may be applied in the following manner.

For $|z| \leqq r<1$, $\epsilon>0$, $M(r)>|(e^{it}+z)/(e^{it}-z)|$, there exists a k_0 such that

$$|f(z) - f_{kj}(z)| \leqq M(r)\left|\int_0^{2\pi} d\phi(t) - \int_0^{2\pi} d\phi_{kj}(t)\right| < \epsilon \tag{6}$$

$$\text{for } kj > k_0, \quad \text{and} \quad |z| \leqq r < 1,$$

which implies that $\{f_{kj}\}$ converges uniformly to $f(z)$ on $|z| \leqq r<1$. By (5), $f(z)$ is in R^+, therefore R^+ is compact.

THEOREM 1. *The class S^* is compact.*

Proof. Let $\{f_k\}$ be an infinite sequence of functions each of which is in S^*. Obtain an infinite sequence $\{F_k\}$ by setting

$$F_k(z) = (zf_k'(z)/(f_k(z)). \tag{7}$$

Each function $F_k(z)$ is in R^+ [5, p. 221]. By (1), S^* is compact in S and there exists a subsequence $\{f_{kj}\}$ which converges uniformly on $|z| \leqq r<1$ for each $r<1$ to $f(z)$ in S. This implies that $\{f_{kj}'\}$ converges uniformly on $|z| \leqq r<1$ to $f'(z)$. From (7), the subsequence $\{F_{kj}\}$ converges pointwise on $|z| \leqq r<1$ to $F(z)$, where

$$F(z) = (zf'(z))/(f(z)), \qquad |z| \leqq r < 1. \tag{8}$$

Since R^+ is compact, there exists a subsequence of $\{F_{kj}\}$ which converges uniformly and therefore pointwise on $|z| \leqq r<1$ to a function $G(z)$ in R^+. But every pointwise convergent subsequence of $\{F_{kj}\}$ converges pointwise to $F(z)$ on $|z| \leqq r<1$. Therefore $F(z)\equiv G(z)$, $F(z)$ is in R^+, and $(zf'(z))/(f(z))$ is in R^+ which implies that $f(z)$ is in S^*. Therefore S^* is compact.

THEOREM 2. *The class S^c is compact.*

Proof. From (1) and Theorem 1, S^c is compact in S^*. For an infinite sequence $\{f_k\}$ of functions, each of which is contained in S^c, there exists a subsequence $\{f_{kj}\}$ which converges uniformly on $|z| \leqq r<1$ for every $r<1$ to a function $f(z)$ in S^*. This implies that the sequence $\{zf_{kj}'\}$ converges uniformly on $|z| \leqq r<1$ to $zf'(z)$.

Since each term of the sequence $\{zf_{kj}'\}$ is in S^* [5, p. 223] and S^* is compact,

the limit function $zf'(z)$ is also in S^*. Therefore $f(z)$ is in S^c and S^c is compact.

THEOREM 3. *The class* ₵ *is compact.*

Proof. Let $\{f_k\}$ be an infinite sequence of functions each of which is in ₵. For each $f_k(z)$, there exists a function $g_k(z)$ in S^c such that $f_k'(z)/g_k'(z) = F_k(z)$ is in R^+ [3]. Since S^c is compact, there exists a subsequence $\{g_{k_j}\}$ which converges uniformly on $|z| \leqq r < 1$ to a function $g(z)$ in S^c. This implies that the subsequence $\{g_{kj}'\}$ converges uniformly on $|z| \leqq r < 1$ to a function $g'(z)$.

A necessary and sufficient condition that $g_k(z)$ is in S^c is that $zg_k'(z)$ is in S^* [5, p. 223]. Since the function $zg_k'(z)$ has only one zero on $|z| \leqq r < 1$, namely at $z=0$, $g_k'(z)$ is zero-free on $|z| \leqq r < 1$. This implies that each function of the subsequence $\{g_{kj}'\}$ is zero-free and by Hurwitz's Theorem [8, p. 119], the limit function $g'(z)$ is also zero-free.

For each $g_{kj}(z)$, there corresponds an $f_{kj}(z)$ in ₵ and an $F_{kj}(z)$ in R^+. From the sequence $\{F_{kj}\}$, there exists a subsequence $\{F_{k_j i}\}$ which converges uniformly to $F(z)$ in R^+, and the corresponding subsequence $\{g_{kji}'\}$ converges uniformly to $g'(z)$. Thus, the sequence $\{F_{kji}g_{kji}'\}$ converges uniformly on $|z| \leqq r < 1$ to $F(z)g'(z)$. Let

$$h(z) = F(z)g'(z), \tag{9}$$

then the subsequence $\{f_{kji}'\}$ converges uniformly to the analytic function $h(z)$ on $|z| \leqq r < 1$. This implies that

$$\int_0^z h(z)dz \qquad \text{is in ₵}. \tag{10}$$

Because of the uniform convergence of $\{f_{kji}'\}$ to $h(z)$,

$$\int_0^z h(z)dz = \int_0^z \lim f_{kji}'(z)dz = \lim \int_0^z f_{kji}'(z)dz = \lim f_{kji}(z) = f(z). \tag{11}$$

From (10) and (11), the subsequence $\{f_{kji}\}$ converges uniformly on $|z| \leqq r < 1$ to $f(z)$ in ₵. Therefore ₵ is compact.

In conclusion, it is observed* that the function classes S^*, S^c and ₵ have the representation of the form $F[p(z)]$, where $p(z)$ is in class R^+. Since R^+ is compact and each of S^*, S^c and ₵ is compact, it is interesting to establish the conditions satisfied by the function F which will imply the compactness of the class $F(R^+)$.

From the fundamentals of function theory, the following two lemmas and theorem are obtained which identify the conditions satisfied by F.

LEMMA 2 [4, p. 360, Thm. 27]. *With P as a compact metric space and Q as a metric space, if the function F is continuous on P to Q, then $F(P) = Q$ is compact.*

LEMMA 3. *Let P be a complex analytic function class with the function F defined on P to a complex function class Q. A necessary and sufficient condition that Q be a complex analytic function class is that F be analytic.*

THEOREM 4. *Let P be a compact complex analytic function class with the function F defined on P to a complex function class Q. If F is an analytic function, then $F(P)=Q$ is a compact analytic function class.*

Thus, if F is an analytic function, then $F(R^+)$ is a compact analytic function class.

By determining the precise form of the function F which yields S^*, S^c and $\mathbb{C}$, one can observe that F is analytic and thereby show that these subclasses of S are compact.

* The author is indebted to the referee for suggesting this representation and the investigation associated with it.

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MATHEMATICAL NOTES

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REFLECTIONS OF THE NOTES EDITOR

It has been a year since the present editorial policy was announced, and we hope in particular that our readers are finding the Notes sections relevant and interesting.

The change has created some confusion among our authors and potential authors, and for this reason we wish to discuss the standards of these sections.

I. Mathematical Notes. The MONTHLY is not a journal of research. However, its very wide circulation, especially beyond the major universities, presents an audience which should appreciate short articles that are not too technical or specialized. We also accept specialized articles when they are well-written and do not make excessive demands on the reader, and offer insights of interest to more than a small group of workers in the field. In so doing, we hope to provide glimpses of current trends in the science.

In general, however, we are interested in short papers dealing with material familiar to a reasonable number of MONTHLY readers. We are able to afford more space than is common in research journals, and authors are urged to add an occasional paragraph if that will increase readability. But most important, there must be a sound mathematical reason for an article. One theorem with real content and an elegant proof is worth twenty-five new definitions or minor generalizations. Results without significant examples are always judged inferior to those with meaningful applications.

II. Classroom Notes. Contributions to this section are few in number. This is not surprising since many undergraduate fields have been so worked over that new insights useful in the classroom are not easy to produce. Good Classroom Notes are greatly valued and will be published promptly.

A PROPERTY OF GRADIENTS

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The object of this note is to prove the following:

THEOREM. *Let f be a continuous real-valued function on the unit Euclidean n-ball $B=\{x\colon \|x\|\leqq 1\}$, let the first order partial derivatives of f exist at every point in the interior of B, and let $|f(x)|\leqq 1$, whenever $\|x\|=1$. Then there is a point v in the interior of B for which $\|\operatorname{grad} f(v)\|\leqq 1$.*

The function f, defined by $f(x)=a\cdot x$ where $\|a\|=1$, shows that the final inequality in the Theorem cannot be sharpened. The Theorem thus provides the answer to H. S. Shapiro's question concerning the best value of the constant in Problem E 1986 [this MONTHLY, 75 (1968) p. 787]. In addition, the Theorem shows that the hypotheses of the problem, that f be differentiable on an open set containing B, and that $|f(x)|\leqq 1$ for every x in B, can be relaxed.

Proof of the Theorem. Let

$$g(x)=\|x\|^2-f(x)^2,\qquad \lambda=\min_{\|x\|\leqq 1} g(x).$$

Then $g(x)\geqq 0$ whenever $\|x\|=1$, and $\lambda\leqq g(0)\leqq 0$. There is thus a point v such that $\|v\|<1$ and $g(v)=\lambda$. Consequently

$$\operatorname{grad} g(v)=2v-2f(v)\operatorname{grad} f(v)=0,$$

and so

$$f(v)^2\|\operatorname{grad} f(v)\|^2=\|v\|^2=f(v)^2+\lambda\leqq f(v)^2.$$

The required conclusion follows either if $\lambda<0$, or if $\lambda=0$ and $g(v)=0$ for some point v such that $0<\|v\|<1$.

In the one remaining case we have $g(x)>\lambda=0$ whenever $0<\|x\|<1$; so that $|f(x)|<\frac{1}{2}$ whenever $\|x\|=\frac{1}{2}$ and, consequently, there is a positive ϵ for which

$$\max_{\|x\|=1/2} |f(x)+\epsilon| < \tfrac{1}{2}.$$

Setting

$$h(x) = \|x\|^2 - \{f(x)+\epsilon\}^2, \qquad \mu = \min_{\|x\|\leq 1/2} h(x),$$

we observe that $h(x)>0$ whenever $\|x\|=\frac{1}{2}$, and $\mu \leqq h(0) = -\epsilon^2<0$. Hence there is a point w such that $\|w\|<\frac{1}{2}$ and $h(w)=\mu$. As above we deduce that

$$\{f(w)+\epsilon\}^2\|\text{grad } f(w)\|^2 = \|w\|^2 = \{f(w)+\epsilon\}^2+\mu < \{f(w)+\epsilon\}^2,$$

from which it follows that $\|\text{grad } f(w)\|<1$.

PIVOTAL ROLE OF THE TRIPLE CROSS PRODUCT

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This paper gives a proof of the following:

THEOREM. *Let $(R, \cdot)$ be a real inner product space and let $(\mathrm{x} \circ y)$ be a vector product defined in R. If R is at least two dimensional and if the vector product satisfies the identity*

$$(\mathrm{x}\circ y)\circ z = (\mathrm{x}\cdot z)y - (y\cdot z)\mathrm{x} \tag{1}$$

for all x, y, z *in* R, *then*

(A) $(\mathrm{x}\circ y)$ *is linear in both* x *and* y,
(B) $(\mathrm{x}\circ y)+(y\circ \mathrm{x})=0$ *for all* x, y *in* R,
(C) R *is exactly three dimensional, and*
(D) *if* (i, j, k) *is an orthonormal basis for* R, *then* $(i\circ j)=ck$, $(j\circ k)=ci$, *and* $(k\circ i)=cj$, *where* c *is either* $+1$ *or* -1.

By a vector product, $(\mathrm{x}\circ y)$, is meant a not necessarily linear function defined for all ordered pairs, (x, y), of elements of R and having values again in R. The identity (1) is the "triple cross product identity" and the conclusion of the Theorem is that $(\mathrm{x}\circ y)$ can only be the usual three-space cross product (either right handed or left handed). That is, the triple cross product identity is sufficient to single out the usual cross product(s) from among all possible vector products, and it even forces the dimension of the space on which it operates to be three.

The proof of the Theorem makes use of some lemmas.

LEMMA 1. $\mathrm{x}\circ y = y\circ(-\mathrm{x})$ *for all* x, y *in* R.

Proof. Let $\mathrm{x}\circ y=u$ and $y\circ(-\mathrm{x})=u'$ and apply (1) to show that $u\circ v = u'\circ v$ for all v in R, so that

$$(2)\qquad (u \circ v) \circ w = (u' \circ v) \circ w \qquad \text{for all } v, w \text{ in } R.$$

For any w in R a nonzero v can be selected orthogonal to w. This done, (1) applied to (2) yields

$$-(u \cdot w)v = -(u' \cdot w)v \quad \text{or}$$
$$(u \cdot w) = (u' \cdot w) \qquad \text{for all } w \text{ in } R.$$

This implies that $u = u'$ and the lemma is proved.

LEMMA 2. *For given* x, y, z *in* R *and scalars* a *and* b, *suppose that*

$$(3)\qquad (z \circ w) = a(x \circ w) + b(y \circ w)$$

or all w *in* R. *Then* $z = ax + by$.

Proof. In equation (3) put $-w$ for w and apply Lemma 1 to obtain

$$(4)\qquad (w \circ z) = a(w \circ x) + b(w \circ y).$$

Now in (4) replace w by $(u \circ v)$ and apply (1) to obtain, after some rearranging,

$$(5)\qquad (u \cdot [z - ax - by])v = (v \cdot [z - ax - by])u$$

for all u, v in R. As before, for specified u, v can always be selected linearly independent of u, so that (5) implies that

$$(u \cdot [z - ax - by]) = 0 \qquad \text{for all } u \text{ in } R$$

and it follows immediately that $z = ax + by$ and the lemma is proved.

Now parts (A) and (B) of the Theorem can be proved. Use (1) to show that

$$(6)\qquad [(ax + by) \circ z] \circ w = a[(x \circ z) \circ w] + b[(y \circ z) \circ w],$$

$$(7)\qquad [x \circ (ay + bz)] \circ w = a[(x \circ y) \circ w] + b[(x \circ z) \circ w],$$

$$(8)\qquad (x \circ y) \circ w = -((y \circ x) \circ w).$$

Lemma 2 applied to (6) and (7) yields part (A) of the Theorem and applied to (8) yields part (B).

From the linearity of $(x \circ y)$ it follows in particular that

$$(9)\qquad 0 \circ x = x \circ 0 = 0 \qquad \text{for all } x \text{ in } R;$$

from the anticommutativity of $(x \circ y)$ it follows in particular that

$$(10)\qquad x \circ x = 0 \qquad \text{for all } x \text{ in } R.$$

These results will be used in proving the final lemma.

LEMMA 3. *Let* i *and* j *be unit orthogonal vectors in* R *and let* $k' = (i \circ j)$. *Then*

$$(11)\qquad i = (j \circ k') \quad \text{and} \quad j = (k' \circ i).$$

Furthermore (i, j, k') *is an orthonormal set, and if* k *is any unit vector orthogonal*

to i and j, then

(12) $$k = ck', \quad \textit{where } c \textit{ is either } +1 \textit{ or } -1.$$

Proof. To prove (11) write

$$\begin{aligned} j \circ k' &= j \circ (i \circ j) && \text{(Definition of } k'\text{)} \\ &= -(i \circ j) \circ j && \text{(Anticommutativity)} \\ &= i && \text{(Equation (1))} \\ k' \circ i &= j && \text{(Equation (1)).} \end{aligned}$$

To prove that k' is orthogonal to i and j write

$$\begin{aligned} 0 &= k' \circ k' && \text{(Equation (10))} \\ &= (i \circ j) \circ k' && \text{(Definition of } k'\text{)} \\ &= (j \cdot k')i - (i \cdot k')j && \text{(Equation (1)).} \end{aligned}$$

Since i and j are linearly independent this implies that $(i \cdot k') = (j \cdot k') = 0$.

To prove that k' is a unit vector write

$$\begin{aligned} i &= j \circ k' && \text{(Equation (11))} \\ &= (k' \circ i) \circ k' && \text{(Equation (11))} \\ &= (i \cdot k')k' - (k' \cdot k')i && \text{(Equation (1)).} \end{aligned}$$

Since $(i \cdot k') = 0$, this implies that $(k' \cdot k') = 1$ and k' is a unit vector.

Finally, let k be a unit vector orthogonal to i and j. Then

$$\begin{aligned} k' \circ k &= (i \circ j) \circ k && \text{(Definition of } k'\text{)} \\ &= (i \cdot k)j - (j \cdot k)i && \text{(Equation (1))} \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= (k' \circ k) \circ k && \text{(Equation (9))} \\ &= (k' \cdot k)k - (k \cdot k)k' && \text{(Equation (1)).} \end{aligned}$$

Since k and k' are both unit vectors, this implies that $k = ck'$ with c either $+1$ or -1, and the proof of the lemma is complete.

Now to prove part (C) of the Theorem, observe that R contains unit orthogonal vectors i and j, because it is at least two dimensional. By Lemma 3 it contains a third unit vector, k', orthogonal to both, so that it is at least three dimensional. Again by Lemma 3 it is at most three dimensional because any other unit vector k orthogonal to i and j is a multiple of k'.

To prove part (D) of the Theorem use the multiplication table for (i, j, k') given in Lemma 3 and the fact that $k = ck'$.

With this the proof of the Theorem is complete.

A NOTE ON A THEOREM OF SAGLE

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Sagle ([2], Corollary 5.3) proved for the special case of Malcev algebras the following theorem.

THEOREM. *Let $\mathfrak{A}$ be a nonassociative algebra over a field F, without absolute divisors of zero such that $\mathfrak{A}^2=\mathfrak{A}$. Let $\mathfrak{L}$ be the Lie multiplication algebra of $\mathfrak{A}$, i.e., the Lie algebra generated by the right and left multiplications R_x, L_x. If $\mathfrak{L}$ is simple, then $\mathfrak{A}$ is simple.*

In this note we adapt techniques of Albert [1] to give a strikingly simple proof of the general theorem. In this connection we point out that Sagle's proof for the special case, besides involving techniques applicable only to Malcev algebras, is apparently not so simple, even for this special case, as the proof given below for the general theorem.

Proof of the theorem. If $\mathfrak{B}$ is a nonzero ideal of $\mathfrak{A}$, then $\mathfrak{B}=\mathfrak{A}E$ for a fixed idempotent projection E of the vector space $\mathfrak{A}$ onto $\mathfrak{B}$. The set $\mathfrak{T}$ of all S in $\mathfrak{L}$ such that $S=SE$ is an ideal of $\mathfrak{L}$ (see [1], Section 7). Furthermore, $R_x\neq 0$ or $L_x\neq 0$ for each x in $\mathfrak{B}$, since there are no absolute divisors of zero in $\mathfrak{A}$. Also $R_x=R_xE$, $L_x=L_xE$, since $\mathfrak{B}$ is an ideal of $\mathfrak{A}$. Consequently $\mathfrak{T}=\mathfrak{L}$ by the simplicity of $\mathfrak{L}$, i.e., $S=SE$ for each S in $\mathfrak{L}$. In particular, $\mathfrak{A}=\mathfrak{A}^2=\mathfrak{A}R(\mathfrak{A})=\mathfrak{A}R(\mathfrak{A})E \subseteq\mathfrak{B}$. In other words, the only nonzero ideal of $\mathfrak{A}$ is $\mathfrak{A}$ itself, or, equivalently, $\mathfrak{A}$ is simple.

REMARK. In the case of a Lie algebra $\mathfrak{A}$, $\mathfrak{L}=\{\operatorname{ad} x \mid x\in\mathfrak{A}\}$ ($\operatorname{ad} x$ is the map: $y\to[xy]$). Since there are no absolute divisors of zero, $x\to\operatorname{ad} x$ is an isomorphism, hence $\mathfrak{A}$ is simple; $\mathfrak{A}^2=\mathfrak{A}$ is a consequence in this case.

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NOTE ON TWO PROBLEMS ON MATRICES

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In a recent issue of the Monthly [1] appeared two problems on matrices which we rewrite here:

1. If each A_i is symmetric and $\sum_{i=1}^{m} A_i=I$, then the three following conditions are equivalent:

(a) Each A_i is idempotent,

(b) $A_iA_j = 0, \qquad i\neq j,$

(c) $\sum_{i=1}^{m} n_i = n,$

where n_i is the rank of A_i and n the dimension of the A_i.

2. Let A_i $(i=1, \cdots, m)$ be a collection of $n\times n$ symmetric matrices where the rank of A_i is n_i. Let $A=\sum_{i=1}^m A_i$ have rank p. Consider the four conditions

$$\mathrm{C}_1.\ \text{Each } A_i \text{ is idempotent,}$$

$$\mathrm{C}_2.\ A_iA_j=0, \qquad i\neq j,$$

$$\mathrm{C}_3.\ A \text{ is idempotent,}$$

$$\mathrm{C}_4.\ p=\sum_{i=1}^m n_i.$$

Then

(1) Any two of the three conditions C_1, C_2, C_3 imply all four of the conditions C_1, C_2, C_3, C_4.

(2) C_3 and C_4 imply C_1 and C_2.

In a discussion of solutions to a related problem E 1933 the editor asked whether the requirement of symmetry is necessary in these two problems. In this note we shall give a complete answer to this question. We mention that both problems are taken from [2].

For our purpose the elements of all matrices are assumed to belong to an arbitrary field F of characteristic 0.

THEOREM 1. *If $\sum_{i=1}^m A_i=I$ then the conditions* (a), (b), (c) *are equivalent.*

Proof. (i) (b)$\Rightarrow$(a). Multiply $\sum A_i=I$ by A_i to get $A_i^2=A_i$.

(ii) (a)$\Rightarrow$(c). Since A_i are idempotent matrices we have $n_i=\operatorname{tr} A_i$. Take traces in $\sum A_i=I$ to get (c).

(iii) (c)$\Rightarrow$(b). To prove this let us consider A_i as linear operators on an n-dimensional vector space V over F. Let V_i be the range of A_i. Since $\sum A_i=I$ we must have $\sum V_i=V$. By (c) we have $\sum \dim V_i=\dim V$. Therefore we have a direct sum $V=V_1\oplus\cdots\oplus V_m$. Using $\sum A_i=I$ we get

$$\left(\sum A_i\right)A_jx = A_jx$$

for all $x\in E$ and all $j=1, \cdots, m$. Since $A_iA_jx\in V_i$, $A_jx\in V_j$ and the sum of V_i is direct we must have

$$A_iA_jx = 0$$

for all $x\in E$ if $i\neq j$. This means that $A_iA_j=0$ $(i\neq j)$.

THEOREM 2. *If $\sum_{i=1}^m A_i=A$ then*

$$\mathrm{C}_1 \textit{ and } \mathrm{C}_2 \Leftrightarrow \mathrm{C}_3 \textit{ and } \mathrm{C}_4, \qquad \mathrm{C}_1 \textit{ and } \mathrm{C}_3 \Rightarrow \mathrm{C}_2 \textit{ and } \mathrm{C}_4.$$

It is not true that C_2 *and* C_3 *imply* C_1 *or* C_4.

Proof. (i) C_1 and $\mathrm{C}_2\Rightarrow\mathrm{C}_3$ and C_4. Assume C_1 and C_2. C_3 follows by straightforward computation. Now, C_4 follows by taking traces in $\sum A_i=A$.

(ii) C_3 and $\mathrm{C}_4\Rightarrow\mathrm{C}_1$ and C_2. The proof of this implication is essentially the same as the part (iii) of the proof of Theorem 1. We shall not repeat the argument.

(iii) C_1 and $C_3 \Rightarrow C_2$ and C_4. Assume C_1 and C_3. Then C_4 follows by the same argument as in (i). Now, C_2 follows by (ii).

(iv) For the last assertion it is sufficient to give an example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This work was supported in part by N.R.C. Grant A-5285.

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ON THE WEYL CRITERION IN THE THEORY OF UNIFORM DISTRIBUTION MODULO 1

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Let $\{u_n\}$ be a sequence of distinct real numbers lying in the interval (0, 1), and let x be a real number with $0 \leqq x \leqq 1$. If N is any positive integer, consider the first N elements of the sequence $\{u_n\}$. By $A(N) = A(x, N)$ denote the number of those elements among the $u_1, u_2, \cdots, u_N$, that belong to the interval $[0, x)$. Furthermore, set

$$f(x, N) = \frac{A(x, N)}{N} - x. \tag{1}$$

The sequence $\{u_n\}$ is said to be *uniformly distributed* if for each x in $[0, 1]$, we have $\lim_{N\to\infty} f(x, N) = 0$. The well-known fundamental theorem, due to H. Weyl [1], provides a necessary and sufficient condition for a sequence of real numbers to be uniformly distributed modulo one.

THEOREM 1. (WEYL'S CRITERION). *A sequence $\{u_n\}$ of real numbers is uniformly distributed* (mod 1) *if and only if for each positive integer h,* $\lim_{N\to\infty} (1/N) \sum_{n=1}^{N} e^{2\pi i h u_n} = 0$.

The purpose of this paper is to give a new proof of the theorem, based on the following relation (2) (which is of importance in that part of uniform distribution mod 1 theory where the notion of the discrepancy mod 1 of a sequence is explored [2], p. 90). This relation, which can be shown by elementary integration, was found by J. F. Koksma and communicated by him to the second author in 1963. A simple proof of (2) can be given by labeling the first N points of the given sequence so that $0 < u_1 < \cdots < u_N < 1$. The relation mentioned above is

$$\int_0^1 [f(x, N)]^2 dx = \left[\int_0^1 f(x, N)dx\right]^2 + \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i h u_n)\right|^2. \tag{2}$$

Proof of Theorem 1. First, let $\{u_n\}$ be uniformly distributed modulo 1, i.e., assume that

$$\lim_{N\to\infty} f(x, N) = 0. \tag{3}$$

From (3) and the Lebesgue Bounded Convergence Theorem, it follows that both integrals occurring in (2) tend to zero as $N\to\infty$. Therefore each term of the series in (2) tends to zero as $N\to\infty$, so that we get (1).

Next assume that for every positive integer h, (1) holds. Since the series in (2) is uniformly convergent with respect to N, it follows that this series converges to zero as $N\to\infty$. This implies that for $N\to\infty$ we have

$$\int_0^1 [f(x, N)]^2 dx - \left(\int_0^1 f(x, N)dx\right)^2 \to 0. \tag{4}$$

We now show that (4) implies $\lim_{N\to\infty} f(x, N) = 0$.

First Step. The functions $f(x, N)$ ($N = 1, 2, \cdots$) are all of bounded variation on $[0, 1]$ and the set of all total variations of the $f(x, N)$ is bounded (in fact each total variation is equal to 2). It follows from Helly's Theorem [**3**], p. 233, that the sequence $\{f(x, N)\}$ ($N = 1, 2, \cdots$) contains a subsequence $\{f(x, N_k)\}$ ($k = 1, 2, \cdots$), which converges to a value, called $f(x)$, for each $x \in [0, 1]$. Moreover, $f(x)$ is of bounded variation on $[0, 1]$.

Second Step. Since $\lim_{k\to\infty} f(x, N_k) = f(x)$ everywhere, it follows by the Lebesgue Bounded Convergence Theorem that

$$\lim_{k\to\infty} \int_0^1 f(x, N_k)dx = \int_0^1 f(x)dx \text{ and } \lim_{k\to\infty} \int_0^1 (f(x, N_k))^2 dx = \int_0^1 (f(x))^2 dx.$$

Hence

$$\lim_{k\to\infty} \left[\int_0^1 (f(x, N_k))^2 dx - \left(\int_0^1 f(x, N_k)dx\right)^2\right] = \int_0^1 (f(x))^2 dx - \left(\int_0^1 f(x)dx\right)^2.$$

Hence by (4) we get

$$\int_0^1 (f(x))^2 dx = \left(\int_0^1 f(x)dx\right)^2. \tag{5}$$

On the other hand, by the Cauchy-Schwarz-Buniakovski inequality we always have

$$\left(\int_0^1 f(x)dx\right)^2 \leqq \int_0^1 (f(x))^2 dx$$

and the equality in (5) implies that $f(x) = \lambda =$ constant, for almost all x in $[0, 1]$. Now, the assumption $\lambda \neq 0$ leads to a contradiction. To see this we need only consider the shape of the graph of $f(x, N)$ in the neighborhood of $x = 0$ and $x = 1$.

In fact, for arbitrary N we have $f(x, N) = -x$ for $x \in [0, \delta_N)$ and $f(x, N) = 1-x$ for $x \in (1-\delta'_N, 1]$, where δ_N and δ'_N are some positive numbers. Obviously this is possible only if $\lambda = 0$, so that

$$\lim_{k\to\infty} f(x, N_k) = 0 \text{ almost everywhere in } [0, 1]. \tag{6}$$

Third Step. We now show that $f(x) = \lim_{k\to\infty} f(x, N_k) = 0$ for all x in $[0, 1]$. Assume, on the contrary, that for some $x_0 \in [0, 1]$ we have $\lim_{k\to\infty} f(x_0, N_k) = f(x_0) = a \neq 0$. Set $A(x, N_k)/N_k = \phi(x, k)$. We have $\lim_{k\to\infty} \phi(x_0, k) = x_0 + a \neq x_0$. It follows from the definition of $A(x, N)$ that $\phi(x, k)$ is a nondecreasing function of x. Therefore the limit function to which the sequence $\{\phi(x, k)\}$ converges must also be nondecreasing. But, since by (6) there are values of x to the right and to the left of x_0, arbitrarily close to x_0, at which $\phi(x, k) \to x$, therefore a must be zero (otherwise the monotonicity of the limit function would be destroyed). This proves that $\lim_{k\to\infty} \phi(x, k) = x$ everywhere in $[0, 1]$.

Fourth Step. Assume now that $\lim_{N\to\infty} f(x, N) = 0$ for all x in $[0, 1]$ is not true. This means that there is an $\epsilon_0 > 0$, an $x_0 \in [0, 1]$, and a strictly increasing sequence of positive integers $\{n_1, n_2, \cdots, n_k, \cdots\}$ such that for $k = 1, 2, \cdots$ we have $|f(x_0, n_k)| \geqq \epsilon_0$. Consider the sequence of functions $\{f(x, n_k)\}$ $(k = 1, 2, \cdots)$. By the same Helly's Theorem mentioned before, there is a subsequence of $\{f(x, n_k)\}$, say $\{f(x, m_k)\}$ $(k = 1, 2, \cdots)$, which converges for all x in $[0, 1]$. By the preceding arguments, it converges to zero. Since $\{m_k\}$ is a subsequence of $\{n_k\}$ we are led to a contradiction. This completes the proof of Theorem 1.

REMARK. The above argument can also be applied to more general cases, e.g., that of sequences $\{x_n\}$ which have a modulo 1 continuous distribution function. As is known, a necessary and sufficient condition for a sequence $\{x_n\}$ to have a continuous distribution function $z(x)$ $(0 \leqq x \leqq 1, z(0) = 1, z(1) = 1)$ is that for each $h = 1, 2, \cdots$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i h x_n) = \int_0^1 \exp(2\pi i h t) dz(t).$$

In the present paper the case $z(x) \equiv x$ has been considered. See also [4].

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NONPERIODICITY OF SOLUTIONS OF AN nth ORDER EQUATION

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W. R. Utz [this MONTHLY, 74 (1967) 705–6] has considered the differential equation

$$y^{(n)} = p(x)f(y) \tag{1}$$

for the case $n=3$. It is assumed that $p(x)$ is continuous on the interval $[a, b]$ and $f(y)$ is continuous for all y.

The purpose of this paper is to notice that his results and methods of proof generalize to the nth order case. There is also a minor flaw in his first theorem that states if $n=3$, if $p(x)$ is positive on $[a, b]$ and if for all real $y\neq 0$ $yf(y)>0$, then no solution $y\not\equiv 0$ of (1) in (a, b) can be tangent to $y=0$ more than once.

The flaw is seen by taking $p(x)\equiv 24$ on $[0, 2]$ and

$$f(y) = \begin{cases} y^{1/4} & \text{for } y \geqq 0, \\ -|y|^{1/4} & \text{for } y < 0. \end{cases}$$

We have that $yf(y)>0$ for $y\neq 0$; however,

$$y(x) = \begin{cases} 0 & \text{for } x \in [0, 1], \\ (x-1)^4 & \text{for } x \in (1, 2], \end{cases}$$

is a solution to (1) for $n=3$, is not identically zero on $[0, 2]$ and is tangent to $y=0$ on $[0, 1]$.

This problem can be overcome if one assumes that a solution $y(x)$ has separated zeros on (a, b). That is, about each zero x_0 of y there is a deleted neighborhood on which $y(x)\neq 0$. Unique solutions to initial value problems will assure separated zeros.

We shall divide our results into the categories of n odd and n even.

THEOREM 1. *If $p(x)$ is continuous and nonzero on $[a, b]$, if $yf(y)>0$ for all $y\neq 0$ and if $n=2m+1\geqq 3$, then no solution $y\not\equiv 0$ of (1) on (a, b) having separated zeros can satisfy*

$$y(x_0) = y^{(2m-1)}(x_0)y'(x_0) = \cdots = y^{(m+1)}(x_0)y^{(m-1)}(x_0) = y^{(m)}(x_0) = 0 \tag{2}$$

for more than one $x_0\in(a, b)$.

Proof. Unless there is one x_0 satisfying (2) we are done, and so we make that assumption. Then $y(x_0)=0$ and there is a deleted neighborhood of x_0 on which $y(x)\neq 0$.

Define

$$v = \sum_{j=0}^{m-1} (-1)^j y^{(2m-j)} y^{(j)} + \frac{(-1)^m}{2}\left[y^{(m)}\right]^2.$$

One can easily verify that $v' = y^{(2m+1)}y = p(x)f(y)y$ so that by the assumptions either $v' \geqq 0$ or $v' \leqq 0$ holds everywhere on (a, b) with strict inequality wherever $y(x) \neq 0$. Thus since $v(x_0) = 0$ we have $v(x) \neq 0$ for all $x \neq x_0$. Consequently (2) cannot hold anywhere except at x_0. This proves the theorem.

THEOREM 2. *If $p(x)$ is continuous with $(-1)^m p(x) < 0$ on $[a, b]$ when $n = 2m \geqq 2$ and $yf(y) > 0$ for $y \neq 0$, then no solution $y \not\equiv 0$ of* (1) *on (a, b) having separated zeros can satisfy*

$$(3) \qquad y(x_0) = y^{(2m-2)}(x_0)y'(x_0) = \cdots = y^{(m)}(x_0)y^{(m-1)}(x_0) = 0$$

for more than one $x_0 \in (a, b)$.

Proof. Here define $v = \sum_{j=0}^{m-1} (-1)^j y^{(2m-1-j)} y^{(j)}$ where $y(x)$ is as in the theorem and assume there is an x_0 such that (3) holds.

Here we have

$$\begin{aligned} v' &= y^{(2m)}y + (-1)^{m-1}[y^{(m)}]^2 \\ &= p(x)f(y)y + (-1)^{m-1}[y^{(m)}]^2 \end{aligned}$$

so that $(-1)^m v' = (-1)^m p(x)f(y)y + (-1)^{2m-1}[y^{(m)}]^2 \leqq 0$ with strict inequality holding where $y(x) \neq 0$. By the same argument as in Theorem 1 the desired conclusion follows.

THEOREM 3. *Under the hypotheses of Theorem* 1 *for n odd and of Theorem* 2 *for n even, no solution of* (1) *not identically zero is periodic.*

Proof. In either case, by the properties of f, $f(0) = 0$; so that $y \equiv 0$ is a periodic solution of (1).

On the other hand if y were a periodic solution to (1) then in each case v would be periodic and monotonic. This is impossible unless v is identically constant. In each case $v'(x) \neq 0$ when $y(x) \neq 0$ so that v cannot be identically constant for nontrivial y. This proves the theorem.

AN ALGEBRAIC STRUCTURE FOR GRAPHS ON THE FACTORIZATION OF GROUPOIDS

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I. Neutral Partial Groupoids: Representation. The linear graphs have been until now considered as combinatorial objects without well-defined bonds with the classical algebraic structures.

It would be interesting, from the conceptual point of view, to succeed in defining graphs by means of an algebraic structure, in order to throw some light on the nature of combinatorics. This is the purpose of the note.

Let us denote by $p(E, \circ) = p(B \rightarrow E)$, where $B \subseteq E \times E$, the set E with a law of composition $\circ$ defined on some couples B of its elements. $p(E, \circ)$ is called a

partial groupoid or p-groupoid. If x (at the left) and y (at the right) cannot be composed, we write $x \circ y = \varnothing$. We define the valency $v(x)$ of an element x of a p-groupoid as $v(x) = |\{y | x \circ y \text{ or } y \circ x \text{ are defined}\}|$. It can be seen that this definition is analogous to the definition of valency of a vertex in a graph.

If the law of composition is defined on any couple of elements of E, we say that the groupoid is a complete groupoid, or simply a groupoid. The groupoid is a special case of the p-groupoid.

If given x and y of E, we write $x \circ y$ without further qualification, it means that $x \circ y$ is defined.

A graph $G(V, \Gamma(V))$, where V is the set of vertices, $\Gamma(V)$ the set of edges of the graph, is called the transformation graph of a left-neutral p-groupoid $p(E, \circ)$ if:

(i) there exists a bijection $f: E \to V$,

(ii) $e \circ e'$ is defined if and only if there exists a unique oriented edge (vv') $\in \Gamma(V)$ with $f(e) = v$, $f(e') = v'$.

As an immediate consequence of this definition, we have

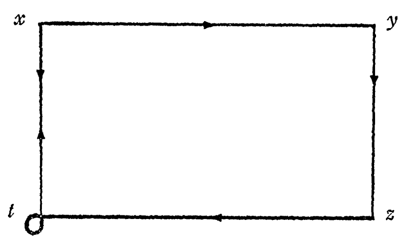
THEOREM 1. *Every graph is the transformation graph of a left-neutral p-groupoid; conversely, any left-neutral p-groupoid has a transformation graph.*

Example of left-neutral p-groupoid: The elements of E are: x, y, z, t. The rules of composition are the following:

$$x \circ y = y, \quad y \circ z = z, \quad t \circ z = z, \quad x \circ t = t, \quad t \circ x = x, \quad t \circ t = t,$$

$$z \circ y = z \circ z = z \circ t = x \circ x = y \circ y = y \circ x = z \circ x = y \circ t = t \circ y = x \circ z = \varnothing.$$

The transformation graph of $p(E, \circ)$ is the following:



A p-cycloid of length 3 is a p-groupoid such that $x_1 \circ (x_2 \circ x_3) = (x_1 \circ x_2) \circ x_3$. An element g is left-cancellable if $g \circ x = g \circ y$ implies $x = y$; a p-groupoid is left cancellable if every element is left-cancellable. A left p-semi-group is a left cancellable p-cycloid of length 3. It is immediately verified that:

PROPOSITION: Any left-neutral (complete) groupoid is a left-semi-group.

This property is interesting because the representation of the left-neutral complete groupoid is a complete graph. It agrees with the use by Hammersley of semi-groups to proceed to enumeration on complete graphs [1].

II. Factorization of Groupoids. Let us call an f-groupoid, a p-groupoid such that:

(i) $\forall x \in E,\ v(x) \geqq 1$,

(ii) $x \circ y = x' \circ y' \Rightarrow x = x',\ y = y'$,

(iii) $x \circ y \neq \varnothing \Rightarrow y \circ x = \varnothing$.

Let $p(E, \circ)$ be a p-groupoid such that $\forall x \in E,\ v(x) \geqq 1$.

$f(E_i, \circ_i)$ is a partial replica of $p(E, \circ)$ iff:

(i) $E_i \subseteq E$,

(ii) $f(E_i, \circ_i)$ is an f-groupoid,

(iii) $x \circ_i y = z \Rightarrow x \circ y = z$, but $x \circ_i y = \varnothing$ does not necessarily imply $x \circ y = \varnothing$.

Finally, we say that $p(E, \circ)$ is a simple product of p-groupoids $p(E_i, \circ_i)$ if (i) $E_i \subseteq E$ and (ii) the mappings $(x, y) \rightarrow x \circ y$ for x, y are uniquely determined by the mappings $(x, y) \rightarrow x \circ_i y$ for x, y, i. We write

$$p(E, \circ) = S \prod_i p(E_i, \circ_i).$$

Remark: We can better see the difference between the notions of direct product, sum and simple product, by means of an example.

Let $P = p(E, \circ)$ be a p-groupoid defined on $E = \{x, y, z\}$; only $x \circ y = z$, $z \circ y = y$, $y \circ z = z$ are defined. Two partial replicas are the following:

$\mathrm{I} = p(E_1, \circ_1)$: $E_1 = \{x, y, z\}$ only $x \circ_1 y = z$ $z \circ_1 y = y$ are defined;

$\mathrm{II} = p(E_2, \circ_2)$: $E_2 = \{y, z\}$ only $y \circ_2 z = z$ is defined.

The direct product of $p(E_1, \circ_1)$ and $p(E_2, \circ_2)$ is usually defined on the cartesian product $E_1 \times E_2$ of the underlying sets; the simple product of $p(E_1, \circ_1)$ and $p(E_2, \circ_2)$ is defined on $E = E_1 \cup E_2$.

Besides, suppose that h: $P' \rightarrow P$ is a morphism of p-groupoids. Let us choose $P' = p(E', \circ')$ where: $E' = \{y', z'\}$, only $y' \circ z' = z'$ is defined, $h(y') = y$, $h(z') = z$. Then

$$\begin{tikzcd} P' \arrow[r] \arrow[dr] & P \arrow[d] \\ & \mathrm{II} \end{tikzcd} \quad \text{has a sense,}$$

$$\text{while} \quad \begin{tikzcd} & \mathrm{I} \\ P' \arrow[r] \arrow[ur] & P \arrow[u] \end{tikzcd} \quad \text{has no sense.}$$

The usual direct product (and sum for a similar reason) does not agree with the simple product.

The decomposition of $p(E, \circ)$ into partial replicas is not unique in general. But there is a unique maximal one obtained by choosing every $p(E_i, \circ_i)$ such that $E_i = \{x, y, z\}$, only $x \circ_i y = z$ is defined (we do not necessarily have $x \neq y \neq z$).

THEOREM 2. *If the valency of any element of a p-groupoid $p(E, \circ)$ is greater or equal to* 1, *$p(E, \circ)$ is a simple product of left-neutral p-groupoids.*

Proof. Let us first suppose that the given p-groupoid is an f-groupoid $f(E, \circ)$. We define a left-neutral p-groupoid $p(E, 1)$ by the condition: $e\,1\,e'=e'$ iff $e \circ e'$ is defined. (This specifies the order between the factors of the product $e \circ e'$.)

Let $S(e')=\{e_i\colon\ e_i \circ e_j=e'\}\cup\{e_j\colon\ e_i \circ e_j=e'\}$. We define the left-neutral p-groupoid $p(E, 2)$ by $e'2e=e$ iff $e\in S(e')$.

$p(E, 1)$ and $p(E, 2)$ are defined in a unique way.

Conversely, $p(E, 1)$ and $p(E, 2)$ define $p(E, \circ)$ in a unique way: $e\,1\,e'=e'$ implies that $e \circ e'$ exists, in this order. Then in $p(E, 2)$ we look for an e'' such that $e''2e'=e'$. When these conditions are satisfied they imply, by the definition of an f-groupoid, that $e''=e \circ e'$. In that case $f(E, \circ)=S\prod p(E, 1)p(E, 2)$.

If now $p(E, \circ)$ is a general p-groupoid, it is a simple product of its partial replicas $f(E_i, \circ_i)$; hence $p(E, \circ)=S\prod_i p(E_i, 1_i)p(E_i, 2_i)$.

Example: Let us consider $P=\mathrm{I}\cdot\mathrm{II}$, where P, I, II are the p-groupoids defined in the previous paragraph.

I is the simple product of the two following left-neutral p-groupoids:

$p(E_1, 1_1)\colon E_1=\{x, y, z\}$, only $x\ 1_1\ y=y$, $z\ 1_1\ y=y$ are defined;

$p(E_1, 2_1)\colon E_1=\{x, y, z\}$, only $z\ 2_1\ x=x$, $z\ 2_1\ y=y$, $y\ 2_1\ z=z$, $y\ 2_1\ y=y$ are defined.

II is the simple product of the two following left-neutral p-groupoids:

$p(E_2, 1_2)\colon E_2=\{y, z\}$, only $y\ 2_1\ z=z$ is defined;

$p(E_2, 2_2)\colon E_2=\{y, z\}$, only $z\ 2_2\ y=y$, $z\ 2_2\ z=z$ are defined.

This paper was prepared while the author was at the University of Waterloo, Ontario, Canada, ported by the National Council of Canada.

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THE KISS PRECISE

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In a poem whose title we have borrowed for this paper, Soddy [**1**] popu- ized a theorem relating the curvatures of four mutually tangent spheres. sset [**2**] extended the poem, and the theorem, to the case of n mutually tan- it $(n-3)$-dimensional spheres in $(n-2)$-dimensional space. Mauldon [**3**] ved the extended theorem and its converse as special cases of a more general orem about sets of equally inclined spheres. Pedoe [**4**] traced the circle the- m back to Descartes, and gave three proofs of the extended theorem, includ- g an algebraic proof by A. Aeppli.

This paper presents brief, elementary, self-contained proofs of the extended orem and its converse, and also an interpretation of the basic formula. Al- ough our proof of the extended theorem is similar to Aeppli's, the starting int is more familiar.

THEOREM 1. *Let $S_1, \cdots, S_n$ be $(n-3)$-dimensional spheres in $(n-2)$-dimensional Euclidean space $(n \geqq 3)$, such that each is tangent to all of the others at distinct points. (Our n is the same as the $n+2$ of [2]–[4].) Let $C_1, \cdots, C_n$ be the centers of these spheres; $r_1, \cdots, r_n$ their radii; and $\epsilon_1, \cdots, \epsilon_n$ their curvatures (that is, the reciprocals of the radii). Then*

(a) *either all of the spheres touch each other externally, or one of them contains the others;*

(b) *the curvatures satisfy the formula*

$$\left(\sum_{i=1}^{n} \epsilon_i\right)^2 = (n-2)\sum_{i=1}^{n} \epsilon_i^2,$$

provided that the radius and curvature of the containing sphere, if any, are taken to be negative.

Proof of (a). If one sphere S contains any other, then it must contain all of the others, since a sphere inside S and a sphere outside S, touching S at distinct points, could not touch each other.

Proof of (b). Let $x_1, \cdots, x_{n-1}$ be vectors from C_n to $C_1, \cdots, C_{n-1}$ respectively. Since the spheres are mutually tangent, we have

$$x_i^2 = (r_i + r_n)^2 \qquad (i = 1, \cdots, n-1),$$
$$(x_i - x_j)^2 = (r_i + r_j)^2 \qquad (i \neq j).$$

On the other hand, since the vectors lie in a space of dimension $n-2$, they are linearly dependent, and therefore

$$\det[x_i \cdot x_j] = 0 \qquad (i, j = 1, \cdots, n-1).$$

From the above tangency conditions, we find

$$x_i^2 = \frac{(\epsilon_i + \epsilon_n)^2}{\epsilon_i^2 \epsilon_n^2} \qquad (i = 1, \cdots, n-1), \tag{1}$$

$$x_i \cdot x_j = \frac{(\epsilon_i + \epsilon_n)(\epsilon_j + \epsilon_n) - 2\epsilon_n^2}{\epsilon_i \epsilon_j \epsilon_n^2} \qquad (i \neq j). \tag{2}$$

The determinant can be reduced to

$$\det[x_i \cdot x_j] = \frac{2^{n-2}}{\epsilon_1^2 \cdots \epsilon_n^2}\left[\left(\sum_{i=1}^{n} \epsilon_i\right)^2 - (n-2)\sum_{i=1}^{n} \epsilon_i^2\right], \tag{3}$$

and our conclusion follows immediately.

THEOREM 2. *Let $\epsilon_1, \cdots, \epsilon_n$ be nonzero real numbers, all or all but one positive, such that* (b) *is satisfied, and let $S_1, \cdots, S_n$ be $(n-3)$-dimensional spheres in $(n-2)$-dimensional space, with curvatures $\epsilon_1, \cdots, \epsilon_n$ respectively. Then $S_1, \cdots,$*

S_n can be located in such a way that each is tangent to all of the others at distinct points.

LEMMA. *Let K be any subset of k of the integers* $1, \cdots, n$ $(k \geqq 2)$. *If* (b) *holds, then*

$$\left(\sum_{i\in K} \epsilon_i\right)^2 - (k-2)\sum_{i\in K} \epsilon_i^2 \geqq 0.$$

Proof of the Lemma. For $k=2$ and $k=n$ the lemma is trivial. Let $E(K)$ denote the left hand side of the inequality, and suppose the lemma is false. Then there is a maximal set K with $E(K)<0$, and we have $2<k<n$. Adjoining to K an integer j not in K, we obtain a set K', with $E(K')\geqq 0$ since K is maximal. But

$$(k-2)E(K') - (k-1)E(K) = -\left[(k-2)\epsilon_j - \sum_{i\in K} \epsilon_i\right]^2 \leqq 0,$$

so $E(K')<0$, which is a contradiction.

Proof of Theorem 2. It is sufficient to construct vectors $x_1, \cdots, x_{n-1}$ satisfying (1) and (2). Let M be the real symmetric matrix of these right hand sides. By (b) the determinant of M is zero, and by the lemma its principal minors are nonnegative. Therefore it is positive semidefinite and has a decomposition ([**5**], page 256) of the form

$$XX^T = M$$

with X real. Taking the i-th row of X as the vector x_i, for $i=1, \cdots, n-1$, we are finished.

Interpretation. Let $\bar{\epsilon}$ and σ be the root mean square and the standard deviation, respectively, of the curvatures $\epsilon_1, \cdots, \epsilon_n$. Then (b) can be rewritten as

$$\sigma = \bar{\epsilon}\sqrt{2/n}; \qquad n \geqq 3.$$

This formula specifies the degree of nonuniformity among the curvatures, and shows that this nonuniformity decreases with increasing n.

The author thanks Prof. H. S. M. Coxeter for calling his attention to [3].

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A NEW RELATION BETWEEN PRIMITIVE ROOTS AND PERMUTATIONS

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1. Introduction. A prime p is said to have a as a primitive root if and only if $a^{p-1} \equiv 1 \bmod p$ and $a^d \not\equiv 1 \bmod p$ for all positive integers $d < p-1$.

In his *Disquisitiones Arithmeticae,* Gauss conjectured that there are infinitely many primes having 10 as a primitive root. A generalization of this conjecture did not come until 1927 when Emil Artin proposed that there are infinitely many primes having a as a primitive root, where a is not equal to -1, 1, or a perfect square. In this paper we show that the problem for the case $a=2$ is equivalent to one involving permutations. More precisely, we define

$$(m!) = \prod_{k=0}^{m-1} (1, 2, \cdots, (m-k))$$

to be the product of the first m cycles. We shall then prove the following theorem:

THEOREM. *The permutation* $(m!)$ *is transitive if and only if* $2m+1$ *is a prime number having* 2 *as a primitive root.*

2. Proof of the theorem. For each odd integer $n \geqq 3$, we define the symbols $P(n)$ and $A_n(j)$ for $j=1, 3, 5, \cdots, n-2$ by the following:

$$(1)\quad P(n) = \prod_{k=1}^{(n-1)/2} (1, 3, 5, \cdots, (n-2k)) = \begin{pmatrix} 1 & 3 & \cdots & n-2 \\ A_n(1) & A_n(3) & \cdots & A_n(n-2) \end{pmatrix}.$$

We now prove by induction on n that for any odd integer j such that $1 \leqq j \leqq n-2$,

$$(2)\qquad A_n(j) = \frac{j+n}{(j+n, 2^n)}\,.$$

First, the assertion is obvious when $n=3$. Second, if we assume the assertion for n, then it also holds for $n+2$ because

$$P(n+2) = \begin{pmatrix} 1 & 3 & \cdots & n-2 & n \\ A_n(3) & A_n(5) & \cdots & n & A_n(1) \end{pmatrix}.$$

which implies for $j < n-2$ that

$$A_{n+2}(j) = A_n(j+2) = \frac{(j+2)+n}{((j+2)+n, 2^n)} = \frac{j+(n+2)}{(j+(n+2), 2^{n+2})},$$

and for $j = n-2$ that

$$A_{n+2}(j) = n = \frac{n+n}{(n+n, 2^n)} = \frac{j+(n+2)}{(j+(n+2), 2^{n+2})},$$

and finally for $j=n$ that

$$A_{n+2}(j) = A_n(1) = \frac{1+n}{(1+n,\,2^n)} = \frac{j+(n+2)}{(j+(n+2),\,2^{n+2})}.$$

Next we define $C_n(1)=(A_n^0(1),\ A_n^1(1),\ \cdots,\ A_n^{r-1}(1))$, where

$$\overset{0}{A}_n(1) = \overset{r}{A}_n(1) = 1 \quad\text{and}\quad \overset{k+1}{A}_n(1) = A_n(\overset{k}{A}_n(1)),$$

to be the cyclic factor of $P(n)$ containing 1, and it follows directly from (2) that

$$\overset{k+1}{A}_n(1) = \frac{\overset{k}{A}_n(1)+n}{(A_n^k(1)+n,\,2^n)}.$$

We finally consider the set S, whose elements are derived from the cyclic factor C_n (1).

$$S = \left\{\overset{0}{A}_n(1), \frac{\overset{0}{A}_n(1)+n}{2}, \cdots, \frac{\overset{0}{A}_n(1)+n}{2^{s_1}};\ \overset{1}{A}_n(1), \frac{\overset{1}{A}_n(1)+n}{2}, \cdots, \frac{\overset{1}{A}_n(1)+n}{2^{s_2}};\right.$$
$$\left.\overset{2}{A}_n(1), \cdots;\ \overset{r-1}{A}_n(1), \frac{\overset{r-1}{A}_n(1)+n}{2}, \cdots, \frac{\overset{r-1}{A}_n(1)+n}{2^{s_r}}\right\},$$

where $s_k=(A_n^{k-1}(1)+n,\ 2^n)-1$.

We shall presently establish that each number in S has a unique index to base 2 relative to mod n (a is said to have d as an index to base 2 relative to mod n if and only if d is the smallest positive integer such that $2^d\equiv a \bmod n$). The fact that the elements of S have unique indices to base 2 relative to mod n is all that is necessary to complete the proof.

The reasoning in the if direction of the theorem's proof is trivial and will be presented at the end of the paper. The reasoning in the only if direction is more difficult. It is as follows. If $P(n)$ is transitive, then S contains all $n-1$ positive integers less than n. Now, because each s in S has a unique index to base 2 relative to mod n, the set T of these unique indices must contain $n-1$ positive integers also. Furthermore, using Fermat's General Theorem, it is a simple matter to show that each t in T is such that $t<n$. These two preceding statements suffice to show $2^d\not\equiv 1 \bmod n$ for positive $d<n-1$. If not, there is a positive $s<n-1$ such that $2^s\equiv 1 \bmod n$, which implies that $2^{s+1}\equiv 2 \bmod n$. Now because $2^1\equiv 2 \bmod n$ and $1<s+1$, we have that 2 in S has 1 as its index, and $s+1<n$ is not an index and, therefore, not in T. Hence, T has at most $n-2$ elements, which is a contradiction. So $2^d\not\equiv 1 \bmod n$ for $d<n-1$. Considering this preceding statement and the fact that 1 is in S, we must have $2^{n-1}\equiv 1 \bmod n$. Now by the converse to Fermat's Theorem with extended hypothesis, n is a prime. Furthermore, by the definition of primitive root of 2, n has 2 as a primitive root.

To prove, therefore, that each number in S has a unique index to the base 2 relative to mod n, we rewrite S as follows:

$$S = \{t_1, t_2, \cdots, t_w\}, \quad \text{where} \quad t_1 = 1, \qquad t_2 = (1+n)/2, \cdots.$$

We now prove by induction on v that

$$\text{(3)} \qquad \operatorname{ind} t_v = (\operatorname{ind} 1) - (v - 1).$$

The assertion is obvious for $v=1$. If we assume the assertion for v, then it also holds for $v+1$ because when t_v is even

$$\operatorname{ind} t_{v+1} = \operatorname{ind} (t_v/2) = (\operatorname{ind} t_v) - 1 = (\operatorname{ind} 1) - v,$$

and when t_v is odd

$$\operatorname{ind} t_{v+1} = \operatorname{ind} ((t_v + n)/2) = (\operatorname{ind} t_v) - 1 = (\operatorname{ind} 1) - v.$$

It follows by equation (3) that each number t_v in S has a unique index to base 2 relative to mod n. Hence, by our previous reasoning if $P(n)$ is transitive or equivalently S contains all positive integers less than n, then n is a prime having 2 as a primitive root. On the other hand, if n is a prime having 2 as a primitive root, then $(\operatorname{ind} 1) = n-1$, so that by equation (3) S would contain all the positive integers less than n and $P(n)$ would therefore be transitive. A simple transformation of $P(n)$ into $(m!)$ by

$$m = (n-1)/2, \quad \text{where} \quad n \geqq 3$$

proves the theorem.

EXTENSIONS OF OLIVIER'S THEOREM

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The purpose of this note is first to point out that an extension of Olivier's Theorem [1] recently published in this MONTHLY [2] could be given this simpl-- and more general formulation:

THEOREM 1. *Let* $\sum_1^\infty u_n$ *be a convergent or oscillating real series,* $\sum_1^\infty 1/\alpha_n$ *divergent series of positive terms and*

$$\alpha_n u_n - \alpha_{n-1} u_{n-1} \leqq v_n, \qquad n > 1,$$

where $\sum_1^\infty v_n$ *is convergent. Then* $\alpha_n u_n \to 0$.

Proof. The last two assumptions imply that $r_n = \sum_{n+1}^\infty v_k \to 0$ and $\alpha_n u_n + r_n \downarrow$ $(-\infty \leqq l < +\infty)$, so that $\alpha_n u_n \to l$. By the divergence of $\sum 1/\alpha_n$, $l \neq 0$ wou imply the divergence of $\sum u_n$ (to $-\infty$ if $-\infty \leqq l < 0$ and to $+\infty$ if $0 < l < +\infty$ which is excluded by hypothesis.

Olivier's Theorem is the special case of Theorem 1 in which $\sum u_n$ is conve gent, $\alpha_n = n$ and $v_n = u_n$. Another noteworthy special case [3], more general tha Olivier's Theorem, is:

THEOREM 2. *Let* $p_n>0$, $P_n=\sum_1^n p_k\to\infty$, $a_n\geqq a_{n+1}>0$ *and* $\sum_1^\infty p_n\ a_n<+\infty$. *Then* $P_na_n\to 0$.

Proof. Theorem 1 with $v_n=u_n=p_na_n$ and $\alpha_n=P_n/p_n$.

It should be noted that the above proof of Theorem 2 assumes the divergence of the Abel series $\sum p_n/P_n$. On the other hand, the simple direct proof contained in that of Theorem 3 below ($\alpha=2$) yields this useful fact as a corollary, even as a direct proof of Olivier's Theorem establishes the divergence of $\sum 1/n$. It may also be noted that the more restrictive form of Theorem 1 as given in [2] does not cover Theorem 2 without this additional condition on the p_n: $\sum\left|p_{n+1}/p_n-p_n/p_{n-1}\right|<+\infty$.

We now extend Olivier's Theorem in another way suggested by an interesting Putnam problem [4] by replacing the pair of conditions on the a_n in Theorem 2 by a single weaker condition:

THEOREM 3. *Let* $p_n>0$, $P_n=\sum_1^n p_k\to\infty$, $a_n\geqq 0$ *and*

$$\sum_{k=1}^{\infty} p_k \max\{a_n:\ P_k\leqq P_n<\alpha P_k\}<+\infty$$

for some $\alpha>1$. *Then* $P_na_n\to 0$.

Proof. Setting $b_k=\max\{a_n: P_k\leqq P_n<\alpha P_k\}$, $k=1,2,\cdots$, we have

$$a_n\leqq\min\{b_k:\ \alpha^{-1}P_n<P_k\leqq P_n\},\qquad n=1,2,\cdots. \tag{1}$$

By assumption, $0=P_0<P_1<P_2<\cdots$ and $P_n\to\infty$. Hence the sequence of indices $0=k_1\leqq k_2\leqq k_3\leqq\cdots$ is well defined by

$$P_{k_n}\leqq\alpha^{-1}P_n<P_{k_n+1},\qquad n=1,2,\cdots \tag{2}$$

and $k_n\to\infty$. By (1) and (2), $\sum_{k_n+1}^n p_kb_k\geqq(P_n-P_{k_n})a_n\geqq(1-\alpha^{-1})P_na_n$, so that

$$0\leqq P_na_n\leqq\frac{\alpha}{\alpha-1}\sum_{k_n+1}^{\infty}p_kb_k. \tag{3}$$

The conclusion is immediate from (3) by the assumed convergence of $\sum p_k\ b_k$.

The special case of Theorem 3 in which $p_n=1$ and $\alpha=2$ is essentially the content of the Putnam problem cited above.

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EVEN AND ODD PERMUTATIONS

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In proving that a permutation cannot be expressed both as a product of an even number and an odd number of transpositions, it is customary to refer to the polynomial $\prod_{i<j}(x_i-x_j)$. As Herstein [1, p. 67] points out, this polynomial "seems extraneous to the matter in hand." The following simple proof avoids its use.

Suppose the permutation θ (on the n symbols $1, 2, \cdots, n$) is expressible both as a product of an even and an odd number of transpositions. Then θ^{-1} has the same property, and so the identity permutation $e(=\theta\theta^{-1})$ is expressible as a product of transpositions

$$e = (a_1b_1)(a_2b_2)\cdots(a_kb_k) \tag{1}$$

with k odd. We may suppose without loss of generality that $1 \leqq a_r < b_r \leqq n$ for all r.

Now whenever $a_r \neq 1$, replace (a_rb_r) in (1) by $(1a_r)(1b_r)(1a_r)$. We obtain

$$e = (1c_1)(1c_2)\cdots(1c_l), \tag{2}$$

where l is odd and $c_i \neq 1$ for all i. Since the identity permutation maps the symbol s $(1 < s \leqq n)$ onto itself, the expression (2) for e must involve an even number of transpositions $(1s)$. This contradicts the oddness of l and the proof is complete.

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AN APPLICATION OF DIOPHANTINE APPROXIMATION

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1. Introduction. Consider the function f defined by:

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n,\ (m, n) = 1, \quad \text{and} \quad n > 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \tag{1}$$

It is well known that f is continuous at all the irrationals and discontinuous at all the rationals. From this it also follows that f is Riemann integrable on any interval. In this paper the differentiability of f and functions related to f is discussed. It is clear that f is not differentiable at any rational number (since it is not even continuous there), so we focus our attention on the set of irrationals.

It will be shown that f^a (i.e., replace n by n^a in (1)) is nowhere differentiable if $a \leqq 2$, but f^a is differentiable at all (irrational) algebraic numbers if $a > 2$. Also, if $a > 2$, there exist (transcendental) numbers where f^a is not differentiable.

Furthermore if n is replaced in (1) by an arbitrary positive valued function of n, there will still exist irrational numbers at which the resulting function will not be differentiable.

2. Preliminary theorems. The following theorems will be needed in what follows:

THEOREM 1. (Hurwitz) [2; p. 68] *For any irrational number ξ there are infinitely many rational numbers h/k such that*

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}\,k^2} < \frac{1}{k^2}. \tag{2}$$

Note that it can be assumed that h/k is irreducible; for if $(h, k) = d > 1$, $h = dh_1$, $k = dk_1$ where $(h_1, k_1) = 1$, then

$$|\xi - (h_1/k_1)| = |\xi - (h/k)| < 1/k^2 < 1/k_1^2.$$

THEOREM 2. (Roth) [3; p. 2] *If ξ is an irrational algebraic number and $\delta > 0$, then for all but a finite number of rational numbers h/k*

$$|\xi - (h/k)| \geqq 1/k^{2+\delta}.$$

THEOREM 3. [1; p. 68] *Let a be any real number. Then there are (transcendental) numbers ξ for which the inequality*

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{k^a} \tag{3}$$

has infinitely many rational solutions h/k.

3. Results.

THEOREM 4. *Let f be defined as in* (1). *If $a \leqq 2$, then f^a is nowhere differentiable.*

Proof. Let ξ be an irrational number and consider the difference quotient

$$\Delta = \frac{f^a(\xi + t) - f^a(\xi)}{t} = \frac{f^a(\xi + t)}{t}. \tag{4}$$

If t is rational, Δ is 0, hence if f^a is differentiable at ξ, its derivative must be 0. Therefore to prove the theorem it suffices to exhibit a sequence $\{t_n\}$ tending to zero for which $|\Delta|$ is bounded away from 0. Choose $t_n = (h_n/k_n) - \xi$ where h_n/k_n satisfies (2) for $n = 1, 2, \cdots$. Then

$$|\Delta| = \left|\frac{1}{k_n^a((h_n/k_n) - \xi)}\right| \geqq \frac{k_n^2}{k_n^a} = k_n^{2-a} \geqq 1,$$

since $2 - a \geqq 0$.

THEOREM 5. *With f defined as in* (1) *and $a>2$, f^a is differentiable at all irrational algebraic numbers.*

Proof. Let ξ be an irrational algebraic number and again consider the difference quotient Δ in (4). Choose $\delta>0$ such that $2+\delta<a$. Let t be restricted to be so small that $\xi+t$ is never equal to one of the finite number of exceptional rational numbers mentioned in Theorem 2. Then $|\Delta|=0$ if $\xi+t$ is irrational and if $\xi+t=h/k$ is rational,

$$|\Delta| = \left|\frac{1}{k^a((h/k)-\xi)}\right| \leqq \frac{k^{2+\delta}}{k^a} = \frac{1}{k^{a-2-\delta}}$$

which can be made arbitrarily small by choosing k large. Since k becomes large as t becomes small we have $\Delta\to 0$ as $t\to 0$. Hence f^a is differentiable at ξ and the value of the derivative is zero.

THEOREM 6. *For all a there are transcendental numbers at which f^a is not differentiable.*

Proof. Let ξ be a transcendental number for which Theorem 3 holds and consider the difference quotient Δ. Choose a sequence $t_n=(h_n/k_n)-\xi$ such that the rational numbers h_n/k_n satisfy (3). Then

$$\Delta| = \left|\frac{1}{k_n^a((h_n/k_n)-\xi)}\right| \geqq \frac{k_n^a}{k_n^a} = 1.$$

Hence for this sequence $\{t_n\}$ approaching 0, $|\Delta|$ is bounded away from 0, while for a sequence of rationals $\{t_n'\}$ approaching 0, $\Delta=0$. Hence f^a is not differentiable at ξ.

Let λ be a positive valued function defined for all positive integers and define g_λ as follows:

$$g_\lambda(x) = \begin{cases} 1/\lambda(n) & \text{for } x = m/n,\ (m,n)=1,\ n>0 \\ 0 & \text{for } x \text{ irrational.} \end{cases} \tag{5}$$

It might be conjectured that for $\lambda(n)=n^n$, or some other function that increases very rapidly, g_λ is differentiable at all irrationals. That this is *not* the case is shown by the following result.

THEOREM 7. *With g_λ defined as in* (5) *there are uncountably many transcendental numbers at which g_λ is not differentiable no matter how λ is chosen.*

Proof. Theorem 3 remains valid if on the right hand side of inequality (3), k^a is replaced by $\lambda(k)$ [1; p. 68]. Now the proof continues as that of Theorem 6. That the set of points, where g_λ is not differentiable, is uncountable can be easily verified by checking through the proof of the generalization of Theorem 3 mentioned above.

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ON FERMAT'S LAST THEOREM

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J. M. Gandhi [1] proved that if p and $4p+1$ are primes with $p>3$ and $a^p+b^p+c^p=0$, where a, b, c are nonzero, pairwise prime integers, then precisely one of the integers a, b, c is divisible by $4p+1$. It may be remarked that in the theorem proved by Gandhi, the condition that p is a prime is unnecessary, because at the end of the proof he obtained the congruence $\pm 4\equiv 1$ or 9 (mod $4p+1$), and this is false for any integer $p>3$.

In this paper we prove a theorem similar to the one proved by Gandhi, an extension of D. E. Stone's Theorem [2] and some other theorems relating to Fermat's Last Theorem.

Theorem 1. *If for any integer $p>2$, $3p+1$ is a prime and $a^p+b^p=c^p$, where a, b, c are nonzero, pairwise prime integers, then precisely one of the integers a, b, c is divisible by $3p+1$.*

Proof. Suppose $(abc, 3p+1)=1$. Then $(3p+1, a)=1$, $(3p+1, b)=1$ and $(3p+1, c)=1$. Hence, by Fermat's Theorem

$$a^{3p}\equiv b^{3p}\equiv c^{3p}\equiv 1 \pmod{3p+1}. \tag{1}$$

Since $a^p+b^p=c^p$, it follows that

$$a^{3p}+b^{3p}+3a^pb^pc^p=c^{3p}. \tag{2}$$

Therefore, (1) and (2) imply $3a^pb^pc^p\equiv -1$ (mod $3p+1$). Hence, we obtain $27a^{3p}b^{3p}c^{3p}\equiv -1$ (mod $3p+1$). Now using (1) it follows that $27\equiv -1$ (mod $3p+1$), which is false, since $p>2$ and $3p+1$ is a prime. Hence, Theorem 1 is proved.

D. E. Stone [2] proved that if p and $2p+1$ are odd primes and $a^p+b^p+c^p=0$, where a, b, c are nonzero, pairwise prime integers, then precisely one of the integers a, b, c is divisible by p. Now we shall prove the following:

Theorem 2. *If p and $2p+1$ are odd primes and a, b, c are nonzero pairwise prime integers such that $a^p+b^p+c^p=0$, then exactly one of the integers a, b, c is divisible by p^2.*

To prove Theorem 2, the following lemma is needed.

LEMMA 1. *If x and y are integers such that $x+y\equiv 0 \pmod{p}$ and*

$$\phi(x, y) = \sum_{r=1}^{p} (-1)^{r-1}x^{p-r}y^{r-1}, \qquad \textit{then } \phi(x, y) \equiv px^{p-1} \pmod{p^2}.$$

Proof. Since $x+y\equiv 0 \pmod{p}$, $y=kp-x$ for some integer k, so that

$$\phi(x, y) = \sum_{r=1}^{p} (-1)^{r-1}x^{p-r}(kp - x)^{r-1}.$$

Collecting coefficients of $x^{p-r}(kp)^{r-1}$ and writing

$$\binom{n}{0} = 1,$$

for every integer $n\geqq 0$, it follows that

$$\phi(x, y) = px^{p-1} + \sum_{r=2}^{p} (-1)^{r-1}x^{p-r}(kp)^{r-1}\left[\sum_{n=r-1}^{p-1}\binom{n}{r-1}\right].$$

But

$$\sum_{n=r-1}^{p-1}\binom{n}{r-1} = \binom{p}{r}, \qquad \text{since} \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r};$$

and

$$\binom{p}{r} \equiv 0 \pmod{p}, \qquad 1 \leqq r < p,$$

since p is a prime. Hence, the lemma is proved.

Proof of Theorem 2. Already by Stone's Theorem exactly one of the integers a, b, c is divisible by p. So without loss of generality let p divide c. Then we can write

$$c = p^s\lambda, \tag{2'}$$

where s is a positive integer and λ an integer such that $(p, \lambda)=1$. Now, since $a^p+b^p=-c^p$, it follows that $a^p+b^p=-p^{sp}\lambda^p$, which implies

$$(a + b)\phi(a, b) = - p^{sp}\lambda^p. \tag{3}$$

But $(a, p)=(b, p)=1$, therefore, $a^p+b^p\equiv(a+b) \pmod{p}$, so that we get $-c^p \equiv(a+b) \pmod{p}$. Since p divides c, it follows that $0\equiv(a+b) \pmod{p}$. Hence, by Lemma 1, $\phi(a, b)\equiv pa^{p-1} \pmod{p^2}$, which shows that $\phi(a, b)\equiv 0 \pmod{p}$, but $\phi(a, b)\not\equiv 0 \pmod{p^2}$. Hence (3) implies $(a+b)=p^{sp-1}u$ and $\phi(a, b)=pv$, where u and v are integers such that $(p, u)=(p, v)=1$ and

$$uv = - \lambda^p. \tag{3'}$$

Now we observe that $(u, v)=1$, for, if q is a prime divisor of u and v, $b=tq-a$ for some integer t, and hence $\phi(a, b)\equiv pa^{p-1}$ (mod q), which shows that $0\equiv pa^{p-1}$ (mod q), and this is false. Therefore, from (3′) it follows that $u=c_0^p$ and $v=-c_1^p$. But $(b+c, \phi(b, c))=1$, for, if q_1 is a prime divisor of $b+c$ and $\phi(b, c)$, we have $b=q_1t_1-c$ for some integer t_1, so that $0\equiv\phi(b, c)\equiv pc^{p-1}$(mod q_1) and hence $pc^{p-1}\equiv 0$ (mod q_1). Therefore, it follows that $q_1=p$ or q_1 divides c. If $q_1=p$, then p divides $b+c$ which together with (2′) shows that p divides b and thus we get $(b, c)\neq 1$, which contradicts the hypothesis of Theorem 2. If q_1 divides c it follows that q_1 divides b and again $(b, c)\neq 1$ and this is false. Likewise it follows that $(c+a, \phi(c, a))=1$. Hence, we have the following relations:

(4) $$a+b=p^{sp-1}c_0^p,$$ (4′) $$\phi(a, b)=-pc_1^p,$$

(5) $$b+c=a_0^p,$$ (5′) $$\phi(b, c)=-a_1^p,$$

(6) $$c+a=b_0^p,$$ (6′) $$\phi(c, a)=-b_1^p,$$

where $a=a_0a_1$, $b=b_0b_1$, $\lambda=c_0c_1$ and $a_0, a_1, b_0, b_1, c_0, c_1$ are pairwise prime integers. Now (5) and (6) imply

(7) $$a+b+2c=(a_0+b_0)\phi(a_0, b_0).$$

But $(p, a_0)=(p, b_0)=1$, so that

(8) $$a_0^p+b_0^p\equiv a_0+b_0 \pmod{p}.$$

Since p divides c and also $a+b$, from (5) and (6) we get $a_0^p+b_0^p\equiv 0$ (mod p); hence (8) implies $a_0+b_0\equiv 0$ (mod p), from which follows $\phi(a_0, b_0)\equiv 0$ (mod p), by Lemma 1. Therefore, (7) implies $a+b+2c\equiv 0$ (mod p^2). Now using (4) it follows that $c\equiv 0$ (mod p^2). Hence, Theorem 2 is proved.

THEOREM 3. *If a, b, c are positive integers such that $c\geqq$ Min (a^2, b^2), then for any integer $p\geqq 3$, the equation $a^p+b^p=c^p$ is impossible.*

Proof. Suppose $a^p+b^p=c^p$, then

(9) $$a^p=c^p-b^p=(c-b)\sum_{r=1}^{p}c^{p-r}b^{r-1}.$$

Using the well-known result that the arithmetic mean of n positive numbers is not less than their geometric mean, we get that $\sum_{r=1}^{p}c^{p-r}b^{r-1}>p(cb)^{(p-1)/2}>(cb)^{(p-1)/2}$. Hence, since $c-b\geqq 1$, (9) shows that $a>(cb)^{\frac{1}{2}-\frac{1}{2}p}$, which implies

(10) $$a>(cb)^{1/3}, \qquad \text{since } p\geqq 3.$$

Likewise it follows that

(11) $$b>(ca)^{1/3}.$$

Hence, from (10) and (11) we get $a^9 > c^4 a$ and $b^9 > c^4 b$. Therefore, $c < a^2$ and $c < b^2$ and hence $c < \mathrm{Min}(a^2, b^2)$, which contradicts the hypothesis. Thus Theorem 3 is proved.

THEOREM 4. *If a, b, c are positive integers such that $a^p + b^p = c^p$, where p is any integer > 4, then* $\sqrt{2}\,\mathrm{Min}\,(a, b) > \mathrm{Max}\,(\sqrt{a}, \sqrt{b})\,(1 + (2p \log 2p)^{-1})$.

To prove Theorem 4 we need the following:

LEMMA 2. *For any integer $p > 4$, $2^{1/p} > 1 + (p \log p)^{-1}$.*

Proof. If $p > e^{1/\log 2}$, that is, if $p > 4$, we get $\log p > (\log 2)^{-1}$, which implies $(p \log p)^{-1} < p^{-1} \log 2$. But

$$\log(1 + (p \log p)^{-1}) < (p \log p)^{-1}.$$

Hence, we get $\log(1 + (p \log p)^{-1}) < p^{-1} \log 2$, which proves the lemma.

Proof of Theorem 4. It is well known that $a^p + b^p > 2^{1-p}(a+b)^p$. Since $a^p + b^p = c^p$, it follows that $c > L(a+b)$, where $L = 2^{(1/p)-1}$. Using Theorem 3 we get $a^2 > L(a+b)$ and $b^2 > L(a+b)$. $a^2 > L(a+b)$ implies that $a > (L + \sqrt{L^2 + 4Lb})/2$. Hence, we find that $a > \sqrt{Lb}$ so that $\sqrt{2}a/\sqrt{b} > 2^{1/2p}$. Therefore $\sqrt{2}a/\sqrt{b} > 1 + (2p \log 2p)^{-1}$ by Lemma 2. Likewise $b^2 > L(a+b)$ implies $\sqrt{2}b/\sqrt{a} > 1 + (2p \log 2p)^{-1}$. Hence, Theorem 4 is proved.

THEOREM 5. *If $a^p + b^p = c^p$ has positive integral solutions for any prime $p > 2$, then $p < \mathrm{Min}\,(a, b) < c$.*

Proof. It is plain that $\mathrm{Min}\,(a, b) < c$. R. Sauer [3] proved that $a^p + b^p = c^p$ does not hold if either a or b or c is a prime. Hence, $p \neq \mathrm{Min}\,(a, b)$. Suppose $\mathrm{Min}\,(a, b) < p$. Then $a < b < p$ or $a < p < b$ or $b < a < p$ or $b < p < a$. Let $a < b < p$. Now $a^p = c^p - b^p \geqq (b+1)^p - b^p > pb^{p-1}$ and we find that $a > p(b/a)^{p-1} > p$ so that $a > p$, and this is false. Likewise $a < p < b$, $b < a < p$ and $b < p < a$ lead to contradictions. Hence Theorem 5 is proved.

NOTE. Theorem 5 directly gives the following corollary. *For a given positive integer a or b or c, there exists a positive integer m, (in fact, $m = \mathrm{Min}\,(a, b)$) such that for every integer $p \geqq m$, the equation $a^p + b^p = c^p$ is impossible, a, b, c being positive integers.*

THEOREM 6. *If $a^p + b^p = c^p$ has positive integral solutions for infinitely many primes p, then there exist sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ of positive integers and a sequence $\{p_n\}$ of primes such that* (i) $a_n^{p_n} + b_n^{p_n} = c_n^{p_n}$ *and* (ii) $c_n \sim \mathrm{Max}\,(a_n, b_n)$, *more precisely,*

$$c_n = \mathrm{Max}(a_n, b_n)(1 + O(n \log n)^{-1}).$$

Proof. Suppose $a^p + b^p = c^p$ has positive integral solutions for infinitely many primes p. Then there is a sequence of primes $\{p_n\}$, such that for each p_n there

exist positive integers A_n, B_n, C_n such that

$$A_n^{p_n} + B_n^{p_n} = C_n^{p_n}. \tag{12}$$

Let a_n, b_n, c_n be the least of the sets of positive integers $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ satisfying (12), for a given p_n.

Now suppose that $b_n < a_n$. Then (i) implies $c_n^{p_n} < 2\ a_n^{p_n}$, which implies $c_n < 2^{1/p_n} a_n$. But it is easy to see that $2^{1/m} < 1 + 1/m$ for any integer $m > 1$. Hence, we get $1 < c_n/a_n < 1 + 1/p_n$. If Q_n is the nth prime, then plainly $p_n \geqq Q_n$. But it is well known that $Q_n = O\ (n \log n)$. Hence, it follows that $c_n/a_n = 1 + O((n \log n)^{-1})$. Likewise it follows that $c_n/b_n = 1 + O((n \log n)^{-1})$, if $a_n < b_n$. Hence, Theorem 6 is proved.

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THE LONG LINE AS A SUBSET OF $\mathcal{P}$ (R)

J. PELHAM THOMAS, Western Carolina University

The "long line" is described by Hocking and Young ([1], p. 55). It is a totally ordered set having certain interesting properties. In this paper we shall show that any two sets having these properties are homeomorphic to each other; that is, up to homeomorphism, there is only one long line. We shall then construct the long line as a set of subsets of the real numbers.

THEOREM 1. *Let M and N be totally ordered sets, each with a least element and having the following properties*: (*a*) *each countable subset is bounded, and* (*b*) *with the interval topology, each closed interval is an arc. Then M is homeomorphic to N.*

Proof. Let δ be the just uncountable ordinal. Define $m_0(n_0)$ to be the least element of $M(N)$. If $m_\alpha(n_\alpha)$ has been defined for each $\alpha < \beta$, where $\beta < \delta$, choose $m_\beta(n_\beta)$ so that, if β is not a limit ordinal, $m_\beta > m_{\beta-1}(n_\beta > n_{\beta-1})$, and if β is a limit ordinal, $m_\beta = \text{l.u.b.}\{m_\alpha | \alpha < \beta\} (n_\beta = \text{l.u.b.}\{n_\alpha | \alpha < \beta\})$. (Such a choice is always possible since the sets in question are countable, hence bounded.) Then $\{m_\alpha\}_{\alpha<\delta}$ $(\{n_\alpha\}_{\alpha<\delta})$ is uncountable and therefore unbounded, since, if b were a bound, $\{m_{\alpha+1}\}_{\alpha<\delta}(\{n_{\alpha+1}\}_{\alpha<\delta}$)would be an uncountable set of isolated points in $]0, b[$, which is homeomorphic to R.

Now $\bigcup_{\alpha<\delta}[m_\alpha, m_{\alpha+1}] = M(\bigcup_{\alpha<\delta}[n_\alpha, n_{\alpha+1}] = N)$; for let $m \in M(n \in N)$ and let $m_\gamma(n_\gamma)$ be the least element of $\{m_\alpha\}$ $(\{n_\alpha\})$ such that $m \leqq m_\gamma(n \leqq n_\gamma)$. If $m = m_\gamma(n = n_\gamma)$ then $m \in \bigcup_{\alpha<\delta}[m_\alpha, m_{\alpha+1}]$ $(n \in \bigcup_{\alpha<\delta}[n_\alpha, n_{\alpha+1}])$ so suppose $m \neq m_\gamma$

$(n \neq n_\gamma)$. This implies that $m_\gamma(n_\gamma)$ is not a limit ordinal, hence $m \in [m_{\gamma-1}, m_\gamma]$ $(n \in [n_{\gamma-1}, n_\gamma])$. Thus in either case $m \in \bigcup_{\alpha<\delta}[m_\alpha, m_{\alpha+1}]$ $(n \in \bigcup_{\alpha<\delta}[n_\alpha, n_{\alpha+1}])$.

Now for each $\alpha<\delta$ define f_α: $[m_\alpha, m_{\alpha+1}] \to [n_\alpha, n_{\alpha+1}]$, one-one, onto, and order preserving. Define f: $M \to N$ such that if $x \in [m_\alpha, m_{\alpha+1}]$, $f(x) = f_\alpha(x)$. Then f is the required homeomorphism. This completes the proof.

Note. The long line has the properties of M in the above theorem.

We now proceed to construct the long line as a subset of the set of subsets of the reals. Let X be the set of all countable subsets of the reals. Partially order X by $x <_1 y$ if and only if $x \subset y$ and $y-x$ is infinite, where $x, y \in X$. It is clear that there exist unbounded chains in $(X, <_1)$, for otherwise, by Zorn's Lemma, there would exist a maximal countable subset of the reals, an obvious impossibility.

Now no unbounded chain in $(X, <_1)$ has an unbounded (cofinal) countable subset, for if $\{x_\alpha\}_{\alpha\in A}$ is a chain with an unbounded countable subset, say $\{x_n\}_{n\in N}$, then each x_α is a subset of some x_n, hence $x = \bigcup_{\alpha\in A} x_\alpha = \bigcup_{n\in N} x_n$ is countable and therefore an upper bound for $\{x_\alpha\}_{\alpha\in A}$.

Let Y be the set of all unbounded chains in $(X, <_1)$. Order Y by $C <_2 D$ if $C \subset D$ where $C, D \in Y$. Let $\{C_\beta\}_{\beta\in B}$ be a chain in $(Y, <_2)$; note that each C_β is itself an unbounded chain in $(X, <_1)$. Let $C = \bigcup_{\beta\in B} C_\beta$. Let $x, y \in C$, $x \neq y$. Then $x, y \in C_\beta$ for some $\beta \in B$, so either $x <_1 y$ or $y <_1 x$. Thus C is a chain in $(X, <_1)$. Furthermore C is unbounded, for otherwise each C_β is bounded. Thus $C \in Y$ and C is an upper bound for $\{C_\beta\}_{\beta\in B}$. Hence, by Zorn's Lemma, there is a maximal element D of $(Y, <_2)$. Thus D is a set of countable subsets of the reals ordered by $<_1$, and is maximal in $(Y, <_2)$. We shall show that D has the properties of M in Theorem 1 and is hence homeomorphic to the long line.

We have already shown that D is totally ordered and that every countable subset is bounded. The fact that D is maximal in Y assures us that there is an empty or finite set which is a least element of D and which we shall call 0. Hence we only need to show that every closed interval in D is an arc. We shall do this by showing that if $c \in D$, $c \neq 0$, then $]0, c[$ is homeomorphic to the reals. We shall use the following theorem which is stated as an exercise by Hall and Spencer ([2], p. 169).

THEOREM 2. *Let S be a nondegenerate totally ordered set. Then S, in its interval topology, is homeomorphic to the set of all real numbers if and only if the following conditions hold:*

(i) *S has no first point and no last point.*

(ii) *S is connected.*

(iii) *S is separable.*

LEMMA 1. *D is dense in itself.*

Proof. This follows immediately from the maximality of D in $(Y, <_2)$ and the fact that if $x <_1 y$ then $y-x$ is infinite.

LEMMA 2. *D is order complete.*

Proof. Let $A \subset D$, $A \neq \emptyset$, c an upper bound for A; thus A is a set of countable subsets of the reals and c is a countable subset of the reals which is an element of D and such that $x \leqq_1 c$ for every $x \in A$. Let B be the set of all upper bounds for A and assume that A has no least upper bound. Then A has no greatest element and B has no least element. Let $a = \bigcup_{x \in A} x$; note that a is a countable subset of R since $a \subset c$ and $c \in D$. If $y \in D$ and $y <_1 x$ for some $x \in A$, then $y <_1 a$. If $y \nleqq x$ for any $x \in A$ then $x <_1 y$ for each $x \in A$ so $y \in B$. But then there is a $y' \in B$ with $y' <_1 y$, so clearly, since $a \subset y'$, $a <_1 y$. Thus, since we have shown that either $y <_1 a$ or $a <_1 y$ for each $y \in D$, $D \cup \{a\}$ is totally ordered. But D is maximal under $<_2$, so $a \in D$. But then $a \in B$ and is a least element of B, a contradiction. Thus A has a least upper bound. The fact that every nonempty subset of D which is bounded below has a greatest lower bound follows by an argument which is often given for the reals with the usual order.

LEMMA 3. *Let $c \in D$, $c \neq 0$. Then $[0, c]$ with the induced topology is separable.*

Proof. Recall that c is a countable set of real numbers. For each $\alpha \in c$ let x_α be the greatest lower bound of $A_\alpha = \{x \in D \text{ such that } \alpha \in x\}$. (Such a greatest lower bound exists by Lemma 2, since 0 is a lower bound and c is an element of A_α.) Let $E = \{x_\alpha\}_{\alpha \in c}$. Then E is countable, since c is countable. E is dense in $[0, c]$, for let a, $b \in [0, c]$, $a <_1 b$. Choose a', b' such that $a <_1 a' <_1 b' <_1 b$ (Lemma 1). Let $\beta \in b' - a'$. Then $a <_1 a' \leqq x_\beta \leqq b' < b$.

LEMMA 4. *In order that a subset A of D be connected it is necessary and sufficient that A be an interval, bounded or not.*

The proof that the real numbers have these properties as given by Dieudonné ([3], p. 64), uses only those properties of the reals which we have established for D.

PROPOSITION. *Let $c \in D$, $c \neq 0$. Then $]0, c[$ is homeomorphic to the reals.*

Proof. Apply Lemmas 3 and 4 and Theorem 2. Thus by Theorem 1, D is the long line.

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

CAN ALL CONVEX BOREL SETS BE GENERATED IN A BORELIAN MANNER WITHIN THE REALM OF CONVEXITY?

VICTOR KLEE, University of Washington

For any class S of subsets of Euclidean d-space E^d, let $B(S)$ denote the smallest class of sets satisfying the following three conditions:

(i) every member of S belongs to $B(S)$;

(ii) for every increasing sequence $S_1 \subset S_2 \subset \cdots$ of members of $B(S)$, the union $\bigcup_1^\infty S_i$ belongs to $B(S)$;

(iii) for every decreasing sequence $S_1 \supset S_2 \supset \cdots$ of members of $B(S)$, the intersection $\bigcap_1^\infty S_i$ belongs to $B(S)$.

Now let G^d denote the class of all open subsets of E^d, G_c^d the class of all open convex subsets of E^d, F^d the class of all closed subsets of E^d, and F_c^d the class of all closed convex subsets of E^d. Then $B(G^d)=B(F^d)$ and $B(G_c^d)=B(F_c^d)$. $B(G^d)$ is the class of all Borel sets in E^d and every member of $B(G_c^d)$ is a *convex* Borel set. The title question asks whether every convex Borel set in E^d belongs to $B(G_c^d)$. This problem was first raised in [4, p. 451] and for $d=2$ was answered affirmatively in [5, pp. 109–111]. (The answer is trivially affirmative when $d=1$.) I conjectured in [6] that the answer is negative when $d=3$ and suggested in a conversation with D. G. Larman that $B(G^3)\sim B(G_c^3)$ should include a convex set whose closure is a cylinder. Though Larman [7] disproved the specific suggestion, the title problem remains open in the 3-dimensional case. Results in [5] and [7] impose certain restrictions on the facial structure of any member K of $B(G^d)\sim B(G_c^d)$. In particular, K must contain a line segment that lies in the boundary of K [5]. And when $d=3$, there is an extreme point of the closure cl K that lies in two 1-faces F_1 and F_2 of cl K such that neither F_1 nor F_2 is contained in a 2-face of cl K [7].

For some related infinite-dimensional problems see [4] and Alfsen [1, 2, 3].

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

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THE UNCOUNTABILITY OF THE REALS

B. R. WENNER, University of Missouri at Kansas City

The real numbers can be defined to be the set R of "equivalence classes of Cauchy sequences on Q" (we use Q to denote the rationals, and N for the natural numbers); we make this precise by means of a few basic definitions. A sequence $\{a_n\}$ on Q is Cauchy [resp., converges to $r \in Q$] iff for every rational $\epsilon > 0$ there exists $N \in N$ such that $m, n \geqq N$ [resp., $n \geqq N$] implies $|a_m - a_n| < \epsilon$ [resp., $|a_n - r| < \epsilon$]. The collection of all Cauchy sequences on Q will be denoted by $\mathcal{C}$, and we define the equivalence relation $\sim$ on $\mathcal{C}$ as follows: $\{a_n\} \sim \{b_n\}$ iff $\{a_n - b_n\}$ converges to zero. For any $\{a_n\} \in \mathcal{C}$ we denote its equivalence class by $\langle a_n \rangle$, and define $R = \{\langle a_n \rangle : \{a_n\} \in \mathcal{C}\}$. The algebraic operations and order on R can be defined naturally from the structure of Q [cf. 1, Chapter 9]; the standard proof of the uncountability of R, however, involves decimal expansions. Our aim in this note is to give a proof entirely within the above framework.

THEOREM. *R is uncountable.*

Proof. If not, then $R = \{\alpha^i : i \in N\}$, where for each $i \in N$ we have $\alpha^i = \langle a_n^i \rangle$. We shall now define a sequence $\{b_n\} \in \mathcal{C}$ such that for each $i \in N$, $\{a_n^i\} \nsim \{b_n^i\}$; hence $\beta = \langle b_n \rangle$ will be a member of R which is distinct from each α^i, and this contradiction will complete the proof of the theorem.

For each $i \in N$ we define $N_i \in N$ and $b_i \in Q$ inductively as follows. Let N_0 be such that $m, n \geqq N_0$ imply $|a_m^0 - a_n^0| < 1/4$, and let b_0 be such that $|b_0 - a_{N_0}^0| \geqq \frac{1}{2}$. Now assume N_{k-1} and b_{k-1} are defined; let $N_k > N_{k-1}$ be such that $m, n \geqq N_k$ imply $|a_m^k - a_n^k| < 1/2^{3k+2}$, and choose b_k such that both

$$|b_{k-1} - b_k| < 1/2^{3k} \quad \text{and} \quad |b_k - a_{N_k}^k| \geqq 1/2^{3k+1}.$$

(Such a choice of b_k is always possible; e.g., if $a_{N_k}^k \leqq b_{k-1}$, define $b_k = b_{k-1} + 1/2^{3k+1}$, otherwise $b_k = b_{k-1} - 1/2^{3k+1}$.) We note the strict monotonicity of the N_k implies that for all $k \in N$, $k \leqq N_k$.

To see that $\{b_n\} \in \mathcal{C}$, we let $\epsilon > 0$ and choose $N \in N$ such that $1/2^{3N} < \epsilon$. Then $n \geqq m \geqq N$ implies

$$\left|b_m - b_n\right| \leqq \sum_{i=m+1}^{n} \left|b_{i-1} - b_i\right| < \sum_{i=m+1}^{n} 1/2^{3i} < 1/2^{3N} < \epsilon.$$

Now let $k \in N$; we shall show that $\{a_n^k\} \nsim \{b_n\}$ by showing that the "tails" of the two sequences are separated by the positive distance $(3/28)(1/2^{3k})$. This follows from the fact that $n \geqq N_k$ implies

$$\begin{aligned} \left|b_n - a_n^k\right| &= \left|(b_n - b_k) + (b_k - a_{N_k}^k) + (a_{N_k}^k - a_n^k)\right| \\ &\geqq \left|b_k - a_{N_k}^k\right| - \left|b_n - b_k\right| - \left|a_{N_k}^k - a_n^k\right| \\ &> 1/2^{3k+1} - \sum_{i=k+1}^{n} \left|b_{i-1} - b_i\right| - 1/2^{3k+2} \\ &> 1/2^{3k+2} - \sum_{i=k+1}^{n} 1/2^{3i} > 1/2^{3k+2} - \sum_{i=k+1}^{\infty} 1/2^{3i} \\ &= 1/2^{3k+2} - (1/7)(1/2^{3k}) = (3/28)(1/2^{3k}), \end{aligned}$$

which means that $\{a_n^k - b_n\}$ cannot converge to zero, and the proof is complete.

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ANOTHER VIEW OF ROLLE'S THEOREM

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The celebrated Theorem of Rolle states:

If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b), and $f(a) = f(b)$, then there exists an $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

The geometrical significance which always accompanies Rolle's Theorem is that the tangent line to the graph of $f(x)$ at $x = x_0$ is parallel to the x-axis. We give another geometrical implication of Rolle's Theorem—its proof is immediate.

POLAR FORM OF ROLLE'S THEOREM. *If $f(\theta)$ is continuous, nowhere vanishing in $[\theta_1, \theta_2]$, differentiable in (θ_1, θ_2), and $f(\theta_1) = f(\theta_2)$, then there exists a $\theta_0 \in (\theta_1, \theta_2)$ such that the tangent line to the graph of $r = f(\theta)$ at $\theta = \theta_0$ is perpendicular to the radius vector at that point.*

If we translate the setting of the polar form of Rolle's Theorem as stated above back into cartesian coordinates, we obtain the following different form of Rolle's Theorem:

THEOREM. *If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f^2(a) - f^2(b) = b^2 - a^2$, then there exists an $x_0 \in (a, b)$ such that*

$$f'(x_0) \cdot f(x_0) + x_0 = 0.$$

Of course, the above form can be obtained more directly from the usual form of Rolle's Theorem by using the auxiliary function

$$F^2(x) = f^2(x) + x^2.$$

One can also reinterpret Lagrange's Mean-Value Theorem and some other theorems of analysis in similar ways.

MATHEMATICAL EDUCATION

EDITED BY J. G. HARVEY AND M. W. POWNALL

Material for this Department should be sent to either of the editors: J. G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706; M. W. Pownall, Department of Mathematics, Colgate University, Hamilton, NY 13346.

FROM THE EDITORS

The editors of the Mathematical Education section of the MONTHLY welcome articles covering professional aspects of education pertinent to our work as teachers of mathematics.

We are sure that there are many topics of current interest in mathematics education which should be discussed in the pages of this section of the MONTHLY. Our readers have indicated an interest in (1) the educational role of computers in the teaching of mathematics, (2) mathematics programs in the two-year college, (3) the problems of finding well-qualified staff for the mathematics faculty of the two-year college and the small college, (4) improving mathematics programs at colleges having a small staff, and (5) mathematics education for the culturally disadvantaged. Obviously this is an incomplete listing of topics.

We encourage the readers of the MONTHLY to (1) suggest other topics which you would like to see discussed and (2) to submit articles for the Mathematical Education section which you consider appropriate. Our space is extremely limited, and it is highly likely that we will be unable to publish all of the manuscripts we receive. But the submission of articles would be of great assistance to the editors as we continually strive to improve this section and to reflect the mathematics education interests of the community.

THE EFFECT OF THE RECOMMENDATIONS OF THE COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS UPON THE MATHEMATICS CURRICULA OF THE COLLEGES OF MARYLAND

J. E. LIGHTNER, Western Maryland College

The past decade has witnessed many changes in the mathematics curricula of the public schools and colleges of the nation. One body associated with this change in higher education is the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America.

A research study [1] was conducted by the author during the academic year 1967–1968, the purpose of which was to examine the mathematics curricula of the undergraduate colleges in Maryland, to determine the curricular changes which had occurred over a five year period from 1962 to 1967, and to ascertain the amount of influence the CUPM recommendations had on the collegiate mathematics curricular changes: (1) in the program for the general curriculum in mathematics; (2) in the program for the training of teachers of mathematics; (3) in the program for the preparation for graduate study in mathematics.

The study involved personal interviews with the chairmen of the mathematics departments in colleges across the state. Twenty of the twenty-one four-year colleges offering a major or concentration in mathematics agreed to participate in the study, which had been given backing by the Maryland School-College Mathematics Association. The interviews were tape-recorded and later transcribed for analysis. The college catalogs were collected and studied. Information was sought concerning the academic background of the teaching staff, the curricular changes over the five year period as well as those proposed for the future, and the perceptions of the interviewees as to the influence of the CUPM recommendations on these changes.

The investigator consistently felt that he received from knowledgeable, professional individuals honest and candid comments to all his questions. While maintaining an objective approach to the recommendations himself, he felt that the participants provided willingly a wealth of subjective information for comparison and analysis.

Major Findings and Conclusions. The survey revealed what one sensibly could consider the "average" college and the "average" mathematics department in the state, using in all cases the median as the average. This "typical" college had 850 students, and awarded 166 undergraduate degrees in 1967, of which eight were in mathematics. Of these eight graduates, two went to graduate school in mathematics, four entered teaching and two found employment in industry and/or government. The mathematics department had four full-time and one part-time staff members, and of these five teachers, one held a doctorate while the other four held at least master's degrees. The teaching load of this staff was thirteen hours per week per semester. The college library housed the mathematics collection of about 750 volumes and subscribed to six journals in mathematics and mathematics-education.

A total of 335 mathematics majors were graduated from all the colleges in the study. The total number of entering graduate students in mathematics was seventy-one, or 21 per cent of the graduates. The total number of graduates entering teaching was 120, or 36 per cent of the mathematics graduates. The total number of graduates entering industrial and/or governmental work was eighty, or 24 per cent of the graduates. Sixty-four graduates or 19 per cent of the total entered some other field of work, usually military service.

The study further revealed that, for the five year period (1962–1967) under consideration, there had been considerable curricular change. There had been a definite and marked increase in the amount and depth of linear algebra being offered by the mathematics departments. This change was almost matched by the increasing offerings in real and complex analysis. Expanded interest in abstract algebra was reflected by the increase in the offerings of a second course in the area. To a lesser degree there had been a move toward more logic, geometry (especially for prospective teachers), numerical analysis, topology, and computer science. On the other hand there seemed to have been a slight decrease in the offerings in probability and statistics, and in the areas of applied mathematics, but no reasons for this decrease were forthcoming from the chairmen interviewed.

From all that he could observe, the investigator concluded that the spirit in which the mathematics courses were taught was a modern one, and that in most cases the staff was reasonably well prepared academically to teach the courses in the modern curricula. These courses were generally not as rigorous or as deep, however, as those recommended by CUPM, and usually employed less sophisticated and rigorous textbooks.

With regard to the three types of programs given special consideration in this study, it was found that offerings of courses listed in the general curriculum proposal had increased in number over the five year period, as had the courses and requirements for teacher training. Less change seemed to have occurred in the Level I area than in Level III, while Level II, as a distinct program, was almost nonexistent. It also became evident to the investigator that very few colleges prepare students in any special way for graduate work in mathematics, due primarily to the size of the institution and the number of majors in each program. Hence, no conclusions could be drawn about the changes in the offerings in the pre-graduate program.

The study also revealed that there was a universal positive attitude toward the CUPM recommendations, each chairman expressing the feeling that there was value to be gained from having them. A number of reservations about the recommendations were expressed, however, the major one being that CUPM had not considered the smaller institution and its problems. There was no feeling on the part of any participant that he was pressured to adopt or implement the recommendations, but in almost every instance the interviewee stated that he had used the recommendations as guidelines, choosing from them the most appropriate elements in light of his particular students and staff. Often he used the recommendations as a lever when dealing with administrators concerning curricular changes.

From all that was heard and observed, the investigator concluded that the CUPM recommendations *did* have an effect on the collegiate mathematics departments in Maryland. The effect seemed to be more indirect, informal and sometimes incomplete than direct and formal. In only a few instances in large

panels as they strive to create realistic recommendations for undergraduate colleges.

6. The *Basic Library List* and other similar publications should be revised and updated every two years to keep them in tune with the many developments in the constantly growing field of mathematics.

In conclusion, one cannot escape the fact that collegiate mathematics must continue to change to keep pace with the changing world. The Committee on the Undergraduate Program in Mathematics has, in at least Maryland and probably throughout the nation as well, served a very worthwhile purpose of guiding change. In fact, it appears that this organization has had more overall, longrange effect on collegiate mathematics curricula than any other single national group. It has raised our sights and brought the mathematics teaching profession together for meaningful discussions. It can and should continue to be a strong, viable force in the uncertain future.

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NEW ADVANCED PLACEMENT COURSES IN MATHEMATICS

D. T. FINKBEINER, Kenyon College and J. D. NEFF, Georgia Institute of Technology

Undergraduate mathematics faculties and administrative officers should be alerted to important changes in the Advanced Placement Program in Mathematics of the College Entrance Examination Board. Initiated in 1952 as a pilot study, the program proposed that secondary schools offer college-level instruction in calculus to qualified students, and that individual performance be measured by means of a nationwide examination set and graded by committees of examiners and readers from universities, colleges, and secondary schools [1].

Since 1955 the program has been under the aegis of CEEB. The number of examination papers in mathematics has risen rapidly—from 925 in 1955 to 11,623 in 1968. More significantly, the high standards of the program have been recognized by a steadily increasing number of universities and colleges which use the results of the examination as a basis for awarding advanced placement or degree credit or both.

For the first time this year, advanced placement candidates in mathematics will have a choice of writing either one of *two* examinations, denoted by Calculus AB and Calculus BC. Calculus AB includes topics in elementary functions and introductory calculus; its coverage is comparable to the combined content of Mathematics 0 and Mathematics 1 (second version), as described in the CUPM report *A General Curriculum in Mathematics for Colleges* (GCMC) [2]. Calculus BC is a year course in single variable calculus, including topics in infinite series

and differential equations, comparable to GCMC courses Mathematics 1 and Mathematics 2 (first version).

The new system of two examinations is intended to provide college and university mathematics departments with a more precise measure of both the quality and quantity of preparation in calculus of students who enter after completing one of these advanced placement mathematics courses. This should enable each college to place such students more appropriately within its own mathematics curriculum and should facilitate awarding degree credit for demonstrated competence in college-level mathematics studied in secondary school.

Persons who have the responsibility for making placement decisions in colleges and universities are urged to distinguish between candidates who present examination scores for Calculus AB and those who present scores for Calculus BC. The two course descriptions can be obtained from CEEB [3]. As in the past, examination grades will be reported on an ascending scale, 1 through 5. Corresponding grades on the two examinations are intended to indicate a difference of one semester in the calculus preparation of the candidates.

References

1. R. S. Pieters and E. P. Vance, The Advanced Placement Program in Mathematics, The Mathematics Teacher, Vol. 54, No. 4, April 1961.

2. A General Curriculum in Mathematics for Colleges. CUPM Central Office, Box 1024, Berkeley, California, 94701.

3. 1968–69 Advanced Placement Mathematics. College Entrance Examination Board, Publications Order Office, Box 592, Princeton, New Jersey, 08540.

A NEW GRADUATE DEGREE IN MATHEMATICS

J. M. LAIBLE, Eastern Illinois University

Beginning with the fall quarter 1969, Eastern Illinois University, Charleston, Illinois, will offer a new graduate degree in mathematics. The new degree is called the Specialist in College Teaching (S.C.T.) with a major in mathematics. (The program leading to this degree was approved by the Illinois Board of Higher Education on December 2, 1968.) The degree program is the result of more than two years of planning by the Mathematics Department and the University. The idea originated with Dean Lawrence Ringenberg (then also Head of the Mathematics Department) and was initially developed by him. Later revisions and the final proposal were prepared by the current Department Head, Dr. A. J. DiPietro. The outline used in preparation of this program is in line with proposals being made by the National Committee on Graduate Studies of the American Association of State Colleges and Universities.

The S.C.T. degree program is intended to help provide an adequate number of well-trained junior and senior college teachers in mathematics. The S.C.T.

and differential equations, comparable to GCMC courses Mathematics 1 and Mathematics 2 (first version).

The new system of two examinations is intended to provide college and university mathematics departments with a more precise measure of both the quality and quantity of preparation in calculus of students who enter after completing one of these advanced placement mathematics courses. This should enable each college to place such students more appropriately within its own mathematics curriculum and should facilitate awarding degree credit for demonstrated competence in college-level mathematics studied in secondary school.

Persons who have the responsibility for making placement decisions in colleges and universities are urged to distinguish between candidates who present examination scores for Calculus AB and those who present scores for Calculus BC. The two course descriptions can be obtained from CEEB [3]. As in the past, examination grades will be reported on an ascending scale, 1 through 5. Corresponding grades on the two examinations are intended to indicate a difference of one semester in the calculus preparation of the candidates.

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degree is not a master's degree. The course requirements are double those for the usual M.A. or M.S. degree in mathematics. Nor is the S.C.T. degree a doctoral degree. No dissertation is required. The characteristic feature of the S.C.T. degree is a *supervised* teaching internship at the college level.

Prerequisites. The S.C.T. degree program assumes an undergraduate background of mathematics equivalent to the successful completion of two quarters of modern algebra and a year of calculus. Students will be required to take undergraduate courses at no credit to remove any existing deficiencies. No teaching experience is required for the program.

In the description which follows, we assume the minimal background indicated above. The student with course work beyond the bachelor's degree or with teaching experience will be placed in the program at a level appropriate to his background.

Program. The S.C.T. program normally requires three years for completion. An approved program for the degree consists of 12 courses (48 quarter hours) in addition to a master's degree in mathematics. For those students beginning the program with only a bachelor's degree, the M.A. degree requirements must be completed by the end of the second year of residence. The 48 quarter hours of course work together with the work for the master's degree must include the following.

1. Twelve quarter hours in Advanced Calculus, Real Variables, Complex Variables.
2. Twelve quarter hours in Modern Algebra (survey), Groups, Rings, Fields, Topological Groups.
3. Twelve quarter hours in Higher Geometry, Topology, Differential Geometry, Introduction to Differentiable Manifolds, Algebraic Topology.
4. Twelve quarter hours in Computer Programming, Numerical Analysis, Probability, Statistics.
5. Twelve quarter hours in Independent Study and/or Thesis.
6. From twelve to twenty quarter hours in mathematics courses chosen by the student in consultation with his adviser.
7. From twenty to twelve quarter hours in an approved minor.
8. Four quarter hours in Education 596, The Junior College Movement.

A student enrolled in the independent study referred to in requirement 5 above, studies a topic in mathematics which is not among those covered in ordinary university courses. He makes periodic reports to his supervising professor on his progress. The student is examined over the material studied, and a grade is determined. Each independent study carries four quarter hours of credit. The thesis mentioned as a partial alternative to the independent study is merely an expository paper written on a topic chosen by the student and his adviser. A grade is earned, and the thesis counts for four quarter hours credit. Thus the student must do at least eight quarter hours of independent study.

Teaching Internship. The unifying concept of the S.C.T. degree program is the teaching internship. This requirement may be relaxed only if a student has college and/or high school teaching experience. The graduate mathematics faculty will evaluate any teaching experience presented by the student.

During the fall quarter of the first year of participation in the program, the student will observe and assist his adviser in teaching one class. The student will meet with his adviser regularly to discuss teaching plans and the problems associated with the instruction of the class. The student will aid the adviser in preparing examinations and in evaluation of the students in the class. During the winter and spring quarters, the student will teach one class under the supervision of his adviser. The student will be under contract as a Faculty Assistant and will receive $270 per month (the same stipend as a graduate assistant) during the academic year.

During the second and third years of the program, the student will teach two classes under the supervision of his adviser. Naturally, the intensity of supervision will diminish during the three years the student is in the program. None-the-less, the student will continue to meet regularly with his adviser to discuss problems related to the student's teaching. Some of the teaching may be done at a nearby junior college. The student will be under contract as a part-time Instructor and will be paid a minimum of $600 per month during each academic year. Fellowships at $270 per month will be available for the summer quarters.

The mathematics faculty at Eastern is enthusiastic about the S.C.T. degree program, and it is felt that the degree provides excellent training for both junior and senior college teachers. The S.C.T. degree will probably be joined in a few years by a Doctor of Arts degree which will differ from the S.C.T. degree in that a worthwhile expository dissertation will also be required. The decision to offer a doctoral degree will undoubtedly be contingent on the success of the S.C.T. program. Preliminary reaction to the S.C.T. degree here in Illinois has been excellent.

Persons desiring more detailed information concerning the Specialist in College Teaching degree may contact J. M. Laible, Coordinator of the Graduate Program in Mathematics, Department of Mathematics, Eastern Illinois University, Charleston, Illinois 61920.

$$\sum_{k=1}^{\infty} \{[nr^{-k} + ar^{-1}] - [nr^{-k}]\}$$

digits which are $\geqq r-a$.

E 2180. *Proposed by Norman Schaumberger, Bronx Community College, New York*

Show that $\sum_{j=0}^{n-1} (-1)^j \cos^n(\pi j/n) = n/2^{n-1}$.

E 2181. *Proposed by Jack Garfunkel, Forest Hills High School, New York*

Given any triangle ABC and a given segment BP on side BC, determine (by geometric construction) segments CQ, AT on sides CA and AB respectively, so that equilateral triangles erected outwardly on these three segments have vertices that are the vertices of an equilateral triangle.

E 2182.* *Proposed by N. P. Salz, Bangkok, Thailand*

Let a K-sequence be a block of K consecutive odd integers, each of which is divisible by at least one of the n odd primes, $p_1=3$, $p_2=5, \cdots, p_{n-1}, p_n$. Prove or disprove: If p_{n-2} divides at most one term of a K-sequence, then $K \leqq p_{n-1}-1$.

E 2183.* *Proposed by Stevan Silverman, University of British Columbia*

We say that two points in the plane determine an *admissible* line if the line they determine is vertical, horizontal, has slope 1, or has slope -1. What is the maximum number of admissible lines n points can determine?

SOLUTIONS OF ELEMENTARY PROBLEMS

An Inequality

E 2032 [1967, 1134; 1968, 1124]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove the following inequality

$$\min[(b-c)^2, (c-a)^2, (a-b)^2] \leqq \tfrac{1}{2}(a^2+b^2+c^2),$$

with a, b, c real numbers.

Study the analogous problem for min $[(a_k-a_i)^2]$, $k<i$; $k, i=1, 2, \cdots, n$.

Corrected solution by Morris Newman, National Bureau of Standards and Joseph Lehner, University of Maryland. Theorem: If $a_1, a_2, \cdots, a_n$ are real numbers, then

$$\min_{a_i \neq a_j} (a_i - a_j)^2 = \mu^2(a_1^2 + \cdots + a_n^2), \quad \mu^2 = \frac{12}{n(n^2-1)} \cdot$$

Proof: We may assume $a_1 \leqq a_2 \leqq \cdots \leqq a_n$ and, by homogeneity, $\sum_{i=1}^{n} a_i^2 = 1$. Now

$$(*) \qquad \sum_{1 \leqq i < j \leqq n} (a_i - a_j)^2 = n \sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2.$$

Assume $a_{i+1}-a_i>\mu>0$, $i=1, 2, \cdots, n-1$. Then $(a_j-a_i)^2>(i-j)^2\mu^2$, $i, j=1, \cdots, n$ and

$$\sum_{1\leq i<j\leq n} (a_i - a_j)^2 > \mu^2 \cdot \sum_{1\leq i<j\leq n} (i-j)^2 = \mu^2 \cdot \frac{n^2(n^2-1)}{12} = n.$$

Inserting this in (*) we get

$$n < n \sum_{i=1}^n a_i^2 - \left(\sum_{i=1}^n a_i\right)^2 \leqq n \sum_{i=1}^n a_i^2,$$

or $\sum_{i=1}^n a_i^2 > 1$, a contradiction to our normalization. Hence $\min_{a_i\neq a_j}(a_i-a_j)^2 \leqq \mu^2$, as asserted. For each n the inequality is sharp.

Editorial Note. The original solution was incorrect for the general case, although it works when all a_i are of the same sign. The error was pointed out by Thomas Hughes who also provided a solution.

A Determinant Evaluation

E 2064 [1968, 190]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let A_n be an $n\times n$ determinant in which the entries, 1 to n^2, are put in order along the diagonals. For example

$$A_4 = \begin{vmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 8 & 11 \\ 6 & 9 & 12 & 14 \\ 10 & 13 & 15 & 16 \end{vmatrix}.$$

Show that if $n=2k$ then $A_n = \pm k(k+1)$, and if $n=2k+1$, $A_n=\pm(2k^2+2k+1)$.

Solution by Anon, Erewhon-upon-Wabash. By permuting columns,

$$A_n = (-1)^{n(n-1)/2} \begin{vmatrix} \cdot & & & \cdot & 4 & 2 & 1 \\ & \cdot & & & \cdot & 5 & 3 \\ & & \cdot & & & \cdot & 6 \\ n^2 & & & \cdot & & & \cdot \end{vmatrix}.$$

Subtract row $(n-1)$ from row n, then subtract row $(n-2)$ from row $(n-1)$, $\cdots$, finally subtract row 1 from row 2. After that do the same operations with columns; the result is

$$A_n = \epsilon \begin{vmatrix} a & -u \\ v & B \end{vmatrix},$$

where

$$\epsilon = (-1)^{n(n-1)/2}, \qquad a = (n^2 - n + 2)/2,$$
$$u = (n-1, n-2, \cdots, 3, 2, 1),$$

$$v = \begin{bmatrix} n \\ n-1 \\ \cdot \\ \cdot \\ \cdot \\ 4 \\ 3 \\ 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdot & \cdot \\ \cdot & & \cdot & \cdot \; \cdot & \cdot \\ \cdot & & & \cdot \; \cdot \; \cdot & \cdot \\ \cdot & & & \cdot \; \cdot & -1 \\ 1 & \cdot & \cdots & 1 & 0 \end{bmatrix}.$$

By standard formulas,

$$A_n = \epsilon[a \det B + u(\operatorname{cof} B)v].$$

Case $n=2k$. Then $\epsilon=(-1)^k$ and $\det B=0$ since B is skew of odd order. By calculation

$$\operatorname{cof} B = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \\ -1 & 1 & -1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot \end{bmatrix},$$

so $u(\operatorname{cof} B)=k(1, -1, 1, -1, \cdots)$ and $u(\operatorname{cof} B)v=k(k+1)$. Hence $A_{2k}=(-1)^k k(k+1)$. Case $n=2k+1$. Then $\epsilon=(-1)^k$, $a=2k^2+k+1$, and $\det B=1$. Also

$$\operatorname{cof} B = \begin{bmatrix} 0 & 1 & -1 & 1 & -1 & \cdot & \cdot & \cdot \\ -1 & 0 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & 0 & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & \\ \cdot & & & \cdot & \cdot & \cdot & & \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & 1 \\ & & & & & & -1 & 0 \end{bmatrix},$$

hence $u(\operatorname{cof} B)=(-k, (k+1), -k, (k+1), \cdots)=-k(1, -1, 1, -1, \cdots)+(0, 1, 0, 1, \cdots)$ and $u(\operatorname{cof} B)v=-k^2+k(k+1)=k$. Finally $A_{2k+1}=(-1)^k(2k^2+2k+1)$.

Also solved by L. Carlitz, C. Gardner, Norman Miller, Gregory Wulczyn, Alexander Zujus, and the proposer.

Dissection Diameters

E 2102 [1968, 671]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Given an equilateral triangle of side 1. Show how, by a straight cut, to get two pieces which can be rearranged so as to form a figure with maximal diameter (a) if the figure must be convex; (b) otherwise.

Solution by Michael Goldberg, Washington, D. C. To obtain convex arrangements it is necessary that the joining edges be equal. If the cut is made through a vertex, and not through the midpoint of the opposite side, then the only convex arrangements are those obtained by joining the two unit edges, when a maximum diameter of unity is attained. The only other convex arrangements are made by taking a cut through the midpoint of a side.

A convex polygon can be formed from the triangle ABC by the cut DE in which D is the midpoint of AB. The moved triangle ADE can be placed so that E is at E' (which is on the line BC', where C' is the reflection of C across AB) or at E'', as shown in Figure 1. The maximum diameter $E'C$ is obtained when E coincides with C (and E' coincides with C'). Then the diameter is $\sqrt{3}$.

But as E approaches C, the distance CE'' is also increased (see Figure 2). Hence the maximum diameter obtained is $\sqrt{13}/2$.

Nonconvex polygons can be formed by adjoining the two pieces as shown in Figure 3. The maximum diameter approaches 2 as a limit.

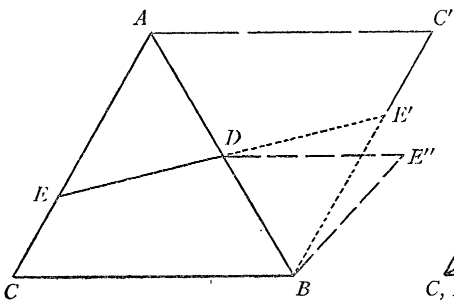

FIG. 1

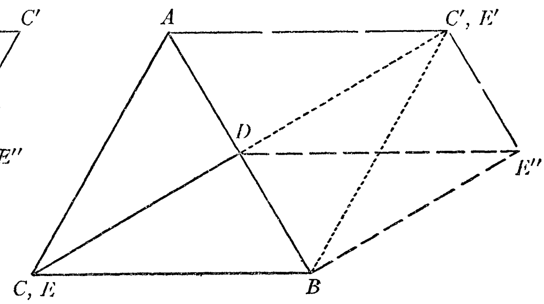

FIG. 2

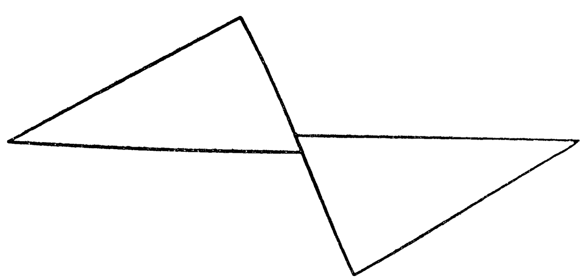

FIG. 3

Point in a Cube

E 2103 [1968 671]. *Proposed by Simeon Reich, The Technion, Haifa, Israel*

Find the seven smallest numbers a_k ($k=1, \cdots, 7$) with the following property: If a point P is inside a unit cube $A_1A_2 \cdots A_8$ at most k of the eight distances PA_j ($j=1, \cdots, 8$) are greater than a_k (thus at most one of these distances will be greater than a_1, at most two greater than a_2, etc.).

Solution by Michael Goldberg, Washington, D. C. At most seven distances will be greater than zero. Hence $a_7=0$.

If P is the midpoint of an edge, then two of the distances will be 1/2. Hence at most six distances will be greater than 1/2; that is, $a_6=1/2$. The 1/2 cannot be reduced.

If P is the midpoint of a face, then four of the distances are $\sqrt{2}/2$. Hence, at most four distances will be greater than $\sqrt{2}/2$; that is, $a_4=\sqrt{2}/2$.

If P is the midpoint of an edge, then the distances to the vertices on the nearby parallel edges are $\sqrt{5}/2$. The distances to the vertices on the opposite parallel edge are greater. Hence, at most two distances are greater than $\sqrt{5}/2$; that is $a_2=\sqrt{5}/2$.

If P is at a vertex, then the most remote vertex is at distance $\sqrt{3}$, but the next nearer vertices are at distance $\sqrt{2}$. Hence, $a_1=\sqrt{2}$.

If P is at distances $\sqrt{2}/2$ from three vertices, then it is at distance $\sqrt{2}/2$ from a fourth vertex also. Hence, $a_3=a_4$.

Similarly, $a_5=a_6$. The complete tabulation follows:

a_7	a_6	a_5	a_4	a_3	a_2	a_1
0	$\frac{1}{2}=.500$	$\frac{1}{2}=.500$	$\sqrt{2}/2=.707$	$\sqrt{2}/2=.707$	$\sqrt{5}/2=1.118$	$\sqrt{2}=1.414$

Also solved by H. V. Monks.

A Volume with Elliptic Cross Sections

E 2104 [1968, 671]. *Proposed by F. Dapkus, Seton Hall University, South Orange, N. J.*

If a line segment AB of fixed length is moving in such a way that A and B are sliding along two perpendicular, non-intersecting lines, determine the volume bounded by the surface swept out by AB.

Solution by Don N. Page, sophomore, William Jewell College, Liberty, Mo. Let $L_A(x=k,\ y=0,\ z=0)$ and $L_B(x=0,\ y=t,\ z=a)$ be the perpendicular lines separated by a fixed minimum distance a. Then the endpoints of segment AB are $A(k,\ 0,\ 0)$ and $B(0,\ t,\ a)$, with $(k^2+t^2+a^2)^{1/2}=c$, the length of AB. Thus $t=\pm(c^2-a^2-k^2)^{1/2}$, and the equations of segment AB for a given k are

$$x = -ku + k, \qquad y = \pm u(c^2-a^2-k^2)^{1/2}, \qquad z = au,$$

with parameter u. In the plane produced by giving z a fixed value, $0\leqq z\leqq a$, we have $x=-kz/a+k$ and $y=\pm z(c^2-a^2-k^2)^{1/2}/a$, with k the parameter as AB is moved, thus forming the ellipse

$$\frac{x^2}{(c^2-a^2)((a-z)/a)^2}+\frac{y^2}{(c^2-a^2)(z/a)^2}=1$$

of area $E=\pi(c^2-a^2)(a-z)(z)/a^2$. Integrating this cross-sectional area between the limits of the two perpendicular lines produces

$$V = \int_0^a E\, dz = \pi a(c^2 - a^2)/6,$$

the volume bounded by the surface swept out by AB.

Also solved by P. R. Chernoff, L. E. Clarke (England), H. Demir (Turkey), Jordi Dou (Spain), Michael Goldberg, J. F. Golightly and W. F. McGrath, C. M. Jensen, Lew Kowarski, D. C. B. Marsh, Bohuslav Mišek (Czechoslovakia), G. N. Reddy, Judith Richman, A. A. Sardinas, J. A. Tierney, Jan Velmsheia (Norway), A. Zujus, and the proposer.

Functions such that $f^{-1} = f'$

E 2105 [1968, 779]. *Proposed by H. L. Nelson, Livermore, California*

Find all continuous real-valued functions f defined on the positive reals for which $f^{-1} = f'$.

Solution by A. C. Hindmarsh, Livermore, California. The class of functions in question contains $f(x) = Ax^c$ provided $c = (1+\sqrt{5})/2$ and $A = c^{1-c}$. It contains no other functions.

Any function f in the class must be monotone and C^1, and must map $(0, \infty)$ onto itself with $f' > 0$. Repeated differentiation of $f'(f(x)) = x$ shows that $f \in C^\infty$ and that $f'' > 0$, $f''' < 0, \cdots, (-1)^k f^{(k)} > 0$. By Bernstein's theorem, f is real-analytic on $(0, \infty)$.

Integration of $f'(f(x))f'(x) = xf'(x)$ gives

$$f(f(x)) = \int_0^x yf'(y)dy.$$

If $f(x) > x$ for all x, then $f'(y) < y$ and we get $x < \int_0^x y^2 dy = x^3/3$, a contradiction for small x. If $f(x) < x$ for all x, we get a similar contradiction for large x, and conclude that f has a fixed point a. But $f(x) - x$, being strictly convex, has at most two zeros, counting the one at $x = 0$. Thus in $(0, \infty)$ f has exactly one fixed point a, with $f(x) < x$ in $(0, a)$ and $f(x) > x$ in (a, ∞). The reverse inequalities hold for f'.

Suppose f_1, f_2 are two functions in the class, having fixed points a_1, a_2 respectively, with $a_1 \geqq a_2$. Let $g = f_1 - f_2$. If $a_1 = a_2 = a$, then $g(a) = 0$, $g'(a) = 0, \cdots$, and g (which is real-analytic) vanishes identically.

If $a_1 > a_2$, then in $[a_2, a_1)$ we have $f_1(x) < x \leqq f_2(x)$, $f_1'(x) > x \geqq f_2'(x)$, or $g < 0$, $g' > 0$. Since $g(0^+) = 0$, there is a point $b \in (0, a_2)$ for which $g'(b) = 0$, $g' > 0$ in (b, a_1), and $g < 0$ in $[b, a_1)$. For this b we have $f_1'(b) = f_2'(b) = b'$ with $b' \in (b, a_2)$ (since $a_2 > f_2'(x) > x$ for $x < a_2$). But this implies that $g(b') < 0$, contrary to $f_1(b') = f_1(f_1'(b)) = b = f_2(f_2'(b)) = f_2(b')$.

We conclude that no pair of distinct functions exists, or that the class contains only the function stated at the start (which has its fixed point at $x = c$).

Also solved by T. E. Elsner. Partial solutions by D. J. Johnson, Lew Kowarski, and the proposer.

Unions of Subsets of a Finite Set

E 2106 [1968, 779]. *Proposed by Bernt Lindström, University of Stockholm, Sweden*

Let S be a set with n elements and $M_1, M_2, \cdots, M_{n+1}$ be nonempty subsets of S. Prove that one can find r, s and $r+s$ distinct indices $i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_s$ such that

$$M_{i_1} \cup \cdots \cup M_{i_r} = M_{j_1} \cup \cdots \cup M_{j_s}.$$

Solution by H. S. Hahn, West Georgia College, Carrollton, Ga. The number of unions of sets taken from $M_1, M_2, \cdots, M_{n+1}$, except the void one, is $2^{n+1}-1$. Since there are only 2^n-1 nonempty subsets of S, these unions cannot all be distinct sets. Taking two unions, identical as sets, and deleting common M-sets, we get the result.

Also solved by M. G. Greening (Australia), J. F. Leetch, Dan Marcus, C. B. A. Peck, Eddy Smet, and the proposer.

Coin Weighing Problem

E 2107 [1968, 779]. *Proposed by Bernt Lindström, University of Stockholm, Sweden*

Counterfeit coins weigh a and genuine coins weigh b, $a \neq b$. One is given two samples of three coins each and knows that each sample has one counterfeit coin. How many weighings are needed to isolate the two counterfeit coins by the aid of an accurate scale (not a balance)?

Solution by the proposer (assuming a and b are known quantities). There are three possible results of each weighing as the number of counterfeit coins on the pan can be 0, 1 or 2. Hence m weighings give at most 3^m possible combinations of results. The counterfeit coins can be isolated only if $3^2 \leqq 3^m$, hence only if $m \geqq 2$. In fact $m > 2$. For, clearly, at least two coins must be weighed in each trial if we hope to succeed in two trials. If the result of both weighings is one counterfeit on the pan there remains ambiguity: either two coins in a sample were weighed simultaneously, or there were at least two pairs of coins with one from each sample weighed simultaneously.

We shall prove that three weighings suffice. Give the coins in each sample labels 0, 1, 2. In the first weighing weigh the coins labelled 1, in the second, coins labelled 2. Then there are at most two possible pairs of counterfeit coins. By one weighing one can determine the counterfeit coin which belongs to the first sample, then also the pair is known.

Also solved by Merrill Barnebey, Jordi Dou (Spain), Michael Goldberg, C. V. Heuer, Thomas Hughes, Dan Marcus, D. C. B. Marsh, P. D. Matthews, Jr., B. McMillan, D. N. Page, James Scandale, B. L. Schwartz, D. E. Searls, W. A. Smith, and Steven Szabo.

Two respondents, Victor Abad, and S. D. Joglekar (India), solved the problem assuming that a and b are unknown, concluding that four weighings are needed. Many variations and extensions of both problems suggest themselves.

Triangle Construction

E 2109 [1968, 780]. *Proposed by H. Demir, Middle East Technical University, Ankara, Turkey*

Let ABC be a triangle and A' be any fixed point on the side BC. Construct the inscribed triangle $A'B'C'$ which is directly similar to a given triangle XYZ.

Note by A. W. Walker, Toronto, Canada. The required construction will be found in N. A. Court, *College Geometry*, ed. 1, 1925, p. 47. It is a simple application of the following theorem, established on p. 46: *If one vertex of a triangle of variable size and given shape remains fixed and a second vertex moves on a given straight line, then the locus of the third vertex is also a straight line.*

Also solved by Anders Bager (Denmark), Walter Bluger, C. W. Eliason, Jr., Michael Goldberg, M. G. Greening (Australia), Beckham Martin, D. N. Page, and the proposer.

Similar Triangles

E 2110 [1968, 780]. *Proposed by H. Demir, Middle East Technical University, Ankara, Turkey*

If, in a plane, the triangles AUV, VBU, UVC are directly similar to a given triangle, then so is ABC.

Solution by M. G. Greening, University of New South Wales, Australia. Represent the points by complex numbers using the appropriate lower case letters and take the given triangle as $Z_1Z_2Z_3$. Let the direct similarities be $z\to\alpha_i z+\beta_i$ $(i=1, 2, 3)$. Then $u=\alpha_i z_{i+1}+\beta_i$, $v=\alpha_i z_{i+2}+\beta_i$ $(i=1, 2, 3)$, taking subscripts modulo 3. Then $\alpha_i z_i(z_{i+1}-z_{i+2})=z_i(u-v)$ and

$$\beta_i(z_{i+1} - z_{i+2}) = z_{i+1}v - z_{i+2}u$$

so that

$$\sum_{i=1}^{3} (\alpha_i z_i + \beta_i)(z_{i+1} - z_{i+2}) = 0.$$

As $\sum_{i=1}^{3} (z_{i+1}-z_{i+2})=0$ and $\sum_{i=1}^{3} z_i(z_{i+1}-z_{i+2})=0$ we get

$$0 = \begin{vmatrix} \alpha_1 z_1 + \beta_1 & z_1 & 1 \\ \alpha_2 z_2 + \beta_2 & z_2 & 1 \\ \alpha_3 z_3 + \beta_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} a & z_1 & 1 \\ b & z_2 & 1 \\ c & z_3 & 1 \end{vmatrix}$$

which is a sufficient condition for a direct similarity: $z_1\to a$, $z_2\to b$, $z_3\to c$ to exist.

Also solved by Leon Bankoff, Jordi Dou (Spain), C. W. Eliason, Jr., Michael Goldberg, Norman Miller, Simeon Reich (Israel), A. W. Walker, and the proposer.

Walker points out that the result may be found on p. 289 of R. A. Johnson, *Modern Geometry* (1929).

An Arithmetic Inequality

E 2111 [1968, 780]. *Proposed by R. D. Jenks, Brookhaven National Laboratory, New York*

Is it true that for any odd number $2n+1 (n \geqq 1)$ of positive numbers $x_1, x_2, \cdots, x_{2n+1}$,

$$\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_4} + \cdots + \frac{x_{2n}x_{2n+1}}{x_1} + \frac{x_{2n+1}x_1}{x_2} \geqq x_1 + x_2 + \cdots + x_{2n+1}$$

with equality only if all are equal?

Solution by L. E. Ward, Sr., Escondido, California. The answer is in the negative. The stated result is true when $n=1$ and false when $n=2$ and the x's have the values 5, 20, 4, 2, 200, respectively.

Also solved by L. Carlitz, W. F. Fox, Toyomasa Fujinawa (Japan), and D. C. B. Marsh. Compare Problem No. 4603 [1956, 191].

Arranging Odd Squares in Even Groups

E 2112 [1968, 780]. *Proposed by D. E. Daykin and D. G. Neal, University of Malaya, Kuala Lumpur*

The n squares of side $1, 3, 5, \cdots, 2n-1$ respectively are closed on two edges and open on two edges. They are to be arranged without overlapping on the x, y plane so that their edges are parallel to the x, y axes and so that no line parallel to an axis is to pass through an odd number of the squares.

(i) For which integers n does such an arrangement exist?

(ii) With n as small as you can, find an arrangement which also satisfies the line condition for lines equally inclined to the axes.

Solution to part (i) by the proposers. Suppose we have an arrangement, and let us draw lines parallel to the x axis through the corners of the squares. We can group these lines into pairs in such a way that the area of the parts of the squares between each pair of lines is an even integer. It follows that the total area, and hence n, is even.

Next we note that any eight consecutive squares of sides $2k-1, 2k+1, \cdots, 2k+13$ can be arranged to satisfy the conditions. The first and last squares are put in opposite corners of a $4k+12$ square, and then the remaining six squares are paired off in like manner. The resulting four $4k+12$ squares are set out as one $8k+24$ square. This fact shows that, if we can arrange n squares, then we can arrange $n+8$ squares, and so we are interested only in the first few values of n. It is easy to show that $n=2, 4, 6$ are impossible, and it is not difficult to give an arrangement for $n=8, 10, 12, 14$. Thus an arrangement exists if and only if n is even and $n \geqq 8$.

The arrangements referred to can be identified on a standard coordinate lattice. Let $s(a, b)$ mean an $s \times s$ square whose lower left-hand corner is at the point

(a, b). Thus, for $n=8$, take 5(0, 0), 11(5, 5), 7(16, 0), 9(23, 7), 1(0, 16), 15(1, 17), 3(16, 16), 13(19, 19).

For $n=10$: 15(0, 0), 13(15, 0), 11(28, 0), 1(39, 10), 7(40, 3), 3(47, 0), 19(0, 20), 9(19, 30), 17(28, 13), 5(45, 15).

For $n=12$: 1(0, 0), 23(1, 1), 11(24, 0), 13(35, 11), 3(0, 24), 21(3, 27), 9(48, 24), 15(57, 33), 5(24, 48), 19(29, 53), 7(48, 48), 17(55, 55).

For $n=14$: 1(0, 0), 5(1, 0), 21(6, 1), 19(27, 5), 9(46, 0), 17(55, 9), 11(72, 0), 15(83, 11), 27(0, 22), 25(27, 24), 3(52, 26), 7(55, 29), 13(62, 36), 23(75, 26).

Part (i) also solved by Michael Goldberg, and by Norman Miller.

Note. No correct solution to part (ii) was received, although Goldberg and Miller solved the problem without the parallel line condition. Miller conjectures that no solution exists.

An Endpoint Maximum

E 2113 [1968, 780]. *Proposed by Francis Sand, Princeton, N. J.*

Given an arbitrary finite set of n pairs of positive numbers $\{(a_i, b_i): i = 1, \cdots, n\}$, show that

$$\prod_{i=1}^{n} [xa_i + (1-x)b_i] \leqq \max\left[\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i\right]$$

for all $x \in [0, 1]$, with equality attained only at $x=0$ or $x=1$; if and only if

$$\left(\sum_{i=1}^{n} \frac{a_i - b_i}{a_i}\right)\left(\sum_{i=1}^{n} \frac{a_i - b_i}{b_i}\right) \geqq 0.$$

Solution by R. M. Meyer, State University of New York at Fredonia. Let

$$f(x) = \prod_{i=1}^{n} [xa_i + (1-x)b_i], \qquad g(x) = \log f(x).$$

Then

$$g'(x) = \sum_{i=1}^{n} \frac{a_i - b_i}{xa_i + (1-x)b_i}, \qquad g''(x) = -\sum_{i=1}^{n} \left(\frac{a_i - b_i}{xa_i + (1-x)b_i}\right)^2.$$

Since $g''(x) < 0$ for all $x \in [0, 1]$, the maximum of $g(x)$ [hence of $f(x)$] is attained at $x=0$ or $x=1$ if and only if $g'(0)$ and $g'(1)$ do not differ in sign, i.e., $g'(0)g'(1) \geqq 0$. Since $f(0) = \prod_{i=1}^{n} a_i$, $f(1) = \prod_{i=1}^{n} b_i$, $g'(0) = \sum_{i=1}^{n} (a_i - b_i)/a_i$, and $g'(1) = \sum_{i=1}^{n} (a_i - b_i)/b_i$, the assertion is proved.

Also solved by Anders Bager (Denmark), T. J. Cullen, Beatriz Margolis (Argentina), and the proposer.

A Covering Problem

E 2114 [1968, 780]. *Proposed by P. Richman and P. Rosenthal, Stanford University*

For each integer $n \geqq 1$, does there exist a fixed, finite set of $s \times s$ squares (s an

integer, $s \geqq n$) such that for all integers $k \geqq n$, any $k \times k$ square can be covered without overlapping by some or all of the $s \times s$ squares (allowing repetitions)? (In other words, is there a finite $s \times s$ basis for each n?) For example, if $n=2$, the $s \times s$ squares for $s=2, 3, 5, 7$ will do.

Solution by Neal Felsinger, Yale University. The set of squares with side s, where $n \leqq s < n^2$ or s is prime, $n^2 < s < 2n^2+n$, will suffice, although it need not be minimal. For, consider a $k \times k$ square with $k \geqq n^2$. Assume that an $m \times m$ square can be covered for all m such that $n \leqq m < k$. If k is composite, $k=pq$ where $k > p \geqq n$ so we can cover $k \times k$ with q^2 $p \times p$ squares and proceed inductively. On the other hand, if k is prime, $k > 2n^2+n$. Subdivide $k \times k$ into two squares of size $m \times m$ and $(k-m) \times (k-m)$ and two rectangles of size $m \times (k-m)$ each, where $m = n(n+1)$. Since $k-m > n^2$, $m \times (k-m)$ can be covered by nonoverlapping strips of dimensions either $m \times n$ or $m \times (n+1)$ and each of these strips can be covered as desired.

Also solved by Norman Miller, B. M. Stewart, and the proposers. Stewart also solved the more general problem for cubes.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before November 30, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

The asterisk (*) will be used to indicate that the proposer did not supply a solution. The editors solicit readers' solutions for these and for all problems (proposers' solutions are frequently not "best possible" and solutions by others will be given preference).

5677. *Proposed by B. W. Levinger, Case Western Reserve University.*

Let $A=(a_{i,j})$ be a real $n \times n$ matrix with $a_{i,j} \geqq 0$, $1 \leqq i, j \leqq n$. Prove that $r(A) \leqq r\left[\frac{1}{2}(A+A^T)\right]$, where $r(C)$ denotes the spectral radius of a matrix C.

5678.* *Proposed by S. Abhyankar, Purdue University*

Find necessary and sufficient conditions on the whole numbers p, q, r so that the rational curve $x=t^p$, $y=t^q$, $z=t^r$ in complex affine three space is the intersection of two surfaces.

5679. *Proposed by H. Kestelman, University College, London, England*

$\{\lambda_n\}$ and $\{\epsilon_n\}$ are real number sequences with $\lim_{n\to\infty} \lambda_n = \infty$. Prove that the set of x for which the sequence $\{\sin(\lambda_n x + \epsilon_n)\}$ converges has Lebesgue measure zero but may have cardinal c.

5680. *Proposed by Kamlesh Wasan, University of Delhi, India*

Consider a Noetherian integrally closed domain R in which each semi-primary ideal (i.e. an ideal whose radical is a prime ideal) is irreducible. Prove R is a Dedekind domain.

5681. *Proposed by C. L. Sabharwal, St. Louis University.*

Let $H_n(x)$ be a monic Hermite polynomial of degree n, given by

$$H_n(x) = (-1)^n e^{x^2/2} \cdot \frac{d^n}{dx^n} (e^{-x^2/2}),$$

and let q and m be nonnegative integers with $q+m$ even. Show that

$$P(x) = \sum_{\nu=0}^{\min(q,m)} (-1)^\nu \binom{q}{\nu} \frac{m!}{(m-\nu)!} x^{m-\nu} H_{q-\nu}(x),$$

has $q+m$ distinct nonzero real roots if $q>m$, and $2q$ distinct nonzero real roots if $q \leqq m$.

5682. *Proposed by Robert Spira, Michigan State University*

What is the relation of the radius of convergence of $\sum a_n z^n$ and the abscissa of convergence of $\sum a_n n^{-z}$?

SOLUTIONS OF ADVANCED PROBLEMS

Infinite Unitary Matrices

5593 [1968, 552]. *Proposed by David Shelupsky, The City College of New York*

Let (U_{rs}), $1 \leqq r, s < \infty$, be a unitary operator on l^2, and let $\{z_r\}$, $1 \leqq r < \infty$, be an arbitrary sequence of complex numbers. Prove that if $\sum_{s=1}^\infty U_{rs} z_s = 0$, $1 \leqq r < \infty$ (that is, if formally we have $Uz=0$), then $z_r = 0$ for each r.

Solution by P. R. Chernoff, University of California, Berkeley. The assertion is false. To see this it is convenient to reformulate it as follows: To say that

$$(*) \qquad \sum_{s=1}^\infty U_{rs} z_s = 0, \qquad r = 1, 2, \cdots$$

is to say that

$$\lim_{n\to\infty} \left(U\left(\sum_{s=1}^n z_s e_s \right), e_r \right) = 0, \qquad r = 1, 2, \cdots,$$

that is

$$\lim_{n\to\infty} \left(\sum_{s=1}^n z_s e_s, e_r' \right) = 0$$

where $e_r' = U^{-1} e_r$. Hence

$$(**) \qquad \lim_{n\to\infty}\left(\sum_{s=1}^{n} z_s e_s, y\right) = 0$$

for all y in the dense subspace spanned by the $\{e_r'\}$. Conversely, if (**) holds for all y in some dense subspace D, then, by extracting an orthonormal basis $\{e_r'\}$ from D via Gram-Schmidt, one can construct a unitary operator U such that (*) holds.

Now for the example. Let $\{z_r\}_1^\infty$ be any sequence such that $\sum_1^\infty |z_r|^2 = \infty$. Let D be the subspace of l_2 consisting of the finitely nonzero sequences $y = \{y_s\}_1^\infty$ such that $\sum_{s=1}^\infty z_s\bar{y}_s = 0$. Then (**) holds by construction for $y \in D$. It remains to show that D is dense in l_2, or, equivalently, that its orthogonal complement is (0). To see this, suppose that $x = \{x_r\}_1^\infty$ is orthogonal to D. Then in particular x is orthogonal to the vectors $\bar{z}_s e_r - \bar{z}_r e_s$, $r, s = 1, 2, \cdots$, which are obviously in D. Hence $x_r z_s - x_s z_r = 0$, so that there is a constant c such that $x_r = c\cdot z_r$. Because $\{x_r\}_1^\infty \in l_2$ and $\sum_1^\infty |z_r|^2 = \infty$, c must be 0.

Generators of the Symmetric Group

5610 [1968, 791]. *Proposed by K. S. Menger, Jr., Cambridge, Mass.*

(i) Show that any $n-1$ cycles, of lengths $2, 3, \cdots, n$ respectively, generate S_n, the symmetric group of degree n.

(ii) Show that if $m = rs$ then there exist $r+s-2$ cycles, no two of the same length, which do not generate S_m.

(iii) Show that any transposition and any cycle of length n generate S_n if and only if n is a prime.

Solution by C. V. Heuer and G. A. Heuer, Concordia College, Moorhead, Minnesota. In place of (i) and (ii) we establish the stronger results: (i′) Any two cycles of lengths 2 and 3, respectively, generate S_3. For $n > 3$, any three cycles of lengths 2, $n-1$ and n, respectively, generate S_n. (ii′) For any $m \geqq 3$, there exist $m-2$ cycles, no two of the same length, which do not generate S_m.

It is well known that the transpositions $(1\ 2), (1\ 3), \cdots, (1\ n)$ generate S_n.

(i′) Clearly any two cycles of length 2 and 3 respectively generate S_3. For $n > 3$ let M be the subgroup generated by the given three cycles. There is no loss of generality in assuming the given cycle of length $n-1$ is $\beta = (2, 3, \cdots, n)$. If $(i\ j)$ is the given transposition and α the given cycle of length n, then some power α^q of α maps i to 1 and $\alpha^{-q}(i\ j)\alpha^q = (1\ k) \in M$ for some k. For any t, $2 \leqq t \leqq n$, some power of β, say β^m, maps k to t. Hence $\beta^{-m}(1\ k)\beta^m = (1\ t) \in M$, so $M \doteq S_n$.

(ii′) The $m-2$ cycles $(1\ 2), (1\ 2\ 3), \cdots, (1\ 2 \cdots m-1)$ clearly do not generate S_m since they all fix m. (We assume, of course, that $m = rs$ is a proper factorization.)

(iii) Suppose n is prime. It is sufficiently general to assume that the given cycles are $\alpha = (1,\ k+1)$ and $\beta = (1\ 2 \cdots n)$. Let M be the subgroup generated

by α and β. For any t, $1 \leqq t \leqq n$, conjugating α by an appropriate power of β yields $(t, t+k)$ which then is in M. (Assume $t+k$ and all other integers in the argument are reduced mod n.) Then

$$(k+1, 2k+1)(1, k+1)(k+1, 2k+1) = (1, 2k+1) \in M.$$

Continuing in this manner we have that $(1, rk+1) \in M$ for all $r \geqq 1$. Since k and n are coprime, it follows that $(1\ i) \in M$ for $i = 2, \cdots, n$. Hence $M = S_n$.

Suppose $n = rs$, $r, s > 1$. Then the cycles $\alpha = (1, r+1)$ and $\beta = (1\ 2 \cdots n)$ do not generate S_n. Indeed if $A = \{(i\ j) : i \equiv j \bmod r\}$, then A is closed under conjugation by α and β and hence closed under conjugation by any element in the subgroup generated by α and β. Since A does not contain all transpositions it follows that α and β do not generate S_n.

Also solved by M. G. Greening (Australia), George Whitson, Kenneth Yanosko, and by the proposer.

A Polynomial Congruence

5611 [1968, 791]. *Proposed by L. Carlitz, Duke University*

Let D denote the polynomial domain $GF[p, x]$, p odd prime. Let $P(x)$ be a monic irreducible polynomial in D of degree n. Show that the congruence

$$U^2 \equiv P'(x) \pmod{P(x)}$$

is solvable with $U \in D$ if and only if

$$n(n-1)(p-1)/4 \equiv n-1 \pmod{2}.$$

Solution by the proposer. The congruence is solvable if and only if $(P'(x))^{(p^n-1)/2} \equiv 1 \pmod{P(x)}$. Put

$$P(x) = \prod_{j=0}^{n-1} (x - \theta^{p^j}) \qquad (\theta \in GF(p^n)).$$

Then $P'(x) \equiv P'(\theta) \pmod{x-\theta}$, so that

$$\begin{aligned}(P'(x))^{(p^n-1)/(p-1)} &\equiv (P'(x))^{1+p+\cdots+p^{n-1}} \\ &\equiv \prod_{j=0}^{n-1} P'(\theta^{p^j}) \equiv (-1)^{n(n-1)/2} d \pmod{x-\theta},\end{aligned}$$

where d is the discriminant of $P(x)$. Hence

$$(P'(x))^{(p^n-1)/2} \equiv ((-1)^{n(n-1)/2} d)^{(p-1)/2} \equiv \left(\frac{(-1)^{n(n-1)/2} d}{p}\right) \pmod{x-\theta},$$

where (d/p) is the ordinary Legendre symbol. It follows from the above that the congruence is solvable if and only if

$$((-1)^{n(n-1)/2} d/p) = 1.$$

On the other hand, by a theorem of Pellet, Voronoi and Stickelberger (see C. R. Acad. Sci. Paris 86 (1878), 1071–1072), if $P(x)$ is irreducible (mod p) and of degree n, the discriminant of $P(x)$ satisfies $(d/p)=(-1)^{n-1}$. Thus the condition becomes

$$(-1)^{n(n-1)(p-1)/4} = (-1)^{n-1},$$

which completes the proof.

Also solved by M. G. Greening (Australia).

Some Binomial Identities

5612 [1968, 791]. *Proposed by Franklin C. Smith, Minneapolis, Minn., and H. W. Gould, West Virginia University*

Evaluate the summations

$$\sum_{k=0}^{n} \binom{2x+i}{2k+j}\binom{x-k}{n-k}, \qquad n \geqq 0,$$

where x is any real number, for $i=0$, 1 and $j=0$, 1 (four sums). Is there a general closed formula valid for all integers i, j?

Solution by M. T. L. Bizley, London, England. Since

$$\binom{2x+i}{2k+j}$$

is the coefficient of θ^{2k+j} in $(1+\theta)^{2x+i}$, and

$$\binom{x-k}{n-k}$$

is the coefficient of $\theta^{2(n-k)}$ in $(1-\theta^2)^{-(x-n+1)}$, it follows that, provided $j=0$ or 1,

$$\sum_{k=0}^{n} \binom{2x+i}{2k+j}\binom{x-k}{n-k}$$

is the coefficient of θ^{2n+j} in $(1+\theta)^{2x+i}(1-\theta^2)^{-x+n-1}$ which equals

$$\begin{aligned}(1+\theta)^{2x+i}&(1+\theta)^{-x+n-1}(1-\theta)^{-x+n-1}\\ &= \left(\frac{1+\theta}{1-\theta}\right)^{x+n+i-1}(1-\theta)^{2n+i-2} = \left\{1+\frac{2\theta}{1-\theta}\right\}^{x+n+i-1}(1-\theta)^{2n+i-2}\\ &= \sum_{t=0}^{\infty}\binom{x+n+i-1}{t}2^t\theta^t(1-\theta)^{-t+2n+i-2}\end{aligned}$$

(where the upper limit would be finite if x happened to be an integer).

Case $i=0, j=0$. We require the coefficient of θ^{2n} in

$$\sum_{t}\binom{x+n-1}{t}2^t\theta^t(1-\theta)^{-t+2n-2}.$$

On the other hand, by a theorem of Pellet, Voronoi and Stickelberger (see C. R. Acad. Sci. Paris 86 (1878), 1071–1072), if $P(x)$ is irreducible (mod p) and of degree n, the discriminant of $P(x)$ satisfies $(d/p)=(-1)^{n-1}$. Thus the condition becomes

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is the coefficient of θ^{2n+j} in $(1+\theta)^{2x+i}(1-\theta^2)^{-x+n-1}$ which equals

$$\begin{aligned}
&(1+\theta)^{2x+i}(1+\theta)^{-x+n-1}(1-\theta)^{-x+n-1}\\
&\quad=\left(\frac{1+\theta}{1-\theta}\right)^{x+n+i-1}(1-\theta)^{2n+i-2} = \left\{1+\frac{2\theta}{1-\theta}\right\}^{x+n+i-1}(1-\theta)^{2n+i-2}\\
&\quad=\sum_{t=0}^{\infty}\binom{x+n+i-1}{t}2^t\theta^t(1-\theta)^{-t+2n+i-2}
\end{aligned}$$

(where the upper limit would be finite if x happened to be an integer).

Case $i=0, j=0$. We require the coefficient of θ^{2n} in

$$\sum_{t}\binom{x+n-1}{t}2^t\theta^t(1-\theta)^{-t+2n-2}.$$

$$\sum_{k=0}^{n}\binom{2x+1}{2n+1}\binom{x-k}{n-k} = 2^{2n}\binom{x+n}{2n} + 2^{2n+1}\binom{x+n}{2n+1}.$$

The above method yields an expression for the sum for general i, provided $j=0$ or 1, but fails for $j>1$ because the given sum is then no longer the coefficient of θ^{2n+j} in the product exhibited. It does not appear, therefore, that there is a closed expression in the general case.

Also solved by R. E. Shafer, and the proposers.

Harmonic Functions of an Entire Function

5613 [1968, 791]. *Proposed by Robert Goldstein, The Northern Polytechnic, London, England*

Let $\omega(z)$ be an entire function such that there exists a nonconstant function $u(z)$, harmonic for all z, for which $u(z)=u(\omega(z))$ for all z. Show that this implies $\omega(z)=\zeta z+a$, where ζ is a root of unity. This is a generalization of Problem 5329 [1966, 904].

Solution by the proposer. Let $v(z)$ be the conjugate harmonic function of $u(z)$. Then $f(z)=u(z)+iv(z)$ is an entire function, and so is $F(z)=e^{f(z)}$. Then

$$\begin{aligned}|F(\omega(z))| &= \left|e^{f(\omega(z))}\right| = \left|e^{u(\omega(z))+iv(\omega(z))}\right| = e^{u(\omega(z))}\\ &= e^{u(z)} = \left|e^{u(z)+iv(z)}\right| = \left|e^{f(z)}\right| = |F(z)|.\end{aligned}$$

As in the first solution of Problem 5329 [1966, 904] we use the result of Polya [J. London Math. Soc., I(1926) pp. 12–15] that for entire functions $f(z)$, $g(z)$ with $g(0)=0$, there is a constant c with $0<c<1$ such that $M_h(r)\geqq M_f(cM_g(r/2))$, where $h(z)=f(g(z))$ and $M_h(r)=\max_{|z|=r}|h(z)|$ with similar meanings given to $M_f(r)$, $M_g(r)$. Applying this result to $F(z)$, $\omega(z)$ we deduce that $\omega(z)$ must be a linear function of z. Let $\omega(z)=\zeta z+a$. Let $U(z)=u(z+a/(1-\zeta))$ if $\zeta\neq 1$. (If $\zeta=1$ there is nothing to prove.) Then

$$\begin{aligned}U(\zeta z) &= u\left(\zeta z+\frac{a}{1-\zeta}\right) = u\left(\zeta\left[z+\frac{a}{1-\zeta}\right]+a\right)\\ &= u\left(z+\frac{a}{1-\zeta}\right) = U(z).\end{aligned}$$

Thus U takes the same values at the points $z, \zeta z, \zeta^2 z, \cdots, \zeta^n z, \cdots$.

(i) $|\zeta|<1$. Then $\zeta^n z\to 0$ as $n\to\infty$, for all z, so by the continuity of $U(z)$ at 0 it follows that $U(z)=U(\zeta z)=\cdots=U(\zeta^n z)=\cdots=U(0)$, that is, $U(z)=U(0)$ for all z. Hence, $U(z)$ is a constant so also $u(z)$ is constant, contrary to hypothesis. So $|\zeta|<1$ is impossible.

(ii) $|\zeta|>1$. Now $|1/\zeta|<1$ and we use $U(z/\zeta)$ instead of $U(z)$ as in (i) to deduce that $|\zeta|>1$ is impossible.

(iii) $|\zeta|=1$. Then the set of points $z_0, \zeta z_0, \zeta^2 z_0, \cdots, \zeta^n z_0, \cdots$ is either

dense on the circle with center O and radius $|z_0|$, or $\zeta^p=1$ for some integer p. In the first case, because of the continuity of $U(z)$ we have that $U\equiv$constant on the above circle and so, using the maximum and minimum principles for harmonic functions, $U(z)\equiv$constant for $|z|\leqq|z_0|$. As z_0 is arbitrary, $U(z)\equiv$constant throughout the plane. Hence $\zeta^p=1$ for some integer p.

REMARK 1. Given any ζ with $\zeta^p=1$, and a constant a, there is a harmonic function $u(z)$ such that $u(z)=u(\zeta z+a)$. If $\zeta=1$ we may take $u(z)=I(z/a)$. If $\zeta\neq 1$, $\zeta^p=1$, we may take $u(z)=R\{(z-a/(1-\zeta))^p\}$.

REMARK 2. Another proof is possible proceeding from the fact that $R\{f(\omega(z))-f(z)\}\equiv 0$, with $f(z)$ as above.

Also solved by D. A. Hejhal.

Fast Tunnels Through the Earth

5614 [1968, 791]. *Proposed by Tung-Po Lin, San Fernando State College, California*

Assume that the earth is a sphere of homogeneous density. An initially stationary object at a point A on the surface of the earth slides under gravitational force through a frictionless tunnel leading to another point B on the surface. Show that the time needed to slide from A to B is a minimum when the tunnel takes the shape of a hypocycloid. Furthermore, show that this minimum time is equal to $(1-b^2/a^2)^{\frac{1}{2}}T$, where a is the radius of the earth, b is the distance from earth center to the nearest point on the tunnel, and T is the time needed to slide from A to B through a frictionless straight-line tunnel.

Solution by R. C. Lyness, Great Singleton (nr. Blackpool), England. Within the sphere, center O, the object at P is subject to a gravitational force $-\mu mr$, where $r=\overline{OP}$. The energy equation gives $\mu(a^2-r^2)=v^2$, and $\mu^{1/2}dt=(a^2-r^2)^{-1/2}ds$. We have to find the path C from A to B so that $\int_C(a^2-r^2)^{-1/2}ds$ is minimized. Now $ds=(1+r^2\theta_1^2)^{1/2}dr$, where $\theta_1=d\theta/dr$. Euler's equation gives

$$\frac{\partial}{\partial\theta_1}(1+r^2\theta_1^2)^{1/2}(a^2-r^2)^{-1/2}=k,$$

from which $r^2\theta_1(1+r^2\theta_1^2)^{1/2}=k(a^2-r^2)^{1/2}$; and on C, $r\sin\phi=k(a^2-r^2)^{1/2}$, where ϕ is the angle from the direction of r to the direction in which s is increasing. This is the (r,ϕ) equation of a hypocycloid. ϕ decreases from π at A to O at B and since r has its least value b when $\phi=\pi/2$, it follows that $k=b(a^2-b^2)^{-1/2}$. From the equation of C we have

$$dr\sin\phi+r\cos\phi d\phi=-kr(a^2-r^2)^{-1/2}dr,$$

$$-k(a^2-r^2)^{-1/2}\sec\phi dr=\tan\phi dr/r+d\phi=d(\theta+\phi)=d\psi.$$

So $k\mu^{1/2}dt=-d\psi$, and the time to traverse any arc is proportional to the angle between the tangents at its extremities. In particular, the time from A to B is

$$\frac{(a^2-b^2)^{1/2}}{\mu^{1/2}b}(\pi - \widehat{AOB}) = \frac{\pi}{\mu^{1/2}}\left(1-\frac{b^2}{a^2}\right)^{1/2},$$

for $\widehat{AOB}=\pi(a-b)/a$. Resolving along a straight line tunnel gives $\mu x=-\ddot{x}$, $T=\pi/\mu^{1/2}$, and the further result follows.

Also solved by M. G. Beumer (Netherlands), Sidney Glicksman, J. R. Hatcher, Frank Herlihy, Jernej Polajnar (Yugoslavia), L. E. Ward, and the proposer.

Chains in the Power Set

5615 [1968, 791]. *Proposed by G. F. Schumm, University of Chicago*

Suppose S is any infinite set of cardinality m, and let $\mathcal{P}(S)$ denote the power set of S. Using the Generalized Continuum Hypothesis, E. S. Wolk (*A theorem on power sets*, this MONTHLY, 72 (1965) 397–398) proves that there exists a chain $\mathcal{C}$ in $\mathcal{P}(S)$ with card $(\mathcal{C})=2^m$. How many such chains are there?

Solution by the proposer. If we denote by Φ the desired collection of chains, then clearly $\text{card}(\Phi)\leqq\text{card}(\mathcal{P}(\mathcal{P}(S)))=2^{2^m}$. On the other hand, put $Z=\{X:X\subseteq\mathcal{C}\text{ and card}(X)\leqq m\}$ and $W=\{X:X\subseteq\mathcal{C}\text{ and card}(X)=2^m\}$. Then, with the aid of the Axiom of Choice (which, as is well known, is entailed by the GCH) it can be shown that $\text{card}(Z)=(\text{card}(\mathcal{C}))^m=(2^m)^m=2^m$ (cf. H. Bachmann, *Transfinite Zahlen*, Berlin, 1955, p. 136). Moreover, according to the GCH there is no p such that $m<p<2^m$; whence $\mathcal{P}(\mathcal{C})=z\cup W$. Now, since $Z\cap W=\varnothing$, we have $\mathcal{P}(\mathcal{C})\backslash Z=W$, from which it follows that there are $2^{2^m}-2^m=2^{2^m}$ subcollections of $\mathcal{C}$ of cardinality 2^m. But each subcollection of $\mathcal{C}$ is again a chain in $\mathcal{P}(S)$; and therefore $\text{card}(\Phi)\geqq 2^{2^m}$. Hence, $\text{card}(\Phi)=2^{2^m}$.

Condition for a Commutative Ring

5616 [1968, 792]. *Proposed by E. R. Gentile, University of Buenos Aires, Argentina*

Let K be a ring with an identity and without zero divisors $\neq 0$. Then, if there is a natural number n such that $(ab)^k=a^kb^k$ for $k=n,\ n+1$, K is a commutative ring. (Cf. I. N. Herstein, *Topics in Algebra*, 1964, p. 31, nos. 4, 5.)

Solution by Geoffrey Kandall, Emanuel College, Boston, Mass. K has no nonzero nilpotent elements and hence no nonzero nil ideals. Thus it suffices to show that $a^n\in Z$ for any $a\in K$, where Z is the center of K (cf. Herstein, *Noncommutative Rings*, 1968, p. 79).

We have to show that $ba^n=a^nb$ for any $a,\ b\in K$. This is trivial if either a or b is 0. If $a,\ b\neq 0$:

$$\begin{aligned} a(ba^n - a^nb)b^n &= aba^nb^n - aa^nbb^n = ab(ab)^n - a^{n+1}b^{n+1} \\ &= (ab)^{n+1} - (ab)^{n+1} = 0, \end{aligned}$$

and therefore $ba^n-a^nb=0$. (The assumption that K has an identity is unnecessary.)

Also solved by E. P. Del Norte, M. A. Ettrick, W. F. Fox, Robert Gilmer, M. G. Greening (Australia), G. A. Heuer, Henry Lieberman, O. P. Lossers (Netherlands), Jiang Luh, W. S. Martindale, Marion Moore, C. F. Stephens, Jr., and the proposer.

The Partial Sums of a Fourier Series

5617 [1968, 792]. *Proposed by C. J. Mozzochi, Trinity College, Hartford, Conn.*

Prove (by elementary methods): If f is continuous on $I=[-\pi, \pi]$, then for every $\epsilon>0$ there exists an integer N such that for all $n\geqq N$ we have $|S_n(x)-f(x)|\leqq(n+1)\epsilon$ for all x in I where $S_n(x)$ is the nth partial sum of the Fourier series for f.

Solution by Dennis Henkel, Milwaukee, Wisconsin. Let $\epsilon>0$ be given. Since the Cesàro averages of the partial sums of the Fourier series converge uniformly to the function, there is some M such that $n\geqq M$ implies

$$\left|\frac{1}{n}\sum_{k=1}^{n} S_k - f\right| < \epsilon/2.$$

Take $N=M+1$. Then for $n\geqq N$, $n-1\geqq M$, so

$$\left|\frac{1}{n}\sum_{k=1}^{n} S_k - f\right| < \epsilon/2, \qquad \left|\frac{1}{n-1}\sum_{k=1}^{n} S_k - f\right| < \epsilon/2;$$

and the required inequality follows directly.

Also solved by Jon Barkhurst, J. L. Brown, Jr., M. A. Ettrick, D. A. Hejhal, and the proposer.

An Average of Sums of Power

5618 [1968, 792]. *Proposed by David Boyd, University of Alberta*

Let $0\leqq r<1$, and define

$$F(r) = \lim_{n\to\infty} 2^{-n} \sum \left| \pm 1 \pm r \pm \cdots \pm r^{n-1} \right|,$$

where the summation is over all 2^n possible choices of sign. Show that

$$\text{(a)}\quad F[(\sqrt{5}-1)/2] = 4/(6-\sqrt{5}), \qquad \text{(b)}\quad F(2^{-1/2}) = 7/6.$$

Solution by F. Göbel and F. W. Steutel, Technische Hogeschool Twente, Enschede, Netherlands. $F(r)$ can be interpreted as follows:

$$F(r) = E\left|\sum_{j=0}^{\infty} X_j\right|,$$

where the X_j are independent random variables taking the values $\pm r^j$ with probability $\frac{1}{2}$, and E denotes expectation.

For $0\leqq r\leqq\frac{1}{2}$ we have $|X_0|\geqq\left|\sum_1^\infty X_j\right|$ and therefore, conditioning on X_0, we have

$$F(r) = \frac{1}{2} E\left(1 + \sum_{j=1}^{\infty} X_j\right) + \frac{1}{2} E\left(1 - \sum_{j=1}^{\infty} X_j\right) = 1.$$

If $r=\frac{1}{2}(\sqrt{5}-1)$ we have $1-r-r^2=0$, $|X_0|+|X_1| \geqq |\sum_2^\infty X_j|$ and $|X_0|-|X_1|+|X_2| \geqq |\sum_3^\infty X_j|$. Conditioning on X_0, X_1 and X_2 and using symmetry we now have

$$F(r) = \frac{1}{2} E\left(1 + r + \sum_{j=2}^{\infty} X_j\right) + \frac{1}{4} E\left(1 - r + r^2 + \sum_{j=3}^{\infty} X_j\right)$$

$$+ \frac{1}{4} E\left|1 - r - r^2 + \sum_{j=3}^{\infty} X_j\right| = 1 + \frac{1}{4} r^3 F(r).$$

It follows that $F(\frac{1}{2}(\sqrt{5}-1)) = 4(6-\sqrt{5})^{-1}$.

Generally we have $E\exp(i\sum_0^\infty X_j t) = \prod_0^\infty \cos r^j t$. If $r=\frac{1}{2}$, then $\sum_0^\infty X_j$ has a rectangular distribution on $(-2, 2)$. For $r=\frac{1}{2}\sqrt{2}$, $\sum_0^\infty X_j$ is distributed as $U+V$, where U and V are independent and rectangularly distributed on $(-2, 2)$ and $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$ respectively. Consequently, $F(r) = E|U+V| = 7/6$.

Taking $r=2^{1/n}$, we have in the same way $F(r) = |\sum_1^n U_k|$, where the U_k are independent and rectangularly distributed on $(-2^{k/n}, 2^{k/n})$. Using the Central Limit Theorem we find $F(2^{-n}) \sim C\cdot\sqrt{n}$ if $n\to\infty$.

Also solved by L. Carlitz, L. E. Clarke (England), G. J. Foschini, M. G. Greening (Australia), O. P. Lossers (Netherlands), Jernej Polajnar (Yugoslavia), and the proposer.

With his solution, Lossers provided the following asymptotic expression for $F(r)$: $F(r)$ $\pi^{1/2}(1-r)^{-1/2}+O(1-r)^{-1/4}$, $r\to 1^-$.

UNSOLVED PROBLEMS

The editors would like to call the attention of interested readers to the following problems for which no acceptable solution has been submitted. Comments and solutions are solicited.

5028	[1962, 438]	5320	[1965, 914]	5429	[1966, 897]
5036	[1962, 570]	5359	[1966, 89]	5432	[1966, 1019]
5078	[1963, 216]	5379	[1966, 312]	5437	[1966, 1019]
5119	[1963, 673]	5382	[1966, 420]	5440	[1966, 1124]
5124	[1963, 765]	5385	[1966, 420]	5441	[1966, 1124]
5218	[1964, 801]	5405	[1966, 674]	5443	[1966, 1124]
5244	[1964, 1047]	5413	[1966, 783]	5465	[1967, 207]
5278	[1965, 324]	5415	[1966, 783]	5489	[1967, 447]
5307	[1965, 674]	5418	[1966, 784]	5497	[1967, 599]
5314	[1965, 795]	5427	[1966, 897]	5503	[1967, 728]
5319	[1965, 796]			5540	[1967, 1269]
E 1822	[1965, 903]	E 1893	[1966, 539]	E 1917	[1966, 891]
E 1847	[1966, 81]	E 1903	[1966, 666]	E 1959	[1967, 198]
E 1853	[1966, 83]	E 1905	[1966, 774]	E 1980	[1967, 438]

REVIEWS

EDITED BY KENNETH O. MAY

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

*All unsigned material is written by the editors. A boldface capital **C** in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.*

EDITORIAL: ON CHOOSING TEXTS

Many readers have expressed a desire for prompt guidance on newly published texts. We would certainly like to oblige, but there are insurmountable difficulties. In the first place, a textbook cannot be validly judged without using it in a class. Such reviews cannot appear prior to about a year and a half after publication. It is of course possible to write a review that makes a judgement about mathematical soundness, possible uses, and probable pedagogical success. We try to get such reviews, but because of the slowness of reviewers and the time required in the publication pipeline, it is seldom possible to publish such reviews earlier than eight months after publication. In the telegraphic reviews, we give indications of possible use. Even though these are written within one week of receipt of the book, pipeline-time delays their publication at least three months. Books published in January are usually reviewed in the June–July issue. We could speed up this process if publishers sent us advance copies, but often they send us books months after the publication date or neglect to supply essential information (such as the price!).

Under these circumstances, it seems that the best procedure for a department is to order examination copies of new textbooks, relying on publishers' advertisements and on listing in telegraphic reviews. They can then decide whether to take a flyer on a new book or to use an older book with which they have had experience or which has been more thoroughly reviewed. Finding such reviews for older books is made easy by the index in each December issue, which covers telegraphic as well as extended reviews.

These remarks are not intended at all to discourage the use of new books, but merely to point out that there is no way to guarantee results in advance. Those who do use new books could help us and their colleagues greatly by volunteering to write reviews based on their experiences.

Theory and Problems of Group Theory. By B. Baumslag and B. Chandler. Schaum's Outline Series, McGraw-Hill, New York, 1968. 279 pp. $3.95 (paper). (Telegraphic Review, March 1969)

A Schaum's Outline in group theory! The idea did not at first appeal to me. I did not believe that the Outline Series could provide a vehicle for the presen-

tation of graduate level group theory. I have come away from the book however, impressed. Theorems are carefully stated; their proofs are readable—brief but complete. I will take the publisher at his word when he states that there are 600 solved problems. Generally speaking, the problems are interesting, challenging, and clarify the theory presented. This book would be excellent for self-study, but the large number of worked problems would make it awkward to use as a text.

Somehow the authors manage to achieve both breadth and depth in the 270 pages of text. The first fifty pages deal with preliminary concepts and a gradual development from groupoids to groups. Thirty pages are then devoted to a worthwhile examination of groups of isometries, Möbius transformations, and automorphisms of algebraic structures, structures with which the reader should have some familiarity. The remaining topics are standard for a first graduate course in group theory.

The book possesses one major defect. There are several curious inversions and lapses in the order of presentation of the material. These peculiarities would make use of the book as a reference text difficult. Symmetric groups and alternating groups are introduced on page 56, but the cycle notation for permutations is not used until page 167. The discussion of permutation representations is stalled to page 214. The work on isometry groups, etc., is presented quite early (page 64), but we find the definition of a cyclic group on page 101! The definition of a normal subgroup is on page 111. That is too long to wait for such an important concept.

Most textbook authors insist that a student mature mathematically at a predetermined rate (the author's). Thus, in chapter A, proofs are complete and ideas discussed fully but in chapter Z, proofs are sketchy and the student must decide what is significant for himself. The concepts naturally get more difficult throughout a course in mathematics, so why add to the student's frustration by having the form of the presentation deteriorate? Baumslag and Chandler have maintained throughout their book essentially the same clear, detailed style. One generally finds this true of a Schaum's Outline Series Text.

J. M. LAIBLE, Eastern Illinois University

Projective Plane Geometry. By John W. Blattner. Holden-Day, San Francisco, 1968. xi+297 pp. $10.95. (Telegraphic Review, March 1969.)

This is a very good text book. The first chapter, Introduction to Projective Geometry, presents the incidence axioms and other necessary material. Excellent use is made of finite models in discussions and exercises. Central collineations are emphasized in the second chapter, Transformations of Projective Planes, preparatory to Desarguesian Planes, chapter 3. A homology is a central collineation in which the pencil of fixed points on a line l and the pencil of fixed points on a point H are such that l and H are not incident. If P and P' are collinear with H, different from H, and not on l Desargues' Theorem is equivalent to the existence of a homology with l as the pencil of fixed points and H as the

center of the pencil of fixed lines which carries P into P'. Desargues' Theorem is taken as an axiom in this form and the properties of the central collineations corresponding to addition and multiplication are studied. Thus each Desarguesian plane gives rise to a division ring. In chapter 4, coordinates are introduced, the relation between matrices and projective collineations is studied, and the construction of a Desarguesian Plane starting from a division ring is given. Chapter 5, Pappian Planes, assumes the axiom that for any fixed l and H, the group of homologies is commutative. This is equivalent to Pappus' Theorem or the Fundamental Theorem. Some applications to affine geometry are given and the book ends with a study of conics.

Throughout the book required algebraic concepts are carefully introduced. There are many exercises. The content of this book seems well suited to an undergraduate of honors caliber. On the other hand, the reviewer would not recommend using this as a text in a one semester introductory geometry course open to all junior and senior majors. The average undergraduate would have difficulty with the heavy emphasis on transformations.

BERT MENDELSON, Smith College

Linear Algebra and Analysis. By Andre Lichnerowicz. Translated by Alison Johnson. Holden-Day, San Francisco, 1967. xv+304 pp. $10.75. (Telegraphic Review, May 1968.)

This is a translation from *Algèbre et Analyse Linéaires*, published in 1947. The book is divided into two parts as indicated by the title. The first part develops linear algebra, including linear equations, Hermitian forms and tensor algebra. The second part considers exterior differential forms, Stokes' Theorem, orthogonal series and integral equations. Most of the book may be read by a fairly mature reader with only a knowledge of elementary calculus. But some reference is made to Lebesgue integration near the end.

Georges Bruhat, who died in a concentration camp near the end of the Second World War, conceived the idea for this book as the start of a series of books on mathematical physics. The book succeeds admirably in its intention of making more modern mathematics accessible to physicists. It covers a great deal of territory, and does so clearly and rigorously.

There are a few difficulties in using the book. Since there are no problem sets, it could not be used easily as the sole text for a college course. The index should be expanded to include, for example, the word *distinct*, as defined formally on page 192, and used several times afterward. In connection with the definition of *piecewise continuity* (page 185, instead of 158, as given in the index), domain and subdomain are not defined. The author denotes a function sometimes by f, and sometimes by $f(x)$. A bibliography of related books would be a useful addition.

But these are essentially quibbles, and we must conclude that this is a valuable book for any physicist or young mathematician.

E. B. LEACH, Case Western Reserve University

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E. B. LEACH, Case Western Reserve University

theory (to arbitrary structures) which are undergoing development and evaluation in current research. In the meantime, the main value of this book lies in its providing a fairly up-to-date collection of material previously available only in research papers, presented with great care and attention to clarity of exposition. The student of mathematical logic is well-advised to familiarize himself with its main contents.

SOLOMON FEFERMAN, Stanford University

Linear Algebra. Richard E. Johnson. Prindle, Weber and Schmidt, Boston, 1967. ix+223 pp. $7.95. (Telegraphic Review, May 1968.)

This is a brief, but well organized, introduction to linear algebra suitable for a one semester course at the sophomore level. The table of contents is standard for such a course, although it does include the Jordan normal form. The text is arranged with some thought to pedagogy: sections are brief and worked examples plentiful. The opening section introduces ordered fields and may be a hurdle, but the pace eases almost immediately.

The most unusual feature is the treatment of the minimal polynomial $m(T)$ of a linear transformation T. By introducing the minimal polynomial of a vector and exploring its properties, the author easily shows that the degree of $m(T)$ is less than or equal to the dimension of the vector space (rather than its square).

The author has made a wise choice in placing functions to the left of their arguments at the beginning (so nobody gets scared) and placing them to the right when it's especially convenient. The problem sets are adequate and the layout easy to read. This text is a clearly written presentation of the standard material.

NEIL GRABOIS, Williams College

Symbolic Logic and the Real Number System. By A. H. Lightstone, Harper and Row, New York, 1965. 225 pp. $7.50. (Telegraphic Review, March 1967.)

Chapter 1 gives a semantic development of the language of formal logic used throughout most of the book. An instructor's feeling about using this text might depend on whether he likes the Division Theorem for the integers in the form:

$$\forall x \forall y[y \neq 0 \rightarrow \exists! q \exists! r[x = q\cdot y + r \wedge 0 \leqq r < |y|]].$$

The next two brief chapters present basic intuitive theory of sets and mappings and use these to give the definitions of large numbers of algebraic structures ranging from semigroups through vector spaces. Chapter 4 derives the natural numbers, integers and rational numbers from the Peano Postulates. Chapter 5 defines a real number to be an infinite decimal sequence, eventually constant in the ambiguous case. The author asserts that this comes closer than do the standard constructions to the intuitions learned in school. This seems plausible, and the construction appears to be a valid *tour de force*. The last chapter applies the techniques developed in the construction to convergence of real sequences.

Throughout the book the informal discussions are relevant, informative and pleasant to read. Good examples and exercises abound. Definitions and theorems are correct, and the proofs usually very clear. A beginning student should, for example, be able to learn from the text why and how quantifiers are used, or be able to follow the development from the Peano Postulates.

A logician might point to some small lapses in the first chapter. This reviewer finds the discussion of recursive definitions on pp. 124–125, not in accord with the current dogma as given in Henkin's *On Mathematical Induction* (this MONTHLY, 67 (1960) 323–338). The fact that the formal language tends to disappear in the difficult parts of the construction of the reals might be considered in conjunction with the statement in the preface that formal logic facilitates communication.

BURROWES HUNT, Reed College

Partial Differential Equations of Mathematical Physics. By A. N. Tychonov and A. A. Samarski. Two volumes, translated by S. Radding. Holden-Day, San Francisco, 1964, 1967. Vol. I. 380 pp. $11.75, Vol. II. x+621 pp. $10.75. (Telegraphic Review, May 1968.)

It is astonishing how much applied mathematics the authors of these two volumes were able to lucidly set forth in only 621 pages, especially so, since over 100 pages are devoted to appendices. The first volume begins in the standard fashion with the classification of second order partial differential operators as to type: hyperbolic, parabolic, elliptic, ultrahyperbolic, etc. In the main body of the volume the authors prove the well-posedness (i.e., existence, uniqueness and continuity with respect to data) and investigate the properties of a wide variety of boundary and initial value problems associated with certain hyperbolic and parabolic operators in the two-dimensional case, and with certain elliptic operators in the two- and three-dimensional case. The techniques employed range from the standard method of separation of variables to the more sophisticated techniques of potential theory and Green's functions. Brief mention is made of difference methods (approximation of differential equations by difference equations) for solving Laplace's equation and the heat conduction equation; variational techniques are skipped altogether. Volume II treats the 3-dimensional case for problems associated with the wave and heat equations as well as the theory of the reduced wave equation, $\Delta u+k^2u=f$, in 2- and 3-dimensions.

But far and away the most interesting feature of these works is the extraordinarily large and diverse collection of applications to mathematical physics—to name a few: gas dynamics and the theory of shock waves, Brownian motion, electrostatics, theory of double and single layer potentials, problems in geophysics, mathematical theory of diffraction, theory of wave guides, electromagnetic theory, and cavity resonators. The basic differential equations for these physical phenomena are derived in sufficient detail to give a mathematician a feeling for the processes involved, and the treatment of the techniques

of solution is done in a style which is appealing to the mathematician as well as the physicist.

As in any work of this kind, however, some compromises must be made. In this case the compromises come in the way of prerequisites. To read these volumes profitably would require at the very least a good knowledge of advanced calculus, some complex analysis, a first course in ordinary differential equations, some linear analysis and more than a passing acquaintance with the theory of Fourier Series. The other well known special functions of mathematical physics (Bessel functions, spherical harmonics, Hermite, Legendre and Laguerre polynomials) are treated in some considerable detail in the long appendix at the end of Volume II along with some discussion of integral transforms. The rather long list of prerequisites almost precludes the use of these volumes as a text for an undergraduate course unless the students are unusually mature—this in spite of the fact that almost everything in these works can be included at some time or other in an undergraduate curriculum. On the other hand, these volumes would be ideal as a text for a first or second year graduate course in applied mathematics (there is an ample supply of good problems in both volumes). Moreover, in view of the fact that both volumes are well indexed and that the editor gives a long list of references to other works on partial differential equations as well as potential theory and the theory of special functions, these books can be highly recommended as reference works.

These volumes are translations from the Russian and, as translations go, seems to read rather smoothly with relatively few typographical errors (the reviewer noticed four misprints in more than a cursory glance). But, more seriously, there also exist instances of bad or even misleading translations (see for example the second sentence in 2–2.6); however, everything considered, Mr. Radding's translation is an excellent one.

R. L. BORRELLI, Harvey Mudd College

TELEGRAPHIC REVIEWS

The following abbreviations indicate suggested uses: T (textbook), S (supplementary student reading), P (professional reading for the teacher), TT (teacher training), L (library purchase), 13 (freshman level) —18 (second graduate year). A boldface star (★) marks a notable book that might be overlooked.

Algebra

Introduction to Modern Algebraic Concepts. By Max D. Larsen (Univ. of Nebraska). Addison-Wesley, Reading, Mass., 1969. 143 pp. $6.95. An introductory one-semester text dealing with groups, rings, the integers, integral domains and fields, polynomials, and the real number system. T (13–15), TT.

Invitation to Number Theory. By Oystein Ore. Random House, New York, and L. W. Singer Co., 1967. viii+129 pp. $1.95 (paper). This is number 20 of the New Mathematical Library, of which every mathematical library should have a set. Though designed for high school enrichment, these booklets are valuable as supplementary reading at the college level, and many of them are first class expositions. One can hardly imagine a better man to write this one than the late Oystein Ore. S (13), TT.

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Permutation Groups. By Donald Passman (Yale Univ.). Benjamin, New York, 1968. ix+310 pp. \$12.50 (cloth) \$4.95 (paper). These are lecture notes on certain classification theorems that are "well known" but usually inaccessible. The book is produced directly from author's typescript, thus saving composition and editing costs. This is a good idea, and the paperback price seems to reflect these costs savings. However, more imagination could be shown in the use of the typewriter. Double spacing is used here throughout. Often single or one-and-one-half spacing would be adequate and would make the page less monotonous and easier to read. Also more underlining and display could be utilized to enable the reader to pick out definitions. A little money spent on design and possibly on the use of more handwritten elements on the page might make a product that would be in some ways better than any typeset book. T (16–17), S, P, L.

Substitutional Analysis. By Daniel Edwin Rutherford (Facsimile of the 1947 edition). Hafner, New York, 1968. xi+102 pp. \$4.00. The subject is the application of Alfred Young's Tableaux to the representation of the symmetric group, and the reprint is motivated by the increasing use of related ideas in physics. S, P, L.

Analysis

Almost Periodic Functions. By C. Corduneanu with the collaboration of N. Gheorghiu and V. Barbu (all of the Univ. of Jassy, Romania). Translated from the Romanian edition by Gitta Bernstein and Eugene Tomer. Interscience, New York, 1968. x+237 pp. \$13.50. It has been said that the theory of almost periodic functions reached an almost complete state at the time of its founding in the twenties by Harald Bohr. This book shows that such a conception is completely false. There is a brief history, an exposition of current knowledge, and a bibliography of 704 titles, which does not include papers on applications of the theory or those only incidentally related to it! P, L.

Representation Theory and Automorphic Function. By I. M. Gel'fand, M. I. Graev and I. I. Pyatetskii-Shapiro (Academy of Sciences, USSR). Translated from the Russian by K. A. Hirsch. Saunders, Philadelphia, 1969. xvi+426 pp. \$18.00. This is a translation of the 6th volume in a distinguished Russian series of monographs on generalized functions, but familiarity with previous volumes is not assumed. Chapter headings are Homogeneous spaces with a discrete stability group, Representations of the group of unimodular matrices of order 2 with elements from a locally compact topological field, and Representation of adele groups. There are a guide to the literature and a bibliography. This book illustrates very nicely the merging in recent times of the traditional so called "branches" of mathematics, since it involves classical analysis, algebra, number theory, abstract analysis, and topology. Much of the material has previously been available only in journals. T (17–18), S, P, L.

Rings of Operators. By Irving Kaplansky (Univ. of Chicago). Benjamin, New York, 1968. vi+151 pp. \$12.50 (cloth) \$3.95 (paper). Another in the growing Mathematics Lecture Note Series, this book provides an account of the algebraic part of the theory of rings of operators. Originally issued in mimeograph form in 1955 and intended for a graduate course or seminar, the notes have been expanded, revised, and brought up to date so as to take the reader from the foundations of the subject to the research frontier. There are exercises. T (17–18), S, P, L.

Functional Equations in a Single Variable. By Marek Kuczma. Polish Scientific Publishers, Warsaw, 1968. Distributed by Hafner, New York. 383 pp. \$10.00. This is a very substantial contribution to the task of systematizing the enormous amount of miscellaneous information on functional equations that has been accumulating since

the time of Euler. It covers functional equations in a single variable exclusive of difference equations (and of course differential and integral equations which are not usually considered to be part of the field at all). Even for this limited field, the author found it impossible to be complete and made various choices such as concentration on real rather than complex variables and maintenance of an elementary level. The bibliography still covers 63 pages! We need more books of this type! P, L.

The Special Functions and their Approximations, Vol. 1. By Yudell L. Luke (Midwest Research Inst., Kansas City). Academic, New York, 1969. xx+349 pp. $19.50. The functions here considered are the gamma function, the hypergeometric function and their generalizations. There follow some general considerations on asymptotic expansions and orthogonal polynomials, a bibliography and a notational index. The use of the term "special functions" should be abandoned since it was poorly chosen in the first place and is both ambiguous and confusing at the present time. P, L.

Induced Representations of Groups and Quantum Mechanics. By George W. Mackey (Harvard Univ.). Benjamin, New York, 1968, and Editore Boringhieri, Torino, 1968. viii+167 pp. $12.50 (cloth) $3.95 (paper). Lectures given at the Scuola Normale, Pisa, in April 1967. No previous knowledge of group representations or quantum mechanics is assumed. The printing is by photo offset of double space typescript, yet the cloth price per page is as high as it would be for first class printing from type. In paper the cost is less than 3¢ a page. S, P, L.

Equations in Linear Spaces. By Danuta Przeworska-Rolewicz and Stefan Rolewicz. Polish Scientific Publishers, Warsaw, 1968. Distributed by Hafner, New York. 380 pp. $15.00. The issue here is the solvability of linear equations in infinite dimensional spaces. The problem is a mixture of algebra, topology, classical analysis and functional analysis. There are historical comments and a bibliography. P, L.

Functions of a Complex Variable. Constructive Theory. By V. I. Smirnov and N. A. Lebedev. Translated by Scripta Technica. MIT Press, Cambridge, Mass., 1968. 9+488 pp. $12.00. Chapters are on uniform approximation by polynomials and rational functions, Faber polynomials and problems of representation of regular functions by polynomial series, Quadratic approximation, Orthogonal functions with respect to a domain and to a contour, and Best uniform approximation. There is a fourteen page bibliography and citations to it in the text. T (17), P.

Generalized Integral Transformation. By A. H. Zemanian (SUNY at Stony Brook). Interscience, New York, 1968. xvi+300 pp. $16.00. The subject is the extension to generalized functions (distributions) of the classical theory of integral transforms. The work is self contained in the sense that it requires only advanced calculus and a little knowledge of real and complex variables. The concepts of generalized function and topological linear space are developed in the first two chapters. Though the emphasis is on theory the book is based on a course designed for both mathematics and engineering students. T (17), P.

Applications

Ergodic Problems of Classical Mechanics. By V. I. Arnold (Univ. of Moscow) and A. Avez (Univ. of Paris). Benjamin, New York, 1968. ix+286 pp. $14.75 (cloth) $6.95 (paper). This international collaborative effort of two young mathematicians is based on the lectures of the first author and many proofs supplied by the second. It is not claimed as an exhaustive treatment, but it appears to supply a very readable survey as well as new results. There are excellent illustrations and a good bibliography. T (17–18), S, P, L.

Mathematics of the Decision Sciences. Part 1. Edited by George B. Dantzig and Arthur F. Veinott. American Mathematical Society, Providence, R. I. 1968. ix+429 pp. $16.40. This is the first of two volumes containing the proceedings of the Fifth Summer Seminar on the Mathematics of the Decision Sciences sponsored by the AMS at Stanford University in the summer of 1967. It testifies to the breadth of applications of mathematics in the social sciences. P, L.

★*Battelle Rencontres. 1967 Lectures in Mathematics and Physics.* Edited by Cecile M. DeWitt (Univ. of N.C. at Chapel Hill) and John A. Wheeler (Princeton Univ.). Benjamin, New York, 1968. xvii+557 pp. $14.50. This volume grew out of a meeting of 33 mathematicians and physicists for 6 weeks in the summer of 1967 under the auspices of the Battelle Memorial Institute. It includes both general exposition and technical discussions as follows: Lie Groups and Symmetric Spaces (Sigurdur Helgason), *Commutativité de l'algèbre des opérateurs différentiels invariants sur un espace symétrique* (André Lichnerowicz), Hyperbolic Partial Differential Equations on a Manifold (Yvonne Choquet-Bruhat), Topics on Space-Time (Andre Lichnerowicz), Relativistic Fluids in Cosmology (Charles W. Misner), Structure of Space-Time (Roger Penrose), The Structure of Singularities (Robert Geroch), Superspace and the Nature of Quantum Geometrodynamics (J. A. Wheeler), Boundary Conditions for the State Functional in Quantum Theory of Gravity (H. Leutwyler), The Everett-Wheeler Interpretation of Quantum Mechanics (Bryce S. DeWitt), Progress and Goals in Renormalization Theory (Klaus Hepp), Perturbation Theory in Quantum Field Theory and Homology (Jean Lascoux), Landau Singularities in the Physical Region (Frederic Pham), Algebraic Topology Methods in the Theory of Feynman Relativistic Amplitudes (Tullio Regge), Topics in Topology and Differential Geometry (Raoul Bott and John Mather), Continuous Solutions of Linear Equations—Some Exceptional Dimensions in Topology (Beno Eckmann), Characterization of Stable Mappings (John N. Mather), One-Parameter Subgroups do not Fill a Neighborhood of the Identity in an Infinite-Dimensional Lie (Pseudo-) Group (Charles Freifeld), Eversion of the 2-Sphere (Bryce S. DeWitt). P, L.

Lagrangian Dynamics. An Introduction for Students. By C. W. Kilmister (Univ. of London). Plenum Press, New York, 1968. vii+136 pp. $7.50. A nice exposition with some attention to historical developments. S, P, L.

Optimization by Vector Space Methods. By David G. Luenberger (Stanford Univ.). Wiley, New York, 1969. xvii+326 pp. $13.95. In his preface, the author writes "the primary objective of the book is to demonstrate that a rather large segment of the field of optimization can be effectively unified by a few geometric principles of linear vector space theory." In order to keep the mathematical prerequisite to familiarity with linear algebra, the early chapters of the book constitute an introduction to functional analysis. The intended audience is graduate students in engineering and operations research as well as mathematics. T (17), S, P.

Error Correcting Codes. Proceedings of a Symposium conducted by the Mathematics Research Center, United States Army at the University of Wisconsin, Madison, May 6–8, 1968. Edited by Henry B. Mann. Wiley, New York, 1968. ix+231 pp. $7.95. The topic here is algebraic coding theory, which developed from problems arising in communication theory. The book begins with an amusing historical survey of 13 pages and continues with 13 papers given at the symposium. P, L.

Systèmes échantillonnés non linéaires. By Pierre Vidal. Gordon and Breach, New York, 1968. xiv+362 pp. $27.50. The subject is nonlinear sampling systems. The mathematics is difference equations. The work is international to an interesting extent. It is printed in Germany, distributed by Dunod of Paris, and has a preface by Stefan Wegrzyn, Scientific Director of the Institute of Automatization of the Polish Acad-

emy of Sciences. The price is about 8¢ a page. It appears to be a useful summary of results previously available only in journal form. P.

Calculus

The Calculus with Analytic Geometry. Part I: Functions of One Variable and Plane Analytic Geometry. Part II: Vectors, Functions of Several Variables, and Infinite Series. By Louis Leithold (California S.C. at Los Angeles). Harper and Row, New York, 1969. I: 648 pp. $9.95. II: 384 pp. $7.95. The two volumes were originally published in one in 1967. $12.95. T (13–14).

Introduction to Modern Calculus. By Herman Meyer (Univ. of Miami). McGraw-Hill, New York, 1969. 521 pp. $10.50. This book introduces significant innovations that appear promising: First, a fairly rigorous treatment of calculus of one variable based on the Moore-Smith theory of limits is developed systematically in chapters 1 through 8 (252 pages). Second, the technique and applications of differentiation and integration are given by a programmed series of problems in chapters 9 through 11 (132 pages), followed by over 100 pages of detailed answers and solutions. Assignments can be made simultaneously on a two track basis from both parts of the book, with the manipulative and applied aspects of the subject left largely to the student and the self-teaching text. The book deserves a wide trial and it will be interesting to see how this use of programming and of modern limit theory work out. T (13)!

Elementary Differential Equations, 4th ed. By Earl D. Rainville and Phillip E. Bedient (Franklin and Marshall College). Macmillan, New York, 1969. xiv+466 pp. $8.95. This is a revision by Bedient of previous editions jointly authored by him and the late Professor Rainville, whose textbooks have been standbys for decades. This edition gives additional emphasis to existence and uniqueness, introduces the Picard approximation, the Runge-Kutta method, the method of Milne, and the use of Taylor series. However, the treatment remains essentially traditional. T (14–15).

A Short Course in Differential Equations, 4th ed. By Earl D. Rainville and Phillip E. Bedient (Franklin and Marshall College). Macmillan, New York, 1969. xi+281 pp. $7.50. The first sixteen chapters of *Elementary Differential Equations*, fourth edition. The principal omissions are those methods dependent on infinite series. T (14–15).

Modern Calculus and Analytic Geometry. By Richard A. Silverman, Macmillan, New York, 1969. xv+1034 pp. $12.95. The topics are those traditional for the calculus compendia, which here goes over the megapage size. The author's philosophy is stated as "motivate all new ideas, both mathematical and physical" and "prove all theorems." These would be innovations! We invite reviews by users! T (13–14).

Computers

Mathematical Programming in Practice. By E. M. L. Beale (Scientific Control Systems). Wiley, New York, 1968. xi+191 pp. $5.50. This book is concerned primarily with "methods of organizing real problems so that they can be solved numerically using standard computer codes." S, L.

Computation by Electronic Analogue Computers. By V. Borsky and J. Matyas. Translated by J. Smizanska, I. Bebarova, and I. Santar. English translation edited by C. C. Ritchie and G. F. Moxon. Publishers of Technical Literature, Prague, Iliff Books Ltd., London, and American Elsevier, New York, 1968. 421 pp. $10.75. A reference book. Chapter headings are Function of electronic analogue computers, General programming diagrams, Detailed programming diagrams, Linear problems, Non-linear problems, Simulation of physical systems, Special problems, Organization of computing works, and Examples of problems solved on analogue computers. There is a short bibliography. P, L.

Machine Intelligence. 1. Edited by N. L. Collins and D. Michie (Univ. of Edinburgh). With a preface by Sir Edward Collingwood. American Elsevier, New York, 1967. x+278 pp. $12.50.

Machine Intelligence. 2. Edited by E. Dale and D. Michie (Univ. of Edinburgh), American Elsevier, New York, 1968. ix+251 pp. $13.75.

Machine Intelligence. 3. Edited by Donald Michie. With a preface by the Earl of Halsbury. American Elsevier, New York, 1968. x+405 pp. $11.50. These volumes report the first three Machine Intelligence Workshops at the University of Edinburgh in 1965, 1966, and 1967. They cover a vast range including abstract foundations, theorem proving, machine learning and heuristic programming, pattern recognition, mechanized mathematics, problem oriented languages, man-machine direction. P, L.

Introduction to Computational Linguistics. By David G. Hays (Rand Corp.). American Elsevier, New York, 1967. xvi+231 pp. $9.75. Intended primarily for introductory university courses in the application of computers to linguistics. The book is also directed to linguists and those concerned with information processing. T, S, P.

Interactive Systems for Experimental Applied Mathematics. Proceedings of the Association for Computing Machinery Inc. Symposium held in Washington, D. C., August 1967. Edited by Melvin Klerer (New York Univ.) and Juris Reinfelds (Univ. of Georgia). Academic, New York, 1968. xiv+472 pp. $19.50. These proceedings report on developments in an area on the edge of new mathematical thinking, in which terms are still ill-defined and work is "an expression of faith that a computer, used to explore ill-defined mathematical constructs and problems, might yield powerful insights and a fruitful methodology." P, L.

An IBM 1130 FORTRAN Primer. By Richard A. Mann (Wright State Univ.). International Textbook Co., Scranton, Penn., 1969. vii+216 pp. $4.95 (paper). No prerequisites. Text and exercises. T, S.

Emerging Concepts in Computer Graphics. 1967 University of Illinois Conference. Edited by Don Secrest and Jurg Nievergelt (both of Univ. of Illinois). Benjamin, New York, 1968. ix+418 pp. $12.50. The conference was concerned with the relatively new field of the representation of computer output in visual forms. The publisher states that the book is printed directly from typescript prepared by the editors without the publisher having reviewed, edited, type set or proof read. Under these conditions, the price is unreasonably high. P, L.

Fortran Programming. By Fredric Stuart (Hofstra Univ.). Wiley, New York, 1969. xix+353 pp. $7.95. The author, who is professor of business statistics, writes that his main purpose is to enable the student to reply "Yes (unless it is a rather small computer)" when the boss asks "can you use a computer?" T (13).

Education

Delta. Official Organ of the Waukesha Mathematical Society, Vol. I, No. I, Fall 1968. To be issued semi-annually at a cost of $1.00 per year. Editor: R. S. Luthar, University of Wisconsin, Waukesha, Wisconsin 53186. This little journal represents an interesting development, a regional mathematical magazine whose purpose is to stimulate local mathematical activity by students and faculty. Its first issue begins with an article on *P*-adic numbers, includes a problem section, and ends with a brief history of the Waukesha Mathematical Society (since its founding in October 1968!). P, L.

★*The Teaching of Mathematics.* Essays by A. Ya. Khinchin. Edited by B. V. Gnedenko. Translated from the Russian by W. Cochrane and D. Vere-Jones. American Elsevier, New York, 1968. xx+167 pp. $9.50. In addition to one substantial essay and three

smaller ones there are extended comments by the editor, a biography and bibliography of Khinchin by Gnedenko and A. I. Markushevich, a forty-seven page essay on mathematics teaching in Soviet schools, and a list of references. It is hard to imagine anyone who should not be interested in the thoughts on mathematical teaching and mathematical creativity of one of the greats of this century. TT, S, P, L.

General

Introduction to Mathematical Ideas. By David G. Crowdis and Brandon W. Wheeler (Sacramento City College). McGraw-Hill, New York, 1969. xii+352 pp. $7.95. For a general education course requiring as little as two years of high school mathematics. Topics include numbers, computations, computers, symbolic logic, proof, sets, boolean algebra, probability and statistics. The format and printing are attractive and there are interesting illustrations. In spite of the minimal prerequisites, there is some material to challenge the student whose background is strong. T (13), TT.

Mainstreams of Mathematics. By John B. Fraleigh (Univ. of Rhode Island). Addison-Wesley, Reading, Mass., 1969. xiii+513 pp. $9.50. The "mainstreams" are foundations (sets, counting, cardinal numbers), algebra (number theory, groups, rings and fields), geometry (real analytic geometry, affine and projective planes, topology), analysis (functions, differential calculus, integral calculus), and probability. Designed for future elementary or secondary teachers, social scientists, or students taking a single course for "cultural enrichment," the book is also a possible starter course. Its motivating idea is the introduction of material new to the student rather than rehashing old ideas or dwelling at length on technicalities. There is a great deal of information on each topic and many exercises. T (13), TT.

Fibonacci and Lucas Numbers. By Verner E. Hoggatt, Jr. Houghton Mifflin, Boston, 1969. iv+92 pp. $1.40 (paper). Though part of an enrichment series for high school students, this booklet can be a nice supplement for college students and especially for future teachers who will find much useful material in it. S, TT.

Modern Math Simplified. A Programmed Review Based on Your Course. Barnes & Noble, New York, 1968. 194 pp. $2.50 (paper). Mathematicians are rightly suspicious of programmed material and cramming devices, but this work book is a good one in both content and presentation. It covers an extraordinary amount of material, virtually everything the student needs to "know" around the grade 13 level, including logic, sets, algebraic systems, non-Euclidean geometry, number systems, some calculus, and so on. The material is presented by self-tests with solutions, exercises, summaries of basic facts and additional information, a list of symbols and an index-dictionary (including definitions of terms in the index itself), and a final examination. This would be a useful supplement for most college freshmen and in many other courses where a variety of gaps need to be filled in by the student. Its use would save much teacher time. S!

Elementary Mathematics. A Logical Approach. 2nd ed. By Paul Sanders and Arnold D. McEntire (both of Appalachian State Univ.). International Textbook Co., Scranton, Penn., 1969. xi+351 pp. $7.50. From logic and proof through real numbers, algebra, analytic geometry, vectors and matrices, programming, to probability and beginning statistics. Developed so that "a student using the book will not be caught unprepared by developments in 'new math' for a long time." T (13), TT.

Reflections on Big Science. By Alvin M. Weinberg. MIT Press, Cambridge, Mass., 1967. ix+182 pp. $1.95 (paper). Because of the author's leading position in American Big Science, what he has to say is of general interest. Mathematics and mathematicians get an inadequate and partly unfavourable treatment, but this is all the more reason

for mathematicians to learn more about the bigger context in which they work and the misunderstandings that they face even within the scientific community. P, L.

Topology Conference. Arizona State University, 1967. Edited by E. E. Grace. Tempe, Arizona. vii+339 pp. $4.50 (paper). This volume publishes papers given at a conference on point set topology (especially abstract topological spaces and structure of continuum) in March 1967 under the direction of R. W. Heath and E. E. Grace. The book ends with some brief research announcements and a list of participants. Individual mathematicians may obtain a free copy by writing to the editor at Arizona State University. P, L.

Algebraic Geometry: Introduction to Schemes. By I. G. Macdonald (Oxford Univ.). Benjamin, New York, 1968. vii+113 pp. $12.50 (cloth) $3.95 (paper). An understandable introduction of interest to the specialist or an outsider wanting a survey. The dust jacket says it is of interest to "postgraduates in classical geometry," by which is apparently meant modern topological methods in algebraic geometry of the last decade or so! S, P, L.

Singular Points of Complex Hypersurfaces. By John Milnor. Annals of Mathematics Studies Number 61. Princeton Univ. Press, Princeton, N. J. and Univ. of Tokyo Press, 1968. 122 pp. $3.25 (paper). Prerequisites are some knowledge of basic algebra and topology. P, L.

Geometry, a Perspective View. By Myron F. Rosskopf (Columbia Univ.) Joan L. Levine (Newark S.C. Union, N. J.) and Bruce R. Vogeli (Columbia Univ.). McGraw-Hill, New York, 1969. xii+306 pp. $8.95. Designed for the training of elementary school teachers, but also suitable for general education courses, this book presents geometry as a fusion of logic, algebra, and geometric ideas. The parts are labelled Mathematics: creativity and formality, Measurement of sets of points (real numbers, coordinate, angles, areas), Congruence, parallelism, and similarity of sets of points, Geometry of transformations. T (13), TT.

Notes on Cobordism Theory. By Robert E. Stong. Mathematical Notes Series. Princeton Univ. Press, Princeton, N. J. 1968. Also the Univ. of Tokyo Press. ii+354 pp. $7.50. These are the outgrowth of a course at Princeton University. The standard of exposition seems high. There are interesting general observations and historical notes as well as a bibliography. The reader is presumed to know algebraic topology "fairly thoroughly." There is an appendix giving results from advanced calculus that are needed and a second appendix on differential manifolds. The 11-th and last chapter is entitled "Spin, Spin^c and Similar Nonsense." T(18) P, L.

Introduction Élémentaire à la Géométrie Lobatschewskienne. By Albert Turc, with a preface by J. Itard. Albert Blanchard, Paris 1967. 170 pp. 15 F. This is a reprint of a book first published in Geneva and Paris in 1914. In his preface Itard describes the book as old fashioned in style but a fine introduction to synthetic Lobachevskyan geometry. P, L.

History

The Great Art, or the Rules of Algebra. By Girolamo Cardano. Translated and edited by T. Richard Witmer, with a foreword by Oystein Ore. MIT, Cambridge, Mass., 1968. xxiv+267 pp. $10.00. As the late Oystein Ore points out in his foreword, this work of 1545, though considered to be one of the great masterpieces of the period, has not previously been available in an acceptable edition. S, P, L.

Greek Mathematical Thought and the Origin of Algebra. By Jacob Klein (St. John's College, Annapolis, Md.). Translated by Eva Brann. With an appendix containing Vieta's

Introduction to the Analytical Art. Translated by the Rev. J. W. Smith. MIT Press, Cambridge, Mass., 1968. xv+360 pp. $12.50. The original German text appeared in *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abt. B: Vol. 3, fasc. I. (Berlin, 1934), pp. 18–105 (Part I); fasc. 2 (1936), pp. 122–235 (Part II). The appendix was translated in 1955. The study is an important one and well deserves the translation and present publication. The author argues that during the 16th century, and particularly in the work of Vieta, a crucial change occurred in the concept of number and in the nature of algebra which amounted to the birth of mathematics in the modern sense. (C. B. Boyer reviews it favorably in *Science*, vol. 162 (6 Dec. 1968), pp. 1112–1113.)

Ferdinand Georg Frobenius Gesammelte Abhandlungen. Edited by J. P. Serre. Springer-Verlag, New York, 1968. 3 volumes. I: vi+650 pp. II: iii+733 pp. III: iv+740 pp. Three volumes $34.00. A very welcome publication of the papers of an enormously influential mathematician. Possibly because he died during the first world war, only the skimpiest of biographies appeared, and to this date nothing substantial has been written on his life or work. The collected works contain only two pages of reminiscences by T. L. Siegel. The editor is aware of the lack but excuses it because of its difficulty and possible uselessness, justifying himself by quoting R. Brauer: "If the reader wants to get an idea about the importance of Frobenius' work today, all he has to do is look at books and papers on groups." The fact is that many people would like to know more about Frobenius and understand his contributions without having to search the literature on group theory. Besides he made other important contributions. It is too bad that these collected works do not contain analysis and biographical information, but let us hope that someone takes up this interesting and worthwhile task independently. Incidentally, the price is extraordinarily reasonable, less than 2¢ a page for a quality printing and binding at a time when publishers often charge three or four times as much per page for equally specialized books of poorer content and production quality. P, L.

★*A Source Book in Mathematics, 1200–1800.* Edited by D. J. Struik (MIT). Harvard University Press, Cambridge, 1969. xiv+427 pp. $11.95. Seventy-five excerpts from authors of the Latin world between 1200 and 1800, accompanied by introductory remarks and explanatory footnotes. They are grouped in five chapters: arithmetic, algebra, geometry, analysis before Newton and Leibniz, and Newton, Leibniz and their School. All excerpts are in English, some in translations by the editor and his wife (who also holds the doctorate in mathematics) and some old translations corrected and amended as necessary. S, P, L.

Logic

Logique à trois valeurs. Logique à seuil. By Michel Carvallo (Univ. of Paris). Gauthier-Villars, Paris, 1968. xii+152 pp. 38 F. This book is concerned with three-valued logic and threshold logic. By placing it in the *Collection de Mathématiques Économiques* (of which it is volume 5) the publishers continue an unfortunate but a very widespread practice of publishing works on mathematics as though they were on some applied field, merely because the author happens to be interested in the field of application or was motivated by it. P, L.

Zermelo-Fraenkel Set Theory. By Seymour Hayden (Lehman College) and John F. Kennison (Clark University). Merrill, Columbus, Ohio, 1968. xi+164 pp. $7.50. Intended for an advanced undergraduate or first year graduate seminar-style course, this book encourages the student to work many things out for himself. It is intended for the general student as well as the specialist in logic. In an appendix other axioms and approaches to set theory are discussed. T (16–17).

Grundlagen der Mathematik I, 2nd ed. By D. Hilbert and P. Bernays. Springer-Verlag, New York, 1968. xv+473 pp. $17.00. The first edition, an established classic, appeared in 1934. The present edition is the result of extensive revision by P. Bernays. It is number 40 in the series *Die Grundlehren der mathematischen Wissenschaften*. P, L!

The Theory of Sets and Transfinite Numbers. By B. Rotman (Univ. of Bristol) and G. T. Kneebone (Bedford College, London). American Elsevier, New York, 1966. 144 pp. $6.50. American edition (see telegraphic review November 1967).

Pre-Calculus

Intermediate Algebra, 2nd ed. By Roy Dubisch (Univ. of Washington) and Vernon E. Howes (American Coll. in Paris). Wiley, New York, 1969. xi+349 pp. $6.95. As in the first edition of 1960 the emphasis is on manipulative skill, but conceptual questions are not neglected. The book is intended for students with as little as a year of high school algebra. While agreeing with the emphasis on manipulation, I would like to see a treatment more in keeping with mathematical developments of the last century and educational development of the last 20 years. T (13–).

Essentials of Algebra. By William L. Hart (Univ. of Minnesota). Prindle, Weber & Schmidt, Boston, 1969. xi+223 pp. $7.50. An abbreviated version of the author's *Intermediate Algebra* (1968). Elementary algebra, linear and quadratic equations with some set theory ideas and terminology. T (13–).

Elementary Algebra. By George E. Wallace (College of San Mateo, Calif.). McGraw-Hill, New York, 1969. xiii+366 pp. $8.95. Designed for college students whose previous work in algebra is so distant or unsuccessful that "no previous algebraic experience" may be assumed, this book in its last chapter reaches complex numbers and quadratics. The subject matter is traditional but the spirit is "modern"; that is, set concepts are used and there is emphasis on concepts and proofs. T (13), TT.

Probability

The Estimation of Probabilities: An Essay on Modern Bayesian Methods. By I. J. Good. Research Monograph No. 30. MIT Press, Cambridge, Mass., 1968. xii+109 pp. $2.45 (paper). This is a paperback edition of the book published first in 1965. It contains a review of the relevant literature (with a bibliography of 115 items), and also much material new at the time. S, P.

Probability and Statistics with Applications. By Y. Leon Maksoudian (Calif. State Polytechnic College, San Luis Obispo). International Textbook, Scranton, Penn., 1969. xi+416 pp. $9.95. The author in his preface does not explain why another textbook in this field is needed or how this book differs from previous efforts, though he does suggest that it grew out of his experience in industry and some years of teaching statistics. The book is designed for use following or concurrently with a calculus course. T (13–14).

NOTABLE PAPERS

The February 1969 issue of *The Mathematics Teacher* (Vol. 62, No. 2) contains many unusually interesting articles, among which may be mentioned "The Chambery Plan—Stages and Perspectives in the Reform of Mathematics Instruction" by the Association of Teachers of Mathematics in Public Instruction, France; "Research in Programmed Instruction in Mathematics" by Edward J. Zoll; and three interesting papers on geometry.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. W. Ball, Auburn University, represented the Association at the inauguration of President A. K. Jackson of Huntingdon College on October 24, 1968.

Professor Grace E. Bates, Mount Holyoke College, represented the Association at the Convocation in observance of The One Hundred Twenty-Fifth Anniversary of the founding of the College of the Holy Cross at Worcester on October 26, 1968.

Professor E. D. Eaves, University of Tennessee, represented the Association at the inauguration of Dr. Culp as President of East Tennessee State University on October 23, 1968.

Professor W. E. Ekman, University of South Dakota, represented the Association at the inauguration of President R. H. Timmins of Huron College on November 15, 1968.

Professor T. L. Hicks, University of Missouri at Rolla, represented the Association at the inauguration of President J. L. Sells of Southwest Baptist College on October 22, 1968.

Professor J. D. E. Konhauser, Macalester College, represented the Association at the inauguration of President G. R. Field of Wisconsin State University, River Falls, on October 24, 1968.

Professor E. A. Maier, University of Oregon, represented the Association at the observance of the Centennial of Oregon State University on October 27, 1968.

Professor Samuel McNeary, Drexel Institute of Technology, represented the Association at the inauguration of Very Reverend Terrence Toland, S.J., as President of Saint Joseph's College on November 14, 1968.

Professor D. C. Moore, Emory University, represented the Association at the inauguration of President J. Whitney Bunting of Georgia College on October 18, 1968.

Professor J. M. Osborn, University of Wisconsin, represented the Association at the inauguration of Sister Mary Cecilia Carey, O.P., as President of Edgewood College on October 5, 1968.

Professor B. E. Rhoades, Indiana University, represented the Association at the inauguration of Sister Mary Gregory Knoerle, S.P., as President of Saint Mary-of-the-Woods College on October 20, 1968.

Mr. J. F. Rieger, Federal Aviation Agency, Oklahoma City, represented the Association at the inauguration of President J. H. Hollomon of the University of Oklahoma on October 18, 1968.

Professor J. L. Smith, Muskingum College, represented the Association at the inauguration of President J. G. Drushal of the College of Wooster on October 11, 1968.

Professor G. L. Spencer, Williams College, represented the Association at the inauguration of President T. D. Lockwood of Trinity College on October 12, 1968.

Professor D. B. Teague, Western Carolina University, represented the Association at the inauguration of President R. M. Bost of Lenoir Rhyne College on November 16, 1968.

Professor R. J. Wisner, New Mexico State University, represented the Association at the inauguration of President Ferrel Heady of the University of New Mexico on November 9, 1968.

Bates College: Professor S. P. Hoffman, Jr., SUNY College at Cortland, has been appointed Professor and Chairman of the Mathematics Department; Assistant Professor R. W. Sampson has been promoted to Associate Professor.

Bowling Green State University: Assistant Professor R. R. Eakin has been promoted to Associate Professor; Dr. R. P. Finkelstein, Arizona State University, has been appointed Assistant Professor.

California State College at Long Beach: Assistant Professor J. R. Baugh has been promoted to Associate Professor; Dr. S. G. Councilman, UCLA, has been appointed Assistant Professor; Associate Professor Chien Wenjen has been promoted to Professor.

Carleton College: Assistant Professors P. S. Jorgensen and R. B. Kirchner have been promoted to Associate Professors; Associate Professor Seymour Schuster, University of Minnesota, has been appointed Professor.

Kent State University: Assistant Professor Edward Bethel has been promoted to Associate Professor; Dr. R. H. Lohman, University of Iowa, has been appointed Assistant Professor; Dr. James Richards has been promoted to Assistant Professor.

University of Minnesota: Dr. J. E. Huneycutt, Jr., University of North Carolina at Chapel Hill, has been appointed Assistant Professor; Assistant Professor J. T. Joichi has been promoted to Associate Professor; Dr. C. D. Meyer, Colorado State University, has been appointed Visiting Assistant Professor; Associate Professor Marian B. Pour-El has been promoted to Professor; Dr. N. J. Rose, Stevens Institute of Technology, has been appointed Head of the Mathematics Department; Dr. Robert Silber, Clemson University, has been appointed Assistant Professor.

Queensborough Community College: Messrs. Bernard Bernstein and W. S. Harris, Jr., have been promoted to Assistant Professors; Dr. A. G. Anderson, Parsons College, has been appointed Professor.

University of the South: Associate Professor J. T. Cross has been promoted to Professor; Dr. S. F. Ebey, Mercer University, has been appointed Assistant Professor.

Texas A & M University: Dr. J. R. Boone, Texas Christian University, has been appointed Assistant Professor; Dr. L. F. Guseman, NASA Manned Spacecraft Center, Houston, has been appointed Assistant Professor; Dr. J. R. Mosher, Texas Christian University, has been appointed Assistant Professor; Dr. N. W. Naugle, NASA Manned Spacecraft Center, Houston, has been appointed Associate Professor.

University of Toledo: Associate Professor A. A. Johnson has been promoted to Professor; Assistant Professor H. W. Vayo has been promoted to Associate Professor.

Associate Professor Emeritus R. L. Caskey, Oklahoma State University, has been appointed Visiting Professor at Cumberland College.

Dr. Seth Catlin, Arizona State University, has been appointed Assistant Professor at Southern Oregon College.

Assistant Professor G. L. Crumley, The Citadel, has been promoted to Associate Professor.

Assistant Professor C. A. Davis, Meredith College, has been promoted to Associate Professor.

Assistant Professor D. L. Deever, Westmar College, has been promoted to Associate Professor.

Dr. A. E. Filano, West Chester State College, has been appointed Director, Division of Sciences and Mathematics, and continues as Chairman of the Mathematics Department.

Associate Professor J. F. Jakobsen, University of Iowa, has been appointed Assistant Dean of the Graduate College. (Correction to item in February 1969 issue.)

Professor Meyer Jerison, Purdue University, has been appointed Head of the Department of Mathematics and Chairman of the Division of Mathematical Sciences.

Associate Professor J. R. Johnson, Wake Forest University, has been promoted to Professor.

Assistant Professor H. A. Krieger, California Institute of Technology, has been appointed Assistant Professor at Harvey Mudd College.

Associate Professor C. W. Leininger, University of Dallas, has been appointed Professor and Chairman of the Mathematics Department at SUNY College at Cortland.

CONFERENCE ON INTERNATIONAL MATHEMATICS PROGRAMS—UNIVERSITY OF ARKANSAS

The University of Arkansas will sponsor a Conference on International Mathematics Programs, August 7–9, 1969, at Fayetteville, Arkansas. The conference will deal with secondary school curriculum materials and teacher training programs in various foreign countries and regions including India, Latin America, Africa, Spain, and Russia. Speakers will include Professors J. N. Kapur, Jose Tola, Albert Dou, Howard Fehr, Bruce Vogeli, Izaak Wirszup, and W. T. Martin. All interested persons are invited to attend. For further information write W. R. Orton, Department of Mathematics, University of Arkansas, Fayetteville, Arkansas 72701.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA convened at Goucher College in Baltimore, Maryland, on November 23, 1968. Professor George Trytten, Chairman of the Section, presided over the 92 in attendance, 83 of whom were members of the MAA. After a welcome address by Dean Rhoda Dorsey and a short business meeting, the following papers were presented:

1. *A formula for the enumeration of $3\times n$ Latin rectangles*, by F. W. Light, Jr., Bel Air, Maryland.
2. *Real solutions of classes of polynomial equations*, by S. B. Jackson, University of Maryland.
3. *Parades and geometry*, by Miss Marguerite Lehr, formerly of Bryn Mawr College, now of Salisbury, Maryland.
4. *Self-interchange graphs*, by B. L. Schwartz, The Mitre Corporation, Bailey's Crossroads, Virginia.
5. *Pythagorean triads and square triangular numbers*, by R. H. Anglin, Danville, Virginia.

The main invited address, "Mathematics in Topography," was given by G. F. Temple, Sedleian Professor of Natural Philosophy at Oxford University and Visiting Research Professor at the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

The meeting concluded with a panel discussion entitled, "The Relevance of the MAA to our Section Colleges and Junior Colleges," with the following panel members: Geraldine Coon, Goucher College; Earl Embree, Morgan State College; Jorg Mayer, George Mason College; William Swyter, Montgomery Junior College; and Alfred Willcox, Executive Director of the MAA.

S. L. GULICK, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fiftieth Summer Meeting, University of Oregon, Eugene, Oregon, August 25–27, 1969.

Fifty-third Annual Meeting, Miami, Florida, January 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN

FLORIDA, Rollins College, Winter Park, Spring 1970.

ILLINOIS

INDIANA

IOWA

KANSAS, Kansas State Teachers College, Emporia, March 1970.

KENTUCKY

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, April 4, 1970.

MISSOURI

NEBRASKA

NEW JERSEY, Seton Hall University, South Orange, November 1, 1969.

NORTH CENTRAL

NORTHEASTERN, Wheaton College, Norton, Massachusetts, November 29, 1969.

NORTHERN CALIFORNIA, Diablo Valley College, Concord, February 7, 1970.

OHIO

OKLAHOMA-ARKANSAS, Southwestern State College, Weatherford, Oklahoma, March 1970.

PACIFIC NORTHWEST, University of Oregon, Eugene, Oregon, August 25–27, 1969.

PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.

ROCKY MOUNTAIN

SOUTHEASTERN, Clemson University, Clemson, South Carolina, Spring 1970.

SOUTHERN CALIFORNIA, University of California, Irvine, March 21, 1970.

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE

WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, University of Oregon, Eugene, Oregon, August 26–29, 1969.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Ohio State University, June 22–25, 1970.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Milwaukee, Wisconsin, November 27–29, 1969.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, New York City, August 19–22, 1969.

MU ALPHA THETA, University of Oregon, Eugene, Oregon, August 27, 1969.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami, Florida, November 10–12, 1969.

PI MU EPSILON, University of Oregon, Eugene, Oregon, August 26–27, 1969.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Disneyland Hotel, Anaheim, California, October 26–30, 1969.

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11. *Irrational Numbers,* by Ivan Niven.
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Additional copies and copies for nonmembers of Monographs 1-4, 6-8 are priced at $6.00 each and are available from the Open Court Publishing Company, La Salle, Illinois 61301.

Similarly, Monographs 9-15 are available at $6.00 each from John Wiley and Sons, 605 Third Avenue, New York, N. Y. 10016.

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VOLUME 76 NUMBER 7

CONTENTS

(continued on inside cover)

AUGUST-SEPTEMBER 1969

NOTICE TO AUTHORS

Articles should be typewritten on good quality paper, triple spaced with wide margins. Submit the original and a duplicate; keep a complete copy for protection against loss. Please follow the format used in current issues of the MONTHLY. The manuscript must make the author's intent clear to the printer. A paper accepted for publication by the editor but not acceptable to the printer will be returned to the author for retyping. Please use the American Mathematical Society *Manual for Authors of Mathematical Papers* as your guide and please conform to correct English usage. A *Manual for Monthly Authors* is available on request.

Prospective MONTHLY authors are advised to consult the Statement of Policy in the January 1969 issue, p. 2. Specialized research is usually unsuitable. Backlog: Main Articles 10 months, Math. Notes 7 months, Research Problems 4 months, Classroom Notes 5 months, Math. Education 4 months.

EDITORIAL CORRESPONDENCE: to HARLEY FLANDERS, American Mathematical Monthly, Purdue University, Lafayette, IN 47907; ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D. C. 20036.

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PRINTED IN THE UNITED STATES OF AMERICA

PARTITION IDENTITIES—FROM EULER TO THE PRESENT

H. L. ALDER, University of California, Davis

1. Introduction. A *partition* of a positive integer n is defined as a way of writing n as the sum of positive integers. Two such ways of writing n in which the parts merely differ in the order in which they are written are considered the same partition. We shall denote by $p(n)$ the number of partitions of n. Thus, for example, since 5 can be expressed as the sum of positive integers by 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$, we have $p(5)=7$. An explicit formula for $p(n)$ valid for all positive integers n was discovered by Rademacher in 1937, but since it is a complicated infinite series and is not needed for the purposes of this paper, it will not be given here. On the other hand, there exists a simple generating function for $p(n)$, that is, a function which, when expanded into a power series $\sum_{n=0}^{\infty} c_n x^n$ has as its general coefficient $c_n = p(n)$.

THEOREM 1. *The generating function for $p(n)$ is given by*

$$f(x) = \frac{1}{\prod_{\nu=1}^{\infty}(1-x^\nu)}, \qquad \textit{where } |x|<1. \tag{1}$$

Proof of Theorem 1. We have to show that the right hand side of (1), when expanded into a power series, has as its general coefficient $p(n)$. To do this, we rewrite the right hand side of (1) as

$$\begin{aligned}(1 + x^1 + x^{1\cdot 2} + x^{1\cdot 3} + x^{1\cdot 4} + \cdots)&\\ \cdot(1 + x^2 + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + \cdots)&\\ \cdot(1 + x^3 + x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + \cdots)&\\ \cdot(1 + x^4 + x^{4\cdot 2} + x^{4\cdot 3} + x^{4\cdot 4} + \cdots)&\\ \cdot(1 + x^5 + x^{5\cdot 2} + x^{5\cdot 3} + x^{5\cdot 4} + \cdots)&\cdots .\end{aligned} \tag{2}$$

If we multiply this out and calculate, for example, the coefficient of x^5, we find that the term x^5 is obtained in the following ways:

$$1\cdot 1\cdot 1\cdot 1\cdot x^5\cdot 1\cdots,\quad x^1\cdot 1\cdot 1\cdot x^4\cdot 1\cdots,\quad 1\cdot x^2\cdot x^3\cdot 1\cdots,$$
$$x^{1\cdot 2}\cdot 1\cdot x^3\cdot 1\cdots,\quad x^1\cdot x^{2\cdot 2}\cdot 1\cdots,\quad x^{1\cdot 3}\cdot x^2\cdot 1\cdots,\quad x^{1\cdot 5}\cdot 1\cdots.$$

Prof. Alder received his Ph.D. under D. H. Lehmer at Berkeley and remained one year as an instructor. Since (except for a Zürich sabbatical) he has been at the University of California, Davis. His main research interest is the subject of the present paper. In addition to many articles, he has published (with E. B. Roessler) a popular text, *Introduction to Probability and Statistics.*

Alder has devoted enormous energy to serving the mathematical community. (Actually I first met him through our mutual interest in high school contests and visiting lecturer programs when I was in California in the fifties; he considerably influenced my participation in MAA activities.) Alder served as National President of Mu Alpha Theta from 1956–1959 and received its Distinguished Service Award in 1965. As Secretary of the MAA since 1960, he has probably contributed more than anyone else to the present vitality of this Association. *Editor.*

Each of these products corresponds to a partition of 5, indeed in exactly the order in which the partitions of 5 are listed above. Since there is a one-to-one correspondence between the number of times the term x^n is obtained in the product (2) and the number of partitions of n, the coefficient of x^n in (2) is $p(n)$.

The function $p(n)$ is also referred to as the number of *unrestricted partitions* of n, to make clear that no restrictions are imposed upon the way in which n is partitioned into parts. A very interesting—perhaps the most interesting—part of the theory of partitions concerns restricted partitions, that is, partitions in which some kind of restrictions is imposed upon the parts. The fascination in this study lies in the fact that there exist numerous surprising identities valid for all positive integers n of the general type

$$p'(n) = p''(n), \tag{3}$$

where $p'(n)$ is the number of partitions of n where the parts of n are subject to a first restriction and $p''(n)$ is the number of partitions of n where the parts of n are subject to an entirely different restriction. It is the object of this paper to give a survey of the existence and nonexistence of such identities as known up to date.

Perhaps the simplest identity of the above kind is given by the following theorem:

THEOREM 2. *The number of partitions of n into exactly μ parts (μ a given positive integer) is equal to the number of partitions of n into parts the largest of which is μ.*

Proof of Theorem 2. A partition of n into exactly μ parts can be represented graphically by μ lines of dots, the number of dots in each line equalling the part. Thus, the partition of 23 into the 5 parts $7+6+4+4+2$ can be represented by the following graph:

$$\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & & & & &
\end{array}$$

When read vertically by columns, this represents the partition of 23 into $5+5+4+4+2+2+1$, that is, into a partition, the largest part of which is 5. Thus, to each partition of n into μ parts corresponds a partition of n into parts the largest of which is μ, and, since this is a one-to-one correspondence, we have proved the theorem.

The following theorem follows immediately from Theorem 2.

THEOREM 3. *The number of partitions of n into at most μ parts (μ a given positive integer) is equal to the number of partitions of n with parts not exceeding μ.*

Theorem 2 was proved by means of a *combinatorial proof* in a direct way, that is, a one-to-one correspondence between the two types of restricted partitions was established. Many—or perhaps most—identities involving two kinds of restricted partitions are proved more easily, and up to now in some cases can be proved only by *analytical proofs*, that is, by showing that the generating functions for the two types of restricted partitions involved are identical. We shall now consider examples of such identities.

2. Restricted Partition Functions. In order to be able to prove identities of type (3) by use of generating functions, we need to know how to derive generating functions for certain restricted partitions. We shall list in the following table a few which will be needed later.

PARTITIONS INTO	GENERATING FUNCTION
Distinct parts	$\prod_{\nu=1}^{\infty} (1 + x^{\nu})$
Odd parts	$1 \Big/ \prod_{\nu=0}^{\infty} (1 - x^{2\nu+1})$
Parts not exceeding μ	$1 \Big/ \prod_{\nu=1}^{\mu} (1 - x^{\nu})$
Parts taken from the set $\{a_1, a_2, \cdots\}$	$1 \Big/ \prod_{\nu=1}^{\infty} (1 - x^{a_{\nu}})$

These generating functions can easily be derived by modifying the proof of Theorem 1 appropriately. Thus, since for distinct parts each positive integer is allowed no more than once, each of the infinite sums of (2) has to be reduced to its first two terms only. Similarly, for partitions with odd parts, the second, fourth, sixth, $\cdots$ infinite sums of (2) have to be deleted which immediately yields the generating function in the second line of the table above.

3. The Euler and Rogers-Ramanujan Identities. The most celebrated identity which is very easily proved by means of generating functions is due to Euler [11], who discovered it in 1748.

THEOREM 4. (Euler). *The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Proof of Theorem 4. We have to show that the generating function for partitions into distinct parts, as given in line 1 of the above table, is equal to the generating function of *n* into odd parts, as given in line 2 of the above table. This is easily done as follows:

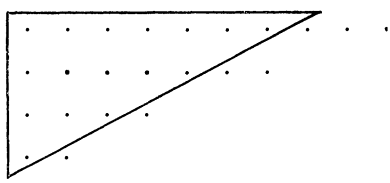
$$\prod_{\nu=1}^{\infty}(1+x^\nu) = (1+x)(1+x^2)(1+x^3)\cdots = \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\cdots$$
$$= \frac{1}{\prod_{\nu=0}^{\infty}(1-x^{2\nu+1})}\,.$$

It is possible—although considerably more difficult—to prove this result by combinatorial methods, see, for example, [26].

Since the Euler identity involves partitions into distinct parts, that is, where parts must differ by at least 1, it is natural to ask whether there exists a corresponding identity involving partitions, where parts must differ by at least 2. Such an identity was discovered by Rogers and Ramanujan around the turn of the century.

THEOREM 5 (Rogers-Ramanujan). *The number of partitions of n into parts differing by at least 2 is equal to the number of partitions of n into parts which are congruent to 1 or 4, modulo 5.*

To express this theorem as an equality of the generating functions for the two kinds of restricted partitions involved, we need to derive the generating function for the number of partitions of n into parts differing by at least 2. We represent such a partition graphically, for example, $23 = 10+7+4+2$, as follows:



Since parts must differ by at least 2, each line must have at least 2 more dots than the one below. Thus, if the partition has exactly μ parts, the graph must have at least $1+3+5+\cdots+(2\mu-1)=\mu^2$ dots (in our graph, they are the dots inside the indicated triangle). Consequently a partition of n into μ parts differing by at least 2 can be graphically represented by a triangle with μ^2 dots and a partition of $n-\mu^2$ into at most μ parts. To obtain the number of all partitions of n into μ parts, we need the generating function for the number of partitions of $n-\mu^2$ into at most μ parts or, which is the same according to Theorem 2, the number of partitions of $n-\mu^2$ into parts not exceeding μ. From line 3 of the above table, we know that the generating function for the number of partitions of n into parts not exceeding μ is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^\mu)}\,.$$

Consequently the coefficient of x^n in

$$\frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)} \tag{4}$$

equals the number of partitions of $n-\mu^2$ into parts not exceeding μ. To find the number of all partitions of n into parts differing by at least 2, we need to sum the coefficients of x^n in (4) for $\mu=1, 2, 3, \cdots$; that is, we determine the coefficient of x^n in

$$\sum_{\mu=1}^{\infty}\frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)}. \tag{5}$$

Accordingly, Theorem (5) is expressed as an identity of generating functions as follows:

$$\sum_{\mu=0}^{\infty}\frac{x^{\mu^2}}{(1-x)(1-x^2)\cdots(1-x^\mu)}=\prod_{\nu=0}^{\infty}\frac{1}{(1-x^{5\nu+1})(1-x^{5\nu+4})}. \tag{6}$$

The proof of (6) and consequently Theorem 5 is somewhat more lengthy than the proof of Euler's identity. The basic tool is the conversion of the infinite product appearing on the right hand side of (6) into the sum on the left hand side by use of Jacobi's identity

$$\prod_{\nu=0}^{\infty}(1-y^{2\nu+2})(1+y^{2\nu+1}z)(1-y^{2\nu+1}z^{-1})=\sum_{\mu=-\infty}^{\infty}y^{\mu^2}z^\mu, \tag{7}$$

and the use of an auxiliary function for which a recurrence equation is derived.

The details of the proof will not be given here, but can be found in [**17**, Chap. 19] and [**20**].

Rogers and Ramanujan found a second identity in which the parts were not only required to differ by at least 2 but also to be all at least equal to 2.

THEOREM 6 (Rogers-Ramanujan). *The number of partitions of n into parts differing by at least* 2, *each part being greater than or equal to* 2, *is equal to the number of partitions of n into parts which are congruent to* 2 *or* 3, *modulo* 5.

To express this theorem as an identity of the generating functions of the two kinds of restricted partitions involved, we proceed exactly as in the derivation following Theorem 5 except that the triangle of that graph is replaced by a trapezoid, the bottom line of which contains 2 dots, the next to last 4 dots, etc., so that the trapezoid would contain inside a total of $2+4+6+\cdots+2\mu=\mu^2+\mu$ dots. Consequently, Theorem 6 is expressed as an identity of generating functions as follows:

$$\sum_{\mu=0}^{\infty}\frac{x^{\mu^2+\mu}}{(1-x)(1-x^2)\cdots(1-x^\mu)}=\prod_{\nu=0}^{\infty}\frac{1}{(1-x^{5\nu+2})(1-x^{5\nu+3})}. \tag{8}$$

4. The Nonexistence of Certain Other Identities of the Euler-Rogers-

Ramanujan Type. If we denote by $q_{d,m}(n)$ the number of partitions of n into parts differing by at least d, each part being greater than or equal to m, then the Euler and Rogers-Ramanujan identities are all of the type

$$q_{d,m}(n) = p_{d,m}(n), \tag{9}$$

where $p_{d,m}(n)$ is the number of partitions of n into parts taken from a fixed set $S_{d,m}$. Thus, in the case of the Euler identity ($d=1$, $m=1$), the set $S_{1,1}$ is the set of odd numbers; in the case of the first Rogers-Ramanujan identity, $S_{2,1}$ is the set of numbers congruent to 1 or 4, modulo 5.

It is natural to ask whether there are any more identities of the type (9). For $d=1$, such an identity exists for every m; that is, there exists the following generalization of the Euler identity:

THEOREM 7. *The number of partitions of n into distinct parts, each part being greater than or equal to m, is equal to the number of partitions of n into parts taken from the set* $\{m, m+1, \cdots, 2m-1, 2m+1, 2m+3, \cdots\}$.

The proof of this theorem is analogous to that of Theorem 4.

Aside from this generalization of the Euler identity and the two Rogers-Ramanujan identities, no other identities of type (9) can exist, so that we have the following theorem.

THEOREM 8. *The number $q_{d,m}(n)$ of partitions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever unless $d=1$ or $d=2$, $m=1, 2$.*

This theorem was proved for the case $m=1$ by D. H. Lehmer [**18**] in 1946, and for the general case by this writer [**1**] in 1948. To prove it, we note that by a slight generalization of the argument used to derive the generating function (5), the one for the number $q_{d,m}(n)$ of partitions into parts differing by at least d, each part being greater than or equal to m, is given by

$$\sum_{\mu=0}^{\infty} \frac{x^{m\mu+d\mu(\mu-1)/2}}{(1-x)(1-x^2)\cdots(1-x^\mu)}, \tag{10}$$

while the generating function for the number of partitions of n into parts taken from a fixed set $\{a_1, a_2, \cdots\}$ is given in the last line of the above table. The proof then consists of showing that no matter how the a_i are chosen, the latter generating function cannot equal that given by (10).

This then proves that the Euler identity and its generalization, given in Theorem 7, together with the two Rogers-Ramanujan identities are indeed the set of all identities of type (9) which can exist.

The question may be raised whether identities of type (9) are possible if $p_{d,m}(n)$ is the number of partitions of n into *distinct* parts taken from a fixed set $S_{d,m}(n)$. That this is not possible is stated in the following theorem:

THEOREM 9. *The number $q_{d,m}(n)$ of partitions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of partitions of n into distinct parts taken from any set of integers whatsoever unless $d=1$.*

The proof of this theorem, also given in [1], consists of showing that for no choice of the elements a_i of the set $S_{d,m}$ the generating function for the number of partitions of n into distinct parts taken from that set, namely

$$\prod_{\nu=1}^{\infty} (1 + x^{a_\nu}),$$

can equal the generating function given by (10).

5. Early Combinatorial Generalizations of the Euler Identity. Although, in accordance with Section 4, certain generalizations of the Euler identity cannot exist, it was already proved in the last century that others do exist. The first remarkable result in this direction was proved by Glaisher [12] in 1883.

THEOREM 10 (Glaisher). *The number of partitions of n into parts not divisible by d is equal to the number of partitions of n of the form $n=n_1+n_2+\cdots+n_s$, where $n_i \geqq n_{i+1}$ and $n_i \geqq n_{i+d-1}+1$.*

For $d=2$, Theorem 10 clearly reduces to Euler's Theorem since the last inequality then requires each part of a partition, when written in nonincreasing order of the parts, to be at least 1 greater than the next one. For $d=3$, the last inequality requires that in any set of three consecutive parts of a partition the first is greater by at least 1 than the last one, thus permitting in this case two consecutive parts to be equal.

Another generalization of Euler's Theorem in an entirely different direction was discovered by Sylvester [26] in 1882.

THEOREM 11 (Sylvester). *The number of partitions of n into odd parts, where exactly k distinct parts appear, is equal to the number of partitions of n into distinct parts, where exactly k sequences of consecutive integers appear.*

Note that in this theorem, a sequence of k consecutive integers may consist of a single integer if $k=1$.

Thus, for $n=13$, the partitions into odd parts where exactly 3 distinct parts appear are $9+3+1$, $7+5+1$, $7+3+1+1+1$, $5+3+3+1+1$, $5+3+1+1+1+1+1$, so that we have 5 partitions of this kind, while the number of partitions of 13 into distinct parts, where exactly 3 sequences of consecutive integers appear, are $9+3+1$ (which consists of the three sequences of 1 integer each), $8+4+1$, $7+5+1$, $7+4+2$, $6+4+2+1$ (which consists of two sequences of 1 integer each and one sequence, namely 2, 1, of two consecutive integers) so that again we have 5 such partitions.

Euler's Theorem is a direct consequence of Theorem 11 by summing over all values of k.

This theorem—and consequently also Euler's Theorem—was proved arithmetically by Sylvester [26, Section 46]. It was proved by the use of generating functions by Andrews [5] in 1966.

6. Analytic Generalizations of the Rogers-Ramanujan Identities. If we consider the Rogers-Ramanujan identities merely as functional equations and disregard their interpretations in terms of partitions, then it is possible to generalize the Rogers-Ramanujan identities. Since Jacobi's identity (7) which plays a vital role in the proof of the Rogers-Ramanujan identities converts an infinite product consisting of *three* different terms into a certain infinite sum, it seems natural to consider the set involved in the first Rogers-Ramanujan identity as the set of numbers *not* congruent to 0, ± 2 (mod 5), rather than the set of numbers congruent to 1 or 4 (mod 5). The first Rogers-Ramanujan identity (6) involving the case where parts are *not* congruent to 0, ± 2 (mod 5) can then be generalized to the case where the parts are not congruent to 0, $\pm k$ (mod $2k+1$) as follows:

THEOREM 12. *The following identity holds:*

$$\prod_{\nu=0}^{\infty} \frac{(1 - x^{(2k+1)\nu+k})(1 - x^{(2k+1)\nu+k+1})}{(1 - x^{(2k+1)\nu+1})(1 - x^{(2k+1)\nu+2}) \cdots (1 - x^{(2k+1)\nu+2k})} = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)}{(1 - x)(1 - x^2) \cdots (1 - x^\mu)}, \tag{11}$$

where the left side is the generating function for the number of partitions into parts not congruent to 0, $\pm k$ (mod $2k+1$) *and the* $G_{k,\mu}(x)$ *are polynomials in* x *and reduce to the monomials* x^{μ^2} *for* $k=2$, *that is, for the Rogers-Ramanujan case.*

The second Rogers-Ramanujan identity (8) involving the case where parts are congruent to 2 or 3, modulo 5, or, what is the same, parts *not* congruent to 0, ± 1 (mod 5) can be generalized to the case where parts are not congruent to 0, ± 1 (mod $2k+1$) as follows:

THEOREM 13. *The following identity holds:*

$$\prod_{\nu=0}^{\infty} \frac{1}{(1 - x^{(2k+1)\nu+2})(1 - x^{(2k+1)\nu+3}) \cdots (1 - x^{(2k+1)\nu+(2k-1)})} = \sum_{\mu=0}^{\infty} \frac{G_{k,\mu}(x)x^\mu}{(1 - x)(1 - x^2) \cdots (1 - x^\mu)}, \tag{12}$$

where the $G_{k,\mu}(x)$ *are the same polynomials as those of Theorem* 12.

More generally, it can be shown that identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(k-r)$ (mod $2k+1$) exist for each r within the range $0 \leqq r \leqq k-1$, so that for a given modulus $2k+1$, there are always k such identities. This agrees with our knowledge that for the modulus 5 two such identities exist.

Theorems 12 and 13 were proved by this writer [2] in 1954 by means of generalizing the proof of the Rogers-Ramanujan identities, again making use of Jacobi's identity (7) and using a generalization of the auxiliary function of the proof of the Rogers-Ramanujan identities. This more general function and a recurrence formula involving it were originally introduced by Selberg in 1936 [22, p. 4, equation 3]. Another proof of Theorems 12 and 13 was given by Singh in 1957 [23]. In [2] some properties of the polynomials $G_{k,\mu}(x)$ were given. Further properties of these polynomials were derived in two papers by Singh in 1957 [24] and in 1959 [25]. An explicit formula for these polynomials was given by Carlitz [9] in 1960. While Theorems 12 and 13 show that it is possible to generalize the Rogers-Ramanujan identities, it has not been possible to describe the right hand sides of (11) and (12) as generating functions for certain types of restricted partitions.

7. Combinatorial Generalizations of the Rogers-Ramanujan Identities. The first success in attempts to generalize the Rogers-Ramanujan identities in a way in which the generalization states an equality between two kinds of restricted partitions was achieved by Gordon in 1961 [15]. This generalization extends the Rogers-Ramanujan identities in a way similar to that in which Glaisher's Theorem 10 extended the Euler identity:

THEOREM 14 (Gordon). *Let $p_{k,r}(n)$ denote the number of partitions of n into parts not congruent to* 0, $\pm r$ (mod $2k+1$), *where* $1 \leqq r \leqq k$. *Let $q_{k,r}(n)$ denote the number of partitions of n of the form $n = n_1 + n_2 + \cdots + n_s$ where $n_i \geqq n_{i+1}$, $n_i \geqq n_{i+k-1} + 2$ and with 1 appearing as a part at most $r-1$ times, then*

$$p_{k,r}(n) = q_{k,r}(n). \tag{13}$$

This theorem reduces to the first Rogers-Ramanujan identity for $k=2$, $r=2$ and to the second for $k=2$, $r=1$. As in the case of the analytic generalizations of the Rogers-Ramanujan identities discussed in Section 6, there are also in this case for a given modulus $2k+1$ exactly k identities.

Gordon in his paper [15] gives a combinatorial proof of Theorem 14 which, therefore, contains as a special case a combinatorial proof of the Rogers-Ramanujan identities. Andrews [6] in 1966 gave an analytic proof of Theorem 14 along the lines of Ramanujan's proof of his identities [20] using the auxiliary function and its recurrence formula introduced by Selberg [22] and also used in the proofs of Theorems 12 and 13 [2].

For some purposes, it is advisable to note that $q_{k,r}(n)$ can also be thought of as the number of partitions of n of the form

$$n = \sum_{i=1}^{\infty} f_i \cdot i,$$

where f_i denotes the number of times the part i appears in the partition, $f_i + f_{i+1} \leqq k-1$, and with 1 appearing as a part at most $r-1$ times. The condition $f_i + f_{i+1} \leqq k-1$ implies that the total number of appearances of two consecutive

integers i and $i+1$ in a partition is at most $k-1$, so that in any set of k consecutive parts in a partition, arranged in nonincreasing order of parts, the first and last part must differ by at least 2, that is, $n_i \geqq n_{i+k-1}+2$.

Using this latter interpretation, Andrews [4] in 1965 was able to generalize Gordon's Theorem further as follows:

THEOREM 15 (Andrews). *Let $p_{d,k,r}(n)$ denote the number of partitions of n into parts not congruent to* 0, $\pm dr$ (mod $d(2k+1)$), *where* $d \geqq 1$, $1 \leqq r \leqq k$. *Let $q_{d,k,r}(n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{\infty} f_i \cdot i$, where if $f_i \equiv \alpha$* (mod d) ($0 \leqq \alpha \leqq d-1$) *then $f_i+f_{i+1} \leqq dk+\alpha-1$, and where* 1 *appears as a part at most $dr-1$ times.*

The above is a corrected version of the abstract which appeared in the *Notices* of the AMS. If $d=1$, Theorem 15 reduces to Gordon's Theorem 14. If $k=1$, $r=1$, Theorem 15 reduces to Glaisher's Theorem 10. This theorem consequently is the first theorem which contains *both* the Euler identity *as well as* the Rogers-Ramanujan identities as special cases.

8. Schur's Identity. If in the Euler identity (see Theorem 4), we replace "odd parts" by "parts congruent to ± 1, modulo 4" and in the first Rogers-Ramanujan identity (see Theorem 5) "parts which are congruent to 1 or 4, modulo 5" by "parts which are congruent to ± 1, modulo 5," the similarity of these two identities becomes even more striking. Let us, therefore, define $p_d(n)$ as the number of partitions of n into parts congruent to ± 1, modulo $d+3$; then the Euler identity can be written as $q_{1,1}(n)=p_1(n)$ and the first Rogers-Ramanujan identity as $q_{2,1}(n)=p_2(n)$. It follows as a consequence of Theorem 8 that $q_{3,1}(n)$ cannot equal $p_3(n)$, but the obvious question arises as to whether there is a relationship between the two. Schur proved that $q_{3,1}(n)-p_3(n)$ is the number of partitions of n into parts differing by at least 3 and containing at least 2 consecutive multiples of 3 so that we have the following theorem, stated in a slightly different form:

THEOREM 16 (Schur). *The number of partitions of n into parts differing by at least* 3 *among which no two consecutive multiples of* 3 *appear is equal to the number of partitions of n into parts which are congruent to* 1 *or* 5, *modulo* 6.

Thus, for example, the number of partitions of 15 into parts differing by at least 3 among which no two consecutive multiples of 3 appear are 15,

$$14+1, 13+2, 12+3, 11+4, 10+5, 10+4+1, 9+5+1, 8+5+2,$$

so that the number of partitions of this kind is 9 which is the number $q_{3,1}(15)$ of all partitions of 15 into parts differing by at least 3, namely 10, minus the number among these partitions in which two consecutive multiples of 3 appears, namely 1 in this case (that is, $9+6$). As the reader may verify, this number, 9, is also the number of partitions of 15 into parts congruent to 1 or 5, modulo 6.

Theorem 16 was proved by Schur in 1926 [21] by means of a lemma concern-

ing recurrence relations for certain polynomials. In 1928, Gleissberg [**13**] gave an intricate arithmetic proof of this theorem. A shorter proof was given by Andrews [**8**] in 1967, based on Appell's Comparison Theorem [**10**, p. **101**].

9. The Nonexistence of Certain Generalizations of Schur's Identity. Using the notation of Section 8, we know that the difference $q_{d,2}(n)-p_d(n)$ is equal to 0 for $d=1$ (Euler identity) and for $d=2$ (the first Rogers-Ramanujan identity) and that for $d=3$ this difference represents the number of partitions of n into parts differing by at least 3 and containing at least 2 consecutive multiples of 3. For $d\geqq 4$, however, there seems to be no simple interpretation of the difference $q_{d,1}(n)-p_d(n)$, even if it could be shown to be nonnegative. The following theorem shows in particular that there cannot be an interpretation exactly like that for $d=3$:

THEOREM 17. *The number of partitions of n into parts differing by at least d, among which no two consecutive multiples of d appear, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever if $d>3$.*

This theorem was proved in 1948 by this writer [**1**]. The method used in that proof can also be used to prove that there cannot exist a dual to Schur's Theorem in the sense that the second of the Rogers-Ramanujan identities is a dual to the first one, so that we have the following theorem:

THEOREM 18. *The number of partitions of n into parts differing by at least* 3, *no part being equal to* 1, *among which no two consecutive multiples of* 3 *appear, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever.*

No results concerning $q_{d,1}(n)-p_d(n)$ for $d\geqq 4$ are available, not even whether this difference is always greater than or equal to 0 although this question was posed by the author as a research problem in the Bulletin of the Amer. Math. Soc. [**3**].

10. Combinatorial Generalizations of Schur's Identity for the Case where $d=2$. Two theorems, both of which are very similar in nature to Schur's Theorem in that they consider partitions of n into parts differing by at least 2 among which no two consecutive multiples of 2 appear (that is, the 3 of Schur's Theorem is replaced by 2) were discovered independently by Göllnitz [**14**] in 1960 and Gordon [**16**] in 1965.

THEOREM 19. (Göllnitz-Gordon). *The number of partitions of n into parts differing by at least* 2 *among which no two consecutive even numbers appear is equal to the number of partitions of n into parts which are congruent to* 1, 4, 7, *modulo* 8.

Thus, for example, the number of partitions of 11 into parts differing by at least 2 and containing no two consecutive even integers are 11, 10+1, 9+2, 8+3, and 7+3+1, so that there are 5 partitions of this kind, while the number

of partitions of 11 into parts congruent to 1, 4, 7, modulo 8, are $9+1+1$, $7+4$, $7+1+1+1+1$, $4+4+1+1+1$, and $1+1+\cdots+1$, again 5 such partitions.

THEOREM 20 (Göllnitz-Gordon). *The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3 is equal to the number of partitions of n into parts which are congruent to* 3, 4, 5, *modulo* 8.

11. A Combinatorial Generalization of the Göllnitz-Gordon Identities. Andrews [7] in 1967 generalized the Göllnitz-Gordon identities in the same manner that Gordon's Theorem 14 generalizes the Rogers-Ramanujan identities.

THEOREM 21 (Andrews). *Let $p_{k,r}(n)$ denote the number of partitions of n into parts not congruent to* 2 (mod 4) *and not congruent to* 0, $\pm(2r-1)$(mod $4k$), *where* $1\leqq r\leqq k$. *Let $q_{k,r}(n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{\infty}f_i\cdot i$, where $f_1+f_2\leqq r-1$ and for all $i\geqq 1$*

$$f_{2i-1}\leqq 1 \quad \textit{and} \quad f_{2i}+f_{2i+1}+f_{2i+2}\leqq k-1.$$

Then

$$p_{k,r}(n) = q_{k,r}(n). \tag{14}$$

Theorem 21 reduces to Theorem 19 for $k=2$, $r=2$ and to Theorem 20 for $k=2$, $r=1$. Let us consider the case $k=3$, $r=3$, then the partitions enumerated by $p_{3,3}(7)$ are $6+1$, $5+2$, $4+3$, $3+3+1$, and $3+1+1+1+1$, so that $p_{3,3}(7)=5$, while the partitions enumerated by $q_{3,3}(7)$ are 7, $6+1$, $5+2$, $4+3$, and $4+2+1$, so that $q_{3,3}(7)=5$.

12. Some Other Identities of the Schur Type. In 1967 Andrews [8] proved a theorem which is similar to Schur's identity, but involves as modulus a multiple of 4.

THEOREM 22 (Andrews). *Let $p_r(n)$ denote the number of partitions of n into parts which are either even and not congruent to $4r-2$* (mod $4r$) *or odd and congruent to $2r-1$ or $4r-1$* (mod $4r$), *where $r\geqq 2$. Let $q_r(n)$ denote the number of partitions of n of the form $n=n_1+n_2+\cdots+n_s$, where $n_i\geqq n_{i-1}$ and, if n_i is odd, $n_i-n_{i+1}\geqq 2r-1$ for $1\leqq i\leqq s$, where we define $n_{s+1}=0$. Then*

$$p_r(n) = q_r(n). \tag{15}$$

In the proof of this theorem, Andrews used a generalization of the method used in his proof of Schur's Theorem.

A student of the author, Elmo Moore [**19**], proved in 1968 that Theorem 22 is also valid for $r=1$, so that we have the following theorem:

THEOREM 23 (Moore). *Let $p_1(n)$ be the number of partitions of n into parts which are either divisible by 4 or odd. Let $q_1(n)$ denote the number of partitions of n of the form $n_1+n_2+\cdots+n_s$, where $n_i\geqq n_{i+1}$ and, if n_i is odd, $n_i-n_{i+1}\geqq 1$ for $1\leqq i\leqq s$. Then*

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SOME MATHEMATICIANS I HAVE KNOWN

GEORGE PÓLYA, Stanford University

Professor Klee, Ladies and Gentlemen,

The occasion requires that I should make a speech. Yet I am very old, my days of invention are over. The little mathematical remarks I have made lately

Prof. Pólya received his Univ. Budapest degree in 1912 and holds honorary degrees from the E. T. H. Zürich, Univ. Alberta, and Univ. Wisconsin. He taught at the E. T. H. until 1940 and has been at Stanford Univ. since. His numerous visiting posts include Cambridge, Oxford, Paris, Göttingen, and Princeton. He is a Correspondent of the Paris Academy of Sciences and holds honorary membership in the Council of the Soc. Math. de France, the London Math. Soc. and the Swiss Math. Soc. Prof. Pólya received the M. A. A. Distinguished Service Award in 1963 and the 1968 N. Y. Film Festival top Blue Ribbon for "Let us teach guessing."

The scientific contributions of George Pólya include over 230 research papers and the books, *Inequalities* (with Hardy and Littlewood), *How to Solve It*, *Isoperimetric Inequalities* (with Szegö), *Mathematics and Plausible Reasoning* (2 v.), and *Mathematical Discovery* (2 v.).

Prof. Pólya's personal influence on three generations of mathematicians has been enormous. Perhaps no book in existence has influenced the direction of thinking of young mathematicians more than his two volume masterpiece with G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*. *Editor.*

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are too little and too few to make a speech about them, and I cannot tell you very well about my former work on which I spent almost sixty years, because too many of you would find it quite unfashionable.

So what shall I do? Make an after dinner speech? Well, it will be a before lunch speech: I shall tell you a few anecdotes about mathematicians I have known. These stories are not printed and perhaps they should not be printed. They are part of an oral tradition—you may find an occasion to tell them to friends or students.

When non-mathematicians discuss mathematicians (when faculty wives discuss their husbands and their husbands' friends) they often ask the same questions: What is particular about mathematicians? How do mathematicians differ from other people? And I often heard the same answers: "Mathematicians are absent-minded" or "Mathematicians are eccentric." Are mathematicians really absent-minded or eccentric? I don't know, but there are infinitely many stories purporting that they are, and I shall quote a few. Probably you will know several of them, but perhaps not all of them.

First, about absent-mindedness. Many such stories are told about Hilbert. Are they true? I doubt it, but some are quite good. Here is one of the very well known ones: There is a party in Hilbert's house and Frau (I mean Mrs.) Hilbert suddenly notices that her husband forgot to put on a fresh shirt. "David," she says sternly, "go upstairs and put on another shirt." David, as it befits a long married man, meekly obeys and goes upstairs. Yet he does not come back. Five minutes pass, ten minutes pass, yet David fails to appear and so Frau Hilbert goes up to the bedroom and there is Hilbert in his bed. You see, it was the natural sequence of things: He took off his coat, then his tie, then his shirt, and so on, and went to sleep.

There is another story which I like even more because it reminds me of the Göttingen I knew where I studied more than half a century ago. Yes, that old-time Göttingen was rather formal. A new member of the faculty was supposed to introduce himself formally to his colleagues. He put on a black coat and a top hat, took a taxi, and made the round of the faculty houses. The taxi stopped in front of each, and the new colleague presented his visiting card at the door. Sometimes he got the answer that the Herr Professor is not at home, but when the Herr Professor was at home the new colleague was supposed to go in and chat for a few minutes. Once such a new colleague came to Hilbert's house and Hilbert decided (or Frau Hilbert decided for him) that he was at home. So the new colleague came in, sat down, put his top hat on the floor, and started talking. This was the proper thing to do, but he did not stop talking. And Hilbert—the visit probably interrupted some mathematical meditation—became more and more impatient. And what did he do finally? He stood up, took the top hat from the floor, put it on his head, touched the arm of his wife, and said: "I think, my dear, we have delayed the Herr Kollege long enough."—and walked out of his own house.

There are some authentic stories about absent-minded mathematicians, for

instance, about Newton who, working intensively at his problems, often forgot to eat his lunch, or, when he ate it, forgot that he had eaten it. Yet other stories are less authentic.

I heard the following from Theodore von Kármán himself. Still, I would not swear that it actually happened; he liked good stories too much, and the best stories do not happen, they are invented. At that time he had a double position: He was professor at Aachen in Germany and also lectured at Cal Tech in Pasadena. As an important aeronautical engineer, he was consultant to several airlines, and so he got free transportation whenever he found an unoccupied seat on a plane of one of these lines. So he commuted more or less regularly between Aachen and Pasadena. He gave similar lectures at both places. Once he was somewhat tired when he arrived in Pasadena, but started lecturing. That was not so difficult: He had his notes which he also used in Aachen. He talked, but as he looked around he had the impression that the faces in the audience looked even more blank than usual. And then he caught himself: He was speaking in German! He became quite upset. "You should have told me—why did you not tell me?" The students were silent, but finally one spoke up: "Don't get upset, Professor. You may speak German, you may speak English, we will understand just as much."

Yet the most beautiful story of my collection is about Norbie—I mean Norbert—Wiener. (The name "Norbie" comes from a conversation I overheard between Wiener and a friend. "Confess," said Wiener, "that you call me Wienie behind my back." "No," said the friend, "we call you Norbie.") Now, here is the story which was widely told, but is hardly true. It is about a student who had a great admiration for Wiener, but never had an opportunity to talk to him. The student walked into a post office one morning. There was Wiener, and in front of Wiener a sheet of paper on the desk at which he looked with tremendous concentration. Suddenly Wiener ran away from, and then back to, the paper, facing it again with tremendous concentration. The student was deeply impressed by the prodigious mental effort mirrored in Wiener's face. He had just one doubt: Should he speak to Wiener or not? Then suddenly there was no doubt, because Wiener, running away from the paper, ran directly into the student who then had to say, "Good morning, Professor Wiener." Wiener stopped, stared, slapped his forehead and said: "Wiener—that's the word."

I cannot tell you a better story on this subject, so let us pass on to the other question: Are mathematicians eccentric? Are they odd, singular, out of the ordinary?

In a way they are, of course. To be really a mathematician, to spend your best effort not in making money, not in working for power, but just in thinking about mathematics is singular behavior. Therefore the question should be put so: Are mathematicians eccentric beyond this point, also in other respects?

Well, I don't know. When I think of the mathematicians I have known not quite superficially, I am inclined to refrain from any general statement. Let me tell you about three mathematicians I have known fairly well: Leopold (Lipót)

Fejér (1880–1959), Adolf Hurwitz (1859–1919), and Godfrey Harold Hardy (1877–1947). I knew each of them for several years, I had the privilege of working with, and I am deeply grateful to, all three. Let me tell you a little about their career, their personality, the style of their work, and (this is a before lunch talk) a few characteristic stories.

Lipót Fejér was born in Hungary. He was about twenty years old when he discovered "Fejér's theorem" (on the arithmetic means of the Fourier series—but I should not enter into mathematical details today). His dissertation (he passed his Ph.D. at the age of twenty-two) deals with that theorem and he came back again and again to his initial discovery: he found sharper formulations, analogies, applications, extensions, he followed up the underlying idea into adjacent domains. Although he also found good remarks on other subjects, his initial discovery remained the center of his work.

His papers are particularly well written, they are very easy to read. This is due to his style of work: When he found an idea, he tended it with loving care; he tried to perfect it, simplify it, free it from unessentials; he worked on it carefully and minutely until the idea became transparently clear. He eventually produced a work of art, not of too large dimensions, but highly finished.

He had artistic talents besides mathematics. He loved music and played the piano. He had a special gift for telling stories, he was a "raconteur." In telling his stories, he acted the part of the persons he was telling about, and underlined the points with little gestures. He liked to talk about his teacher, who was in a rather indirect way responsible for his first discovery, Hermann Amandus Schwarz. When he told about the little misadventures of this great mathematician, he was irresistible, you could not help laughing.

This variety of talents has a bearing on a question which I have often heard: Why did Hungary produce so many mathematicians? Hungary was a small country (it is even smaller today) not much industrialized, and it produced a disproportionately large number of mathematicians, several of whom were active in this country. Why was that so? There is no complete answer, I think—Hungary produced not only mathematicians, but also many musicians and some physicists. Yet, I think, as far as mathematicians are concerned a good part of the answer can be found in Fejér's personality: He attracted many people to mathematics by the success of his own work and by his personal charm. He sat in a coffee house with young people who could not help loving him and trying to imitate him as he wrote formulas on the menus and alternately spoke about mathematics and told stories about mathematicians. In fact, almost all Hungarian mathematicians who were his contemporaries or somewhat younger were personally influenced by him, and several started their mathematical career by working on his problems.

To round out the picture I must quote some of his witty remarks I heard myself.

It happened at a meeting in Germany. At that time I was a "Privatdozent." I cannot completely explain what that is: A financially shaky position, some-

what similar to, but not quite, an Assistant Professor—thank goodness, this institution of Privatdozents has started to disappear nowadays. I was married, and my wife took photographs of the mathematicians. She also stopped Fejér in the company of three or four others, in front of the university on the street car tracks, took a picture and was about to take a second one as Fejér spoke up. "What a good wife! She puts all these full professors on the tracks of the street car so that they may be run over and then her husband will get a job!"

At another meeting (that was several years later) Fejér was very angry (and had some reason to be angry) at a Hungarian mathematician, a topologist whose name I shall not tell you. I walked up and down a long time with Fejér who could not stop talking about the target of his anger and wound up by saying: "And what he says is a topological map of the truth." You must realize how distorted a topological map may be.

Oh yes, let us not forget the question: Was Fejér eccentric? After all these stories, if you could see him in his rather Bohemian attire (which was, I suspect, carefully chosen) you would find him very eccentric. Yet he would not appear so in his natural habitat, in a certain section of Budapest middle class society, many members of which had the same manners, if not quite the same mannerisms, as Fejér—there he would appear about half eccentric.

Adolf Hurwitz was very much like Fejér in one respect, in the style of his work. Felix Klein, in his *History of Mathematics in the Nineteenth Century*, calls Hurwitz an "aphoristician." An aphorism is a concise weighty saying. The aphorism is short, but its author may have worked a long time to make it so short. Also Hurwitz tended his ideas with loving care, until he arrived at the simplest attainable expression, devoid of superfluous ornament or ballast and transparently clear. He was not unlike Fejér in another respect: He preferred not too large problems which are more amenable to perfect clarity. But his range was much wider than that of Fejér. He mastered the whole width of mathematical knowledge of his time as far as that was possible at the beginning of this century. And he learnt much of what he knew at the source: number theory and algebra from Kummer and Kronecker, the Riemannian aspect of complex variables from Felix Klein, the Weierstrassian aspect from Weierstrass himself. His mastery of wide domains of mathematics is described much better than I could, and with infinitely more authority, by Hilbert in the necrology prefixed to Hurwitz's collected works.

Yes, Hurwitz's papers are like aphorisms: In the wide range of his mathematical knowledge he spotted well circumscribed weighty problems capable of a surprisingly simple solution and presented the solution in perfect form. If you wish to have an easily accessible sample, read two pages in his collected works: the proof for the transcendence of the number e.

There was another point of resemblance with Fejér: Music played an important role in Hurwitz's life and he was an excellent pianist. In fact, as a young man he hesitated: should he become a mathematician or a pianist—fortunately he decided against the piano.

Yet here the similarity ends. As a personality, Hurwitz was very different from Fejér. First of all, nothing was farther from Hurwitz than to appear Bohemian or eccentric. He was always correct, reserved, inconspicuous, exceedingly modest, lifting his hat to the servants of the neighbors. A stranger could not suspect that there was more behind this unassuming exterior than middle class respectability. Only those who read his writings or attended his classes could suspect, and only those who knew him better could begin to understand, his strong sense of duty and his deep devotion to truth and clarity.

I never heard Hurwitz utter a sharp sentence in public. Yet in the circle of his family or with good friends he could find a sharp and witty word. I must preface a little what I wish to quote. In discharging conscientiously his duties as a professor, he took care of many Ph.D. candidates, treating them with much consideration and patience. Among so many there were some who needed a lot of help, and even the patient Hurwitz was once led to say: "A Ph.D. dissertation is a paper of the professor written under aggravating circumstances."

G. H. Hardy was very much like Fejér in one respect: Fejér developed mathematics in Hungary by his example, his personal charm, and his personal drive; and Hardy did very much the same in England. Yet the similarity ends there: In other respects, the circumstances and the personalities are very different.

England had a great tradition in applied mathematics, starting with Newton, but did not contribute comparably to pure mathematics which was developed mainly in France and Germany. Hardy insisted on pure mathematics and his insistence changed the trend of mathematical work in England. (That he occasionally misjudged, and was unjust to, applied mathematics is of comparatively little importance.)

Hardy wrote very well and with great facility, but his papers, especially some of his joint papers with Littlewood, make no easy reading: The problems are very hard and the methods unavoidably very complex. He valued clarity, yet what he valued most in mathematics was not clarity but power, surmounting great obstacles that others abandoned in despair. He himself had very great power, and he was fascinated by the Riemann hypothesis.

(Relatively concrete problems, such as the proof of the Riemann hypothesis, are less in vogue nowadays, for reasons partly good and partly bad—"Mostly bad" Littlewood would interject if he were present.)

Yet things were different a few decades ago, and I cannot resist telling here a story, although it needs a long preface.

There is a German legend about Barbarossa, the emperor Frederick I. The common people of Germany liked him and as he died in a crusade and was buried in a far away grave, the legend sprang up that he was still alive, asleep in a cavern of the Kyffhäuser mountain, but would awake and come out, even after hundreds of years, when Germany needed him.

Somebody allegedly asked Hilbert, "If you would revive, like Barbarossa, after five hundred years, what would you do?" "I would ask," said Hilbert, "Has somebody proved the Riemann hypothesis?"

laughing? It is because you don't yet see the underlying theory: If the boat sinks and Hardy drowns, everybody must believe that he has proved the Riemann hypothesis. Yet God would not let Hardy have such a great honor and so he will not let the boat sink.

I believe this story, because almost the same thing happened in my presence. Another summer Hardy stayed in Engelberg, an Alpine valley in Switzerland where we had a chalet. He liked the sunshine, but it rained all the time, and as there was nothing else to do, we played bridge: Hardy, who was quite a good bridge player, my wife, myself, and a friend of mine, F. Gonseth, mathematician and philosopher. Yet after a while Gonseth had to leave, he had to catch a train. I was present as Hardy said to Gonseth: "Please, when the train starts you open the window, you stick your head through the window, look up to the sky, and say in a loud voice: 'I am Hardy.'" Now, some of you are laughing. You have understood the underlying theory: When God thinks that Hardy has left, he will make good weather just to annoy Hardy.

I hope you are not too late for lunch. Thank you.

Meeting of the Northern California Section at the University of Santa Clara, February 8, 1969.

INTERSECTION AND COVERING PROPERTIES OF CONVEX SETS

G. D. CHAKERIAN, University of California at Davis

1. Introduction. This article is a survey of some results in combinatorial geometry pertaining to the intersection and covering properties of finite dimensional convex sets. Our main concern will be with variants of the following theorem, discovered by Eduard Helly in 1913.

HELLY'S THEOREM. *Let $\mathfrak{F}$ be a family of compact, convex subsets of Euclidean n-space R^n, having the property that each $n+1$ or fewer members of $\mathfrak{F}$ have some point in common. Then all the members of $\mathfrak{F}$ have some point in common.*

Much of what we will say overlaps material found in the excellent reference [6], which gives a comprehensive survey of Helly's Theorem, its variants, and diverse applications. Our account is for the most part an abbreviated treatment of some selected topics which appear in [6], including a few results discovered since its publication. (A Russian translation of [6], revised and brought up to date with additional references, is scheduled to appear shortly.) Another excellent reference on this material is the monograph [11].

The selection of topics is naturally a reflection of the author's interests, and those things have been generally emphasized which are related to his joint work with S. K. Stein and G. T. Sallee.

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In the following section we develop some important ideas from the theory of convex sets which, aside from their intrinsic interest, will be useful in later sections. We devote Section 3 to a proof of Helly's Theorem which uses a number of these fundamental ideas. Since much of what we say concerns families of translates of convex sets, we set aside Section 4 to deal with some properties of translates. The later sections get down to the real business of the article, namely variants of Helly's Theorem and covering and intersection properties of convex sets.

The author expresses his gratitude to B. Grünbaum, V. Klee, G. T. Sallee, S. K. Stein, and the referee for a number of helpful comments on the manuscript for this article which have contributed toward its improvement.

2. Some Preliminaries. In the following, R^n, $n \geqq 1$, will be n-dimensional Euclidean space, with the usual inner product between $x, y \in R^n$ denoted by (x, y), and $\|x-y\| = \sqrt{(x-y, x-y)}$ the distance between x and y. The subset $\{x \in R^n : \|x-x_0\| \leqq \rho\}$ is the closed *ball* of radius ρ centered at x_0. We will be considering compact, convex subsets of R^n, that is, those compact subsets K having the property that whenever $x_1, x_2 \in K$, then each point of the segment joining x_1 to x_2 belongs to K. Thus $K \subset R^n$ is convex if and only if whenever x_1, x_2 belong to K, then each convex combination of the form $\lambda_1 x_1 + \lambda_2 x_2$, where λ_1, λ_2 are nonnegative real numbers satisfying $\lambda_1 + \lambda_2 = 1$, belongs to K. In general, if $x_1, \cdots, x_r \in R^n$ then by a *convex combination* of these points we mean a point of the form $\lambda_1 x_1 + \cdots + \lambda_r x_r$, where $\lambda_1, \cdots, \lambda_r$ are nonnegative real numbers satisfying $\lambda_1 + \cdots + \lambda_r = 1$. From the definition, one readily sees that K is convex if and only if any convex combination of points of K again belongs to K. A familiar, and particularly important, convex combination of $x_1, \cdots, x_r$ is their *centroid*, defined as $(x_1 + \cdots + x_r)/r$.

If A is any nonempty subset of R^n, then the *convex hull* of A, written conv A, is the intersection of all convex sets containing A. Since any convex combination of a finite number of points $x_1, \cdots, x_r \in A$ is contained in each convex set containing A, one sees that the set of all such convex combinations is a subset of conv A. On the other hand, it is easy to see that the set of all such convex combinations is itself a convex set containing A, and hence containing conv A. It follows that conv A in fact coincides with the set of all convex combinations of finite subsets of A. The following theorem of C. Caratheodory shows that in constructing conv A, by forming convex combinations of points of $A \subset R^n$, one need never take convex combinations of more than $n+1$ points. In other words, each point of conv A is contained in some simplex whose vertices belong to A.

CARATHEODORY'S THEOREM. *Let $A \subset R^n$. Then any point in* conv A *is expressible as a convex combination of $n+1$ or fewer points of A.*

We will not prove Caratheodory's Theorem here, but it turns out to be equivalent to Helly's Theorem, in the sense that either theorem follows rather easily from the other (for a proof, the reader may consult [7]). However, we

will later use the following immediate consequence of Caratheodory's Theorem in proving Helly's Theorem.

COROLLARY 1. *Let z be the centroid of $z_1, \cdots, z_r \in R^n$, where some, but not all, of the z_α may coincide. Then z is a convex combination of some $n+1$ or fewer of those points of $\{z_1, \cdots, z_r\}$ which are different from z.*

If K is a compact subset and x_0 some point of R^n, then the continuous function $\|x-x_0\|$, with x varying over K, attains a minimum value on K. Any point where this minimum is attained is called a *nearest point* of K. The following theorem of T. S. Motzkin gives a characterization of compact, convex sets in terms of nearest points (for a proof of the theorem, and some interesting related problems, see [**16**, Part VII]).

MOTZKIN'S THEOREM. *Let $K \subset R^n$ be compact. Then K is convex if and only if for each $x \in R^n$ there is a unique nearest point of K.*

Suppose K is a compact, convex subset of R^n. For each $x \in R^n$, let $\nu(x) \in K$ be the unique nearest point of K (so, for example, $\nu(x) = x$ if and only if $x \in K$). Then it is easily verified that

$$\|\nu(x) - \nu(y)\| \leqq \|x - y\|, \qquad \text{for all } x, y \in R^n. \tag{1}$$

It follows, in particular, that $\nu(x)$ depends continuously on x. A property of ν which we will use in the next section is that if $x \notin K$, then K lies entirely to one side of the hyperplane orthogonal to the vector $\nu(x)-x$ and containing $\nu(x)$. This hyperplane is then a *supporting hyperplane* of K. The closed *supporting half-space* containing K and bounded by this hyperplane is expressible as

$$\{y \in R^n: (y - \nu(x), \nu(x) - x) \geqq 0\}. \tag{2}$$

3. A Proof of Helly's Theorem. We now apply some of the ideas of the preceding section to give a proof of Helly's Theorem. This proof is in principle the same as that appearing in [**7**] and the beautiful paper [**14**]. Indeed, we have compounded the sin of this repetition and marred the elementary nature of that proof by appealing at one stage to a powerful topological theorem, but have done so in the hope of obtaining something a bit easier to conceptualize. It should be emphasized here that there are a number of proofs of Helly's Theorem which are much more elementary than this proof. Helly's own proof is among the simplest of these, requiring practically nothing in the way of machinery. Indeed, as Victor Klee pointed out to the author, Helly's Theorem (for the case of finite families of convex sets, where one can remove the restriction of compactness) does not depend on any topological conditions at all and is, in fact, valid in a finite-dimensional vector space over an arbitrary ordered field. For some of these proofs, and references to other proofs, the reader may consult [**6**]. Our proof proceeds as follows.

A simple argument using the compactness of the members of $\mathfrak{F}$ shows that it suffices to prove the theorem in case $\mathfrak{F}$ contains only finitely many members,

say $K_1, \cdots, K_r$. We shall now assume that there is no point common to all members of $\mathfrak{F}$ and prove that some $n+1$ or fewer members have no point in common.

Let B be a closed ball containing all of $\mathfrak{F}$, and define a mapping $f\colon B \to B$ by assigning to each $x \in B$ the centroid of the set of nearest points of members of $\mathfrak{F}$. That is, let $f(x) = (\nu_1(x) + \cdots + \nu_r(x))/r$, where $\nu_\alpha(x)$ is the nearest point of K_α to x. Then f is a well-defined, continuous mapping of B into B, and so, by Brouwer's Fixed Point Theorem, $f(z) = z$, for some z. In other words, some z is the centroid of $\nu_1(z), \cdots, \nu_r(z)$. Since we are assuming there is no point common to all the members of $\mathfrak{F}$, the points $\nu_1(z), \cdots, \nu_r(z)$ do not all coincide (otherwise they would then coincide with their centroid z, and z would be common to $K_1, \cdots, K_r$). Hence Corollary 1 assures us that z is a convex combination of some $m \leqq n+1$ of those $\nu_\alpha(z)$ different from z. Without loss of generality, we may relabel everything and denote these $\nu_\alpha(z)$ by $z_1, \cdots, z_m$ and the corresponding members of $\mathfrak{F}$ by $K_1, \cdots, K_m$. Then we have

$$z = \lambda_1 z_1 + \cdots + \lambda_m z_m,$$

for some $\lambda_1, \cdots, \lambda_m \geqq 0$, $\lambda_1 + \cdots + \lambda_m = 1$, where $z_\alpha \neq z$ is the nearest point of K_α to z, $\alpha = 1, \cdots, m$.

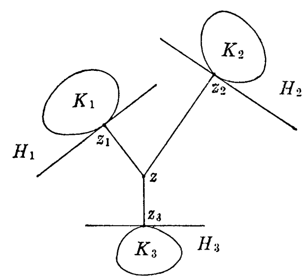

FIG. 1

To complete the proof, we observe that since z is in the convex hull of the set of z_α, and since the closed half-space H_α with z_α on its boundary, orthogonal to $z_\alpha - z$ (and not containing z) is a supporting half-space of K_α, $\alpha = 1, \cdots, m$, then this family of K_α cannot have any point in common (figure 1 illustrates a case where $n = 2$, $m = 3$). To actually prove this, we may assume without loss of generality that $z = 0$. This enables us to write (see equation (2)),

$$H_\alpha = \{y \in R^n\colon\ (y - z_\alpha,\ z_\alpha) \geqq 0\}, \qquad \alpha = 1, \cdots, m, \quad \text{and} \quad \lambda_1 z_1 + \cdots + \lambda_m z_m = 0.$$

Then any point p common to all the K_α would also be common to all the H_α, and hence would satisfy $(p - z_\alpha, z_\alpha) \geqq 0$, or $(p, z_\alpha) \geqq (z_\alpha, z_\alpha)$, $\alpha = 1, \cdots, m$. Thus $0 = (p, 0) = (p, \sum \lambda_\alpha z_\alpha) = \sum \lambda_\alpha (p, z_\alpha) \geqq \sum \lambda_\alpha (z_\alpha, z_\alpha) > 0$, a contradiction which shows there is no such point p. This completes the proof.

The crux of this proof was the existence of the point z. It might be instructive

to point out the following equivalent argument for establishing the existence of z. For each $x \in B$, draw the vector $\nu_\alpha(x) - x$ pointing from x to $\nu_\alpha(x)$, and form the resultant, $\sum(\nu_\alpha(x) - x)$. This yields a continuous vector field on B which points inward on the boundary of B. By another well-known topological theorem, such a field must have a "stagnation point" inside B, that is, a point z where it vanishes. Then $\sum(\nu_\alpha(z) - z) = 0$, which is precisely the condition that z be the centroid of $\nu_1(z), \cdots, \nu_r(z)$.

4. Some Properties of Translates of Convex Sets. The *vector sum*, $X + Y$, of two subsets X and Y of R^n is the set of all points of the form $x + y$, with $x \in X$ and $y \in Y$. If X and Y are compact convex sets, then so is $X + Y$. In the particular case where X consists of a single point, $X = \{x\}$, the vector sum is a *translate* of Y, and we abuse the notation a bit to write $x + Y$ for this translate.

If $X \subset R^n$, and λ is any real number, then λX is the set of all points of the form λx, with $x \in X$. In case $\lambda = -1$, we write $-X = (-1)X = \{-x : x \in X\}$. X is *centrally symmetric* if it coincides with some translate of $-X$. We shall say that X is *homothetic* to Y if there exists $x_0 \in R^n$ and $\lambda > 0$ such that $Y = x_0 + \lambda X$.

The *difference set* of X is defined to be $X + (-X)$. If K is a compact convex subset of R^n, then $K + (-K)$ is a centrally symmetric compact convex set centered at the origin. It is an easy algebraic exercise to verify that $K + (-K)$ is the union of all those translates of K which contain the origin. That is,

$$K + (-K) = \bigcup \{x + K : 0 \in x + K\}. \tag{3}$$

Given $x_1, \cdots, x_r$, let $K(x_1, \cdots, x_r)$ denote the union of those translates of K which contain all the points $x_1, \cdots, x_r$. That is,

$$K(x_1, \cdots, x_r) = \bigcup \{x + K : x_1, \cdots, x_r \in x + K\}. \tag{4}$$

Then $K(x_1, \cdots, x_r)$ is a compact convex set if K is, and coincides with the union of all those translates of K which contain the convex hull of $\{x_1, \cdots, x_r\}$. Note, in particular, that $K(x) = x + (K + (-K))$, which is a translate of the difference set of K.

The *diameter* of a compact subset of R^n is the maximum distance between any pair of its points. It is a well-known fact that conv Q has the same diameter as Q. Observe that this property can be stated in the following equivalent form. If Q is a compact subset of R^n and B is a ball, then each pair of points of conv Q can be covered by a translate of B if and only if each pair of points of Q can be covered by a translate of B. One obtains a (correct) generalization of this last statement by substituting for B any compact convex set K. The statement then amounts to the fact that in the finite dimensional Banach space having $K + (-K)$ for its unit disk, the diameter of any set is equal to the diameter of its convex hull. The following lemma is a generalization of these facts.

LEMMA 1. *Let Q be a subset and K a compact convex subset of R^n. Let r be a fixed positive integer. Then each r points of conv Q can be covered by a translate of K if and only if each r points of Q can be covered by a translate of K.*

Proof. The "only if" part of the lemma is trivial, since $Q\subset \operatorname{conv} Q$. So assume that each r points of Q can be covered by a translate of K, and let $x_1, \cdots, x_r \in \operatorname{conv} Q$. Then if $q_1, \cdots, q_r$ are any points of Q, we have $q_r\in K(q_1, \cdots, q_{r-1})$. Since q_r could be any point of Q, we have $Q\subset K(q_1, \cdots, q_{r-1})$, and so $\operatorname{conv} Q \subset K(q_1, \cdots, q_{r-1})$, since $K(q_1, \cdots, q_{r-1})$ is convex. Thus there is a translate of K containing $q_1, \cdots, q_{r-1}, x_1$, for each $q_1, \cdots, q_{r-1}\in Q$. It follows that $q_{r-1} \in K(q_1, \cdots, q_{r-2}, x_1)$ for any $q_1, \cdots, q_{r-1}\in Q$. Since q_{r-1} could be any point of Q, we have $Q\subset K(q_1, \cdots, q_{r-2}, x_1)$, and so $\operatorname{conv} Q\subset K(q_1, \cdots, q_{r-2}, x_1)$. Thus there is a translate of K containing $q_1, \cdots, q_{r-2}, x_1, x_2$, for each $q_1, \cdots, q_{r-2}\in Q$. Using this, in the same manner, we get $Q\subset K(q_1, \cdots, q_{r-3}, x_1, x_2)$, so $\operatorname{conv} Q \subset K(q_1, \cdots, q_{r-3}, x_1, x_2)$, and thus there is a translate of K containing $q_1, \cdots, q_{r-3}, x_1, x_2, x_3$, for each $q_1, \cdots, q_{r-3}\in Q$. Proceeding in this fashion, we eventually get a translate of K containing $x_1, \cdots, x_r$, and the proof is complete.

5. Helly's Theorem and Universal Covers. The following theorem may be viewed as a generalization of Helly's Theorem (take $P=\{x\}$ in the statement of the theorem). The short, elegant proof is due to V. Klee [**13**].

THEOREM 1. *Let P be a fixed compact convex subset of R^n, and let $\mathfrak{F}$ be a family of compact convex subsets of R^n having the property that each $n+1$ or fewer members of $\mathfrak{F}$ have a translate of P in common. Then all the members of $\mathfrak{F}$ have a translate of P in common.*

Proof. For each $K\in\mathfrak{F}$, define K^* by: $K^*=\{x\in R^n: x+P\subset K\}$, and let $\mathfrak{F}^*=\{K^*: K\in\mathfrak{F}\}$. Then $\mathfrak{F}^*$ is a family of compact convex sets such that each $n+1$ or fewer members have a point in common. By Helly's Theorem, there is a point x^* common to all members of $\mathfrak{F}^*$. Then $x^*+P\subset K$, for all $K\in\mathfrak{F}$. This completes the proof.

The following corollary can be proved directly using Helly's Theorem, but we preferred the proof below because it gives a nice application of Theorem 1 and some ideas of the previous section.

COROLLARY 2. *Let Q be a subset and K a compact convex subset of R^n, and suppose that each $n+1$ points of Q can be covered by a translate of K. Then Q can be covered by a translate of K.*

Proof. By assumption, if $q_1, \cdots, q_{n+1}\in Q$, then there is a translate of K containing $q_1, \cdots, q_{n+1}$; hence $K(q_1), \cdots, K(q_{n+1})$ have a translate of K in common. Thus each $n+1$ members of the family $\mathfrak{F}=\{K(q): q\in Q\}$ have a translate of K in common, so they all have a translate of K in common. This translate contains each $q\in Q$, and the proof is complete.

A compact convex subset $K\subset R^n$ is a *universal cover* if any $Q\subset R^n$ having diameter $\leqq 1$ can be covered by a congruent copy of K. One of the more useful results about 2-dimensional universal covers is the following theorem of J. Pàl (see [**7**]).

PÀL'S THEOREM. *The regular hexagon circumscribed about a circle of diameter* 1 *is a universal cover in* R^2.

It follows immediately from Pàl's Theorem that a circular disk of radius $1/\sqrt{3}$ is also a universal cover. One can establish this same result using Corollary 2 as follows. If $Q \subset R^2$, and the diameter of Q is $\leqq 1$, then each 3 points of Q can be covered by a circular disk of radius $1/\sqrt{3}$; hence Corollary 2 implies Q itself can be covered by such a disk. A similar argument shows that a ball of radius $(n/2n+2)^{1/2}$ is a universal cover in R^n.

While there are numerous scattered results about universal covers, there does not exist any "systematic" method for settling even questions of the following type.

QUESTION 1. For each fixed k, what is the smallest (in diameter) regular k-gon which will serve as a universal cover in R^2?

As far as I know, the answer to this specific question is known only for $k = 3$, 4, and 6. For the case $k = 3$, one has that an equilateral triangle of side $\sqrt{3}$ is the answer, since such a triangle will cover Pàl's hexagon, and no smaller equilateral triangle will serve to cover a circle of diameter 1. For the case $k = 4$, it is easy to show that a square of side 1 is the answer. The answer for $k = 6$ is, of course, Pàl's hexagon.

The most famous unsolved problem of this nature, originally posed by H. Lebesgue [**10**, p. 274], is the following:

THE LEBESGUE COVERING PROBLEM. What is the minimum area that a universal cover in R^2 can have?

An interesting related concept is that of a *minimal universal cover*. A minimal universal cover is a universal cover such that no proper closed convex subset is a universal cover (see [**10**, p. 274]). H. G. Eggleston [8] proved that the union of a circular disk of diameter 1 and a Reuleaux triangle, a side of whose basic equilateral triangle is a diameter of the disk, is a minimal universal cover in R^2. In answer to a question of V. Klee [**10**, p. 274], he also proved the surprising fact that in R^n, with $n \geqq 3$, there exist minimal universal covers of arbitrarily large diameter.

Define a *strong universal cover* to be a compact convex subset K of R^n such that every set $Q \subset R^n$ of diameter $\leqq 1$ can be covered by a translate of K. Roughly speaking, one requires that any set Q of diameter $\leqq 1$ can be placed inside K in *every* orientation, as opposed to the case of ordinary universal covers, where one requires that Q can be placed inside K in *some* orientation. For example, a cube of edge 1 and ball of radius $(n/2n+2)^{1/2}$ are strong universal covers in R^n. Of course, every strong universal cover is in particular a universal cover, but the regular hexagon of Pàl is an example of a universal cover in R^2 which is not a strong universal cover. Note that $K \subset R^2$ is a strong universal cover if and only if every plane set Q of diameter $\leqq 1$ can be "turned" through 360° inside K.

A compact convex subset K of R^n is a *set of constant width* if the distance between parallel supporting hyperplanes of K is constant. The Reuleaux triangle, obtained by intersecting three congruent circular disks, each having its boundary passing through the centers of the other two, is a classic example of a plane set of constant width which is not a circular disk. The following lemma, whose proof can be found in [7], gives a useful characterization of sets of constant width.

LEMMA 2. *Let $K \subset R^n$ be a compact set of diameter λ. Then K is a set of constant width λ if and only if K coincides with the intersection of all balls of radius λ centered in K.*

THEOREM 2. *Let $\triangle \subset R^2$ be a Reuleaux triangle of width λ, and let P be a compact convex subset of R^2. Suppose that each congruent copy of P can be covered by a translate of $\triangle$. Then if K is any set of constant width λ, each congruent copy of P can be covered by a translate of K.*

REMARK. It is easiest to think about Theorem 2 in the following terms. If a plane convex set P can be turned through 360° inside a Reuleaux triangle of width λ, then P can be turned through 360° inside every plane set K of constant width λ.

Proof. Let P' be any congruent copy of P. It is not difficult to verify that the intersection of three circular disks of radius λ, each containing the centers of the other two, contains a Reuleaux triangle of width λ. It follows that the intersection of any three circular disks of radius λ centered in K contains a Reuleaux triangle of width λ, which in turn contains a translate of P'. Thus K is the intersection of a family of circular disks (Lemma 2) such that the intersection of each three contains a translate of P', so, by Theorem 1, K contains a translate of P'. This completes the proof.

Let λ_0 be the smallest value of λ such that the Reuleaux triangle of width λ is a strong universal cover (I do not know the precise value of λ_0). Then any plane set Q of diameter $\leqq 1$ can be turned around inside the Reuleaux triangle of width λ_0; hence Q can be turned around inside any set of constant width λ_0. So we have,

COROLLARY 3. *Any plane set of constant width λ_0 is a strong universal cover.*

6. Variants of Helly's Theorem. Easy examples show that the crucial number "$n+1$" in Helly's Theorem generally cannot be replaced by a smaller number unless further restrictions are placed on the family of sets under consideration. A case where "$n+1$" can be replaced by "2" is the following.

THEOREM 3. *Let $\mathfrak{F}$ be a pairwise intersecting family of rectangular parallelopipeds having their edges parallel to the coordinate axes in R^n. Then there is a point common to all the members of $\mathfrak{F}$.*

We give the reader the pleasure of finding a proof, and merely note that the case $n=1$ is precisely Helly's Theorem for R^1.

The following theorem from [11] is a prototype of results which will be of interest to us in this section.

THEOREM 4. *Let $\mathfrak{F}$ be a pairwise intersecting family of congruent circular disks in R^2. Then there exist 3 points, p_1, p_2, p_3, such that each member of $\mathfrak{F}$ contains at least one of these points.*

Proof. We can write $\mathfrak{F}=\{q+K: q\in Q\}$, where K is a circular disk. Indeed, we may assume without loss of generality that K has diameter 1. Then the fact that $\mathfrak{F}$ is pairwise intersecting implies Q has diameter $\leqq 1$; hence by Pàl's Theorem Q is contained in a regular hexagon of edgelength $1/\sqrt{3}$. This hexagon can be covered by 3 circular disks of diameter 1, whose centers will then serve as the required points p_1, p_2, p_3. This completes the proof.

Theorem 4 suggests that we associate with each compact convex subset K of R^n a number $h(K)$ defined as follows (see Grünbaum [9]):

$h(K)$ is the smallest number r such that whenever $\mathfrak{F}$ is any family of pairwise intersecting translates of K, then there exist r points such that each member of $\mathfrak{F}$ contains at least one of them.

Thus Theorem 4 asserts that if D is a circular disk in R^2, then $h(D)\leqq 3$. Actually, it is easy to see that $h(D)=3$. Grünbaum [9] showed that $h(n) \equiv \sup\{h(K): K\subset R^n\}$ is finite for each n. A basic unsolved problem in this area is to find good bounds on $h(n)$. Even the following question, raised by Grünbaum [9], is still unsettled.

QUESTION 2. Is it true that $h(K)\leqq 3$ for every $K\subset R^2$?

The question has been answered only for some special classes of plane convex sets K. For example, Grünbaum [9] showed it is true for centrally symmetric $K\subset R^2$. G. T. Sallee and the author proved it when $K\subset R^2$ has constant width (see Theorem 5, and [1]. Note that $h(K)$ is an affine invariant of K, so one has $h(K)\leqq 3$ if K is affinely equivalent to a plane set of constant width). Another proof for the centrally symmetric case is given in Section 7.

The reader will find it an easy exercise to verify that $h(K)$ may be defined in the following equivalent fashion:

$h(K)$ is the least number r such that whenever Q is a set having the property that each pair of its points can be covered by a translate of K, then Q can be covered with r translates of K.

With this new interpretation, the assertion that $h(D)\leqq 3$ for a circular disk D of diameter 1 is equivalent to the fact that any plane set of diameter $\leqq 1$ can be covered by 3 translates of D. Note that it was this fact which was implicitly established in the proof of Theorem 4. Theorem 3 implies that $h(P)=1$ if P is a parallelopiped in R^n; hence we have the equivalent covering property that any set, each pair of whose points can be covered by a translate of P, can be covered by a translate of P. This will be used later in establishing the inequality (5). With the new definition of $h(K)$, Grünbaum's question takes the following form.

QUESTION 2*. Let K be a compact plane convex set, and let Q be a plane set, each two of whose points can be covered by a translate of K. May one then conclude that Q can be covered by 3 translates of K?

Using the new definition of $h(K)$, we next prove that the answer to Question 2* is affirmative in case K has constant width. (If K has constant width 1, then the condition that each pair of points of Q can be covered by a translate of K is equivalent to Q having diameter $\leqq 1$. In [**1**], a result stronger than Theorem 5 is proved. It is shown there that there is a number $\eta_0 < 1$ such that given any set Q of diameter $\leqq 1$ and given any 3 sets of constant width η_0, then Q can be covered by translates of these 3 sets. The reader may note that the proof of Theorem 5 can be modified to obtain this result; making an estimate, however, on the best value of η_0, as is done in [**1**], is tedious.)

THEOREM 5. *If* $K \subset R^2$ *has constant width, then* $h(K) \leqq 3$.

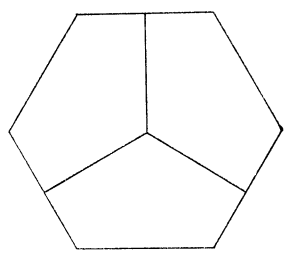

FIG. 2

Proof. We may suppose without loss of generality that K has constant width 1. If $Q \subset R^2$ has the property that each 2 of its points can be covered by a translate of K, then Q has diameter $\leqq 1$. By Pàl's Theorem, $Q \subset H$, where H is a regular hexagon of edgelength $1/\sqrt{3}$. H can be decomposed into 3 congruent irregular pentagons, as indicated in figure 2. One verifies that such pentagons can be "turned" inside a Reuleaux Triangle of width 1; hence Theorem 2 implies that each of the 3 pentagons can be covered by a translate of K. Thus Q can be covered by 3 translates of K, and the theorem is proved.

The reader will see the great number of variants of Helly's Theorem which can be devised. One may consider, for example, families of homothets of a convex set and seek results similar to Theorem 4. Indeed, the following question of T. Gallai, which inspired much of the work in the area, is of this type (see [**6**,‚p. 144]).

GALLAI'S QUESTION. What is the smallest number r such that given any pairwise intersecting family of (not necessarily congruent) plane circular disks, there exist r points having the property that each disk contains at least one of these points?

The answer to this particular question is known to be $r=4$, but the proof,

due to L. Danzer, is unpublished. L. Stachó [15] has proved that 5 points suffice and also obtained some refined results on the possible positioning of the r points.

Motivated by Gallai's question, Grünbaum [9] defined the number $H(K)$ as follows:

$H(K)$ is the smallest number r such that whenever $\mathfrak{F}$ is any family of pairwise intersecting homothets of K, then there exist r points such that each member of $\mathfrak{F}$ contains at least one of them.

Thus, if D is a circular disk in R^2, the answer to Gallai's question can be put in the form: $H(D)=4$. Some other known facts about $H(K)$ are the following. If T is a triangle in R^2, then $H(T)=3$ (see [3] for a proof). Grünbaum [9] proved that $H(K)\leqq 7$ if $K\subset R^2$ is centrally symmetric, and that $H(n)\equiv\sup\{H(K):K\subset R^n\}$ is finite for each n. As far as particular higher dimensional sets are concerned, one has, from Theorem 3, that $H(P)=1$ if P is an n-dimensional parallelopiped, and considerable information on $H(B^n)$, where B^n is an n-dimensional ball, has been obtained by L. Danzer (see [6] for more details). But basic questions of the following type still remain unanswered.

QUESTION 3. What are the precise values of $H(n)$ and $h(n)$, for each n?

QUESTION 4. What are the precise values of $H(K)$ and $h(K)$ when K is an n-dimensional ball, or an n-dimensional simplex, for $n\geqq 3$?

P. Katzarowa-Karanowa [12] has recently established that $h(B^3)=4$, if B^3 is a 3-dimensional ball. See Section 7 for further information on $h(K)$, when K is an n-ball.

With regard to Question 3, various upper bounds on $H(n)$, $h(n)$, and similar "Helly-type" constants are known. It has been pointed out by C. A. Rogers (see the footnotes in [6]) how one can obtain good bounds in many cases using an averaging method due to Erdös and Rogers which is useful in problems involving coverings. The following crude bound is obtained in an elementary fashion:

$$h(n)\leqq n^n,\qquad n\geqq 1. \tag{5}$$

We give the proof, which is also found in [3], since it involves an idea which is often useful in dealing with these problems.

Proof of (5). Let K be a compact convex subset of R^n, and Q a set such that each pair of points of Q can be covered by a translate of K. It is proved in [3] that K is a subset of a parallelopiped P such that a translate of $(1/n)P$ is contained in K. Then each pair of points of Q can be covered by a translate of P; hence (since $h(P)=1$) Q itself can be covered by a translate of P. Thus Q can be covered by n^n translates of $(1/n)P$, and so Q can be covered by n^n translates of K. This completes the proof.

Note that the basic idea of the proof is simply that if $A\subset K\subset B$, then $h(K)$

$\leqq h(B)\nu(B/A)$, where $\nu(B/A)$ is the minimum number of translates of A required to cover B.

It is appealing to think of these Helly-type constants as "piercing numbers." For example, one can put Gallai's question in the following form: suppose one has a family of circular disks (think of them as cut out of paper) such that given any pair of them it is possible to pierce the pair simultaneously with a needle; then how many needles will be required to simultaneously pierce all the disks? Thinking in these terms, one is led naturally to inquire about the following kind of problem. Let s and t be given positive integers. Then given a family of convex sets such that each subfamily of s or fewer members can be pierced with t needles, how many needles will be required to pierce all members of the family? We cannot do justice here to the variety of interesting variants of Helly's Theorem which arise from this question (see [6] for a detailed treatment). We content ourselves with mentioning a recent work [5] of Danzer and Grünbaum, where they settle a question of this nature for families of boxes. As a very special case of their results, one has the following. Let $\mathfrak{F}$ be a family of rectangles in the plane with sides parallel to the coordinate axes, and suppose each 16 or fewer of these rectangles can be pierced with 3 needles. Then the whole family can be pierced with 3 needles.

7. Results on Centrally Symmetric Sets. Let K be a compact convex subset of R^n. Clearly, each pair of points of $-K$ can be covered by a translate of K; hence $-K$ can be covered by $h(K)$ translates of K. But note that the intersection of $-K$ with any translate of K is a centrally symmetric convex set. Thus we have shown that $-K$, and hence K, can be represented as the union of $h(K)$ centrally symmetric convex sets. This leads us to associate with each K a number $g(K)$ defined as follows.

$g(K)$ is the least number r such that K can be represented as the union of r centrally symmetric compact convex sets.

Then what the above argument shows is that,

$$g(K) \leqq h(K). \tag{6}$$

Defining $g(n) \equiv \sup\{g(K): K \subset R^n\}$, it follows that $g(n) \leqq h(n)$ for all n.

THEOREM 6. *Let $S \subset R^n$ be a centrally symmetric compact convex set. Then $h(S) \leqq g(n)$.*

Proof. Let $Q \subset R^n$ be such that each pair of points of Q can be covered by a translate of S. It follows from the compactness of S that each two points of the closure of Q can be covered by a translate of S. Hence, if Q^* is the convex hull of the closure of Q, Lemma 1 implies that each pair of points of Q^* can be covered by a translate of S. Now Q^* is a compact convex set (the compactness of the convex hull of a compact set in R^n follows neatly from Caratheodory's Theorem —see [6, p. 115]) and so $Q^* = K_1 \cup \cdots \cup K_r$, where $K_1, \cdots, K_r$ are centrally

symmetric, and $r \leqq g(n)$. Each pair of points of Q^* can be covered by a translate of S, so in particular each pair of points of K_α can be covered by a translate of S. But one readily sees this implies that each K_α can be covered by a translate of S (place the center of S on the center of K_α). Thus Q^*, hence Q, can be covered by $\leqq g(n)$ translates of S, and the theorem is proved.

Since any plane compact convex set is the union of 3 centrally symmetric sets (see [2] for a proof), we obtain from Theorem 6 the result of Grünbaum that $h(S) \leqq 3$ for centrally symmetric $S \subset R^2$.

Very little is known about $g(K)$ in general. For example, it would be of interest to have an answer to the following question.

Question 5. What is the precise value of $g(T^n)$, where T^n is an n-dimensional simplex?

It is shown in [2] that $g(T^2)=3$; in other words, any triangle is expressible as the union of 3, and no fewer, centrally symmetric convex sets. It is also shown there that any tetrahedron in R^3 can be represented as the union of 8 centrally symmetric convex sets, but not as the union of 6 or fewer such sets; hence $g(T^3)$ is either 7 or 8.

Estimates for $h(B^n)$, where B^n is a closed ball in R^n, are of particular interest because of the relationship to the following famous unsettled (for $n \geqq 4$) problem of K. Borsuk.

Borsuk's Problem. Is every subset of diameter 1 in R^n expressible as the union of $n+1$ sets, each of diameter <1?

We will resist the temptation of digressing into the multitude of covering problems which have their origin in Borsuk's Problem (the reader will find a very complete treatment in [10]). In case B^n is the closed ball of diameter 1 in R^n, $h(B^n)$ is the least number r such that each set of diameter $\leqq 1$ can be covered by r closed balls of diameter 1. Consideration of a few cases in lower dimensions might convince the reader that $h(B^n) \leqq n+1$; however, this is false (at least for large values of n). Indeed, L. Danzer [4] has shown that $h(B^n) > (1.003)^n$, $n \geqq 2$. This means that in high dimensions there exist sets of diameter 1 which require a relatively enormous number (i.e., more than one might suspect) of balls of diameter 1 to cover them. The only precise information we presently have along these lines is that cited in Section 6, that $h(B^2)=3$, and that of P. Katzarowa-Karanowa [12], that $h(B^3)=4$.

This is an expanded version of an address given before the Mathematical Association of America at the San Francisco Meeting, on January 27, 1968.

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INJECTIVE ENVELOPES

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1. Introduction. In this paper we present some conditions which are sufficient to ensure the existence of injective envelopes in a category, and we mention some categories where injective envelopes are known to exist and where these conditions are satisfied. Thus several existing theories are brought under a common categorical roof. Injective envelopes were first introduced [**1**, **5**] in connection with the category $\mathfrak{AG}$ of abelian groups. The result of the existence of injective envelopes was later generalized to certain abelian categories known as Grothendieck categories with a generator; an exposition of this can be found in [**9**], pp. 86–90 or in [**6**], pp. 126–131. Here, after developing the general theory, we look briefly at these cases and more leisurely at the category $\mathfrak{BO}$ of Boolean algebras and unit preserving homomorphisms where the injective envelope of an algebra turns out to be its McNeille completion. As a matter of fact our methods give a new proof of the existence of a completion of a Boolean algebra. We also apply our result to the category $\mathcal{E}_S$ of sets acted on by a semigroup S where it yields a proof of the existence of injective envelopes more direct than that first given by P. Berthiaume [**2**].

Our exposition is for pedestrians. We have tried to make the paper as self-contained as possible by recalling the elementary definitions of category theory on which our work rests and by giving precise references for prerequisites when needed. The reader interested in the application to the McNeille completion

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A MODEL FOR THE FINITE PROJECTIVE SPACES WITH THREE POINTS ON EVERY LINE

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Introduction. Models of finite projective geometries, in particular the 7 point projective plane, are used extensively in discussing and proving the consistency of the axioms of projective geometry. They are also used to impress upon students the advantages in using undefined terms—that is the advantages in employing the abstract method. This is accomplished by constructing several isomorphic models of the geometry, and observing that, while they look different, they all have the same properties.

Most of these models are constructed for 2 and 3 dimensional spaces; while it is not difficult to generalize the construction for spaces of dimension higher than 3, in most cases the actual construction becomes rather cumbersome. Furthermore, in every case it is more or less assumed that the generalized construction leads to a model of n-dimensional space. While it is probably not difficult to prove these facts, the author has not been able to find the proofs in readily available sources.

The purpose of this paper is to introduce a new model which can be easily constructed for any n-dimensional projective space having 3 points on every line, and to prove that such a construction produces the desired model. While constructing the model, well-known properties of certain groups also come up in a natural way.

The 2-dimensional model. Consider a group of order 8 with the property that each element except the identity is of order 2. It is well known that such a group exists, is unique up to isomorphism and is necessarily Abelian. (See Table 1. For more details see [1], pp. 45–55.) Since this group is a natural generalization of "the four group" ([1], pp. 48–49) we shall call it "the eight group."

We exclude the identity element and call the remaining seven elements of the group *points* of our plane. Then a *line* is defined to be the set K of the elements of any subgroup of order 4, excluding the identity element. Finally, a point p is *on a line* K if the element p of the original "eight group" is an element of K.

By inspecting the following table, the multiplication table of "the eight

group," it is easy to see that in this model there exist exactly seven points a, b, ab, c, ac, bc, abc and exactly seven lines $\{a, b, ab\}$, $\{a, c, ac\}$, $\{b, c, bc\}$, $\{a, bc, abc\}$, $\{b, ac, abc\}$, $\{c, ab, abc\}$, $\{ab, bc, ac\}$, each line corresponding to one of the seven subgroups of order 4 of the given "eight group."

TABLE 1

$\times$	e	a	b	ab	c	ac	bc	abc
e	e	a	b	ab	c	ac	bc	abc
a	a	e	ab	b	ac	c	abc	bc
b	b	ab	e	a	bc	abc	c	ac
ab	ab	b	a	e	abc	bc	ac	c
c	c	ac	bc	abc	e	a	b	ab
ac	ac	c	abc	bc	a	e	ab	b
bc	bc	abc	c	ac	b	ab	e	a
abc	abc	bc	ac	c	ab	b	a	e

Now it can be verified that the following three properties, the postulates for plane projective geometry, hold for our model. Therefore, we have constructed a model of a projective plane:

(1) *Two distinct points are contained in exactly one line;*
(2) *Two distinct lines have in common exactly one point;*
(3) *There exist four distinct points (say a, b, c, abc) no three of which are on the same line.*

The n-dimensional model. In this section we shall introduce, and briefly discuss, "elementary Abelian groups," which are obvious generalizations of "the four group." Then, using those groups, we shall construct a model of projective n-space and show that this model satisfies the postulates of such spaces.

The following well-known theorem will be stated without proof. (For a proof of this theorem see [**1**], page 47–48.)

THEOREM 1. *If each element of a group G, other than the identity, is of order 2, then G is Abelian and isomorphic with*

$$C_2 \times C_2 \times \cdots \times C_2,$$

where C_2 denotes the cyclic group of order 2 and $\times$ the direct product. Furthermore, if the group is isomorphic to the direct product of k groups of type C_2, the order of the group is 2^k and it is generated by any independent subset of k elements, each element being of order 2. (A set of elements of a group is said to be independent if no element of the set is the product of two or more distinct elements of that set.) Such a group is called an elementary Abelian group.

It is easy to see that "the eight group" discussed in the previous section is of this type. It is of order 2^3 and it can be generated by independent generators a, b, c or a, b, ac, and so on.

As an immediate consequence of Theorem 1, we observe that an elementary Abelian group of order 2^k exists for each positive integer k. Let G_k stand for such a group. Since G_k is isomorphic to $C_2 \times C_2 \times \cdots \times C_2$, it follows immediately that the only subgroups of this group, besides the identity itself, are elementary Abelian groups G_l, where $1 \leqq l \leqq k$. Furthermore, for any integer l between 1 and k such a subgroup exists.

Next let us adopt a set of postulates for the n-dimensional projective space. Many such sets of postulates have been proposed and the one adopted here is essentially the set given in [2, page 199].

First we postulate the existence of certain objects (undefined terms) called *t-planes*, t being any integer from 0 to n. In particular, 0-plane, 1-plane and 2-plane are sometimes called *point*, *line*, and *plane*, respectively. We also postulate a binary relation which may exist between these objects.

Let S_t stand for a t-plane, t being any nonnegative integer. If the binary relation holds between S_p and S_q, we say that S_p *lies in* S_q, and write $S_p \subseteq S_q$.

This relation is also written as $S_q \supseteq S_p$, and is then translated as "S_q contains S_p."

Next we adopt the following postulates:

I. *For each S_p, $S_p \subseteq S_p$.*
II. *If $S_p \subseteq S_q$ and $S_q \subseteq S_p$, then $S_p = S_q$.*
III. *If $S_p \subseteq S_q$ and $S_q \subseteq S_r$, then $S_p \subseteq S_r$.*

Before adopting any other postulates, we need the following definition:

DEFINITION. A set of $p+1$ points (0-planes) $P_0, P_1, \cdots, P_p$ is *linearly dependent* if there is an S_q (q-plane) containing them, where $q<p$. Points which are not linearly dependent are said to be *linearly independent.*

The following additional postulates characterize the incidence properties of projective space of dimension n.

IV. *Each line (1-plane) contains at least three distinct points (0-planes).*
V. *Given any $p+1$ linearly independent points, there is at least one p-plane which contains them.*
VI. *Each p-plane contains at least one set of $p+1$ linearly independent points.*
VII. *If $P_0, P_1, \cdots, P_p$ are $p+1$ linearly independent points which lie in S_q, each S_p containing them is contained in S_q.*
VIII. *If $P_0, \cdots, P_p$ are $p+1$ linearly independent points of an S_p, and $Q_0, \cdots, Q_q$ are $q+1$ linearly independent points of an S_q, and if the $p+q+2$ points $P_0, \cdots, P_p, Q_0, \cdots, Q_q$ are linearly dependent, there exists at least one point R which lies in both S_p and S_q.*
IX. *There exists an integer $n \geqq 1$ such that there is at least one set of $n+1$ linearly independent points, but any set of m points, where $m>n+1$, is linearly dependent.*

Now we shall construct a model which will satisfy these postulates.

Consider the group G_{n+1}, where n is a fixed positive integer and let G_{k+1} represent a subgroup of G_{n+1}. Consider the set of all elements of this subgroup,

each element of the group G_{n+1} is of order 2, it follows that $P_{\alpha_0} \neq Q_{\beta_0} \cdots Q_{\beta_j} P_{\alpha_1} \cdots P_{\alpha_i}$, where $0 \leqq \alpha_0 \leqq p$.

Similarly, $Q_{\beta_0} \neq P_{\alpha_0} \cdots P_{\alpha_i} Q_{\beta_1} \cdots Q_{\beta_j}$, where $0 \leqq \beta_0 \leqq q$. Therefore $P_0, \cdots, P_p, Q_0, \cdots, Q_q$ must be independent elements of G_{n+1}, and hence they generate a subgroup G_{p+q+2} of G_{n+1}. It follows that these $p+q+2$ points are linearly independent, and this contradicts the fact that these points were given linearly dependent.

This completes the proof, and shows that we have actually constructed a model of projective n-space.

REMARKS: We can interpret in the model certain theorems of projective geometry and obtain immediate proofs of certain properties of the group G_{n+1}. For example, any such group has exactly $2^{n+1}-1$ subgroups of order 2^n. This follows from the fact that any n-space of the type considered in this paper contains $2^{n+1}-1$ $(n-1)$-planes.

Another result of this type is the following:

In G_{n+1} any subgroup of order 2^{n-k+1} has at least one element distinct from the identity in common with any subgroup of order 2^{k+1}. This follows from the fact that in a projective n-space, any $(n-k)$-plane and any k-plane always intersect.

The reader could, of course, discover many more results of this type.

Acknowledgment. I am grateful to the referee for his helpful suggestions in improving the earlier version of this paper.

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THE HOMOTOPY THEOREMS OF FUNCTION THEORY

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1. Introduction. One of the most important topics of complex analysis pertains to the question whether integrals of an analytic function along two different curves, C_1 and C_2, are necessarily equal. Roughly speaking, it turns out that the integrals are equal if C_1 can be continuously deformed into C_2 in such a way that the function remains analytic on all the intermediate curves. When formu-

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lated with precision, this result leads to the homotopy form of Cauchy's Theorem.

Another topic of central importance pertains to the question whether analytic continuation along two curves C_1 and C_2 necessarily leads to the same terminal branch. Roughly speaking, this does happen if C_1 can be continuously deformed into C_2 in such a way that the continuation is possible on all intermediate curves. When formulated with precision, the result leads to the homotopy form of the monodromy theorem.

In this note we prove both theorems by techniques which show a *remarkable similarity of logical structure* in the two cases. To emphasize this similarity is our main objective. As pointed out by the referee, there is a kind of poetic justice in this objective, since it was Cauchy who first formulated the monodromy theorem, and the term *monodromy* is due to him.

In discussing curvilinear integrals there is always the question: What class of curves shall be considered—smooth, piecewise differentiable, rectifiable, or what? As a matter of fact, for analytic functions *no differentiability condition* is needed; it is quite sufficient to suppose that the path of integration is continuous. (This fact has been known to analysts for years, though seldom mentioned in the classroom.) Now continuity is a topological concept and hence, continuous curves are far more appropriate than smooth curves for development of a topic in homotopy theory. By means of simple calculations involving the "distance" between two curves we establish our theorems for the continuous case with no more trouble than usually encountered in the smooth case. Presentation of this technique is a minor objective here.

A second minor objective is to show how the monodromy theorem can be obtained with *no reference to the idea of Riemann surface.* To accomplish this, the functions providing the analytic continuation are indexed by the parameter t of the curve rather than by the point $z(t)$. That a development without Riemann surfaces is possible is also well known to analysts, but is perhaps not as often emphasized as it could be.

Since the results are to be proved in all rigor, without leaning on geometric intuition, it might be thought that a considerable background would be required. Surprisingly, this is not the case. The treatment given here presupposes Cauchy's Theorem for piecewise continuously differentiable paths in a disk. We also assume the fact that if two functions are analytic in open overlapping disks and agree on the intersection, then each provides the unique continuation of the other to the union. Finally, we assume a few elementary facts about continuous functions and closed point sets in the plane. We have restricted the goals of this paper for the express purpose of restricting the necessary background, as outlined here. Actually, the methods carry over to general situations in which one integrates locally exact differentials, or continues a function element on an abstract Riemann surface.

2. Notation. The letter C denotes a curve $z=\zeta(t)$ with ζ continuous, and D

is used if ζ is piecewise continuously differentiable. Unless otherwise stated, the range of t is the interval $0 \leqq t \leqq 1$. Except in Theorem 2, all curves C, C_p, C_q, D_p, D_q, and so on, begin at one and the same point α, and end at one and the same point β. That is, $\zeta(0) = \alpha$ and $\zeta(1) = \beta$.

If C_p and C_q are curves given by $z = \zeta_p(t)$ and $z = \zeta_q(t)$ we define their distance by

$$|C_p - C_q| = \sup_{0 \leqq t \leqq 1} |\zeta_p(t) - \zeta_q(t)|.$$

According to this definition, the curve C is distinguished from the point set $\tilde{C}$ which is the image of the interval $0 \leqq t \leqq 1$ under ζ. If $f(z)$ is analytic in an open set G containing $\tilde{C}$ we define $\rho(C, f)$ to be the distance from $\tilde{C}$ to the boundary of G. (Roughly speaking, $\rho(C, f)$ measures the distance from $\tilde{C}$ to the nearest singularity. The more precise formulation, involving G, is needed to avoid trouble from other branches of f.) If $\rho(C, f) > \delta$ then $f(z)$ is analytic in every disk of radius δ centered at a point $z = \zeta(t)$ of C. Hence, we call $\rho(C, f)$ the *radius of regularity* for f on C. The case in which $\rho(C, f) = \infty$ is trivial and is disregarded.

If C is the curve $z = \zeta(t)$ the *modulus of continuity* of C (or of ζ) satisfies

$$\omega(\eta) = \sup |\zeta(t_1) - \zeta(t_0)| \qquad \text{for } |t_1 - t_0| < \eta.$$

We agree to define $\omega(-\eta) = \omega(\eta)$ for $\eta \geqq 0$, so that we can consider $\omega(t_0 - t_1)$ without introducing absolute values. If instead of C we have another curve C_p or C_q the corresponding modulus of continuity is denoted by ω_p or ω_q, respectively.

Similar considerations apply to functions of two variables; for example, the modulus of continuity of $F(s, t)$ satisfies

$$\omega(\eta) = \sup |F(s_0, t_0) - F(s_1, t_1)| \qquad \text{for } |s_1 - s_0| + |t_1 - t_0| < \eta.$$

Naturally, the variables are supposed to be in the domain of ζ in the first instance and of F in the second.

If C_p and C_q are the curves $z = F(p, t)$ or $z = F(q, t)$ respectively, then

$$|C_p - C_q| = \sup_{0 \leqq t \leqq 1} |F(p, t) - F(q, t)| \leqq \omega(p - q),$$

where ω is the modulus of continuity for F. This relation is used in the sequel.

3. Integration on nonrectifiable paths. We shall establish:

LEMMA 1. *If* $|D_p - C| < \rho(C, f)$ *and* $|D_q - C| < \rho(C, f)$ *then*

$$\int_{D_p} f(z)dz = \int_{D_q} f(z)dz.$$

For proof let ω_p and ω_q be the moduli of continuity for D_p and D_q, respectively, and choose a positive integer n such that

$$\omega_p(1/n) < \rho(C,f) - |D_p - C|, \qquad \omega_q(1/n) < \rho(C,f) - |D_q - C|.$$

We take successive values $t_0 = 0$, $t_1 = 1/n$, $t_2 = 2/n, \cdots, t_n = n/n$. If D_p is given

by $z=\zeta_p(t)$ we let $D_p(k)$ denote the curve $z=\zeta_p(t)$ when t increases from t_k to t_{k+1}, and similarly for $D_q(k)$. Let $L(k)$ denote the directed straight line joining the point $\zeta_p(t_k)$ to $\zeta_q(t_k)$. Then (in an obvious notation)

$$D_p(k) + L(k+1) - D_q(k) - L(k)$$

is a closed contour. We claim that every point of this contour lies in the disk of radius $\rho(C,f)$ centered at the point $z=\zeta(t_k)$ of C.

Indeed, if $\zeta_p(t)$ is a point of $D_p(k)$ then

$$|\zeta_p(t) - \zeta(t_k)| \leqq |\zeta_p(t) - \zeta_p(t_k)| + |\zeta_p(t_k) - \zeta(t_k)| \leqq \omega_p(1/n) + |D_p - C|,$$

and similarly for $D_q(k)$. This shows that the curves $D_p(k)$ and $D_q(k)$ (including their end points) lie in the specified disk. Since the disk is convex, the lines $L(k)$ and $L(k+1)$ joining the end points also lie in the disk.

By Cauchy's Theorem for the disk

$$\int_{D_p(k)} f(z)dz + \int_{L(k+1)} f(z)dz = \int_{D_q(k)} f(z)dz + \int_{L(k)} f(z)dz.$$

If this equation is summed on k from $k=0$ to $k=n-1$, the result is

$$\int_{D_p} f(z)dz + \int_{L_1} f(z)dz = \int_{D_q} f(z)dz + \int_{L_0} f(z)dz, \tag{1}$$

where $L_1=L(n)$ is the line joining $\zeta_p(1)$ to $\zeta_q(1)$ and $L_0=L(0)$ is the line joining $\zeta_p(0)$ to $\zeta_q(0)$. Since the end points of D_p and D_q coincide, these lines have zero length, and Lemma 1 follows.

If $f(z)$ is analytic at every point of C, so that $\rho(C,f)>0$, we can always find a curve D such that $|D-C|<\rho(C,f)$; for example, D could be a polygonal line. We then *define* the integral over C to be the integral over D. By Lemma 1 the result is independent of D, a fact which is formulated for future reference as follows:

LEMMA 2. *If D is any curve such that $|D-C|<\rho(C,f)$ then*

$$\int_C f(z)dz = \int_D f(z)dz.$$

The main result of this section is as follows:

LEMMA 3. *Let $f(z)$ be analytic on curves C_p and C_q and suppose the corresponding radii of regularity satisfy*

$$\rho(C_p,f) > |C_p - C_q| \quad \text{or} \quad \rho(C_q,f) > |C_p - C_q|.$$

Then

$$|\rho(C_p,f) - \rho(C_q,f)| \leqq |C_p - C_q| \quad \text{and} \quad \int_{C_p} f(z)dz = \int_{C_q} f(z)dz.$$

Denoting the equation of C_p by $z=\zeta_p(t)$, let t be a value such that the distance from the point $\zeta_p(t)$ to some singularity γ of f is $\rho(C_p, f)$. Then, at this t,

$$|\zeta_q(t) - \gamma| \leqq |\zeta_q(t) - \zeta_p(t)| + |\zeta_p(t) - \gamma| \leqq |C_p - C_q| + \rho(C_p,f).$$

This shows that

$$\rho(C_q,f) \leqq \rho(C_p,f) + |C_p - C_q|.$$

By symmetry the same holds with p and q interchanged, and we obtain the first assertion in Lemma 3. It will be observed that the hypothesis was not used in this proof.

For the second assertion suppose $\rho(C_p, f) > |C_p - C_q|$, as we may by symmetry. The result just obtained gives

$$\rho(C_q,f) \geqq \rho(C_p,f) - |C_p - C_q| > 0.$$

Hence we can pick curves D_p and D_q such that

$$|D_p - C_p| < \rho(C_p, f), \ |D_q - C_q| < \rho(C_q,f), \ |D_q - C_q| < \rho(C_p,f) - |C_p - C_q|.$$

The first two relations in conjunction with Lemma 2 give

$$\int_{C_p} f(z)dz = \int_{D_p} f(z)dz \quad \text{and} \quad \int_{C_q} f(z)dz = \int_{D_q} f(z)dz.$$

On the other hand the third relation gives

$$|D_q - C_p| \leqq |D_q - C_q| + |C_q - C_p| < \rho(C_p,f).$$

Since also $|D_p - C_p| < \rho(C_p, f)$ Lemma 1 gives

$$\int_{D_p} f(z)dz = \int_{D_q} f(z)dz.$$

This completes the proof of Lemma 3.

4. Cauchy's Theorem. We can now obtain the homotopy version of Cauchy's Theorem. Two curves C_0 and C_1 are said to be *homotopic with fixed end points* if there exists a function $F(s, t)$ of the two real variables s and t with the following three properties:

(i) F is continuous in the unit square $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$.

(ii) C_0 is $z = F(0, t)$ and C_1 is $z = F(1, t)$ $(0 \leqq t \leqq 1)$.

(iii) $F(s, 0) = \alpha$ and $F(s, 1) = \beta$ are constant, $0 \leqq s \leqq 1$.

In other words, C_0 and C_1 are homotopic if there is a family of curves C_s given by $z = F(s, t)$ such that C_s reduces to C_0 or C_1 when $s = 0$ or $s = 1$, respectively, and such that $F(s, t)$ is continuous in the unit square. The curves C_s form a *homotopy family* for C_0 and C_1.

THEOREM 1. *Let $f(z)$ be analytic on curves C_0 and C_1. Suppose further that $f(z)$ is analytic on each curve C_s of a homotopy family for C_0 and C_1. Then*

$$\int_{C_0} f(z)dz = \int_{C_1} f(z)dz.$$

For proof let I denote the image of the unit square $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$ under F and let $\rho(I, f)$ denote the distance from the point-set I to the nearest singularity of f. Since both sets are closed, and since f is analytic at each point of I, we have $\rho(I, f) > 0$. Let n be a positive integer such that $\omega(1/n) < \rho(I, f)$, where ω is the modulus of continuity of F, and choose values

$$s_0 = 0, \qquad s_1 = 1/n, \qquad s_2 = 2/n, \cdots, s_n = n/n.$$

Let $p = s_k$ and $q = s_{k+1}$ be two successive values of s, and denote the corresponding curves

$$z = F(p, t), \qquad z = F(q, t) \qquad (0 \leqq t \leqq 1)$$

by C_p and C_q, respectively. Then

$$|C_p - C_q| \leqq \omega(1/n) < \rho(I, f) \leqq \rho(C_p, f)$$

and, by Lemma 3,

$$\int_{C_p} f(z)dz = \int_{C_q} f(z)dz.$$

Repeated use of this equation for $k = 0, 1, 2, \cdots, n-1$ gives Theorem 1.

So far the condition that all curves have the same end points has played an apparently essential role. There is another form of Theorem 1, however, in which this condition is superfluous. Two curves C_0 and C_1 are said to be *homotopic as closed curves* if there exists a function $F(s, t)$ with the above properties (i) and (ii) and, instead of (iii), with property:

(iii′) $F(s, 0) = F(s, 1)$, $0 \leqq s \leqq 1$.

They are homotopic as closed curves in a region R if the image of the unit square under F is wholly contained in R.

THEOREM 2. *Let C_0 and C_1 be homotopic as closed curves in a region R in which f is analytic. Then*

$$\int_{C_0} f(z)dz = \int_{C_1} f(z)dz.$$

The proof follows by re-examining the discussion leading to Lemma 1. In the fixed end-point case the integrals involving L_0 and L_1 in (1) drop out because each of them is 0. In the closed curve case being considered now, these integrals drop out because L_0 and L_1 coincide, so that

$$\int_{L_1} f(z)dz = \int_{L_0} f(z)dz.$$

The resulting form of Lemma 1 leads to corresponding versions of Lemmas 2 and 3. If these are used as in the proof of Theorem 1, we get Theorem 2.

5. Analytic continuation. From now on our convention about initial and end points is again in force; that is, all curves have one and the same initial point and one and the same end point.

Let C be a given curve $z=\zeta(t)$. For each value of t we consider a function $f(t, z)$ which is analytic at the point $z=\zeta(t)$. This means that it has a power-series expansion with positive radius $r(t)$, the distance from $\zeta(t)$ to the nearest singularity of $f(t, z)$. For the ends in view the case in which $r(t)=\infty$ for some t is trivial, and is disregarded.

Let $z_0=\zeta(t_0)$ and $z_1=\zeta(t_1)$ be points of C. It is said that the point z_1 *belongs to the disk* $|z-z_0|<r_0$ if $\omega(t_1-t_0)<r_0$, where ω is the modulus of continuity for C. Thus, not only $\zeta(t_1)$ but all the points $\zeta(t)$ for t between t_0 and t_1 must lie in the disk. It is true that the curve may return to the disk at some other value t_2, but this need not make $\zeta(t_2)$ "belong to the disk" in the sense of our definition.

The function $f(t, z)$ of the foregoing discussion is said to give an *analytic continuation* of $f(0, z)$ along C when the following two conditions are fulfilled:

(i) $r(t)>0$ for each t, $0\leqq t\leqq 1$.

(ii) If z_1 belongs to the disk $|z-z_0|<r(t_0)$ the values of $f(t_0, z)$ and $f(t_1, z)$ agree in the common part of the two disks $|z-z_0|<r(t_0)$ and $|z-z_1|<r(t_1)$.

We shall now establish:

LEMMA 4. *If $f(t, z)$ gives an analytic continuation of $f(0, z)$ along C then the radius of convergence, $r(t)$, admits the same modulus of continuity as does the curve C.*

Indeed, let t_0 be a value where the continuity is to be verified and choose t_1 so close to t_0 that $\omega(t_1-t_0)<r(t_0)$. The condition (ii) of the definition shows then that $f(t_1, z)$ is analytic at least in the disk $|z-z_0|<r(t_0)$ and hence

$$r(t_1) \geqq r(t_0) - |z_1 - z_0| \geqq r(t_0) - \omega(t_1 - t_0).$$

Although we have assumed $\omega(t_1-t_0)<r(t_0)$, the conclusion actually holds without this restriction, since $r(t_1)>0$. By symmetry the same conclusion holds with t_1 and t_0 interchanged, and hence

$$|r(t_1) - r(t_0)| \leqq \omega(t_1 - t_0). \tag{2}$$

This is the desired result.

Since $r(t)$ is continuous and positive it has a positive lower bound, which we call the *radius of regularity* for f on C and write in the form $\rho(C, f)$. We shall establish:

LEMMA 5. *Let $f_p(t, z)$ and $f_q(t, z)$ give analytic continuations of one and the same function $f_p(0, z)\equiv f_q(0, z)$ along curves C_p and C_q. Suppose the corresponding radii of regularity satisfy*

$$\rho(C_p, f_p) > |C_p - C_q| \quad or \quad \rho(C_q, f_q) > |C_p - C_q|.$$

Then

$$|\rho(C_p,f_p) - \rho(C_q,f_q)| \leqq |C_p - C_q| \quad \textit{and} \quad f_p(1,z) \equiv f_q(1,z).$$

The function $f(0, z)$ in the theory of analytic continuation is sometimes called the initial branch, and $f(1, z)$ is the terminal branch. Lemma 5 asserts that two continuations having the same initial branch will lead to the same terminal branch, provided the curves C_p and C_q are near one another in the sense specified by the hypothesis.

For proof assume that $\rho(C_p, f_p) > |C_p - C_q|$. In this case every point $z = \zeta_q(t)$ is interior to the disk centered at $z = \zeta_p(t)$ in which $f_p(z, t)$ is analytic, and hence there are two candidates for the analytic function on C_q. One is the specified continuation $f_q(z, t)$ and the other is $f_p(z, t)$. We call t a *good value* if $f_q(t, z) = f_p(t, z)$ in the common region of convergence. Let a positive integer n be chosen so that

$$\omega_p(1/n) < \rho(C_p,f_p) - |C_p - C_q| \quad \text{and} \quad \omega_q(1/n) < \rho(C_q,f_q).$$

We shall show that if t_0 is good, and $|t_0 - t_1| \leqq 1/n$, then t_1 is also good.

The inequalities $\omega_q(t_0 - t_1) \leqq \omega_q(1/n) < \rho(C_q, f_q) \leqq r_q(t_0)$ show that the point $\zeta_q(t_1)$ belongs to the disk for $f_q(t_0, z)$ centered at $\zeta_q(t_0)$. The definition of analytic continuation thus gives

$$f_q(t_1, z) = f_q(t_0, z)$$

in the common part of the two disks

$$|z - \zeta_q(t_1)| < r_q(t_1), \qquad |z - \zeta_q(t_0)| < r_q(t_0). \tag{3}$$

In just the same way $\omega_p(t_1 - t_0) < r_0(t_0)$ and hence $f_p(t_1, z) = f_p(t_0, z)$ in the common part of the two disks

$$|z - \zeta_p(t_1)| < r_p(t_1), \qquad |z - \zeta_p(t_0)| < r_p(t_0). \tag{4}$$

Finally, the fact that t_0 is good gives $f_p(t_0, z) = f_q(t_0, z)$ in the common part of the two disks

$$|z - \zeta_p(t_0)| < r_p(t_0), \qquad |z - \zeta_q(t_0)| < r_q(t_0).$$

We assert that the point $\zeta_q(t_1)$ is interior to all four disks mentioned above. Indeed, it is interior to the disks (3) as has already been seen, and it belongs to the first disk (4) because

$$|\zeta_q(t_1) - \zeta_p(t_1)| \leqq |C_p - C_q| < \rho(C_p,f_p) \leqq r_p(t_1).$$

It also belongs to the second disk (4) because

$$\begin{aligned} r_p(t_0) &\geqq \rho(C_p,f_p) > \omega_p(1/n) + |C_p - C_q| \\ &\geqq \omega_p(t_0 - t_1) + |C_p - C_q| \\ &\geqq |\zeta_p(t_0) - \zeta_p(t_1)| + |\zeta_p(t_1) - \zeta_q(t_1)| \\ &\geqq |\zeta_p(t_0) - \zeta_q(t_1)|. \end{aligned}$$

Lemma 5 shows that C_p and C_q lead to the same terminal branch; that is,

$$f_q(1, z) = f_p(1, z).$$

Repeated use of this equation for $k=0, 1, 2, \cdots, n-1$ gives Theorem 3.

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ON THE INDEPENDENCE OF SET THEORETICAL AXIOMS

ALEXANDER ABIAN, Iowa State University

In this paper we prove the consistency and the independence of four axioms of *Extensionality*, *Power-set*, *Sum-set* and *Choice* of the Zermelo-Fraenkel Theory of Sets [**1**]. The proofs are given in a context which requires almost no knowledge of formal Theory of Sets. This circumstance is a main reason for the article since the results mentioned above are well known. Also, it is believed that the present paper would help the reader to develop a set-theoretical abstract approach which is indispensable for further discussions concerning the six axioms of the Zermelo-Fraenkel Theory of Sets (the additional two axioms are: *Infinity* and *Replacement*, which is in fact an axiom scheme).

In what follows the *equality* sign "$=$" is not borrowed from logic and is introduced by its usual set-theoretical definition (i.e., $x=y$ if and only if each element of x is an element of y and vice versa). The set-theoretical indistinguishability between equal sets is secured by the axiom of Extensionality, which states that *equal sets are elements of the same sets*. The Power-set axiom states that *each set s has a power-set* (i.e., a set whose elements are exactly the subsets of s). If s has a power-set and if the axiom of Extensionality is valid, then the power-set of s is denoted by $P(s)$. The axiom of Sum-set states that *each set s has a sum-set* (i.e., a set whose elements are exactly the elements of the elements of s). If s has a sum-set and if the axiom of Extensionality is valid, then the sum-set of s is denoted by $\bigcup s$. If a set w has a unique element in common with each nonempty element of a nonempty set s, and if w has no other elements, then w is called a *selection-set* of s. A selection-set of an empty set is defined to be an empty set. With this in mind, the axiom of Choice states that *each disjointed set has a selection-set*, where a set is called *disjointed* if no two distinct elements

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of it have an element in common, and if it is not a set whose unique element is an empty set.

The independence of the axioms is proved below by means of one finite and three denumerable models. It has been shown [2] that in a finite model the axiom of Power-set alone implies the axioms of Extensionality, Sum-set and Choice.

When no confusion is likely to arise, we shall write

$$a = \{ \cdots \}, \tag{1}$$

where inside the braces we insert exactly all those sets which are elements of a.

We now prove:

THEOREM. *The axioms of Extensionality, Power-set, Sum-set, and Choice form a consistent and independent system of axioms.*

Proof. To prove the consistency, let us consider a model whose domain consists of a single set a, where

$$a \in a. \tag{2}$$

Clearly, in this model the axiom of Extensionality is valid. Also, in this model $P(a) = \bigcup a = a$. Moreover, a is disjointed and is its own selection-set. Thus the remaining three axioms are also valid.

To prove that the axiom of Power-set is independent of the remaining three axioms, it is enough to consider a model whose domain consists of a single set s, where

$$s \notin s, \tag{3}$$

i.e., s is an empty set. Clearly, in this model the axiom of Extensionality is valid. However, since in this model there is no set whose element is s, the axiom of Power-set is not valid. On the other hand, since in this model $\bigcup s = s$, and since s is its own selection-set, the axiom of Sum-set as well as the axiom of Choice is valid.

To prove that the axiom of Extensionality is independent of the remaining three axioms, let us consider a model whose domain consists of the sets a_1, a_2, $a_3, \cdots$, and which is described by (4), where the notation introduced in (1) is used:

$$\begin{aligned}
a_1 &= \{a_1\} \\
a_2 &= \{a_1\} \\
a_3 &= \{a_1, a_2\} \\
a_4 &= \{a_1, a_2, a_3\} \\
&\cdots\cdots\cdots \\
a_n &= \{a_1, a_2, a_3, \cdots, a_{n-1}\}, \quad \text{for } n > 1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned} \tag{4}$$

We see that in the above:

(i) $a_1 = a_2$ and $a_1 \in a_1$, however, $(a_2 \notin a_1)$. Thus, in (4) the axiom of Extensionality is not valid.

(ii) $\bigcup a_1 = \bigcup a_2 = \bigcup a_3 = a_1$ and $\bigcup a_n = a_{n-1}$, for $n > 3$. Thus, in (4) the axiom of Sum-set is valid.

(iii) $P(a_1) = a_3$ and $P(a_n) = a_{n+1}$, for $n > 1$. Thus in (4) the axiom of Power-set is valid.

(iv) In (4) the set a_1 is a selection-set of a_1 and of a_2, which are the only disjointed sets of the model. Thus, in (4) the axiom of Choice is valid.

To prove that the axiom of Sum-set is independent of the remaining three axioms, let us consider a model whose domain consists of the sets $a_1, b_1, a_2, b_2, a_3, b_3, \cdots$ and which is described by (5):

$$
\begin{array}{lll}
 & a_1 = \{a_1\} & b_1 = \{a_1, a_2\} \\
 & a_2 = \{a_1, b_1, b_2\} & b_2 = \{a_1, b_1,\} \\
 & a_3 = \{a_1, a_2, b_2, b_3\} & b_3 = \{a_1, b_2\} \\
(5) & a_4 = \{a_1, a_3, b_1, b_3, b_4\} & \cdots\cdots\cdots \\
 & a_5 = \{a_1, a_4, b_2, b_4, b_5\} & b_n = \{a_1, b_{n-1}\}, \quad for\ n > 1 \\
 & \cdots\cdots\cdots & \cdots\cdots\cdots \\
 & a_n = \{a_1, a_{n-1}, b_{n-3}, b_{n-1}, b_n\}, \quad for\ n > 3 & \\
 & \cdots\cdots\cdots &
\end{array}
$$

Now we see that in the above:

(i) No two differently lettered sets are equal. Thus, in (5) the axiom of Extensionality is valid.

(ii) There is no set whose elements are a_1, a_2, and b_1. Therefore, there is no sum-set of the set a_2. Thus, in (5) the axiom of Sum-set is not valid.

(iii) $P(a_1) = a_1$ and $P(a_n) = a_{n+1}$, for $n > 1$. Also, $P(b_n) = b_{n+1}$, for $n \geqq 1$. Thus, in (5) the axiom of Power-set is valid.

(iv) In (5) the set a_1 is a selection-set of a_1 which is the only disjointed set of the model. Thus, in (5) the axiom of Choice is valid.

To prove that the axiom of Choice is independent of the remaining three axioms, it is enough to consider a model whose domain consists of the sets $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, \cdots$ and which is described by (6):

$$
\begin{array}{lll}
 & a_1 = \{a_2, b_1\} & b_1 = \{c_2\} \\
 & a_2 = \{a_1, b_2\} & b_2 = \{b_1\} \\
(6) & a_3 = \{a_2, b_3\} & b_3 = \{b_2\} \\
 & a_4 = \{a_3, b_4\} & \cdots\cdots\cdots \\
 & \cdots\cdots\cdots & b_n = \{b_{n-1}\}, \quad \text{for } n > 1 \\
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 & \cdots\cdots\cdots & \cdots\cdots\cdots
\end{array}
$$

MATHEMATICAL NOTES

EDITED BY DAVID DRASIN

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

AND WHAT IS YOUR ERDÖS NUMBER?

CASPER GOFFMAN, Purdue University

The great mathematician Paul Erdös has written joint papers with many mathematicians. This fact may lend some interest to the notion of Erdös number which we are about to describe.

Let A and B be mathematicians, and let A_i, $i=0, 1, \cdots, n$, be mathematicians with $A_0=A$, $A_n=B$, where A_i has written at least one joint paper with A_{i+1}, $i=0, \cdots, n-1$. Then $A_0, A_1, \cdots, A_n$ is called a chain of length n joining A to B. The A-number of B, $\nu(A; B)$, is the shortest length of all chains joining A to B. If there are no chains joining A to B, then $\nu(A; B)=+\infty$. Moreover, $\nu(A; A)=0$. Then $\nu(A; B)=\nu(B; A)$ and $\nu(A; B)+\nu(B; C)\geqq\nu(A; C)$.

For the special case $A=$ Erdös, we obtain the function $\nu(\text{Erdös}; \cdot)$ whose domain is the set of all mathematicians.

I was told several years ago that my Erdös number was 7. It has recently been lowered to 3. Last year I saw Erdös in London and was surprised to learn that he did not know that the function $\nu(\text{Erdös}; \cdot)$ was being considered. When I told him the good news that my Erdös number had just been lowered, he expressed regret that he had to leave London the same day. Otherwise, an ultimate lowering might have been accomplished.

A GENERAL CHAINING LEMMA

H. S. BEAR, New Mexico State University

Our purpose here is to develop a setting in which a well-known chaining argument can be used to prove once and for all a standard useable lemma. This lemma has the following consequences:

1. The existence of a maximum pseudo-metric on a set T dominated by a given nonnegative function on $T\times T$. [**2**, p. 48].

2. The existence of a maximum pseudo-metric on a set such that a suitable family of functions on a metric space to the set is distance decreasing. [**3**, p. 462].

3. The existence of a maximum seminorm dominated by a given positive and absolutely homogeneous function on a linear space, [**1**, p. 470].

4. The existence of a maximum measure dominated by a family of nonnegative measures on a given σ-ring. [**4**, p. 427].

Our setting for the lemma is a set X in which abstract "sums" $x_1+\cdots+x_n$, or $x_1+x_2+\cdots$ make sense for *some* finite or infinite sequences. We call these sequences chains. The sum of any chain in X is an element of X. For the purpose

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Our setting for the lemma is a set X in which abstract "sums" $x_1+\cdots+x_n$, or $x_1+x_2+\cdots$ make sense for *some* finite or infinite sequences. We call these sequences chains. The sum of any chain in X is an element of X. For the purpose

of axiomatizing we write $\sigma(x_1, \cdots, x_n)$ or $\sigma(x_1, x_2, \cdots)$ for the sum of a chain.

DEFINITION. *A chained space is a triple $\langle X, S, \sigma\rangle$, where S is a family of finite or infinite sequences in X, and σ is a function on S to X such that:*

I. *For each $x\in X$, $(x)\in S$ and $\sigma(x)=x$.*

II. *If $(x_1, x_2, \cdots)$ is a sequence in S with more than n terms, then $(x_1, \cdots, x_n)\in S$, $(x_{n+1}, \cdots)\in S$, $(\sigma(x_1, \cdots, x_n), \sigma(x_{n+1}, \cdots))\in S$, and $\sigma(\sigma(x_1, \cdots, x_n), \sigma(x_{n+1}, \cdots))=\sigma(x_1, x_2, \cdots)$.*

III. *If $(x_1, \cdots, x_n, \cdots)\in S$ and $x_n=\sigma(y_1, \cdots, y_m)$, then $(x_1, \cdots, x_{n-1}, y_1, \cdots, y_m, x_{n+1}, \cdots)\in S$ and $\sigma(x_1, \cdots, x_{n-1}, y_1, \cdots, y_m, x_{n+1}, \cdots) =\sigma(x_1, \cdots, x_n, \cdots)$.*

The intent of the axioms is clearer in the more usual sum notation. Condition I says that every one-element sequence is a chain with sum equal to the element. Condition II says that if $x_1+x_2+\cdots$ is defined, then so are $x_1, +\cdots+x_n$ and $x_{n+1}+\cdots$ and $(x_1+\cdots+x_n)+(x_{n+1}+\cdots)=x_1+x_2+\cdots$. The third condition allows substitution of any finite chain for its sum in any other chain.

It is easy to deduce all finite associative laws from II. For example, if $x_1+x_2+x_3$ is defined, then $(x_1+x_2)+(x_3)=x_1+x_2+x_3$ and $(x_1)+(x_2+x_3) =x_1+x_2+x_3$.

The following are examples of chained spaces.

(a) X is a group or linear space, S is all finite sequences in X, and σ is the usual sum.

(b) X is all pairs (P, Q) of points in a given set Z, S is all finite sequences of the form $((P_0, P_1), (P_1, P_2), (P_2, P_3), \cdots, (P_{n-1}, P_n))$, and σ of this chain is (P_0, P_n).

(c) X is a σ-ring of subsets of a space, S is all finite or infinite sequences of pairwise disjoint sets in X, and σ is the union.

Example (a) can be modified by assuming that X is a topological group and including in S all infinite sums in the usual sense. Example (b) can be modified by assuming a Hausdorff topology in Z, and including in S all infinite chains of the appropriate form such that P_n converges to some $P\in X$. Then the sum of $((P_0, P_1), (P_1, P_2), \cdots, (P_{n-1}, P_n)\cdots)$ is (P_0, P).

DEFINITION. *A real-valued function q on a chained space $\langle X, S, \sigma\rangle$ is subadditive if and only if $q(\sigma(x, y))\leqq q(x)+q(y)$ whenever $(x, y)\in S$; q is superadditive if the reverse inequality holds.*

It is easy to conclude from II that $q(\sigma(x_1, \cdots, x_n))\leqq q(x_1)+\cdots+q(x_n)$ if $(x_1, \cdots, x_n)\in S$ and q is subadditive.

Now we can state a lemma for which a familiar chaining argument is the proof. We will henceforth write $x_1+\cdots+x_n$ for $\sigma(x_1, \cdots, x_n)$.

LEMMA. *If $\langle X, S, \sigma\rangle$ is a chained space and p is a nonnegative real function on X, then there is a largest subadditive function p^* on X such that $p^*\leqq p$. Explicitly, $p^*(x)=\inf\{p(x_1)+\cdots+p(x_n)\}$ where the infimum is over all finite chains such that $x_1+\cdots+x_n=x$.*

Proof. Since $(x) \in S$ for each $x \in X$, $p^*(x) \leqq p(x)$. If q is any subadditive function dominated by p, and $x_1 + \cdots + x_n = x$, then $q(x) = q(x_1 + \cdots + x_n) \leqq q(x_1) + \cdots + q(x_n) \leqq p(x_1) + \cdots + p(x_n)$. Since the sums on the right approximate $p^*(x)$ arbitrarily closely, $q(x) \leqq p^*(x)$.

To show p^* is subadditive, let (x, y) be a chain. For any $\epsilon > 0$ pick chains such that

$$x = x_1 + \cdots + x_n$$

$$y = y_1 + \cdots + y_m$$

$$p^*(x) + \epsilon > p(x_1) + \cdots + p(x_n)$$

$$p^*(y) + \epsilon > p(y_1) + \cdots + p(y_m).$$

Then by III, $(x_1, \cdots, x_n, y_1, \cdots, y_m) \in S$ and $x + y = (x_1 + \cdots + x_n) + (y_1 + \cdots + y_m) = x_1 + \cdots + x_n + y_1 + \cdots + y_m$. Therefore, $p^*(x+y) \leqq \sum_{j=1}^n p(x_j) + \sum_{j=1}^m p(y_j) < p^*(x) + p^*(y) + 2\epsilon$. Since ϵ is arbitrary, $p^*(x+y) \leqq p^*(x) + p^*(y)$.

COROLLARY 1. *If p is a nonnegative function on $T \times T$ for any set T, then there is a largest semimetric p^* on T such that $p^*(P, Q) \leqq p(P, Q)$.*

Proof. We let $X = T \times T$ and S be all chains $((P_0, P_1), (P_1, P_2), \cdots, (P_{n-1}, P_n))$ as in the example (b), with sum (P_0, P_n). The subadditivity of p^* is exactly the triangle inequality. Symmetry of p^* follows from the fact that any chain from P_0 to P_n can be reversed to give a chain from P_n to P_0.

DEFINITION. *Let F be a family of functions on a set D to a set T. We say F links the points of T if given $x, y \in T$, there are points $x_0 = x, x_1, x_2, \cdots, x_n = y$ of T and functions $f_1, \cdots, f_n$ of F such that $x_0, x_1 \in f_1(D)$, $x_1, x_2 \in f_2(D), \cdots, x_{n-1}, x_n \in f_n(D)$. That is, there is a finite sequence of points from x to y such that any adjacent points are in the range of some $f \in F$.*

COROLLARY 2. *Let F be a family of functions on a metric space (D, δ) to a set T, and assume that F links the points of T. Then there is a largest pseudo-metric on T such that every f in F is distance decreasing.*

Proof. For x, y and T, let $p(x, y) = \sup \{\delta(z, w) : \{x, y\} = f\{z, w\}$ some $f \in F\}$. If there is no $f \in F$ such that $\{x, y\} = f\{z, w\}$, let $p(x, y) = \infty$. Since F links the points of T, the function p^* of Corollary 1 is a (finite) pseudo-metric on T, and the largest $\leqq p$. Since $p^* \leqq p$, every $f \in F$ is distance decreasing with respect to δ and p^*.

Let κ be any pseudo-metric on T such that every $f \in F$ is distance decreasing with respect to δ and κ; i.e., for every $f \in F$, $\kappa(f(z), f(w)) \leqq \delta(z, w)$. Then $\kappa(x, y) \leqq p(x, y)$, since $p(x, y)$ is greater or equal to the supremum of the numbers $\delta(z, w)$ such that $f(z) = x$, $f(w) = y$. Therefore, $\kappa(x, y) \leqq p^*(x, y)$, since p^* is the largest pseudo-metric dominated by p.

COROLLARY 3. *If p is a nonnegative absolutely homogeneous (i.e., $p(ax) = |a| p(x)$*

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AN IRREDUCIBILITY CRITERION FOR POLYNOMIALS OVER THE INTEGERS

W. S. BROWN and R. L. GRAHAM, Bell Telephone Laboratories

1. Introduction. If $P(x)$ is a reducible polynomial of degree $d \geqq 1$ with integer coefficients, we should not expect the sequence

$$\mathcal{S}(P) = (\cdots, P(-1), P(0), P(1), \cdots)$$

to have many noncomposite (that is, prime or unit) elements. By making this idea precise, we shall obtain an irreducibility criterion. A special case of our main result is that if $\mathcal{S}(P)$ contains p primes and u units with $p+2u>d+4$, then P is irreducible.

2. Fatness. Let $P(x)$ be any polynomial of degree $d \geqq 1$ with integer coefficients, and let u be the number of units in $\mathcal{S}(P)$. We define the *fatness* of P to be

$$f(P) = u - d,$$

and we say that P is *fat* if $f(P) > 0$.

If ϵ is a unit (that is, $+1$ or -1), and if $a_1, \cdots, a_d$ are distinct integers, then the polynomial $(x-a_1)\cdots(x-a_d)+\epsilon$ has fatness at least 0. If P is fat, then clearly $\mathcal{S}(P)$ must contain units of both signs.

Note that all polynomials in the set

$$\mathfrak{J}(P) = \{\pm P(\pm x + b)\},$$

where b ranges over the integers and where all possible choices of signs are taken, have the same fatness.

3. Notation. If $P(x)$ is a polynomial, we define

$$\begin{aligned}
d &= d(P) &&= \text{degree of } P\\
p &= p(P) &&= \text{number of primes in } \mathcal{S}(P)\\
u &= u(P) &&= \text{number of units in } \mathcal{S}(P)\\
u_+ &= u_+(P) &&= \text{number of positive units in } \mathcal{S}(P)\\
u_- &= u_-(P) &&= \text{number of negative units in } \mathcal{S}(P)\\
f &= f(P) &&= \text{fatness of } P.
\end{aligned}$$

References

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u_+ &= u_+(P) &&= \text{number of positive units in } \mathcal{S}(P)\\
u_- &= u_-(P) &&= \text{number of negative units in } \mathcal{S}(P)\\
f &= f(P) &&= \text{fatness of } P.
\end{aligned}$$

Thus $u=u_++u_-$, and $f=u-d$.

4. Classification of fat polynomials.

THEOREM 1. *Let $P(x)$ be a fat polynomial (with $d\geqq 1$). Then $u\leqq 4$, $d\leqq 3$, $f\leqq 2$; and one of the following holds:*

(a) $P(x)\in\mathfrak{I}(x)$, $u_+=1$, $u_-=1$, $d=1, f=1$

(b) $P(x)\in\mathfrak{I}(x^2+x-1)$, $u_+=2$, $u_-=2$, $d=2, f=2$

(c) $P(x)\in\mathfrak{I}(x^3+2x^2-x-1)$, $u_\pm=3$, $u_\mp=1$, $d=3, f=1$

(d) $P(x)\in\mathfrak{I}(2x-1)$, $u_+=1$, $u_-=1$, $d=1, f=1$

(e) $P(x)\in\mathfrak{I}(2x^2-1)$, $u_\pm=2$, $u_\mp=1$, $d=2, f=1$.

Proof. We first prove that $u\leqq 4$. Since P is fat, we have seen that $u_+\geqq 1$ and $u_-\geqq 1$. Clearly P may be written

$$P(x) = (x-a_1)\cdots(x-a_{u_+})Q(x)+1,$$

where $a_1<\cdots<a_{u_+}$. Now if $P(b)=-1$, we have $(b-a_1)\cdots(b-a_{u_+})Q(b)=-2$, $\{b-a_1,\cdots,b-a_{u_+}\}\subseteq\{-2,-1,1,2\}$. By the first of these relations, at least u_+-1 of the distinct integers $b-a_1,\cdots,b-a_{u_+}$ must be ± 1. Hence $1\leqq u_+\leqq 3$, and similarly $1\leqq u_-\leqq 3$. If $u_+=3$, there is at most one integer b for which the second relation holds, so $u_-=1$. If $u_+=2$, there are at most two such integers, so $u_-\leqq 2$. Thus in every case $u\leqq 4$.

Since P is fat, $d<u$, and therefore $d\leqq 3$. Since $u\leqq 4$ and $d\geqq 1$, we have $f\leqq 3$; however, we shall see that the case $f=3$ does not occur, and therefore $f\leqq 2$.

Next we prove that $d(Q)=0$. We may assume $u_+\geqq u_-$, (otherwise replace P by $-P$). Since $u\leqq 4$, it follows that $u_-\leqq 2$. Since P is fat, $d(Q)<u_-$, and therefore $d(Q)=0$ or 1. If $d(Q)=1$, then $u_+=u_-=2$. Hence, for some $b_1\neq b_2$,

$$(b_1-a_1)(b_1-a_2)Q(b_1)=(b_2-a_1)(b_2-a_2)Q(b_2)=-2,$$
$$\{b_1-a_1,\ b_1-a_2,\ b_2-a_1,\ b_2-a_2\}\subseteq\{-2,-1,1,2\}.$$

Since $\{b_1-a_1,\ b_1-a_2\}$ is a translate of $\{b_2-a_1,\ b_2-a_2\}$, it follows that $(b_1-a_1)(b_1-a_2)=(b_2-a_1)(b_2-a_2)$ and $Q(b_1)=Q(b_2)$. Hence $Q(x)$ is constant.

We now have

$$P(x) = c(x-a_1)\cdots(x-a_{u_+})+1.$$

Since $u_-\geqq 1$, we may assume $P(0)=-1$; that is, $(-1)^{u_+}ca_1\cdots a_{u_+}=-2$. It follows that $|c|=1$ or 2.

If $|c|=1$, then $a_1\cdots a_{u_+}=\pm 2$, so either $a_1=-2$ or $a_{u_+}=2$. We may assume $a_1=-2$. (Otherwise replace $P(x)$ by $P(-x)$.) If $u_+=1$, then $ca_1=2$, $c=-1$, and $P(x)=-(x+2)+1=-(x+1)$, so $-P(x-1)=x$. If $u_+=2$, then $ca_1a_2=-2$, $ca_2=1$, $a_2=c=\pm 1$, and $P(x)=c(x+2)(x-c)+1$. If $c=1$, then $P(x)=x^2+x-1$. If $c=-1$, then $P(x)=-(x+2)(x+1)+1$, so $-P(x-1)=x^2+x-1$. Finally, if $u_+=3$, then $ca_1a_2a_3=2$, $ca_2a_3=-1$, $a_2=-1$, $a_3=1$, $c=1$, and $P(x)=x^3+2x^2-x-1$.

If $|c| = 2$, then $a_1 \cdots a_{u_+} = \pm 1$, so $u_+ = 1$ or 2. If $u_+ = 1$, then $ca_1 = 2$, $c = \pm 2$, $a_1 = \pm 1$, and $P(\pm x) = 2x - 1$. If $u_+ = 2$, then $ca_1a_2 = -2$, $a_1 = -1$, $a_2 = 1$, $c = 2$, and $P(x) = 2x^2 - 1$. This completes the proof.

COROLLARY 1. *If P is a fat polynomial with $d = 1$ or 2, then there is an integer b such that $P(-x) = (-1)^d P(x-b)$.*

5. Irreducibility criterion.

THEOREM 2. *Let $P(x)$ be a polynomial with $p + 2u > d \geqq 2$. Then either P is irreducible or $P = QR$ with $f(Q) + f(R) \geqq p + 2u - d$.*

Proof. If P is reducible, we can write $P = QR$ with $f(Q) \geqq f(R)$. Now for each integer n such that $P(n)$ is prime, either $Q(n)$ or $R(n)$ must be a unit, while for each n such that $P(n)$ is a unit, both $Q(n)$ and $R(n)$ must be units. Therefore $u(Q) + u(R) \geqq p + 2u$, and $f(Q) + f(R) \geqq p + 2u - d$, as was to be shown.

COROLLARY 2: *If $p + 2u > d + 4$, then P is irreducible.*

6. Example. Let $P(x) = x^5 - x^4 + 2x^3 - x^2 + x - 1$. Then

$$\begin{aligned}
P(0) &= -1\\
P(1) &= 1\\
P(2) &= 29\\
P(4) &= 883\\
P(-1) &= -7\\
P(-2) &= -71\\
P(-4) &= -1429.
\end{aligned}$$

Thus $p \geqq 5$, $u \geqq 2$, and $p + 2u - d \geqq 4$. Hence if P is reducible, we have $P = QR$ with $f(Q) = f(R) = 2$. But this implies $d = 4$, which is a contradiction, so P is irreducible.

If we fail to notice that $P(4)$ and $P(-4)$ are prime, then we have $p \geqq 3$, $u \geqq 2$, and $p + 2u - d \geqq 2$. In this case, if P is reducible, we have $P = QR$ with $f(Q) + f(R) \geqq 2$. Thus either $f(Q) = f(R) = 1$ or $f(Q) = 2$. In the first case we may assume $d(Q) = 2$, and therefore $Q \in \mathfrak{J}(2x^2 - 1)$. But this is impossible because P is monic. Therefore $f(Q) = 2$, and $Q \in \mathfrak{J}(x^2 + x - 1)$. Now by Corollary 1 we have $Q(x) = (x-b)^2 + (x-b) - 1$, and so $x^2 + x - 1$ divides $P(x+b)$. However the remainder of $P(x+b)$ modulo $x^2 + x - 1$ is $R_1(b) + xR_2(b)$, where

$$R_1(b) = b^5 - b^4 + 12b^3 - 17b^2 + 21b - 9$$

$$R_2(b) = 5b^4 - 14b^3 + 32b^2 - 31b + 14.$$

Since R_1 and R_2 have no common integer root, the remainder cannot vanish for any integer b. This contradiction proves that P is irreducible.

We thank the referee for his very helpful suggestions.

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A NOTE ON PERMUTABLE COMPLEMENTS

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A group G is *permutably decomposable* iff there exist subgroups A and B such that $G=AB$, $A\cap B=1$. One says that A is *permutably complemented* by B. Then a set $\mathfrak{A}$ of subgroups would be *permutably complemented*, or $\mathfrak{A}$ is p.c., iff for each $A\in\mathfrak{A}$ there is a permutable complement B, not necessarily in $\mathfrak{A}$. General problems with regard to the above include determination that one subset is p.c. iff another set is p.c. for a given group type. Restrict attention to *finite solvable* groups. C. Christensen [2] has shown that the set of all characteristic subgroups is p.c. iff the set of all normal subgroups $\mathfrak{n}$ is p.c. However, for the latter all that is known further is that $\mathfrak{w}$ is p.c. iff the subgroup lattice is complemented, i.e. the group is a K-group (e.g. see [1]).

THEOREM. *The set of all subnormal subgroups in a finite solvable group is permutably complemented iff the subgroup lattice is permutably complemented.*

Proof. A finite solvable group G is a subdirect product of solvable groups G^* having a unique minimal normal subgroup M. Restrict attention to the non-abelian direct factors. Since this property on subnormal subgroups is preserved under homomorphisms, induction will be used on the group order. In M, denote by P a cyclic subgroup of prime order $p\mid \mathrm{ord}(M)$. Since M is not a direct factor of G^*, there exists a maximal self-normalizing subgroup B such that $G^*=PB$, $P\cap B=1$. The homomorphism $\theta\colon G^*\to S_p$, where S_p is the symmetric group of degree p, induced by P on the cosets of B by translation, is necessarily injective, since $1\neq \mathrm{Ker}\ \theta\subset B$ implies $P\subset B$. Since P must be M, then inductively B has square-free order, and the homomorphism θ implies $p\nmid \mathrm{ord}(B)$. So $\mathrm{ord}(G^*)$ is square-free. Therefore G is a subdirect product of groups of square-free order, and this is a necessary and sufficient condition that the subgroup lattice be permutably complemented (see P. Hall [3]). From the same result of P. Hall's the converse is immediate.

REMARK. It is clear that the condition of the subnormal subgroups of a group G being p.c. iff the subgroup lattice of G is p.c. implies that G is supersolvable, in fact, G is a subdirect product of groups of square-free order (see [3]).

From the remark and the theorem, one concludes that for supersolvable groups the following statements are equivalent:

1. The lattice of characteristic subgroups is p.c.
2. The lattice of normal subgroups is p.c.
3. The lattice of subnormal subgroups is p.c.
4. The lattice of subgroups is p.c. (or equivalently, the group is a supersolvable K-group [3]).

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THE MAGNITUDE OF PARTIAL QUOTIENTS

J. H. JORDAN, Washington State University

For an irrational number ξ let $\langle a_0, a_1, a_2, \cdots, a_n, \cdots \rangle$ be its simple continued fraction expansion and let p_n/q_n be its nth convergent. The question that will be considered in this paper will concern the size of the partial quotients, a_n, for a typical irrational ξ.

The following result due independently to Borel [2] and Bernstein [1] is mentioned in Hardy and Wright [3].

THEOREM A. *If $G(n)$ is a monotonic increasing positive sequence such that $\sum_{n=1}^{\infty} (G(n))^{-1}$ converges and $A = \{\xi: a_n > G(n) \text{ for } \infty \text{ many } n\}$, then the set A is of measure zero.*

It is the purpose of this paper to prove a theorem similar in nature to Theorem A by arguments quite different from those of Borel and Bernstein. The theorem is:

THEOREM 1. *Let $G(n)$ be as in Theorem A and if*

$$C = \{\xi: \text{there is a } c_\xi > 0 \text{ such that} \limsup_{n\to\infty} q_{n+1}/G(q_n) > c_\xi\},$$

then C is a measure of zero.

Proof. It, of course, suffices to prove the theorem restricted to the unit interval. Let f be defined in the following way:

$$\begin{aligned} f(x) &= (qG(q))^{-1} \quad && \text{if } x = p/q,\ (p, q) = 1 \\ &= 0 && \text{if } x \text{ is irrational.} \end{aligned}$$

Now f is of bounded variation on the unit interval since for any partition

$$\sum \left| f(x_i) - f(x_{i-1}) \right| \leqq 2 \sum_{0<x\leqq 1} f(x) = 2 \sum_{0<p/q\leqq 1} (qG(q))^{-1} < 2 \sum_{n=1}^{\infty} (G(n))^{-1}$$

which is a fixed constant since the series converges.

Since f is of bounded variation it has a finite derivative except perhaps on a set of measure zero.

Let ξ be in C. If $f'(\xi)$ were to exist it must be zero since any difference quotient taken at irrational points would be zero. Let n_k be that subsequence of n's for which $q_{n_k+1}/G(q_{n_k}) > c_\xi$. Now $f(p_{n_k}/q_{n_k}) - f(\xi) = (q_{n_k}G(q_{n_k}))^{-1}$ and an inequality

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THEOREM 1. *Let $G(n)$ be as in Theorem A and if*

$$C = \{\xi: \text{there is a } c_\xi > 0 \text{ such that} \limsup_{n\to\infty} q_{n+1}/G(q_n) > c_\xi\},$$

then C is a measure of zero.

Proof. It, of course, suffices to prove the theorem restricted to the unit interval. Let f be defined in the following way:

$$\begin{aligned} f(x) &= (qG(q))^{-1} \quad \text{if } x = p/q,\ (p, q) = 1 \\ &= 0 \quad\quad\quad\quad\ \ \text{if } x \text{ is irrational.} \end{aligned}$$

Now f is of bounded variation on the unit interval since for any partition

$$\sum \left| f(x_i) - f(x_{i-1}) \right| \leq 2 \sum_{0<x\leq 1} f(x) = 2 \sum_{0<p/q\leq 1} (qG(q))^{-1} < 2 \sum_{n=1}^{\infty} (G(n))^{-1}$$

which is a fixed constant since the series converges.

Since f is of bounded variation it has a finite derivative except perhaps on a set of measure zero.

Let ξ be in C. If $f'(\xi)$ were to exist it must be zero since any difference quotient taken at irrational points would be zero. Let n_k be that subsequence of n's for which $q_{n_k+1}/G(q_{n_k}) > c_\xi$. Now $f(p_{n_k}/q_{n_k}) - f(\xi) = (q_{n_k}G(q_{n_k}))^{-1}$ and an inequality

mentioned in Hardy and Wright [3] is that

$$|p_{n_k}/q_{n_k} - \xi| < (q_{n_k}q_{n_k+1})^{-1}.$$

Combining these two results one has

$$\frac{f(p_{n_k}/q_{n_k}) - f(\xi)}{|p_{n_k}/q_{n_k} - \xi|} > \frac{q_{n_k+1}q_{n_k}}{q_{n_k}G(q_{n_k})} = \frac{q_{n_k+1}}{G(q_{n_k})} > c_\xi > 0.$$

Therefore $f'(\xi)$ cannot exist hence C is a subset of a set of measure zero.

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UNIFORM CONVERGENCE AND REARRANGEMENT

D. J. HARRIS and B. L. D. THORP, University College, Cardiff

The printed solution to problem A5390 (this MONTHLY, 74 (1967) p. 600) employed the fact that a series of continuous functions on an interval with the property that every rearrangement is uniformly convergent is necessarily absolutely convergent. In this note it is shown that absolute convergence is not sufficient to guarantee uniform convergence of all rearrangements of a given uniformly convergent series. Explicitly, *an example is given of a series* $\sum_{n=1}^{\infty} f_n$ *of continuous functions on* $[0, 1]$ *with the properties*:

(i) $\sum_{n=1}^{\infty} f_n$ *is uniformly convergent,*

(ii) $\sum_{n=1}^{\infty} |f_n(x)|$ *is convergent for each* $0 \leqq x \leqq 1$,

(iii) $\sum_{k=1}^{\infty} f_{n_k}$ *is not uniformly convergent for some rearrangement* $\{n_k\}$ $(k=1, 2, \cdots)$.

For each positive integer n let

$$u_n(x) = \begin{cases} 3x & \text{for } 0 \leqq x \leqq (1/3n) \\ 1/n & \text{for } (1/3n) \leqq x \leqq (2/3n) \\ (3/n) - 3x & \text{for } (2/3n) \leqq x \leqq (1/n) \\ 0 & \text{for } (1/n) \leqq x \leqq 1 \end{cases} \qquad (0 \leqq x \leqq 1),$$

and set $f_n = (-1)^n u_n$. The sequence $\{u_n(x)\}$ $(n=1, 2, \cdots)$ is monotonically decreasing for each value of x and is uniformly convergent to zero.

Given $n > 0$, $p \geqq 0$, and x, we have

$$|f_n(x) + \cdots + f_{n+p}(x)| = u_n(x) - [u_{n+1}(x) - u_{n+2}(x)] - [u_{n+3}(x) - u_{n+4}(x)] - \cdots \leqq u_n(x),$$

hence the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Clearly $\sum_{n=1}^{\infty}|f_n(x)|$ converges for each x since the series has only finitely many nonzero terms. Consider the rearrangement $\{n_k\}=\{1, (2), 3, (4), 5, (6, 8), 7, (10, 12, 14, 16), 9, (18, 20, 22, 24, 26, 28, 30, 32), 11, (\cdots\}$, where for the pth bracketed block $B_p(p>1)$, the last term β_p and the first term α_p are given respectively by $\beta_p=2^p$, $\alpha_p=2^{p-1}+2$, and the block contains 2^{p-2} terms. For each $p>1$ and each value of x in the nonempty closed interval $[1/3\alpha_p, 2/3\beta_p]$

$$\sum_{k\in B_p} f_{n_k}(x) = \sum_{r=\alpha_p/2}^{\beta_p/2} 1/2r > 2^{p-2}(1/2^p) = 1/4.$$

The series $\sum_{k=1}^{\infty} f_n$ is evidently not uniformly convergent.

ON A FUNCTIONAL EQUATION ARISING IN PROBABILITY

R. A. Horn, Johns Hopkins University and R. D. Meredith, University of Santa Clara

In one section of a recent paper devoted to a classical problem of probability it was necessary to show that the functional equation

$$f(x) = \frac{1}{2}\int_{x-1}^{x+1} f(t)dt \tag{1}$$

has only linear solutions within the class of continuous functions of polynomial growth ([1], Lemma 3.7). The proofs offered for this fact all made essential use of the Fourier transform and were not in harmony with the "elementary" methods of the rest of the paper. In this note we show that a more general result can be proved using a completely elementary method which may be of interest in itself for pedagogical purposes.

Let us denote the set of real numbers by $\mathfrak{R}$, write $f(x)=O(g(x))$ if $f(x)/g(x)$ is a bounded function on $\mathfrak{R}$, and define $S(\alpha)\equiv\{f\colon \mathfrak{R}\to\mathfrak{R} \mid f$ is locally integrable and $f(x)=O(e^{\alpha|x|})\}$ for $\alpha\in\mathfrak{R}$. With this notation, our result can be stated concisely as the

THEOREM. *There exists some $\alpha>0$ such that the functional equation* (1) *has only linear solutions within the class $S(\alpha)$.*

To prove this, observe first that any locally integrable solution of (1) is necessarily continuous and possesses derivatives of every order. In fact,

$$f^{(n)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+n-2k);$$

so if $f\in S(\alpha)$ for some $\alpha\geqq 0$ we have $f^{(n)}(x)=O(e^{\alpha|x|+\alpha n})$. Using this estimate and the Maclaurin expansion with remainder, one sees that the Maclaurin series $\sum_{n=0}^{\infty} x^n f^{(n)}(0)/n!$ converges absolutely for all x to the function $f(x)$. Now insert this power series representation into the functional equation to find that f is a solution of (1) if and only if the vector $(f^{(n)}(0))_{n=2}^{\infty}$ is a solution of the de-

$$Q(y, z) = \sum_{i,j=1}^{\infty} y_i b_{ij} z_j$$

is absolutely convergent if y and z are square summable vectors. Furthermore, v is a square summable vector and $Q(v, v)=0$ since $Bv=0$. Finally, using the arithmetic-geometric mean inequality, we compute

$$\begin{aligned} 0 = Q(v, v) &= \sum_{i,j} v_i b_{ij} v_j = \sum_n b_{nn} v_n^2 + \sum_{i\neq j} b_{ij} v_i v_j \\ &\geqq \sum_n b_{nn} v_n^2 - \sum_{i\neq j} \left| b_{ij} v_i v_j \right| \geqq \sum_n b_{nn} v_n^2 - \frac{1}{2} \sum_{i\neq j} \left| b_{ij} \right| (v_i^2 + v_j^2) \\ &= \sum_{n=1}^{\infty} v_n^2 \left[b_{nn} - \frac{1}{2} \sum_{\substack{k=1 \\ k\neq n}}^{\infty} (\left| b_{nk} \right| + \left| b_{kn} \right|) \right], \end{aligned}$$

and we find from condition (c) that $v_n=0$ for all $n=1, 2, 3, \cdots$.

Thus, if $0 \leqq \alpha < \frac{1}{2} \log 10 \doteq 1.15$, the equations (2) have only the trivial solution and the functional equation (1) has only linear solutions within the class $S(\alpha)$. It is easy to improve this lower bound for the best α of the theorem by being more careful in making the estimates leading to (c′) above, but not much can be gained since there is a solution to (1) of the form $f(x) = e^{ax} \cos bx$ with $a \doteq 2.8$.

This work was done while the second author was a participant in an Undergraduate Research Participation Program at the University of Santa Clara supported by the National Science Foundation under Grant GY-2645.

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A HOMOLOGY THEOREM FOR RINGS OF FUNCTIONS

LI PI SU, University of Oklahoma

Hu proved that for a compact Hausdorff space X, the singular complex $S(X)$ of the space X is simplicially isomorphic with the singular complex $S(C(X))$ of the algebra $C(X)$. (See [2], Theorem 11.3.) It is also proved in the same paper (p. 487) that if A is any closed subspace of X and I_A the ideal of $C(X)$ which consists of the elements $f \in C(X)$ such that $f(A) = \{0\}$, then there is a natural simplicial map which maps the subcomplex $S(A)$ of $S(X)$ onto the subcomplex $S(I_A)$ of $S(C(X))$.

Hu used the fact that a compact Hausdorff space X and the space of maximal ideals of the ring of real-valued continuous functions $C(X)$ are homeomorphic to establish the results. (See [2], Sections 9–10.) We know that a realcompact

$$Q(y, z) \equiv \sum_{i,j=1}^{\infty} y_i b_{ij} z_j$$

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Proof. By definition of the induced homomorphism, for each $f \in I_A$ (as a matter of fact, for each $f \in C(Y)$) and $x \in X$, we have $f(\tau(x)) = (\tau'(f))(x)$. Hence $\tau'(f) = 0$ if and only if $f(\tau(x)) = 0$. In virtue of the complete regularity of the space, however, we have $f(\tau(x)) = 0$ if and only if $\tau(x) \in A$.

Now, having these results in hand, we can prove the following two results by means of the same arguments as the proof of (11.3) in [2].

THEOREM. *For a realcompact space X and any closed subset A of X, $(S(X), S(A))$ and $(S(C(X)), S(I_A))$ are simplicially isomorphic.*

COROLLARY. *For each dimension $q \geqq 0$, and every coefficient group G, then*

$$H_q(X, A, G) \simeq H_q(C(X), I_A, G), \quad H^q(X, A, G) \simeq H^q(C(X), I_A, G).$$

In particular, if A is an empty subset, then

$$H_q(X, G) \simeq H_q(C(X), G), \qquad H^q(X, G) \simeq H^q(C(X), G).$$

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THE DISTANCE BETWEEN SETS

CHAN KAI-MENG, University of Malaya, Malaysia

If A and B are subsets of a metric space (M, ρ), then the distance between them, $\rho(A, B)$ is defined to be the infimum of $\rho(x, y)$, where x runs through A and y runs through B. What is $\rho(A, B)$ if one of the sets happens to be the empty set $\emptyset$? Now $\rho(A, \emptyset) = \inf \emptyset = +\infty$, according to [1]. According to [2], however, $\rho(A, \emptyset) = 0$. We shall relate these definitions to a remark of Saks in measure theory.

DEFINITION. *Let μ be a nonnegative, extended real-valued set function defined for each subset of M. Let μ satisfy the following conditions:*

(1) $\mu(X) \leqq \mu(Y)$ *if* $X \subseteq Y$.

(2) $\mu(\bigcup_{n=1}^{\infty} X_n) \leqq \sum_{n=1}^{\infty} \mu(X_n)$ *for each sequence* $\{X_n\}_{n \geqq 1}$ *of subsets of* M.

(3) $\mu(X \cup Y) = \mu(X) + \mu(Y)$ *if* $\rho(X, Y) > 0$.

Such a μ is called an *outer measure*. Only (3) involves the metric, so that if M is not supplied with a metric, (3) is dropped.

DEFINITION. *If E is a subset of M, then E is measurable (with respect to the outer measure) if and only if*

$$\mu(X) = \mu(X \cap E) + \mu(X \cap \mathcal{C}(E))$$

for each $X \subseteq M$, where $\mathcal{C}(E)$ denotes the complement of E.

Proof. By definition of the induced homomorphism, for each $f \in I_A$ (as a matter of fact, for each $f \in C(Y)$) and $x \in X$, we have $f(\tau(x)) = (\tau'(f))(x)$. Hence $\tau'(f) = 0$ if and only if $f(\tau(x)) = 0$. In virtue of the complete regularity of the space, however, we have $f(\tau(x)) = 0$ if and only if $\tau(x) \in A$.

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DEFINITION. *If E is a subset of M, then E is measurable (with respect to the outer measure) if and only if*

$$\mu(X) = \mu(X \cap E) + \mu(X \cap \mathcal{C}(E))$$

for each $X \subseteq M$, where $\mathcal{C}(E)$ denotes the complement of E.

Saks says that the class of measurable sets includes the empty set $\varnothing$ and all sets E for which $\mu(E)=0$. This is not quite correct. The exact situation may be given by this theorem.

THEOREM. (a) *For an outer measure defined by* (1) *and* (2), *all sets E with* $\mu(E)=0$ *are measurable.*

(b) *For an outer measure defined by* (1) *and* (2), *no set need be measurable.*

(c) *For an outer measure defined by* (1), (2), *and* (3) *with* $\rho(A, \varnothing)=+\infty$, *the empty set is measurable.*

(d) *For an outer measure defined by* (1), (2), *and* (3) *with* $\rho(A, \varnothing)=0$, *the empty set need not be measurable.*

Proof.

$$\text{(a)}\quad \mu(X\cap E)+\mu(X\cap \mathcal{C}(E)) \leqq \mu(E)+\mu(X) = \mu(X)$$
$$= \mu((X\cap E)\cup(X\cap \mathcal{C}(\mathrm{E})) \leqq \mu(X\cap E)+\mu(X\cap \mathcal{C}(E)).$$

(b) Define $\mu(X)=1$ for all X. Then (1) and (2) are satisfied. No set can be measurable as

$$\mu(X) = 1 \neq 1+1 = \mu(X\cap E)+\mu(X\cap \mathcal{C}(E)).$$

(c) Either $\mu(X)=+\infty$ for each X, whence it is clear that every set is measurable, or $\mu(X)$ is finite for at least one X. Then for such an X,

$$\mu(X) = \mu(X\cup\varnothing) = \mu(X)+\mu(\varnothing),$$

whence $\mu(\varnothing)=0$. The measurability of $\varnothing$ now follows from (a).

(d) Let M be a nonempty, finite set with the metric

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Define

$$\mu(X) = \begin{cases} 1/2 & \text{if } X = \varnothing \\ \text{cardinality of } X & \text{if } X \neq \varnothing. \end{cases}$$

Clearly (1), (2), and (3) are satisfied. If X is a one element set,

$$\mu(X) = 1 \neq 1/2+1 = \mu(\varnothing)+\mu(X),$$

whence $\varnothing$ is not measurable. This completes the proof of the theorem.

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$\max\{\deg \phi_j\} \leqq r/n$. This contradicts the minimality of r and proves the theorem.

REMARKS. The proof in [3] actually shows more than this. It shows that (1) has no nontrivial solution in qth roots of rational functions, where $(q, n)=1$ and $n>2$. Further recent work has dealt with solutions to (1) in entire or meromorphic functions [1, 2].

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Correction to: "A Note on Doubly-Stochastic Matrices" by E. J. Bell and H. G. Daellenbach, this MONTHLY, 76 (1969) page 284. The last display, and the sentence preceding it, should read: "To prove sufficiency, assume (3) is satisfied, then it follows immediately that (2) is a solution to $a_i p_j = a_{ij}$, $i, j=1, 2, \cdots, N$."

RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

DOES THERE EXIST A HYPOTRACEABLE GRAPH?

H. V. KRONK, SUNY at Binghamton

A graph G is called *hamiltonian* (*traceable*) if G has a cycle (path) containing all the points of G. A graph G is called *hypohamiltonian* (*hypotraceable*) if G is not hamiltonian (traceable), but $G-v$ is hamiltonian (traceable) for every point v in G. ($G-v$ denotes the maximal subgraph of G not containing v.) Herz, Gaudin, and Rossi [3] have shown that no hypohamiltonian graph of order $p\leqq 9$ exists and that there is exactly one hypohamiltonian graph of order 10; namely, the well-known Petersen graph shown in Figure 1. An infinite family of hypohamiltonian graphs of order $p=6n+4$ was constructed in [2, 5]. Various properties of hypohamiltonian graphs are established in [2], however, there are still several interesting open questions. (For example, can a hypohamiltonian graph have a cycle of length less than five?) A natural question to ask is the following:

Does there exist a connected hypotraceable graph?

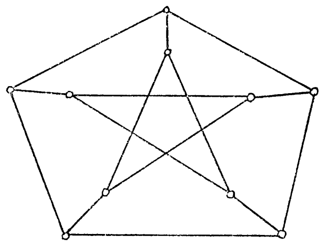

FIG. 1. The Petersen Graph

This problem was first raised in [4], where the authors were investigating properties of longest paths in graphs. It is easy to see that any two longest paths in a connected graph intersect [6, p. 31]. Gallai [1] asked whether or not all the longest paths in a connected graph have a point in common. This was answered in the negative by Walther [7], who gave an example of a connected graph G of order 25 such that every point of G was not on some longest path of G. The longest paths in his example had length 20. An affirmative answer to the following related question would imply that no graph is hypotraceable:

If the longest paths in a graph G of order p have length $p-2$, is there a point common to all the longest paths of G?

The author feels strongly that the above question has an affirmative solution and hopes that some interested reader will supply one.

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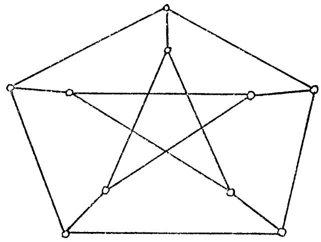
WHAT ARE THE INTERSECTION GRAPHS OF ARCS IN A CIRCLE?

VICTOR KLEE, University of Washington

Let $\mathcal{S}$ be a finite family of nonempty sets. The *intersection graph* $G(\mathcal{S})$ has vertices corresponding to the members of $\mathcal{S}$ and edges corresponding to the pairs of intersecting members of $\mathcal{S}$. That is, two vertices of $G(\mathcal{S})$ are neighbors (joined by an edge of $G(\mathcal{S})$) if and only if the members of $\mathcal{S}$ corresponding to the two vertices have nonempty intersection. Note that $G(\mathcal{S})$ is a combinatorial rather than a topological entity. The following figure shows a family of seven circular arcs along with the intersection graph of that family.



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This problem was first raised in [4], where the authors were investigating properties of longest paths in graphs. It is easy to see that any two longest paths in a connected graph intersect [6, p. 31]. Gallai [1] asked whether or not all the longest paths in a connected graph have a point in common. This was answered in the negative by Walther [7], who gave an example of a connected graph G of order 25 such that every point of G was not on some longest path of G. The longest paths in his example had length 20. An affirmative answer to the following related question would imply that no graph is hypotraceable:

If the longest paths in a graph G of order p have length $p-2$, is there a point common to all the longest paths of G?

The author feels strongly that the above question has an affirmative solution and hopes that some interested reader will supply one.

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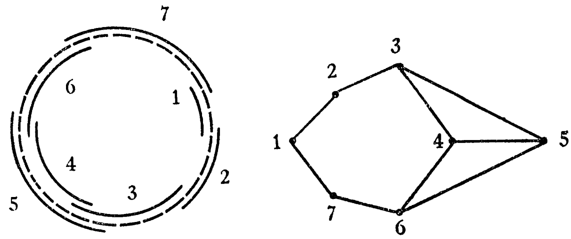
WHAT ARE THE INTERSECTION GRAPHS OF ARCS IN A CIRCLE?

VICTOR KLEE, University of Washington

Let $\mathcal{S}$ be a finite family of nonempty sets. The *intersection graph* $G(\mathcal{S})$ has vertices corresponding to the members of $\mathcal{S}$ and edges corresponding to the pairs of intersecting members of $\mathcal{S}$. That is, two vertices of $G(\mathcal{S})$ are neighbors (joined by an edge of $G(\mathcal{S})$) if and only if the members of $\mathcal{S}$ corresponding to the two vertices have nonempty intersection. Note that $G(\mathcal{S})$ is a combinatorial rather than a topological entity. The following figure shows a family of seven circular arcs along with the intersection graph of that family.



$\mathcal{S}$ $G(\mathcal{S})$

The title question asks for an intrinsic, combinatorial characterization of graphs of the form $G(\mathcal{S})$, where $\mathcal{S}$ is a finite family of arcs in a circle. There would be special interest in a characterization leading to an efficient algorithm for deciding whether a given graph is of the indicated form.

In attacking the above problem, attention may be restricted to families of arcs that cover the circle, for if $\mathcal{S}$ is a family of circular arcs not covering the circle then $G(\mathcal{S})$ is isomorphic to the intersection graph of a family of arcs in the line. Such graphs, the so-called *interval graphs*, have been characterized in various ways. For example, Lekkerkerker and Boland [7] show that a graph G is an interval graph if and only if the following two conditions are satisfied:

(i) G is a rigid circuit graph; that is, any simple circuit of length at least 4 in G admits at least one "diagonal";

(ii) any three vertices of G can be ordered in such a way that every path from the first vertex to the third vertex passes through the second vertex or through a neighbor of the second vertex.

Note that the intersection graph depicted above lacks both of these properties. For other characterizations of interval graphs, see Lekkerkerker and Boland [7], Gilmore and Hoffman [4], and Fulkerson and Gross [3]. The last of these [3] contains the most efficient algorithm known at present for testing the "intervality" of a given graph. For some related results, see Kotzig [7], Wegner [13], Roberts [10], Tucker [12], and Renz [9].

There have been several sources of interest in interval graphs. The problem of characterizing them was proposed by Hajos [5], and independently by Benzer [1] in connection with genetic studies mentioned in the next paragraph. Wegner [13] and Roberts [10] characterized the intersection graphs of families of *unit* intervals in the line as interval graphs in which no vertex has three independent neighbors, and Roberts [10] used these graphs to characterize the indifference systems of psychophysics. Kendall [6] noted a relationship between interval graphs and certain dating problems of archeology. Joel E. Cohen was led by a study of food webs to conjecture that the trophic aspect of any ecological system is one-dimensional in the following sense: If a graph is formed by using a vertex for each species and joining two vertices by an edge if and only if the corresponding species have some prey in common, then the resulting graph is an interval graph.

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

CHARACTERIZATIONS OF CONVERGENCE IN PROBABILITY

N. Y. LUTHER, Washington State University

The author is unable to find the following result explicitly in print: ($A\Delta B$ denotes the symmetric difference $(A-B)\cup(B-A)$ of the sets A and B; other notation and terminology as in the reference [1], especially Chapter 3 thereof).

THEOREM. *Let X_n, X be random variables on the probability space $(\Omega, \mathcal{A}, P)$. Let F_X be the distribution function of X and let $C(F_X)$ denote the set of continuity points of F_X.*

(i) *$X_n \to X$ in probability if and only if, for each $x \in C(F_X)$,*

$$P([X_n < x]\Delta[X < x]) \to 0 \quad as \quad n \to \infty.$$

(ii) *$X_n \to X$ almost surely (a.s.) if and only if, for each $x \in C(F_X)$,*

$$P[\bigcup_{k\geq n} ([X_k < x]\Delta[X < x])] \to 0 \quad as \quad n \to \infty.$$

REMARKS: I. The relationship between convergence in probability and convergence in law (i.e., convergence in distribution) is nicely exposed by (i). Now (i) shows that $X_n \xrightarrow{P} X$ if and only if, for every $x \in C(F_X)$, $E|I_{[X_n<x]} - I_{[X<x]}| \to 0$ as $n \to \infty$ (i.e., $I_{[X_n<x]}$ converges to $I_{[X<x]}$ in L_1) where I_A denotes the indicator function of A and E denotes expectation. Consider the elementary inequality

$$(1) \quad |P(A) - P(B)| \leqq P(A\Delta B); \qquad \text{i.e., } |EI_A - EI_B| \leqq E|I_A - I_B|.$$

($|P(A)-P(B)| = |P(A-B)-P(B-A)| \leqq P(A-B)+P(B-A)$; alternatively, one may obtain (1) by applying the well-known inequality $|EX| \leqq E|X|$ to $X = I_A - I_B$.) Applying the inequality (1) to the sets $A = [X_n < x]$, $B = [X < x]$, we obtain immediately the well-known fact that $X_n \xrightarrow{P} X$ implies $X_n \to X$ in law. If X is the constant c a.s., then $[X < x]$ is a.s. ϕ or Ω according as whether $x \leqq c$ or $x > c$, respectively. In this case, the inequality (1) becomes equality (for

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$A=[X_n<x]$, $B=[X<x]$) and hence we have the well-known result that $X_n \xrightarrow{P} c$ if, and only if, $X_n \to c$ in law.

II. By (i), $X_n \xrightarrow{P} X$ if and only if, for each $x \in C(F_X)$, $[X_n<x]$ converges to $[X<x]$ in the complete metric space [**1**, p. 101; 17] consisting of all $A \in \mathcal{A}$ (identified a.s.) endowed with the metric $d(A, B)=P(A \Delta B)$.

Proof. (i): Suppose $X_n \xrightarrow{P} X$. Let x, x' be real numbers. If $x<x'$, we have $[X_n<x]-[X<x']=[X_n<x] \cap [X \geqq x'] \subset [|X_n-X| \geqq x'-x]$, so that $P([X_n<x]-[X<x']) \to 0$ as $n \to \infty$. Similarly, if $x'<x$, we obtain $P([X<x']-[X_n<x]) \to 0$ as $n \to \infty$.

Suppose $x \in C(F_X)$. Let $\epsilon>0$. Then there is a positive number $\delta>0$ such that $P([X<x+\delta]-[X<x])<\epsilon/4$ and $P([X<x]-[X<x-\delta])<\epsilon/4$. From above, there is a positive integer N such that, for every $n \geqq N$, $P([X_n<x]-[X<x+\delta])<\epsilon/4$ and $P([X<x-\delta]-[X_n<x])<\epsilon/4$. Since

$$\begin{aligned}[X_n < x]\Delta[X < x] \subset [&([X_n < x] - [X < x + \delta]) \\ &\cup ([X < x + \delta] - [X < x]) \cup ([X < x] - [X < x - \delta]) \\ &\cup ([X < x - \delta] - [X_n < x])] \qquad \text{for every } n,\end{aligned}$$

it follows easily that $P([X_n<x]\Delta[X<x])<\epsilon$ if $n \geqq N$.

To prove the converse, we let $\epsilon>0$ and show that $P[|X_n-X| \geqq \epsilon] \to 0$ as $n \to \infty$: Let $\delta>0$. Let $A_m=[-m<X<m]$, $m=1, 2, \cdots$. Since $A_m \uparrow \Omega$, there exists a positive integer M such that $P(\Omega-A_M)<\delta/2$. Since $C(F_X)$ is dense in the reals, we can choose a *finite* set $\{x_1, \cdots, x_k\} \subset C(F_X)$ such that

$$-M \geqq x_1 < x_2 < \cdots < x_{k-1} < x_k \geqq M \quad \text{and} \quad x_i - x_{i-1} < \epsilon, \; i = 2, \cdots, k.$$

Since each $\omega \in A_M \cap [|X_n-X| \geqq \epsilon]$ must be an element of either $[X_n<x_i] \cap [X \geqq x_i]$ or $[X<x_i] \cap [X_n \geqq x_i]$ for some i ($i=1, \cdots, k$), we have

$$A_M \cap [| X_n - X| \geqq \epsilon] \subset \bigcup_{i=1}^{k} ([X_n < x_i]\Delta[X < x_i]).$$

Now by hypothesis there is a positive integer N such that for each $n \geqq N$ and each $i=1, \cdots, k$,

$$P([X_n < x_i]\Delta[X < x_i]) < \delta/2k.$$

It follows easily that $P[|X_n-X| \geqq \epsilon] \leqq P(\Omega-A_M)+P(A_M \cap [|X_n-X| \geqq \epsilon]) < \delta/2+\sum_{i=1}^{k} \delta/2k=\delta$ if $n \geqq N$.

(ii): A proof very similar to that of (i) (but slightly more involved) may be based on the fact [**1**, p. 151] that $X_n \to X$ a.s. if and only if, for every $\epsilon>0$,

$$P(\bigcup_{k \geqq n} [| X_k - X| \geqq \epsilon]) \to 0 \quad \text{as} \quad n \to \infty.$$

The details are left to the reader.

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which is Callebaut's generalization of Schwarz's inequality. Furthermore, the first proof above can also be extended to the continuous case; verification is left to the reader.

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THE RELATION BETWEEN THE DERIVATIVES OF f AND f^{-1}

W. R. JONES AND M. D. LANDAU, Lafayette College

The familiar relation $g'(y_0)=1/f'(x_0)$ in which $g=f^{-1}$ and $y_0=f(x_0)$ is known to be valid under various sets of hypotheses each including, of course, that $f'(x_0)\neq 0$ exists. If for $k\neq 0$, we define $h=g(y_0+k)-g(y_0)$, and note that $k\neq 0$ implies $h\neq 0$ by the one-to-one character of g, then we may write

$$\frac{g(y_0+k)-g(y_0)}{k}=1\Big/\frac{f(x_0+h)-f(x_0)}{h},$$

since $g(y_0+k)=x_0+h$ implies that $y_0+k=f(x_0+h)$. Now if g is defined on an interval containing y_0, possibly as an endpoint, and if we could establish that $h\to 0$ as $k\to 0$, then we could conclude that $g'(y_0)=1/f'(x_0)$. Sufficient conditions on f which yield the desired conclusion are that f be strictly monotone and continuous in an interval about x_0 and that $f'(x_0)\neq 0$.

In several calculus textbooks (e.g., [**1**], [**2**], [**3**], [**4**], [**5**]) it is erroneously stated that if $f'(x_0)\neq 0$ exists and if an inverse function $g=f^{-1}$ exists, then $g'(y_0)$ exists ($y_0=f(x_0)$) and is the reciprocal of $f'(x_0)$. The following example shows that with these hypotheses alone, the domain of g may have empty interior, so that by the usual definition, no derivative exists. Further, the example may be modified so that g is defined on a neighborhood of y_0, removing the above objection, but in such a way as to introduce a discontinuity at y_0. Thus, the stated conditions do not even guarantee continuity of the inverse function g.

EXAMPLE. For $0<x<2$ let $f(x)=x^2$ if x is rational and $f(x)=2x-1$ if x is irrational. Then f is one-to-one on $(0, 2)$, and while f fails to be continuous at any point of $(0, 1)$ or $(1, 2)$, it is clear that $f'(1)=2$. Thus f has an inverse g, and we are concerned with $g'(1)$. The function g is defined at all points of $(0, 3)$ except for those rational numbers with irrational square roots. In fact, the domain of g has no interior points. Hence $g'(1)$ cannot exist in the ordinary sense. To remedy this let

$$A=\{y \mid 0<y<3,\ y \text{ is rational, } \sqrt{y} \text{ is irrational}\} \text{ and}$$

$$B=\{x \mid x=\tfrac{1}{2}(y+4) \text{ for some } y \text{ in } A\}.$$

Then B is a subset of $(2, 7/2)$ and we may extend the domain of f to include B by letting $f(x)=2x-4$ for x in B. Then f remains one-to-one. (In fact, one may, with the aid of a cardinality argument, extend the domain of f to the entire interval $(0, 7/2)$ in such a way that f has the entire interval $(-1, 4)$ as range and still remains one-to-one.) By definition $f(B)=A$, so now the entire interval $(0, 3)$ lies in the domain of g. Since both A and the set of irrational numbers are dense in $(0, 3)$, we see that in each deleted neighborhood of $y=1$ the function g assumes values in B as well as values arbitrarily close to 1. Thus lim g fails to exist at 1.

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CONTINUOUS OPEN MAPS ON THE UNIT INTERVAL

S. S. Mitra, Wilkes College

Let f be a continuous open map from $I=[0, 1]$ onto a nondegenerate Hausdorff space X. Is I necessarily homeomorphic to X? We get an affirmative answer in the following theorem.

THEOREM 1. *Let f be a continuous open map of I onto a Hausdorff space X containing at least two elements. Then X and I are homeomorphic.*

Proof. First we observe that X is compact and connected. Let $[0, a]$, $0<a\leqq 1$, be the smallest closed interval such that $f([0, a])=X$. (The existence of such an interval can be easily proved by using the continuity of f and the fact that X is nondegenerate Hausdorff.) The theorem will be proved as soon as we can establish that $f|\,[0, a]$ is one-to-one. Suppose that $f(x_1)=f(x_2)$, for some x_1, x_2, with $0\leqq x_1<x_2\leqq a$.

Case 1. $0\leqq x_1<x_2<a$. Here, $f[0, x_2]=f[0, x_2)$, i.e., $f[0, x_2]$ is both open and closed. However, by the minimality of $[0, a]$, $f[0, x_2]\neq X$. This is impossible, since X is connected.

Case 2. $0\leqq x_1<x_2=a$. In this case

$$X=f[0, a]=f[0, a)=f\left[\bigcup_{n=1}^{\infty}\left[0, a-\frac{1}{n}\right)\right]=\bigcup_{n=1}^{\infty} f\left[0, a-\frac{1}{n}\right).$$

Since f is open, each set on the right is an open subset of X. By the minimality of $[0, a]$, X cannot be covered by any finite collection of these open sets. How-

ever, this contradicts the fact that X is compact. Thus $f|[0, a]$ is one to one and the theorem is proved.

Consider once again a continuous open map f from I onto a nondegenerate Hausdorff space X. In the following theorem we show that every such function is necessarily a piecewise homeomorphism onto X in the sense that there exists a partition $0=a_0<a_1<\cdots<a_n=1$ of I such that $f|[a_i, a_{i+1}]$ $(i=1, 2, \cdots, n-1)$ is a homeomorphism onto X.

THEOREM 2. *Every continuous open map from I onto a nondegenerate Hausdorff space X is necessarily a piecewise homeomorphism onto X.*

Proof. Let h be a homeomorphism from X onto I (this is possible by virtue of Theorem 1). Evidently, h is a continuous open map from I onto I. It has been proved in [1] that every continuous open map from I onto I is necessarily a piecewise homeomorphism. From this it follows easily that f is also a piecewise homeomorphism onto X.

It should be observed that in Theorem 1, the interval $[0, 1]$ cannot be replaced by an arbitrary compact, connected, T_2, space; e.g., the unit square allows a continuous open projection onto a side.

Added in Proof: The author has been informed that the result of this paper was already obtained by L. F. McAuley. See Topological Conference, Arizona State University (1968) 184-202.

Reference

1. J. R. Porter, A note on common fixpoints for commuting functions, Mathematical Notes, No. 39, 1966, University of Oklahoma.

MATHEMATICAL EDUCATION

EDITED BY J. G. HARVEY AND M. W. POWNALL

Material for this Department should be sent to either of the editors: J. G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706; M. W. Pownall, Department of Mathematics, Colgate University, Hamilton, NY 13346.

ON THE PH.D. IN MATHEMATICS

I. N. HERSTEIN, University of Chicago

At the present time almost all aspects of university education in the United States are being re-examined and re-evaluated. There is a need to take another look at the nature of the Ph.D. in mathematics and at the premises which have brought it to its present form.

My own thinking on these matters is more a consequence of my association with CUPM than of today's student discontent and turmoil in our universities. Via my association with CUPM I have had the opportunity to speak with many

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people teaching in four-year colleges and in the so-called lesser universities and to become aware of the problems they face. However, what I say in no way reflects the thinking or position of CUPM. This is no mere disclaimer; the fact is that I do not know the opinions of the various members of CUPM on the questions I am about to discuss.

The Ph.D. in mathematics is viewed today as a research degree. The training of the student proceeding toward this degree is totally oriented to the production of research scholars. Perhaps more significant, the total emphasis is that of the importance of doing research. How realistic is all this? Even more pertinent, how effective and constructive is it?

To state with a high degree of precision what percentage of our Ph.D.'s end up as research mathematicians is extremely difficult. For one thing, there would be no uniformity of opinion on what constitutes a research mathematician or even research in mathematics. Using the vaguest of criteria, the general estimate is that between 20 and 25 percent of our Ph.D.'s go on to become researchers. What happens to the other 75%? For the most part they end up teaching in four-year colleges and lesser established universities.

In what state do the people making up this 75% arrive in the institutions where they will take up their life's work—teaching? Very often they arrive poorly educated. Since everything in their graduate training has been pointed towards the production of an *original, publishable, research* thesis, too often they have narrowed or have been narrowed in order to learn enough about some small slice of mathematics so as to be able to write something original. They have been given a view of mathematics through a tiny slit and have very little of a global view of mathematics or of their own particular sub-field. Now they are to teach across the whole undergraduate mathematics curriculum, which today involves much non-trivial subject matter, not as algebraists, topologists or analysts but as mathematicians. Are they prepared to do so? On many subjects that they will be required to teach they will know little more than their students and will have relatively little more appreciation of how these things fit into the large picture. How can they guide and imbue their students with a proper feeling for their subject? True, they are more mature, especially if it comes to the structure of hemi-demi groupoids with chain conditions say, the topic of their thesis. Unfortunately this sophistication on a very small corner of mathematics does not help them at all in carrying out their every-day function as good teachers and in speaking with some authority about their chosen discipline to their students.

Unfortunately, this sophistication about hemi-demi groupoids with chain conditions does not even help them very much as researchers, either. How much is there to say about such gadgets and, in fact, is it worth saying anything about them? The specialized knowledge about them is of no use when trying to switch to try one's hand at some other research.

So, for a large part of this 75%, the training given fails on both counts. It does not give the necessary breadth of knowledge and understanding of math-

ematics to teach from a broad point of view nor does it put the person in a position to do mathematical research.

Worse than this, there is something else amiss in the picture. Many of this 75% have been inculcated throughout their whole graduate education with the feeling that being a mathematician is synonymous with being a research mathematician. Anything less is failure. So now one suffers from a sense of guilt, from a sense of not delivering the goods expected, from the feeling of being a failure in one's chosen profession. This frequently leads to a demoralization, an indifference to "lesser" duties—that is, teaching—and imagined failure now becomes actual failure. Why should people doing the honest job of being good teachers and responsible members of their departments have to feel ashamed of what they are doing?

Another factor now enters the scene. Their colleges and universities have been conditioned—by our own profession unfortunately—that they must have research mathematicians in their departments. So these administrations start applying pressure, often economic in nature, that their department members produce research. The net effect of this is to intensify the frustration and sense of failure in a great part of their mathematics staff.

How often I get letters from reasonable institutions, with a fine reputation and tradition for producing good mathematics majors, asking me for the names of some research mathematicians who might be interested in jobs with them. They want them so as to embark on M.A. or Ph.D. program. Why? There is no lack of such programs around; a new program, like the other existing ones, serves no particular purpose. Possibly the economics of research grants enters, to some degree, in their thinking. Of much greater relevance, however, is the desire to acquire the prestige and status which they think a graduate program will give them. Here again, our own profession has been guilty of selling them a bill of goods: namely, that the best (if not only) road to success, honor and all that is to have a lot of hot research mathematicians on their faculty. So one embarks on a graduate program. At best it is doomed to end up as just another, often mediocre, graduate program indistinguishable from so many others around.

Too often, the effect of this is disastrous. In taking on this new activity, a very noble objective—that of training good young mathematics majors—gets distorted or perverted. A fine undergraduate program now plays second fiddle to an n-th rate graduate effort. The essentially undergraduate faculty members get shunted aside and ignored.

Even if the desire to acquire the graduate faculty and to undertake a graduate program were justified, the dice are loaded against such ventures. Where will the necessary faculty be found? Where will reasonable graduate students be recruited? There are too many competing, better established, programs already in being for these new ones to take root and to enter the competition successfully.

In what I have said above we see a distorted set of values at play. Where do these values come from? They come from the highly active research end of our

profession and from the famous high-powered institutions with a long history of producing first-rate, highly successful, Ph.D.'s. They have imposed a set of values, from the top down, on the whole spectrum of institutions teaching mathematics in the United States. This small group of mathematicians and institutions has no difficulty meeting the standards it has set, but it has set a level of standards and performance on the whole of the profession which the profession in the large can not live up to. It is even questionable if it is desirable that it do so.

I believe it is time that we question the objectives of our Ph.D. in mathematics and that we try to attenuate the intensity of this "research only" spirit which permeates it.

Let there be no misunderstanding! I'm not trying to denigrate or minimize, in any way, the importance of mathematical research. Quite the contrary. mathematical research is the very life-blood of mathematics as a living science and discipline. The 25% of our Ph.D.'s who do become researchers provide more than an adequate replenishment for mathematicians aging or phasing out of their research activity. The simple truth of the matter is that there is one thing we do not suffer a lack of, and that is of first-rate, highly creative young mathematicians. Never in the history of our country have we had so large, able and inspiring a group of young research mathematicians as we have today. I am extremely proud of them and their achievements. I am proud of my own students who have gone on to become first-rate research mathematicians. But I also take a deep sense of pride in my other students who have gone on to become excellent teachers and worthy members of their departments and faculties.

If the only responsibility of our community were to produce active, original, creative mathematicians then there is no doubt that we have done a darn fine job. But it isn't our only responsibility. We must take into account the other 75% of our Ph.D.'s and take cognizance of the various other activities—other than doing research—that mathematically trained people are called on to perform in the world of today. If I felt that my suggestions would imperil the production of research scholars then I would not make them. I feel that this we will continue to do. I merely want us to change the attitude and training of a large bulk of other people that we produce.

Now comes the question of how we can go about changing things. First and foremost we must consciously change our attitude and create a different atmosphere in many of our graduate schools. It must become important, and stressed as such, that the production of fine, well-educated college teachers ready and able to teach, is in itself a noble goal. That it is an effort no more or less worthy than the production of other kinds of mathematicians. We must break down the feeling that college teaching, as an end in itself, is not sufficient. Moreover, it seems to me that the place to do this is at the Ph.D. dissertation.

I am not arguing for a new degree; I was and am unalterably opposed to the Doctor of Arts degree proposed several years ago as a nonresearch degree. I do feel that the student should do some research to get his degree, that at least

once in his life he be involved intimately and actively in his own field. This is essential both for his morale and for his attitudes towards his own subject. My argument is with the meaning of the word "research" for the Ph.D. thesis.

It strikes me that the demands we place on the nature of the doctoral dissertation are much more exigent than in any other field. This is reflected in the fact that in our four-year colleges the mathematics departments have a smaller percentage of Ph.D.'s than any other departments. Why must we insist that the thesis be a serious and highly original piece of work? In actual fact it is not—at least judging by the many theses I have seen at many institutions—but we universally give the student this impression, that this is what is expected of him. This leads to a notable attrition, an attrition of people who could serve highly useful mathematical functions, of people who do not even bother to present themselves for their qualifying exams because they feel that they are not up to the exalted standards asked of them in their theses.

As far as most theses written are concerned—I speak here of the 75% rather than the 25%—the thesis is of importance only to its writer. Seldom are its results an important contribution to mathematics in the large. Sure, one can always cite counter-examples to me, but I'm speaking of the large majority. In other words, the purpose of the thesis—not its stated purpose, but in fact how it works out in the end—is to involve the writer for the first, and probably the last, time with his subject not as a learner but as a doer. It is to show his coming to maturity, to represent a break from having been a student all these years to becoming a professional capable of doing something on his own. If this be the case—and I'm convinced it is and should be—then the nature and type of research required of our Ph.D. aspirant is all wrong.

After all, what kind of research is really needed to imbue the thesis-writer with the feeling of being involved with his subject? It doesn't have to be anything highly original, ambitious or, that holy-of-holies, publishable. It should only represent an honest effort by the writer to be functioning independently as a scholar. To achieve this almost any kind of research effort would do. A thesis in which the author gives a new or newish proof of an old theorem, not necessarily some famous old theorem but even some relatively unimportant known result, would serve. It could even be short—a few pages would suffice. All I want is that the thesis have something personal of its author. Why shouldn't a compilative thesis be acceptable?—one where the writer reworks and re-organizes a certain area, not merely copying word by word from the literature but giving this reworking his own personal touches.

In most Ph.D.-granting institutions such types of theses would not be accepted. Far better some unimportant but *new* and *original* result, no matter how poor or special? This, unfortunately, is the prevailing attitude of today.

My first suggestion would thus be that the scientific scope—in general—of the thesis be reduced and that the time that the student spends on it be substantially cut. In the time saved the student could take some special courses designed to give him an integrated view of mathematics. In these, emphasis

unto itself, with little or no inter-play among the various courses he takes. Too often he sees a generalization without realizing where it comes from or without knowing the significance of particular cases of this generalization. I recall the graduate student—a good one, at that—who on an oral exam knew the spectral theorem for normal operators on a Hilbert space cold, but didn't know that a Hermitian matrix could be diagonalized. When it was pointed out to him that the second was a finite-dimensional consequence of the first, he appreciated it immediately. However, when asked to prove the matrix result, he could only repeat the liturgy he knew, namely, to give the proof of the spectral theorem in general. Clearly something was lacking in his education, and it wasn't his fault either.

I would recommend that a group, for instance CUPM, undertake the writing of suggested syllabi for a variety of such over-all courses. Perhaps some distinguished mathematicians could even be induced to write some books in this vein.

My third suggestion is in the nature of follow-up programs for this new kind of Ph.D. It would be the creation of summer institutes such as those that were held once at Cornell and twice at Bowdoin to up-date college teachers on a certain number of mathematical topics. Their purpose should not be to make researchers out of these college teachers; rather, they should treat topics that are both scientifically interesting and relatively current and whose material will help the participants in doing their teaching. The contact with other people in the same boat as themselves is as important as the scientific content of the institute. The feeling of being isolated, unimportant and a failure quickly disappears in such an atmosphere. In the institute in which I took part (at Bowdoin) it was clear that the morale of those attending was tremendously higher at the end of the institute than at the beginning. In fact, for me it was the most gratifying teaching experience I ever had.

All in all, what I have wanted to do in this polemic was to raise doubts about something which we have regarded with some smugness and self-satisfaction, namely, the success of our Ph.D. program in mathematics. I should like to see some discussion about possible changes in this program.

CONSULTATION SERVICE FOR GRADUATE PROGRAMS

With the recent rapid expansion of graduate education in mathematics, many universities are seeking advice and appraisal for new or proposed graduate programs. The editors wish to call attention to a consultation service available through the Council of Graduate Schools in the United States. This organization maintains a list of consultants in mathematics and other fields, makes arrangements for site visits, and handles financial details. Information can be obtained by writing to Gustave O. Arlt, President, Council of Graduate Schools in the United States, 1785 Massachusetts Avenue, N. W., Washington, D. C. 20036.

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E 2187. *Proposed by R. E. Chandler, North Carolina State University*

Let D be the closed unit disk in the complex plane and let $f, g: D \to D$. Suppose g is analytic on some open set containing D and suppose that f and its iterates $f^2, f^3, \cdots$ each have exactly one fixed point on D. If f and g commute $(f(g(z)) = g(f(z))$ for all $z \in D)$ then either g has exactly one fixed point in D or $g(z) \equiv z$.

E 2188. *Proposed by P. R. Chernoff, University of California at Berkeley, and W. C. Waterhouse, Cornell University*

Let $\sum_S a_n$, where S is a subset of the positive integers, mean the sum with elements of S taken in increasing order. Let F be an arbitrary countable family of subsets; construct a conditionally convergent series $\sum a_n$ with $\sum_S a_n$ converging for all S in F. (Several special cases, notably F = all arithmetic progressions and F = all geometric progressions, are problems in Pólya u. Szegö, *Aufgaben und Lehrsätze*, I.3.3.)

E 2189. *Proposed by S. W. Golomb, University of Southern California*

Let $b(n, k)$ be the number of partitions of the positive integer n into k integer parts all of which are powers of two. (We allow parts to be repeated; we allow $2^0 = 1$ as a part; and we disregard the order in which the k parts are listed.) Show that

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} b(n, k) x^n y^k = 1 \Big/ \sum_{n=0}^{\infty} (-y)^{w(n)} x^n,$$

where $w(n)$ is the number of 1's in the binary representation of n. Also, reach a conclusion about these partitions based on the substitution $y = -1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Multiplicative Function

E 1891 [1966, 538; 1967, 1266]. *Proposed by A. E. Livingston, Oregon State University*

Let f be a number-theoretic function, and set

$$F(n) = \sum_{d|n} f(d) f\left(\frac{n}{d}\right).$$

If $f(1) = 1$, and F is multiplicative, then f is multiplicative.

Remark by David Singmaster, American University of Beirut, Lebanon. The method described in solution II is fallacious. The described inversion could be done iff the original was of the form $F(n) = \sum_{d|n} g(d)$, whereas in the given we have $F(n) = \sum_{d|n} f(d) f(n/d)$ and the summand is not a function of d alone. Computation with $f(n) = \phi(n)$ gives $F(1) = 1$, $F(3) = 4$, but $2 = \phi(1)\phi(3) \neq \sum_{d|3} \mu(d) F(3/d) = F(3) - F(1) = 3$.

Conjugate Matrices

E 2018 [1967, 1005; 1968, 1115]. *Proposed by W. A. McWorter, Ohio State University*

Let M_n be the group of all nonsingular $n \times n$ matrices over a field of characteristic 0. Let S_n be the subgroup of M_n consisting of all permutation matrices. Prove that if two elements of S_n are conjugate in M_n, then they are already conjugate in S_n.

Remark by E. L. Spitznagel, Jr., Northwestern University. There are two mistakes in solution II of this problem. First, it is well known that S_6 is not complete; that is, it possesses an automorphism which is not inner. Cf. Burnside, *Theory of Groups of Finite Order*, pp. 209–210.

Second, letting

$$M = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

we find that

$$M \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{but } M \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M^{-1} \notin S_4.$$

Hence his assertion that $MS_nM^{-1} = S_n$ does not follow from the hypotheses of the problem.

An Application of the Fibonacci Numbers

E 2022 [1967, 1006; 1968, 1117]. *Proposed by Richard Parris, Tufts University*

Ordered k-tuples are formed using the two symbols A, B. In how many ways can this be done, if it is required that "A" does not occur in adjacent positions? I.e., for $k=3$, the triples (A, A, A), (A, A, B), (B, A, A) are not counted.

Comment by D. M. Bloom, Brooklyn College. In both of the published solutions the answer is incorrectly stated as F_{k-2}; it should read F_{k+2}.

Removable Squares of Polyominos

E 2108 [1968, 779]. *Proposed by L. J. Lander and T. R. Parkin, Aerospace Corporation, Los Angeles*

A polyomino is a finite rookwise-connected set of squares chosen from an infinite plane chessboard. A square in the polyomino is called removable if after re-

moving it the remainder of the set is still a polyomino. Prove that every polyomino having at least two squares contains at least two removable squares.

Solution by Neal Felsinger, State University of New York at Buffalo. Let P be a polyomino and let s, s' be two squares of P. Define $D(s, s')$ to be the length of the shortest path between s and s' lying in P. Then D is a metric on the squares of P. We shall say a square t is locally maximally distant from a square s if $D(s, t) \geqq D(s, t')$ for any square t' adjacent to t. Let s be any square of P and let t be locally maximally distant from s. Then t is removable for we can get to any square adjacent to t by a path in P not longer than $D(s, t)$, hence this path doesn't use the square t and therefore $P-\{t\}$ is connected. Finally, let t' be locally maximally distant from t. Then t and t' are removable and distinct.

Also solved by Anders Bager (Denmark) T. E. Elsner, Michael Goldberg, B. McMillan' Don M. Page, P. D. Rosenbaum, Azriel Rosenfeld, and the proposers.

A Convergent Sequence

E 2115 [1968, 897]. *Proposed by K. M. Brown, Cornell University*

Let a sequence $\{S_n\}$ be defined by

$$S_n = \frac{n+1}{2^{n+1}} \sum_{i=1}^{n} 2^i/i, \qquad n = 1, 2, \cdots .$$

Show that $\lim_{n\to\infty} S_n$ exists and find the value of this limit.

Solution by G. V. McWilliams, Texas Technological College.

$$S_{n+1} = \frac{n+2}{2^{n+2}} \sum_{i=1}^{n+1} \frac{2^i}{i} = \frac{n+2}{2(n+1)} (S_n + 1), \tag{1}$$

so

$$S_{n+2} - S_{n+1} = \frac{(n+2)^2(S_{n+1} - S_n) - S_{n+1} - 1}{2(n+1)(n+2)} . \tag{2}$$

Since $S_n \geqq 0$ for all n, we see from (2) that $S_{n+1}-S_n \leqq 0$ implies that $S_{n+2}-S_{n+1} \leqq 0$. An easy calculation shows that $S_4-S_3=0$, so $\{S_n\}$ is nonincreasing for $n \geqq 3$. Since $S_n \geqq 0$ for all n, $\lim_{n\to\infty} S_n = S$ exists. From (1) we have

$$S = \lim_{n\to\infty} \frac{n+2}{2(n+1)} (S_n + 1) = \tfrac{1}{2}(S+1), \qquad \text{so } S = 1.$$

Also solved by sixty others including the proposer.

Cubes and Quartics Generate All Residues

E 2116 [1968, 898]. *Proposed by Erwin Just, Bronx Community College, New York*

Prove that every integer, modulo a prime, may be expressed either as a sum

of two cubes, as a sum of two fourth powers, or as a sum of a cube and a fourth power.

Solution by D. A. Hensley, University of Kansas. The group of integers, mod p, with respect to multiplication is cyclic with respect to some element a. It is immediate that there are at least $(p-1)/3$ cubes, and at least $(p-1)/4$ fourth powers.

If $3|p-1$ and $4|p-1$, then there are $(p-1)/3$ cubes, $(p-1)/4$ fourth powers, and $(p-1)/12$ elements which are both cubes and fourth powers. Thus $(p-1)/2$ elements which are either cubes or fourth powers. If $4\nmid p-1$, every fourth element in the chain $a, a^2, a^3, \cdots$ is a fourth power, and there are $(p-1)/2$ of them. If $3\nmid p-1$, every element is a cube. In every case there are at least $(p-1)/2$ cubes and fourth powers in the multiplicative group I mod p.

In the integers mod p, 0 is a cube. So there are in all $(p+1)/2$ cubes and fourth powers. I mod p is an additive group, and whenever a subset of a group contains more than half the elements, every member of the group can be expressed as the sum of two elements of the subset. (This fact was one of the questions on the latest Putnam Prize Examination.) In this case, the subset is $\{b \in I \text{ mod } p \mid b=a^3 \text{ or } b=c^4,\ a \text{ or } c \in I\}$ and every integer mod p can be expressed as the sum of two fourth powers, or as a sum of a cube and a fourth power or as the sum of two cubes.

Also solved by W. J. Blundon, L. Carlitz, E. P. Del Norte, R. B. Eggleton (Australia), M. G. Greening (Australia), E. W. Trost (Switzerland), and the proposer.

An Extension of a Known Inequality

E 2117 [1968, 898]. *Proposed by Michael Schulz, Bell Telephone Laboratories, Murray Hill, N. J.*

Show that the following relation is valid for all real $n \geqq 1$:

$$\int_0^n \frac{n^{[x]}}{[x]!}\,dx \geqq e^{n-1}.$$

Solution by G. A. Heuer, Concordia College. Let

$$I(t) = \int_0^t \frac{t^{[x]}}{[x]!}\,dx;$$

if m is a positive integer and $m<t<m+1$, then

$$I(t) = \sum_{k=0}^{m-1} t^k/k! + t^m(t-m)/m!.$$

Direct calculation yields

$$I'(t) - I(t) = (t^{m-1}/m!)(t-m)(m+1-t),$$

which is positive on $(m, m+1)$. If $G(t) = \log I(t)$, we have, therefore, $G'(t) > 1$ on

$(m, m+1)$, and G is continuous at m, whence by the mean-value theorem, $G(t) > G(m) + (t-m)$. By E 1583 [1964, 208], $I(m) \geqq e^{m-1}$, so $G(m) \geqq m-1$. Thus $G(t) > t-1$, and therefore $I(t) > e^{t-1}$ on $(m, m+1)$.

Also solved by W. D. Bouwsma, Michael Goldberg, Simeon Reich (Israel), H. J. Ricardo, and the proposer.

Maximizing an Exponential Function on Partitions

E 2118 [1968, 898]. *Proposed by S. W. Golomb, University of Southern California*

For $N \geqq 1$, define

$$f(N) = \max A_1^{A_2^{\cdot^{\cdot^{\cdot^{A_k}}}}}$$

where the maximum is extended over all partitions of N into positive integers, $N = A_1 + A_2 + \cdots + A_k$. Thus $f(1)=1$, $f(2)=2$, $f(3)=3$, $f(4)=4$, $f(5)=9$, $f(6)=27$, $f(7)=512$, etc. Determine $f(N)$ in general.

Solution by Judith Richman, Drexel Institute of Technology. First of all, it is clear that for $n > 1$, $A_i > 1$ for all i in the maximum expression. Also, $f(n)$ is increasing, i.e., $f(n+1) \geqq f(n)+1$. If $A_1 = k$ in the maximum expression, then its largest possible exponent is $f(n-k)$. Thus, for $n \geqq 4$,

$$f(n) = \max_{2 \leqq k \leqq n-2} k^{f(n-k)}.$$

It is easy to show that, for $n \geqq 4$, $f(n+1) \geqq 2f(n)$. Since $f(n) = k^{f(n-k)}$ for some $k \geqq 2$, we have $f(n+1) \geqq k^{f(n-k+1)} \geqq k^{f(n-k)+1} = k \cdot k^{f(n-k)} \geqq 2f(n)$.

Next we compare $a^{f(b+1)}$ with $(a+1)^{f(b)}$ or $f(b+1)/f(b)$ with $\ln(a+1)/\ln a$. If $a \geqq 2$, then $a+1 < a^2$, and $\ln(a+1)/\ln a < 2$. If $b \geqq 4$, then $f(b+1)/f(b) \geqq 2$. Thus, $a^{f(b+1)} > (a+1)^{f(b)}$ if $a \geqq 2$ and $b \geqq 4$. A little calculation shows that

$$\frac{\ln 4}{\ln 3} < \frac{f(4)}{f(3)} < \frac{f(3)}{f(2)} < \frac{\ln 3}{\ln 2}\,.$$

Thus, $k^{f(n-k)}$ is maximum when $k=2$ except for $n \leqq 6$. As a final result we have:

$$f(n) = \begin{cases} 2^{\cdot^{\cdot^{\cdot^{2^{3^2}}}}} & \text{if } n \text{ is odd and } n > 3 \\ 2^{\cdot^{\cdot^{\cdot^{2^{3^3}}}}} & \text{if } n \text{ is even and } n > 4. \end{cases}$$

Also solved by W. D. Bouwsma, Slobodan Ćuk (Yugoslavia), R. B. Eggleton (Australia), Neal Felsinger, Michael Goldberg, Doug. Hensley, G. A. Heuer, B. McMillan, L. F. Meyers, Norman Miller, Jernej Polajnar (Yugoslavia), Steven Russ, E. F. Schmeichel, E. W. Trost (Switzerland), and the proposer.

A More General Triangle Inequality

E 2119 [1968, 898]. *Proposed by J. Garfunkel, Forest Hills (N. Y.) High School*

If A, B, C are the angles of an acute triangle ABC, prove that

$$\frac{\sqrt{1+8\cos^2 B}}{\sin A}+\frac{\sqrt{1+8\cos^2 C}}{\sin B}+\frac{\sqrt{1+8\cos^2 A}}{\sin C}\geqq 6.$$

Solution by Joseph Gillis, The Weizmann Institute, Israel. It follows from the theorem of the arithmetic and geometric means that

$$\frac{\sqrt{1+8\cos^2 B}}{\sin A}+\frac{\sqrt{1+8\cos^2 C}}{\sin B}+\frac{\sqrt{1+8\cos^2 A}}{\sin C}$$
$$\geqq 3\left[(1+8\cos^2 A)(1+8\cos^2 B)(1+8\cos^2 C)\right]^{1/6}(\sin A\sin B\sin C)^{-1/3}$$

and it will therefore be sufficient to prove that

$$F(A, B, C)=\frac{(1+8\cos^2 A)(1+8\cos^2 B)(1+8\cos^2 C)}{\sin^2 A\sin^2 B\sin^2 C}\geqq 64$$

if $A+B+C=\pi$.

For extremal values of F, $\partial F/\partial A=\partial F/\partial B=\partial F/\partial C$. But

$$\frac{1}{F}\frac{\partial F}{\partial A}=-\frac{18\cot A}{1+8\cos^2 A}$$

and so $\partial F/\partial B=\partial F/\partial C$ implies

$$\sin B\cos C(1+8\cos^2 B)=\sin C\cos B(1+8\cos^2 C),$$

i.e., $$\sin(B-C)+4\cos B\cos C(\sin 2B-\sin 2C)=0,$$

i.e., $$\sin(B-C)\{1-8\cos A\cos B\cos C\}=0.$$

But it is easily verified that, for any triangle, $\cos A\cos B\cos C\leqq 1/8$, with equality if and only if $A=B=C$. Hence the only possible extremal is the equilateral triangle. The inequality follows, and we note incidentally that equality is possible if and only if $A=B=C$. The entire argument and result still hold if, in the original expression, we permute the denominators in any way. Moreover, we have nowhere used the condition that the triangle is acute-angled.

Also solved by Huseyin Demir (Turkey) and Gojko Kalajdžić (Yugoslavia).

The G.C.D. of $S_n=\{k^n-k \mid k=2, 3, \cdots\}$

E 2120 [1968, 898]. *Proposed by E. B. Wright, Western Washington State College*

Find the greatest common divisor of the set $\{k^n-k\colon k=2, 3, \cdots,\}$ where n is a given, fixed positive integer.

Solution by J. A. Onstad, student, University of Nebraska. Let $n\neq 1$ be a fixed positive integer and let d be the greatest common divisor of the set $\{k^n-k\colon k=2, 3, \cdots,\}$. Consider

$$a = \prod_{p-1|n-1} p,$$

where p is a prime. If p is a prime such that $p-1 \mid n-1$, say $n-1=(p-1)s$, then

$$k^n = k^{(p-1)s+1} \equiv k \pmod{p} \qquad k = 2, 3, \cdots.$$

So $p \mid d$ and hence $a \mid d$. On the other hand, if p is a prime such that $p \mid d$, then $p^2 \nmid d$, since $p^2 \nmid p^n-p$. Furthermore, if g is a positive primitive root modulo p, then $g^n-g\equiv 0 \pmod{p}$ and so $g^{n-1}\equiv 1 \pmod{p}$. Hence $p-1$, the order of g, divides $n-1$, and so $p \mid a$. But this is true for each such prime p and so $d \mid a$, and since $a \mid d$ also, we have $d=a$. So the greatest common divisor is the product of those primes p such that $p-1 \mid n-1$.

Also solved by L. Carlitz, R. B. Eggleton (Australia), M. G. Greening (Australia), M. L. Faulkner, W. F. Fox, Bernard Jacobson, J. L. Johnson & C. V. Heuer, A. Makowski (Poland), Dan Marcus, D. C. B. Marsh, B. J. Parshall & M. D. Parshall, Stephen Pierce, A. P. Shah (India), Stephen Spindler, David Sumner, L. J. Warren, and the proposer. Partial solutions by Lew Kowarski, and Norman Miller.

Makowski indicates that the problem is stated and solved in A. Schinzel, *Sur les diviseurs naturels des polynômes*, Le Matematiche 12 (1957), fasc. 1, pp. 18–22.

Rectangular Areas

E 2121 [1968, 898]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Consider rectangles inscribed in a given rectangle R (i.e., having a vertex on each side of R) and suppose two such inscribed rectangles R_1 and R_2 have a common vertex on one side of R. Show that the sum of the areas of R_1 and R_2 equals the area of R.

Solution by L. Kuipers, Southern Illinois University. Let $ABCD$ be the given rectangle. Let $EFGH$ and $HQFP$ be two inscribed rectangles such that E and Q are on AB, F on BC, P and G on CD, and H on AD. Let O be the common center of the three rectangles. Then O is at equal distances from E, F, P, G, H and Q. Hence $AQ=EB=DG=PC$. We notice that GQ is parallel to DA. So we have area $\Delta HFG+$area $\Delta HFP=$area $\Delta HFG+$area $\Delta HQF=$area $\Delta HQG+$area $\Delta GQF=\frac{1}{2}$ area rectangle $AQGD+\frac{1}{2}$ area rectangle $QBCG=\frac{1}{2}$ area rectangle $ABCD$, as required.

Also solved by the proposer and twenty-nine others.

An Extension of Napoleon's Theorem

E 2122 [1968, 898]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Let D, E and F be points in the plane of a nonequilateral triangle ABC so that triangles BDC, CEA and AFB are directly similar. Prove that triangle DEF is equilateral if and only if the three triangles are isosceles (with a side of triangle ABC as base) with base angles 30°. (The "if" part, Napoleon's theorem, is known. See the MATHEMATICS MAGAZINE, 1966, p. 166.)

Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey. The following lemma is easily proved:

LEMMA. A triangle in the complex plane with vertices a, b and c is equilateral if and only if $a^2+b^2+c^2-bc-ca-ab=0$.

Let a, b, c; d, e, f be the affixes of the vertices of the triangles ABC, DEF. Since the triangles DBC, ECA, FAB are directly similar, then for some t

$$d = b + (c - b)t, \qquad e = c + (a - c)t, \qquad f = a + (b - a)t.$$

Forming the expression $U=d^2+e^2+f^2-ef-fd-de$, we find

$$U = (a^2 + b^2 + c^2 - bc - ca - ab)(3t^2 - 3t + 1).$$

If ABC is equilateral, then by the lemma, $U=0$, and again by lemma, DEF is equilateral. Now suppose that DEF is equilateral, that is $U=0$ by lemma. Since ABC is supposed to be non-equilateral, we must have $3t^2-3t+1=0$. Solving for t, we find $t=\frac{1}{3}\sqrt{3}$ cis $(\pm\pi/6)$ which proves the assertion.

Also solved by Walter Bluger, Slobodan Ćuk (Yugoslavia), M. G. Greening (Australia). L. Kuipers, C. F. Merrill, and the proposer. Jordi Dou (Spain) shows the uniqueness of the solution. A. W. Walker mentions a weaker result given in a paper by Wong, this MONTHLY, 48 (1941), p. 530.

The Generalized nth Derivative

E 2123 [1968, 899]. *Proposed by A. C. Williams, Mobil Research, Princeton, N. J.*

Define the generalized nth derivative as in problem E 1992 [1968, 900] i.e., $f^{(n)}(x)=\lim_{h\to 0}\Delta^n f(x)/h^n$, where

$$\Delta^n f(x) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{n-j} f(x + jh).$$

Show that the existence of the nth generalized derivative does not imply the existence of lower order derivatives. Show, in fact, that for each subset S of the positive integers, there exists a function such that $f^{(n)}(0)$ exists for $n\in S$ and fails to exist for $n\notin S$.

Solution by W. D. Bouwsma, Southern Illinois University. Let

$$f_k(x) = \begin{cases} x^k & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then

$$\triangle^n f_k(0) = \begin{cases} h^k \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{n-j} j^k & \text{if } h \text{ is rational} \\ 0 & \text{if } h \text{ is irrational.} \end{cases}$$

(We shall write A_{nk} for the coefficient of h^k above.) Define the operator E by $E(u) = d(xu)/dx$. It is easily shown (by induction on k) that

$$E^{k-1}\left[\frac{d}{dx}(1-x)^n\right] = \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^k x^{j-1}.$$

Then for $n > k$, we have

$$A_{nk} = (-1)^n E^{k-1}\left[\frac{d}{dx}(1-x)^n\right]_{x=1} = 0.$$

Hence $f_k^{(n)}(0) = 0$ if $n > k$. If $n < k$, we have

$$\left|\frac{\Delta^n f_k(0)}{h^n}\right| \leqq \frac{(nh)^k 2^n}{h^n} \to 0$$

as $h \to 0$, so that $f_k^{(n)}(0) = 0$.

If $n = k$ and h is rational, then

$$\frac{\Delta^k f_k(0)}{h^k} = A_{kk} = (-1)^k E^{k-1}\left[\frac{d}{dx}(1-x)^n\right]_{x=1} \neq 0,$$

so that $f_k^{(k)}(0)$ does not exist.

If S is a subset of the positive integers, let $f(x) = \sum_{k \notin S} f_k(x)$ (which converges for $|x| < 1$). Then $f^{(n)}(0)$ exists iff $n \in S$.

Also solved by the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before December 31, 1969. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

The asterisk (*) will be used to indicate that the proposer did not supply a solution. The editors solicit readers' solutions for these and for all problems (proposers' solutions are frequently not "best possible" and solutions by others will be given preference).

5673 [1969, 565]. **Correction.** The last sentence should read: Then prove that $ABA \cap BAB \neq AB + BA$ if $AB \cap BA = A + B$.

5683. *Proposed by Hugh Warren, University of Oregon*

Let U be the set of functions $f(x)$ defined on $[0, 1]$ with $|f(x)| \leqq 1$ and $|f(x)-f(y)| \leqq |x-y|$. Let E be the set of extreme points of U. It has been shown by A. K. Roy (Canadian Journal of Mathematics 20 (1968), pp. 1150–1164) that if $|f'(x)| = 1$ almost everywhere, then $f \in E$. Show by elementary means that the closure of E in the uniform norm is U.

5684. *Proposed by Steven Minsker, Massachusetts Institute of Technology*

Let $U = \{z : |z| < 1\}$ in the complex plane. We pick a sequence of open disks U_n of radius r_n such that the following three conditions are satisfied:

$$\text{(a) } \overline{U}_n \subset U, \qquad \text{(b) } \overline{U}_i \cap \overline{U}_j = \varnothing, \qquad \text{(c) } \sum_{n-1}^{\infty} r_n < \infty,$$

$n = 1, 2, 3, \cdots, i \neq j$. Let $X = U - \bigcup_{n=1}^{\infty} U_n$. Prove that X has positive Lebesgue measure.

5685. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur*

Let m, n be positive integers and $a_1, a_2, \cdots, a_n$ be positive reals. For $i = 1, 2, 3, \cdots$, put $a_{n+i} = a_i$ and $b_i = a_{i+1} + a_{i+2} + \cdots + a_{i+m}$. Then show that $m^n a_1 a_2 \cdots a_n < b_1 b_2 \cdots b_n$, except if all the a_i are equal.

5686. *Proposed by J. K. Washenberger, Virginia Polytechnic Institute, Blacksburg*

The field $Q(t)$ of quotients of polynomials with rational coefficients is the standard example of a non-Archimedean ordered field. In this field $p(t)/q(t) > 0$ if the leading coefficient of the product $p(t)q(t)$ is positive. Characterize the Archimedean ordered subfields of $Q(t)$.

5687.* *Proposed by Z. Govindarajulu, University of Kentucky*

Prove or disprove:

$$\int_{-\infty}^{\infty} [f^2(x)/\{1 - F(x)\}]dx = 2\sqrt{2}/\pi,$$

where $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$, $-\infty < x < \infty$ and $F(x) = \int_{-\infty}^{x} f(t)dt$.

5688. *Proposed by P. R. Chernoff, University of California, Berkeley*

Let X and Y be normed vector spaces and $f: X \to Y$ a homogeneous isometry; that is $f(t \cdot v) = tf(v)$ and $\|f(v) - f(w)\| = \|v - w\|$ for all scalars t and vectors v, w. Must f be linear? (Cf. Banach, p. 166, Theorem 2.)

SOLUTIONS OF ADVANCED PROBLEMS

Coefficients of Polynomials

5620 [1968, 910]. *Proposed by Simeon Reich, The Technion, Haifa, Israel*

What is the nature of those natural numbers N with the following property:

if $f(x)$ is any polynomial with integer coefficients such that $f(m)\equiv 0 \pmod{N}$ for all integers m, then N divides each coefficient of $f(x)$?

Solution by Ellen Hertz, Columbia University. The only such N is 1, as is seen by considering the polynomial

$$x(x+1)(x+2)\cdots(x+N-1).$$

Also solved by Anders Bager (Denmark), L. Carlitz, P. R. Chernoff, Robert Gilmer, F. Göbel (Netherlands), D. A. Herrero, M. S. Klamkin, M. L. Laplaza (Puerto Rico), Douglas Lind (England), O. P. Lossers (Netherlands), Andrzej Mąkowski (Poland), Brian Parshall, Ira Rosenholtz, and Steven Russ.

Numerical Range and Eigenspaces of a Linear Operator

5621 [1968, 910]. *Proposed by J. P. Williams, Indiana University*

Let T be a bounded linear operator on a complex Hilbert space whose numerical radius

$$|W(T)| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$$

does not exceed 1. Show that each eigenspace corresponding to an eigenvalue of modulus 1 decomposes T.

Solution by P. R. Chernoff, University of California, Berkeley. Suppose $Tx=\mu x$, $|\mu|=1$, $\|x\|=1$. Then if $\langle x, y\rangle=0$, $\|y\|=1$, and t is any number, we have $\|x+t\cdot y\|^2=1+|t|^2$, so that by hypothesis

$$\begin{aligned}1+|t|^2 &\geqq |\langle T(x+ty), x+ty\rangle| \\ &= |\mu+t\langle Ty, x\rangle+|t|^2\langle Ty, y\rangle|.\end{aligned}$$

It follows that $\langle Ty, x\rangle=0$ for all such y; that is, the orthogonal complement of x is T-invariant.

Also solved by S. K. Berberian, Mary R. Embry, C. R. Mac Cluer, Bernd Schmidt (Germany), and the proposer.

Berberian and Schmidt note that the result appears in a paper by S. Hildebrant, Math. Annalen, 163 (1966) pp. 230–247; see Theorem 2, wherein it is proved that the eigenspace under consideration is also an eigenspace of the adjoint operator.

Countable Families of Well-ordered Sets

5622 [1968, 910; 1969, 94]. *Proposed by D. S. Lawrence, Brooklyn Polytechnic Institute, New York*

Let X be a collection of sets of real numbers, each of which is well-ordered by magnitude. Suppose also that X is well-ordered by inclusion. Show that X is denumerable.

Solution by Peter Ungar, New York University. A set of points on the real line is well-ordered by magnitude if and only if it contains no infinite decreasing sequence, and such sets must be countable. If the union of every countable subcollection of X is well-ordered then the union U of all the sets in X is also well-ordered, since if U contains an infinite decreasing sequence so does the union of the countable subcollection of X from which the members of that sequence came.

We now prove that X is denumerable. Since X is well-ordered, it is isomorphic to an initial segment I of the class of ordinals. For $z \in I$, let S_z denote the set of X which corresponds to the ordinal z. The cardinality of U is at least as great as the cardinality of I since we can establish a 1-1 correspondence between the ordinals in I and a subset of U as follows: To the ordinal z assign the least element in S_{z+1} which is not in S_z; there is at most one z for which the above assignment can not be carried out because $z+1$ is not in I and we can assign the least element of S_0 if this arises.

Suppose now that I is not denumerable; then U is not denumerable either. Thus U is not well-ordered by magnitude. Hence X has a countable subcollection $S_{z_1}, S_{z_2}, \cdots$ whose union is not well-ordered either. Let L be the least upper bound of the z_i. Since the least upper bound of a denumerable set of denumerable ordinals is denumerable (each S_z, being countable, has at most a countable number of predecessors), L is denumerable. Since I was assumed nondenumerable, every denumerable ordinal is in I. By the inclusion property of the collection X, the set S_L contains the union of the S_{z_i} and is not well-ordered with respect to magnitude, in contradiction to the hypothesis. Thus X is denumerable as in the assertion.

If it is merely assumed that X is simply ordered, it does not follow that X is denumerable. A simple class may be obtained by indexing the rationals and letting A_t be a set of integers n for which $r_n < t$. The class A_t is clearly nondenumerable. Sierpinski has extended this by establishing the existence of families of increasing sets of real numbers whose cardinality exceeds c. See, e.g., Proposition C_{64} in his *Hypothèse du Continu.*

Also solved by Anders Bager (Denmark), L. F. Botway, R. A. Christiansen, R. O. Davies (England), E. P. Del Norte, M. A. Ettrick, Fred Galvin, G. S. Glazer, J. M. Howard, J. P. Jones, M. L. Laplaza (Puerto Rico), Dan Marcus, W. G. McArthur, Ka Menehune, J. C. Morgan II, Otto Morphy, Hugh Noland, Charles Riley, Joel Spencer, P. van der Steen (Netherlands), R. C. Weger, and the proposer.

Derangement of Series

5623 [1968, 911]. *Proposed by J. M. S. Simões Pereira, Gulbenkian Scientific Computing Center, Lisbon, Portugal*

Let $\{f_n\}$ be a sequence of continuous real functions such that $\sum_{n=0}^{\infty} f_n(x)$ is conditionally convergent in a neighborhood of a number c but the sum is not continuous at c. Is it possible that some rearrangement of the series will have a sum which is continuous at c?

Solution by Charles Riley, Keene (N.H.) State College. The situation can occur. Let $f_n(x) = (-1)^n |x| / n$ for all x, and

$$g_n(x) = \begin{cases} (-1)^n(|x| - 1/n), & x \leqq 1/n \\ 0, & x \geqq 1/n. \end{cases}$$

Consider the series

$$f_1(x) + g_1(x) + f_2(x) + g_2(x) + f_3(x) + g_3(x) + \cdots.$$

If $x \neq 0$, only finitely many of the g_i have nonzero value. For $1/(m+1) \leqq x \leqq 1/m$, the sum becomes

$$\sum_{n=1}^{m} g_n(x) + \sum_{n=1}^{\infty} f_n(x) = S|x| + \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n} + l|x|.$$

where $l = \sum_{n=1}^{\infty}(-1)^n/n$; so $S=0$ or $S=-1$, depending on the parity of m. As $x \to 0$, this sum has limit $= -l$. For $x=0$, $f_n(x)=0$ and the series sums to $\sum_{n=1}^{\infty} g_n(0) = -l$. Thus the sum is continuous at $x=0$. Let $\{k_n\}$ be an arrangement of the positive integers for which $\sum_{n=1}^{\infty}(-1)^k/k_n = 0$, and rearrange the series to

$$f_1(x) + g_{k_1}(x) + f_2(x) + g_{k_2}(x) + f_3(x) + g_{k_3}(x) + \cdots.$$

The sum of this series for $x \neq 0$ is the same as in the original, since only finitely many of the g_i are nonzero. But at $x=0$, the series sums to 0.

Also solved by P. R. Chernoff, M. A. Ettrick, D. A. Hejhal, O. P. Lossers (Netherlands), and the proposer.

Iterates and Derivatives of Entire Functions

5624 [1968, 911]. *Proposed by A. A. Mullin, Taegu, Korea*

Let A be the set of all nonconstant analytic functions, let C be the set of all complex numbers, and let N be the set of all nonnegative integers. Put F^2 for $F(F(\cdot))$ and let F^n be defined recursively. Let $F^{(n)}$ be the nth derivative of F. Does there exist a function $F \in A$ such that $F^2 = F^{(1)}$? Indeed, does there exist $F \in A$ such that for each $n \in N$, $F^{n+1}(z) = F^{(n)}(z)$ for every $z \in C$?

Solution by O. P. Lossers, Technological University, Eindhoven, The Netherlands

We shall show that any function F satisfying $F(F(z)) = F'(z)$ must vanish identically. If $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent $(F(z_1) = F(z_2) \Rightarrow z_1 = z_2)$, then by Picard's Theorem and the fundamental theorem of algebra, F must be linear, and then we have $F \equiv 0$.

Therefore, suppose $F(z_1) = F(z_2)$, $z_1 \neq z_2$. Then, by repeated differentiation of the equation $F(F(z)) = F'(z)$ we obtain

$$F^{(n)}(z_1) = F^{(n)}(z_2), \qquad n = 0, 1, 2, \cdots$$

so that F is periodic with a period equal to $z_2 - z_1$. Write

$$F(z) = \sum_{n=-\infty}^{+\infty} c_n e^{\beta n z},$$

the Fourier expansion being uniformly convergent on any compact set, where $2\pi i/\beta$ is a period with minimum absolute value. We put $e^{\beta z} = w$, so that the fundamental strip of F is mapped one-to-one onto $\{w : w \neq 0\}$, and we have $F(z) = G(w) = \sum_{n=-\infty}^{+\infty} c_n w^n$.

If G is not simple then F has another period in the fundamental strip and therefore F is double-periodic in view of the minimality of $|2\pi i/\beta|$. So F is constant by Liouville's Theorem and then F must be zero.

The remaining case (G is simple) is treated as before, so that G must be linear, $F(z)\equiv c_0+c_1e^{\beta z}$. Substitution in the fundamental equation implies that again F is constant$\equiv 0$.

Also solved by Robert Goldstein (England) who proves more generally that if $F^{n+1}=F^{(n)}$ for some integer n, then $F\equiv 0$. He uses the bound estimate $(R-r)M(r, F^{(1)})\leqq M(R, F)$, $(R>r)$ where $M(r, g)=\max|g(re^{i\theta})|$, $0\leqq\theta\leqq 2\pi$. This leads to $M(r, F^{(n)})\leqq M(z^n r, F)$, r large. This is followed by an application of some results due to G. Polya (Journal London Math. Soc., vi (1926), pp. 12–15) to establish that F^n and finally F are univalent, linear, and identically zero.

The Square-Root Function in a Topological Field

5627 [1968, 912]. *Proposed by Hansjoachim Groh, University of Florida*

Let K be a commutative topological field. Let S be the set of squares in K, and let A be the equivalence relation on $K\times K$ defined by $(k_1, k_2)A(k_1', k_2')$ if and only if $\{k_1, k_2\}=\{k_1', k_2'\}$. Is the square root function $f:S\to K\times K/A$, defined by $f(k^2)=\{k, -k\}$ always continuous?

Solution by P. R. Chernoff, University of California, Berkeley. If x_μ is a net in K, we write $x_\mu\to\pm x$ if and only if for any 0-neighborhood V it is eventually true that either $x_\mu-x$ or $x_\mu+x\in V$. Then the square root function is continuous if and only if $x_\mu^2\to x^2$ implies $x_\mu\to\pm x$.

The square root function need not be continuous. To see this consider the field K of rational functions over $\boldsymbol{R}$, with the topology of convergence in measure on the interval (0, 1). (It is well known that this is a topological field.) Choose a sequence of polynomials P_n such that $|P_n(t)-1|<1/n$ on $(0, \frac{1}{2}-1/n)$ and $|P_n(t)+1|<1/n$ on $(\frac{1}{2}, 1)$.

Then $P_n^2\to 1$ in the topology of K, but obviously P_n does not approach ± 1.

On the other hand, suppose that K has the following property:

(*) If U, V are 0-neighborhoods then there is a 0-neighborhood W such that $W\cdot(k\backslash U)^{-1}\subset V$. (All of the "usual" topological fields have this property.)

Then the square root function is *uniformly continuous* in the following sense: given a 0-neighborhood U there is a 0-neighborhood V such that if $x^2-y^2\in V$ then either $x+y$ or $x-y\in U$. Indeed, by (*) we can choose V such that $V\cdot(K\backslash U)^{-1}\subset U$. Suppose that $x^2-y^2\in V$ but $x+y\notin U$. Then $x-y=(x^2-y^2)\cdot(x+y)^{-1}\in V\cdot(K\backslash U)^{-1}$, i.e., $x-y\in U$.

Also solved by J. O. Kiltinen.

An Elementary Factor Inequality

5628 [1968, 912]. *Proposed by Raymond Redheffer, University of California, Los Angeles*

If z is complex and p is a nonnegative integer, prove that

$$\log|(1-z)\exp(z+\tfrac{1}{2}z^2+\cdots+(1/p)z^p)| \leqq c_p \min(|z|^p, |z|^{p+1}),$$

where $c_p = 1+2\log(1+p)$. Can the optimum c_p be determined?

Solution by D. A. Hejhal, University of Chicago. Define

$$E(z; p) = (1-z)\exp\{z+z^2/2+\cdots+z^p/p\}, \qquad p \geqq 1.$$

According to Hille, *Analytic Function Theory*, vol. 2, p. 195,

$$\log|E(z;p)| \leqq \begin{cases} |z|^{p+1}, & |z| \leqq 1 \\ (2+\log p)|z|^p, & |z| > 1. \end{cases}$$

The result now follows from $2+\log p \leqq 1+2\log(1+p)$ for $p \geqq 1$ which provides also a better estimate for c_p.

REVIEWS

EDITED BY KENNETH O. MAY

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.*

NOTICE

With this issue we are initiating a new printing procedure that permits telegraphic reviews to be inserted just before each issue goes to the press run. When the new system is in full operation, readers of the Monthly will get information about textbooks at approximately the publication date. In particular, the flow of books published early in the year will be listed in plenty of time for consideration before the end of the academic year.

We are continuing efforts to obtain prompt classroom reviews. Once again we ask readers who are using new books to volunteer to perform this very useful service.

It would also be very helpful if readers would call to our attention particularly good or bad examples of exposition in recently published material. Such communications could be published on the responsibility of the editor or over the signature of the contributor. We should also very much appreciate hearing from readers about our own errors of judgement or fact.

Introduction to General Topology. By Helen F. Cullen, Heath, Boston, 1968. x+427 pp. $12.95. (Telegraphic Review, August–September, 1968.)

This text treats material commonly covered in a beginning graduate-level course. In addition to material dealing with basic topological concepts, there is much that greatly enriches the text and introduces the student to many important and interesting ideas. In the treatment of connectedness, for instance, there is, in addition to standard basic material, discussions of quasicomponents, cut (i.e., separating) points, local connectivity, and an extended treatment of locally connected continua. The latter includes characterizations of arcs and of simple closed curves, a proof of the Hahn-Mazurkiewicz Theorem, and an arcwise connectivity theorem. The discussion of paracompactness and metrizability is quite extensive. There is some material on rings of continuous functions, and a proof of the Stone-Weierstrass Theorem. There is a chapter on homotopy theory, the fundamental group, and related topics.

Although this text requires no familiarity with topology and, except for set theory, is self-contained, the reviewer feels that, for most beginning students, a different text would be advisable. The scope of the text and the depth of much of the material demands considerable maturity on the part of the student. The author unfortunately provides no treatment of those set theoretical concepts used. Such material (the Axiom of Choice, for instance, in its various equivalent forms) may present considerable difficulty. This omission is a serious weakness of the text.

There are not enough *illuminating* examples. This, however, is a common failing of topology texts. There is little indication to the student that the theorems, lemmas, etc., answer significant and interesting questions about definite topological objects. There are a few exercises, but they help little in this direction.

The author seems, at times, excessively fussy. Frequently the notation emphasizes precision at the expense of clarity, a questionable practice in a beginning text. Many proofs are written out in enormous detail. Such detail is possibly unnecessary (and probably undesirable) for any student who is studying this text, particularly the last half.

In the reviewer's opinion, this book is best suited for reasonably mature students, either with good preparation in set theory or studying set theory concurrently. Many examples will have to be supplied, either by the student or his instructor.

STEVE ARMENTROUT, University of Iowa

Geometry for Teachers. By G. Y. Rainich and S. M. Dowdy. Wiley, New York, 1968. ix+228 pp. $7.95. (Telegraphic Review, March 1969.)

The subtitle, An Introduction to Geometrical Theories, indicates succinctly both the scope and purpose of the book. The geometries introduced are euclidean, projective, affine, inversive, hyperbolic, equiform, and equiaffine. Following a discussion of euclidean geometry by use of vectors, the authors

present concepts of the succeeding geometries very carefully as generalizations or modifications of familiar concepts in euclidean geometry. Intuitive beginnings are followed by axiomatic systems. The reader is advised to digest the material in the first half of the text in order to cope with the higher level of sophistication in the treatment of foundations of geometry in the concluding sections where the respective axiom systems and groups of transformations are presented. The concluding chapter on Numbers in Geometries develops numbers from the axioms of a geometry instead of, as the authors state, "attaching the numbers to geometry ready-made from the extraneous subject of algebra and analysis." There is an ample list of exercises, with hints for solution of some of them.

This book is based on lectures given at National Science Foundation Academic Year Institutes at The University of Michigan and at The University of Notre Dame. It does not tell teachers how to teach. It provides them with an impetus toward a deeper knowledge that should underly improved teaching.

C. E. SPRINGER, University of Oklahoma

A First Course in Abstract Algebra. By John B. Fraleigh. Addison-Wesley, Reading, Mass., 1967. xvi+447 pp. $9.75. (Telegraphic Review, November 1967.)

C This book admirably satisfies its stated primary objective of providing "a text from which an *average* student of mathematics can acquire as much depth and comprehension in his first study of abstract algebra, exclusive of linear algebra, as is possible in a first course" (vii). The reviewer used this text in a trimester course in abstract algebra for regular (as opposed to "honors") majors in mathematics at Dartmouth. In this course, topics covered included finitely generated abelian groups, the Sylow Theorems, unique factorization and principal ideal domains, the structure theory of finite fields and a considerable amount of Galois theory. Both the students and the instructor must work hard to cover this much material in addition to the usual introductory material.

The exercises in the book are excellent. There are many computational exercises that are designed to give the student an insight into the workings of the theory. About half the exercises request proofs—varying from routine translations into new situations of proofs in the text to rather sophisticated extensions of the theory. Nearly every group of problems includes a problem giving about ten true-false questions. The questions amuse and challenge the student—the reviewer's students felt that these problems contributed much to their understanding of the course.

The exposition is leisurely and informal. Difficult proofs are sometimes omitted; often they are relegated to starred sections. The examples are better than average, but some confuse the students because they turn out to be hard ways of solving easy problems.

There are several real disadvantages to the book. The concept of an equiva-

present concepts of the succeeding geometries very carefully as generalizations or modifications of familiar concepts in euclidean geometry. Intuitive beginnings are followed by axiomatic systems. The reader is advised to digest the material in the first half of the text in order to cope with the higher level of sophistication in the treatment of foundations of geometry in the concluding sections where the respective axiom systems and groups of transformations are presented. The concluding chapter on Numbers in Geometries develops numbers from the axioms of a geometry instead of, as the authors state, "attaching the numbers to geometry ready-made from the extraneous subject of algebra and analysis." There is an ample list of exercises, with hints for solution of some of them.

This book is based on lectures given at National Science Foundation Academic Year Institutes at The University of Michigan and at The University of Notre Dame. It does not tell teachers how to teach. It provides them with an impetus toward a deeper knowledge that should underly improved teaching.

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counts something. It seems to me that most students already had an overdose of computations in their calculus courses, therefore ideas rather than computations and details should be emphasized. Unquestionably, a computation successfully carried out gives some satisfaction but what is this compared to the satisfaction one can get from understanding (or finding) a beautiful idea?

These criticisms notwithstanding, the book is a good one. The author tries very hard to make the student understand the subject matter, and he mostly succeeds. And that is much more than can be said of most books.

GEORGE GRÄTZER, University of Manitoba

Elements of Real Analysis. By Sze-Tsen Hu (University of California, Los Angeles). Holden-Day, San Francisco, 1967. xii+365 pp. $11.50. (Telegraphic Review, December 1967.)

In the past few years numerous textbooks have appeared on real analysis, but the one under review is quite different from others in its organization and style. Written to meet CUPM recommendations for pregraduate preparation, it is more suitable for today's first year graduate students.

Some unconventional topics are covered, e.g., topological groups, Banach algebras and Hilbert spaces with Fourier analysis. Particularly well treated are the Stone-Weierstrass Theorem, Arzelà-Ascoli Theorem, abstract measure and integration, Lebesgue decomposition of measures, Hahn and Jordan decompositions of signed measures, Radon-Nikodym Theorem, construction of a Haar measure and Lebesgue theorem on differentiation. The derivative is defined for functions mapping a pseudo-normed linear space into another, and also in the form of a distribution. On the other hand it is somewhat surprising that many basic notions are completely untouched, e.g., classification of discontinuities, Baire functions, oscillation of a function, singular functions, Lebesgue decomposition of functions of bounded variation, rectifiable curves, metric density and the Stieltjes integral (Riemann or Lebesgue). Other topics such as semicontinuity, the Riesz representation theorem and the fundamental theorem of integral calculus are introduced only in the exercises or without proofs. The four limits of a real function at a point remain undefined, but are used to define the Dini derivates. It is to be wondered if the absence of a single example of discontinuous functions, of functions continuous but not derivable at one or more points and of non-Lebesgue-measurable sets or functions could be really in the spirit of the CUPM recommendations. In the domain of cardinal numbers only countability is discussed. On ordinals and cardinals the author remarks "These are abstract, awkward, and, just like downstreets to a travelling motorist, better bypassed." So are bypassed the Cantor's nondense perfect set and Lebesgue's continuous singular function which go a long way in clarifying so many notions of real analysis. Discussions on the origin or the significance of results and examples to illustrate the necessity of their hypotheses are frequently lacking.

The style is clear and precise. Proofs are detailed and easy to follow. Two

significant slips are: on p. 186, the last line of Proposition 6.4 should read "*complete* pseudo-metric space," and on p. 307, the beginning of the 3rd line of Theorem 5.2 should read "given *σ-finite* measure ν." This book should be useful as supplementary reading, since its unusual topics are important. As a text it may prove useful to an instructor who is willing to fill the gaps.

K. M. GARG, University of Alberta

Introduction to Analysis. By Bernard Kripke. Freeman, San Francisco and London, 1968. vii+274 pp. $8.50. (Telegraphic review, March, 1969.)

This book is intended to lead students through the transition from calculus to functional analysis. The list of chapter titles indicates its scope: 1. Introduction; 2. Axioms for the real number system and their consequences; 3. Vector spaces, inner products, and linear maps; 4. Metric and normed spaces; 5. Introduction to complex numbers; 6. Some applications; 7. Compactness; 8. Connectedness; 9. Further applications. To provide a slightly sharper focus on details, I also list the sections in Chapter 6: (A) Uniform convergence; (B) Absolutely uniform convergence of series, tests of convergence; (C) The radius of convergence of a power series; (D) The contraction mapping theorem; (E) Existence and uniqueness theorems for ordinary differential equations; (F) Linear differential equations; (G) Differentiation of functions of several variables, the inverse and implicit function theorems. Each section is followed by a few exercises. At the end of the book, the reader finds answers and solutions to starred exercises, also a nineteen-page collection of additional problems. The list of 65 references is preceded by four pages of comment (much of it frankly subjective) on their merits.

As far as I can judge without using Kripke's text in the classroom, the author has written a mathematically sound and pedagogically sensible book. However, I take offense at the flippant tone of the text and at the numerous editorial lapses. The remainder of this review concerns the book as a piece of English composition, and it is not intended to reflect on the author's competence as a mathematician or teacher.

To make certain that I do not report a negative reaction that is so strictly personal that nobody else would share it, I showed the book to two intelligent and sensitive undergraduates. One of them, a freshman girl, expressed enthusiastic approval. She was pleased that the author provides clear motivation and that he avoids both stuffiness and condescension. A slightly older student gagged. Some of the emetic items are listed in the index: careful, Cockatoo Peak, Elfenegg, glib, glop, handwaving, mistakes, muckle, nit-picking, Reward and Punishment (Principle of), skeptical student, wise, Wishful Thinking (Principle of). The criteria for successful humor are more severe in print than in the medium of the lecture. In some parts of Kripke's text, deletion would have provided the only salvation. For example, the sentence "as often happens, the truth lies somewhere to the left of both extremes, whatever that means" (page

101) would fall flat even in a lecture, to say nothing of a text aimed at intelligent undergraduates.

The complaint of the preceding paragraph is directed to the author. It concerns a matter of taste, and the author could reasonably reply that he deliberately wrote as he did. However, the text is marred by many small blemishes that are not debatable. In the joint enterprise of producing a textbook, the publisher's contribution includes the task of copy-editing. The second paragraph in the preface begins: "I am one of those teachers who is · · · ." As a slip on the author's part, the passage is excusable; on the editor's part it indicates gross dereliction of duty and calls for demotion to proofreader or to oiler of the hand-trucks in the printing shop. The preface continues: "It has been my custom to write Dittoed lecture notes according to my taste, which I pass out to my students." A competent editor would have changed the capital D to a lower-case letter, and he would have changed the second half of the sentence to read "and to distribute them among my students." On the second page of the preface, the author writes " · · · , I could hardly see any justification for adding one more to the heap of textbooks on calculus that are available." The editor should have replaced the last six words with "calculus texts," or at most with "available calculus texts."

My list stops here, not for want of material, nor because I feel sorry for the publisher, but out of courtesy to the reader. I take the position that the author has written prose of the quality that we can reasonably expect from active and vigorous young mathematicians, that with a moderate effort any good craftsman in the publishing trade could have made it excellent, and that the publisher ignored both an obligation and an opportunity.

I plead guilty to crying "Oh the time, oh the customs!" If our civilization is not to degenerate into a vast oil-slick on the culture of modern man, we must imbue our students with a feeling for the necessity of excellence. This calls for painstaking care in the construction of small sailboats, in the performance of chamber music, and in the production of books.

GEORGE PIRANIAN, University of Michigan

Introduction to Analysis. By Maxwell Rosenlicht. Scott, Foresman, Glenview, Illinois, 1968. 254 pp. $10.75. (Telegraphic Review, April, 1969.)

This book is suitable for a high-powered course in advanced calculus at the junior or senior level. It joins the growing trend of presenting the theory of metric spaces and then polishing off elementary calculus in a few pages.

The content of advanced topics is between that of Goldberg's "Methods of Real Analysis" and Rudin's "Principles of Mathematical Analysis." Goldberg limits himself to one variable, while Rosenlicht also treats differentiation and integration of functions of several variables, and Rudin goes all the way to Stokes' Theorem. Both Goldberg and Rosenlicht prove Picard's Existence Theorem for differential equations by using the fixed-point theorem for contractions. Rosenlicht considers only Riemann integration, while both Goldberg

and Rudin do the Lebesgue theory. Rudin also does Riemann-Stieltjes integration.

The proofs are clear and easy to follow. The book is "readable" by any serious mathematics student with a background in calculus.

There is an ample supply of exercises ranging from the routine to the very interesting and the nearly impossible.

ROBERT ELLIS, University of Maryland

Lectures on Calculus. Edited by Kenneth O. May (Univ. of Toronto). Holden-Day, San Francisco, 1967. vii+180 pp. $5.75 (paper). (Telegraphic Review, April, 1968.)

This is a collection of lectures by nine different mathematicians. Each is self-contained and excellent supplemental reading for the serious calculus student; two or three are real gems, worth anyone's while, ideas from which ought to find a place in subsequent analysis texts. For example, M. K. Fort, Jr., in *Continuous Square Roots of Mappings* proves that the identity map on the circle admits no continuous square root while any continuous map of the closed disk into itself minus 0 does. These facts are then used to prove an *Antipoden Satz*, Brouwer's Fixed Point Theorem in the disk and the Fundamental Theoreom of Algebra. (Essentially a homotopy of each polynomial of degree n to z^n is constructed for the latter proof.) This is an exceptionally elegant job, proceeding from absolutely minimal hypotheses. The only background required is that a continuous complex function on a closed disk is uniformly continuous. Albert Wilansky in *Additive Functions* treats Cauchy's functional equation $f(x+y)=f(x)+f(y)$ on the line. For example, complete proofs are given of the theorems that $f(x)\equiv cx \Leftrightarrow f$ is continuous everywhere $\Leftrightarrow f$ is continuous somewhere $\Leftrightarrow f$ is monotone on some interval $\Leftrightarrow f$ is bounded on some interval $\Leftrightarrow f$ is bounded on some set of positive measure $\Leftrightarrow f$ maps some interval onto a nondense set of reals $\Leftrightarrow$ the graph of f is not dense in the plane. Two proofs of the latter equivalence are given; the second is a surprisingly brief 8 lines, but the first appears to be in error (lines 12, 13, 14, p. 102). Hamel bases and pathological examples are also discussed as well as a few less elementary results, for which abundant references to the literature are offered.

R. B. BURCKEL, Eugene, Oregon

Elementary Theory and Application of Numerical Analysis. By David G. Moursund (Michigan State University) and Charles S. Duris (Michigan State University). McGraw-Hill, New York, 1967. xi+297 pp. $8.95. (Telegraphic Review, May 1968.)

The abundance of worked-out-in-detail examples will appeal to the kind of student for whom the book is primarily designed; namely, sophomore-junior engineers and physical scientists with a background of elementary calculus including an introduction to ordinary differential equations. Commendable, too,

is the clarity of treatment of standard topics in numerical analysis, e.g., fixed point iterations for single equations and systems of equations, Lagrange interpolation, numerical differentiation and integration, numerical solutions of ordinary differential equations. The order of presentation of each topic, *definition-algorithm-examples-theorem-exercises*, is consistent and easy to follow with errors few in number and not too disturbing. Numerical exercises (with answers) are plentiful. Familiarity with FORTRAN is assumed; it is used extensively in sample programs and exercises.

Mathematically oriented students who delight in exploring the theoretical "why" along with the practical "how" will be disappointed by the lack of thought-provoking exercises. They will want to consult the many references the authors give to more demanding texts.

WINIFRED ASPREY, Vassar College

Introduction to Automata. By R. J. Nelson (Case Western Reserve University) Wiley, New York, 1968. xii+400 pp. $12.95. (Telegraphic Review, May 1968.)

This book is a much-needed introduction to the main mathematical ideas used in a variety of new research areas, including the theories of computability and computational complexity, logical design of switching circuits, neural nets, artificial and natural language translation, and computer programming.

The first two chapters cover sets, relations, functions, algebraic concepts, and recursive functions. Chapter 3, Formal Systems, is the heart of the book, in that its terminology is used throughout the remainder. The formal system and semithue system terminology is used to define such diverse concepts as: Turing machine, grammar, sequential machine, and pushdown automaton. Chapter 4 contains many of the main results of recursive function theory. In the preface the author says: "Chapters 1 to 4 may be viewed as independent of the others, although in my opinion they are absolutely basic to automata theory," a sentiment with which this reviewer certainly agrees. Chapter 5 is a rather complete treatment of sequential machines, including Zeiger's proof of the Krohn-Rhodes Theorem. Chapter 6 is concerned with the implementation of sequential machines by networks of switching elements, and Chapter 7 contains the main theorems of finite automata theory, and discusses finite transducers and universal acceptors. Chapter 8 contains the basic closure, containment and decidability theorems about phrase structure grammars, and proofs of the characterizations by finite and push-down automata.

The book brings a much-needed global viewpoint to the theories of automata and formal languages. It has a mathematical rigor not often seen in treatments of its subject matter. The numerous exercises and the definition of many diverse topics in terms of formal systems give the material a unity which aids in study and teaching. A possible disadvantage of the formal system approach is that it is necessary to change some nearly-established definitions and notations in order to fit them into his unified scheme. In summary, the book is a comprehen-

sive, thorough, teachable introduction to many parts of automata and formal language theory, and has every reason to become a fundamental work in the field.

N. D. JONES, Pennsylvania State University

Computer Facilities for Mathematics Instruction. By Computer-Oriented Mathematics Committee of the National Council of Teachers of Mathematics. NCTM, Washington, D. C. 1967. v+47 pp. \$.90. (Telegraphic Review, May 1968.)

It seems odd that a booklet designed to acquaint mathematics teachers with the use of computers in mathematics education contains no mention of the mathematical analysis involved in the five sample problems presented. A great deal of important mathematics can be introduced through the analysis of computer problems, and mathematics teachers need to be made aware of this. The observation that the computer will produce an invalid solution of zero as a root of the equation $x^2+40{,}000x+20=0$ under the algorithm of Sample Problem 1 and that this could be avoided by developing a new algorithm

$$X_1 = -\text{ (sign of } B)\left[\frac{|B| + \sqrt{B^2-4AC}}{2A}\right], \qquad X_2 = \frac{C}{AX_1},$$

would have been a valuable contribution. Still it is not the place of a reviewer to complain that the authors did not write a different book. The choice of material is an author's prerogative.

The five sample problems are well chosen to illustrate the different types of demand, and the authors are to be congratulated on including time runs on various configurations. This may well be the most important feature of the booklet. The emphasis in Chapter II on steps in problem solving by computer is worth the modest purchase price of the entire booklet, and the discussion of the various uses of computers in secondary schools is meritorious. Anyone using the booklet as a source of information should also be aware of PL/I and MAT (a dialect of Iverson's APL language) as possible programming languages for secondary school use.

R. V. ANDREE, University of Oklahoma

Informal Geometry. By Lawrence A. Ringenberg. Wiley, New York, 1967. xi+151 pp. \$5.50. (Telegraphic Review, March 1968.)

Chapters 1–6 (84 pp.) treat basic concepts of plane and solid geometry from a modern point of view: linear and angular measure, betweenness of points and rays, congruence of segments and angles, triangle congruence, parallelism, area, convexity, separation, surfaces, surface area, volume, etc. The treatment is informal, using intuition, experiment and reasoning in the development, but with logical relationships subordinated to the statement of fact. Chapter 4 is

noteworthy for its careful treatment of plane area. Area postulates are stated, from which are derived formulas, including those for arbitrary rectangles and for circles. In Chapters 7–11 (51 pp.), selected concepts are treated with complete rigor: e.g., incidence, distance and the ruler postulate, convexity, angles, betweenness, etc. Chapter 12 presents the principles of ruler-compass construction and provides a long list of construction exercises.

The text is clearly written, but densely packed with ideas. The chapters are divided into short sections, each with exercises. The exercises are well designed to test comprehension of the preceding text and, where practicable, to anticipate the material to follow. The exercises are mostly straightforward and few challenge the imagination. The text is designed for use in the training of elementary teachers, either as a separate course or as a unit in geometry. The author suggests that the first six chapters might be covered in 10 to 25 lessons, depending on the preparation of the students, and the entire text in proportionately more.

S. F. DICE, Wittenberg University

Language and Symbolic Systems. By Yuen Ren Chao. Cambridge, New York, 1968. xv+240 pp. $5.00 (cloth) $1.95 (paper). (Telegraphic Review, August 1968.)

The book under review is the work of one of the few contributors to the mainstream of American linguistics who started his career in the "hard" sciences. His first academic post (according to *Who's Who in America*) was Instructor of Physics at Cornell 1919–1920. In the book he sets out to introduce the general reader—that is "the general reader in the sense that he may be a specialist in some other subject, but new to the field of linguistic inquiries" (p. v)—to the broad area of inquiry called *linguistics*. In regard to this term, he provides a cautionary note: "I . . . devote more attention in this book to the place of language as a part of life and as a special case of symbolizing in general than to schools or theories of language and that is why the word *linguistics* does not appear in the title and occurs with relative infrequency for a book of this nature," (pp. v–vi).

Despite his desire not to take sides in the various controversies of linguistics, the author provides in this book an admirable introduction to the classical aspects of the subject, as well as to the scientific outlook of linguists.

A careful reading of it should be very helpful to those MONTHLY readers who contemplate active research in the applications of mathematics and/or computer science to the study of natural language.

The general mathematical reader should find the book a stimulating introduction to the methodology of one of the newest of the social sciences. Chapter 12, Symbolic Systems, should be of interest since it compares symbolism of natural language with that used in mathematics and the natural sciences. There are some mathematicians who would benefit greatly from Section 72 on the "Ten requirements for good symbols." Section 60, Translation, should interest those of us who are engaged in the translation of texts and papers.

The author provides suggestions for further reading for most of the topics he covers, either when the topic is considered or in a list at the end of the book. However, references are lacking in regard to one topic which might be of interest to MONTHLY readers, namely statistical identification of authorship. An interesting recent book on this subject is *Inference and Disputed Authorship: The Federalist* by F. Mosteller and D. L. Wallace, Addison-Wesley, 1964. Also on p. 64, he mentions the work of Noam Chomsky, but refrains from mentioning his most important work, *Aspects of the Theory of Syntax* (MIT Press, 1965).

The author tells his tale with considerable wit and simplicity, supporting his points with various anecdotes which make the book extremely easy to read.

B. BRAINERD, University of Toronto

Discrete Probability. By R. A. Gangolli and Donald Ylvisaker. Harcourt, Brace and World, New York, 1967. xii+223 pp. $4.50 (paper). (Telegraphic Review, October 1967.)

This extremely well-written text gives the student not just an introduction to but immersion in the elements of probability theory. Discrete (indeed, finite) models constitute the main portion of the text, but there is a brief discussion of some countably infinite sample spaces and of the Poisson and normal approximations to the binomial model (including a statement of the Central Limit Theorem). There are many worked out examples, and problems for the student are well spaced, varied, instructive, and numerous. (Answers to selected problems are included.) The exposition is clear and illuminating.

A feature not common to such texts is the emphasis on partitions of a sample space, the inducing of a partition by a random variable, and the definition of independence in terms of partitions. The reviewer has found these notions helpful and used them increasingly in his own teaching. Whether this is a device that clarifies what is going on, or a sophistication that leaves the average student cold, is a matter for experience to determine.

It is somewhat curious that in a book on discrete probability the only table is that of a continuous distribution function. Binomial and Poisson tables might have been helpful. Since the book appears to be basic enough and good enough to be well used by its owner, the publisher might consider putting out a hard cover edition.

B. W. LINDGREN, University of Minnesota

Combinatorial Theory. By Marshall Hall, Jr. Blaisdell, Waltham, Mass., 1967. x+310 pp. (Telegraphic Review, January 1968.)

Although this book is entitled *Combinatorial Theory*, most of it is confined to the theory of designs. As such it is a comprehensive polished survey. There are extensive discussions of difference sets, finite geometries, orthogonal Latin squares (an example of order ten is illustrated on the dust jacket), Hadamard matrices, and completion and embedding theorems. The treatment is lucid and rigorous, suitable for seniors or graduate students who have had courses in

abstract and linear algebra. This book could well be included in the training of every linear algebraist, to offset the present danger of becoming lost in homological generalities.

As a text the book is reasonably self-contained, but occasionally lacking in motivation. For example, nothing is said of the historical origin of Hadamard matrices. Problems are included only in the introductory chapters. This is unfortunate for the student reading without an instructor and for the mathematician who might be stimulated to new research.

Practising mathematicians will find this an invaluable reference book. Its only limitations are in the paucity of the bibliography and in the varied degree to which the various chapters are up to date. For example, the chapter on Hadamard matrices contains results to 1965, while that on finite geometries does not go beyond chapter 20 of the author's earlier *Theory of Groups.*

It would certainly have been impossible in one book to discuss all branches of combinatorial theory with the same lucidity and rigour devoted here to designs. However, this reviewer questions the wisdom of giving so specialized a study so all embracing a title. At least there should have been token discussions of the other important branches of the subject. For example, the only mention of combinatorial geometry (the term is never explicitly used) is in the four-page (sic) chapter on Ramsey's Theorem, where the familiar result of Erdös-Szekeres is proved. There is no mention of combinatorial results in topology, of matroids, of probabilistic set theory, of the Polya Counting Theorem, or of network flow problems. Rather, the subject matter is confined to some branches of combinatorial theory which have in the past been amenable to algebraic methods. Modulo these reservations, it is to be recommended to the mathematical community as an outstanding contribution to the development and popularization of combinatorial theory.

W. G. Brown, McGill University, Montreal

Mathematics for Management Series. By Clifford H. Springer, Robert E. Herlihy, Robert I. Beggs, Robert T. Mall. Volume I, Basic Mathematics. Volume II. Advanced Methods and Models. Volume III. Statistical Inference. Volume IV. Probabilistic Models. Richard D. Irwin, Homewood, Illinois. xii+225 pp.; ix+273 pp.; x+352 pp. xi+301 pp. Each volume $4.25 (paper). (Telegraphic Review, June 1968.)

These four volumes give a very informal introduction to many branches of mathematics for businessmen and managers. They might be consulted by mathematicians who teach in Business Schools and Industrial Engineering. Many well chosen and vividly presented examples of the application of mathematics to problems of business and management are included.

Gerhard Tintner, University of Southern California

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TELEGRAPHIC REVIEWS

Telegraphic reviews are intended to give prompt notice of books with sufficient information to assist in deciding whether to order an examination copy or suggest library purchase. Possible uses are suggested as follows: T = textbook, S = supplementary student reading, P = professional reading, TT = teacher training, L = library purchase, 13 to 18 = freshman to second graduate year level, 1 to 4 = one to four semesters. An asterisk is used for emphasis. Books covering standard high school material are called "remedial." Publishers are indicated by the standard abbreviations used in *Books in Print* (which gives full names and addresses).

ALGEBRA, P, L. *Representation of Rings by Sections. Memoirs of the AMS, No. 83.* By John Dauns and Karl Heinrich Hofmann. Am Math, 1968. xii + 179. $2.20 (paper). *Bibliography.*

ALGEBRA, T (16-17), P. *Lectures on Forms in Many Variables.* By Marvin J. Greenberg (U of Calif. Santa Cruz). W. A. Benjamin, 1969. 165. $12.50 (cloth) $3.95 (paper). Lecture notes from 1968. Assumes a first course in abstract algebra. Centers on the recent discovery that over certain coefficient fields, if a homogeneous equation has a large number of variables compared to its degree, it automatically admits a non-zero solution. *Bibliography.* No exercises.

ALGEBRA, S (16-17), P. *Infinite Abelian Groups.* Revised edition. By Irving Kaplansky (U of Chicago). U of Mich Pr, 1969. vii + 95. $4.00 (paper). This second edition (first in 1954) includes some corrections, revisions and, unfortunately, the replacement of the complete bibliography up to 1952 by a list of titles mentioned in the text. An updated bibliography such as appeared in the first edition would make the book a must library purchase and of permanent value to every owner, whereas now it is intended only as a less comprehensive "slim volume" to be used with Fuchs' "definitive treatise".

ALGEBRA, T (13-14), TT. *A Programmed Course in Algebra* with a separate *Summary Textbook.* By Ancel C. Mewborn (U of North Carolina), with technical consultation by Wells Hively II (U of Minnesota). A-W, 1969. xvii + 630. $4.95 (cloth), vi + 122. $1.50 (paper). A competent, well-written, effective program of reasonable size and at a modest price! These materials developed from a project initiated by P. C. Rosenbloom at the University of Minnesota in 1963 and are the results of considerable testing on preservice and inservice teachers. I used the materials in a preservice course for teachers with two years of college mathematics and found it very effective and popular (K.O.M.). It requires about 100 hours of student work and can be handled by students as an independent activity carried on in tandem with class-centered activities. This publication contains all of the latest edition of the materials (1966) except a battery of tests, which should be made available to users.

ANALYSIS, APPLICATIONS, P. *Function Theoretic Methods in Partial Differential Equations.* By Robert P. Gilbert (Indiana U). Acad Pr, 1969. xviii + 311. $17.50. One and several complex variables applied to scattering theory and other problems of mathematical physics.

ANALYSIS, P, L. *Theory and Applications of Spline Functions.* Edited by T. N. E. Greville. Proceedings of an Advanced Seminar Conducted

by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, October 7-9, 1968. Acad Pr, 1969. xi + 212. $4.95. "Spline functions are a class of piecewise polynomial functions satisfying continuity properties only slightly less stringent than those of polynomials, ...have highly desirable characteristics as approximating, interpolating, and curve-fitting functions." An introduction and survey requiring no specialized knowledge. *Bibliographies*.

ANALYSIS, *S (13-15). *Theory of Area*. By Marvin I. Knopp. Markham Publishing Co., Chicago, Ill., 1969. xii + 106. $3.50 (paper). A detailed rigorous exposition of Jordan content in the plane, designed for, and used successfully in, an honours course in calculus.

ANALYSIS, (15-16; 2), S. *Topics in Modern Analysis*. By Robert G. Kuller (Northern Illinois U). P-H, 1969. viii + 296. $7.95. Main goal: to bring the student as quickly as possible "within reach of serious study of functional analysis." After three chapters covering the essentials of set theory, real numbers, and metric spaces, there follow chapters on normed vector spaces, integration theory (emphasis on the Lebesgue integral), linear functionals, and an appendix on Zorn's lemma, the axiom of choice, and applications.

ANALYSIS, P, L. *Stochastic Integrals*. By H. P. McKean, Jr. (Rockefeller U). Acad Pr, 1969. xiii + 140. $9.00. A research level monograph, presupposing some knowledge of modern analysis at about the level of a first course in functional analysis and covering the differential-integral calculus based on Brownian motion.

ANALYSIS, T (15; 1). *An Introduction to Real Analysis*. By Burton Randol (Yale U). HB & W, 1969. xi + 112. $6.50. Real numbers, functions, power series, the Weierstrass approximation theorem, fourier series, and an introduction to the Lebesgue integral. The exposition is terse and informal, though following the definition-theorem-proof-remark format.

ANALYSIS, T (17-18), P, *L. *Generalized Functions and Partial Differential Equations*. By Georgi E. Shilov (Moscow State U). Authorized English Edition revised by the author. Translated and edited by Bernard D. Seckler. Gordon, 1968. xii + 345. $21.00. Topics include convolutions, order of singularity, Fourier transforms, Hormander's theory of hypoellipticity and well-posed problems for a half-space.

ANALYSIS, PHYSICS, P, L. *Special Functions. A Group Theoretic Approach*. Based on Lectures by Eugene P. Wigner. By James D.Talman (U of Western Ontario). With an Introduction by Eugene P. Wigner. W. A. Benjamin, 1968. xii + 260. $10.00 (cloth) $5.95 (paper). The badly named topic of special functions has been notoriously resistant to meaningful organization as a part of modern mathematics and has remained largely a collection of particular results. This is another, and partly successful, effort.

ANALYSIS, REPRINT, S, P, *L. *Trigonometric Series*. By A. Zygmund. Two volumes bound together. Cambridge U Pr, 1968. xiii + 383 and vii + 364. $17.50. Corrected reprint of the 1958 edition with a more comprehensive index. The first volume is a complete revision of the classic first edition of 1935, the second volume covers material not in the original edition. An authoritative and definitive treatise on the classical theory of Fourier series. Historical notes, *bibliography*.

APPLICATIONS, T (15), *Mathematical Methods in Physics and Engineering*. 2nd edition. By John W. Dettman (Oakland U). McGraw, 1969. xi + 428. $10.50. A very extensive revision of the first edition of 1962.

*BIBLIOGRAPHY, P, *L, *Contents of Contemporary Mathematical Journals*. Published bi-weekly by the American Mathematical Society. $18.00 per year ($9.00 to individuals). This journal which began publication January 10 1969, reproduces photographically the tables of contents of journals of mathematical interest, along with addresses of the authors.

*BIBLIOGRAPHY, P, MATHEMATICAL OFFPRINT SERVICE, American Mathematical Society. This service (begun in 1968) enables a subscriber to receive prompt notification of articles of primary interest. Cost is nominal. About 300 journals are covered.

BUSINESS MATH, *T (13), *College Mathematics with Business Applications*. By John E. Freund (Arizona State U). P-H, 1969. xiii + 625. $9.95. Assuming only "some high school algebra," this book deals with elementary topics in algebra and analysis, linear programming, matrix algebra, probability, and simulation. A model is likened to a dress form.

BUSINESS MATH, T (13), *College Mathematics for Business*. By Flora M. Locke (Merritt College, Oakland, Calif). Wiley, 1969. ix + 311. $7.95. The prerequisite of only elementary arithmetic testifies to the pathetically weak mathematical background of many college students. The usual topics of mathematics of finance. Unfortunately, no probability and statistics.

*CALCULUS, T (13-14; 3-4), P, L, *Calculus*. By Lipman Bers (Columbia U). HR & W, 1969. xv + 1047. $13.95. Notable for the distinction of its author, the thoughtful and non-routine approach (based on the historically sound point of view that "calculus is the art of setting up and solving differential equations"), the elegant two color printing job, the interesting format with wide margins utilized for notes, titles, figures, and historical comment. It weighs nearly five pounds of which perhaps two are accounted for by unfilled margins. Style is informal. Approach is closely related to applications and to history (the author acknowledges inspiration of Toeplitz and the physicist Zeldovich). Rigor is interpreted as honesty and clarity rather than morbid concern for proving everything in detail. In no field of mathematics is the proportion of distinguished texts as low as in calculus. This book promises to increase the ratio.

CALCULUS, T (13; 2), *Calculus of the Elementary Functions*. By Merrill E. Shanks and Robert Gambill (both of Purdue U). HR & W, 1969. xiii + 545. $10.95. Designed for a one year course covering the usual material through partial derivatives and multiple integrals, this text emphasizes an intuitive approach and problem solving. "Calculus, after all, is a certain kind of calculation; calculus is the solving of problems." Other features are 3000 graded problems, the exclusion of analytic geometry proper, and the separation of differential and integral calculus. There are historical comments. The result is a promising entry in the field of moderate sized texts.

COMPLEX ANALYSIS, *S (14), TT, *Complex Numbers and Their Applications*. By F. J. Budden (The Royal Grammar School, Newcastle Upon Tyne). Longmans, London, 1968. xi + 243. $2.40 (paper). Beginnings of complex analysis with rigor, motivation, and related topics.

COMPLEX ANALYSIS, P, L. *Topics in Several Complex Variables*. By A. Douady, H. Grauert, B. Malgrange, R. Narasimhan, K. Stein. Lectures given at a summer school in Otaniemi, Finland, June 1967. *Monographie No. 17 of L'Enseignement Mathématique*, Genève, 1968. 119. 15 Swiss Francs. This latest in the distinguished series (which began in 1956 with Lebesgue's *La mêsure des grandeurs)* contains 5 articles: *Analytic Spaces* (B. Malgrange), *Meromorphic Mappings* (K. Stein), *Flatness and Privilege* (A. Douady), *Compact Analytic Varieties* (R. Narasimhan), *The Coherence of Direct Images* (H. Grauert). All in English with *bibliographies*.

COMPLEX ANALYSIS, *T (15-17), L. *Introduction to Complex Analysis*. By Rolf Nevanlinna (Academy of Finland) and V. Paatero (U of Helsinki). Translated by T. Kovari and G. S. Goodman. A-W, 1969. ix + 348. $11.50. A solid, high-level introduction presupposing only strong calculus background. The approach is of the "Finnish School", which has been described as "geometric and intuitive".

COMPUTER, S. P. *Perceptrons. An Introduction to Computational Geometry*. By Marvin Minsky and Seymour Papert. M.I.T. Press, Cambridge, Mass., 1969. 258. $4.95 (paper). A tentative exposition of developing ideas relating to pattern recognition. Requires little mathematics beyond the high school level.

COMPUTERS, ANALYSIS, T (16-17), S, P, L. *Computational Solution of Nonlinear Operator Equations*. By Louis B. Rall (Mathematics Research Center, U.S. Army, U of Wisconsin). Wiley, 1969. vii + 225. $14.95. An introduction to powerful computational methods based on concepts of functional analysis. Presupposes elementary numerical analysis, computer programming, linear algebra from a computational point of view, real and complex variables.

DIFFERENTIAL EQUATIONS, T (17), P, L. *Lectures on Ordinary Differential Equations*. By Einer Hille (Yale U). A-W, 1969. xi + 723. $17.50. Based on lectures given by the author over a period of 45 years at many universities. Not a compendium but selected topics for students with training in differential equations, linear algebra, real and complex analysis. Some new material. Topics include linear second order and non-linear equations.

DIFFERENTIAL EQUATIONS, T (15-16; 1-2). *A Course in Ordinary and Partial Differential Equations*. By Zalman Rubinstein (Clark U). Acad Pr, 1969. x + 477. $12.00. Combines substantial work in both ordinary and partial differential equations and could be used for either one, the other being left for self study. Undergraduate calculus and linear algebra, but no differential equations, are expected.

ECONOMICS, P. *Convex Structures and Economic Theory*. By Hukukane Nikaido (Osaka U). Acad Pr, 1968. xii + 405. $19.50. Topics include convexity (86 pp.), simple multisector linear systems, balanced growth in nonlinear systems, efficient allocation and growth, and global univalence. *Bibliography*.

*EDUCATION, P, L. *Aspects of Graduate Training in the Mathematical Sciences. Report of the Survey Committee. Vol. II*. Conference Board of the Mathematical Sciences, 2100 Pennsylvania Avenue, N.W. Washington D.C., 20037. xiv + 140. $2.25. Volume I on undergraduate education was reviewed telegraphically in January 1969. Volume II authored by John Jewett, Lowell J. Paige, Henry O. Pollak and Gail S. Young, presents information on many important matters, such as the production of Ph.D.'s, the composition of graduate de-

partments, the nature of the graduate student population, the character of graduate education, the Ph. D. productivity of professors, and the employment of mathematicians.

*EDUCATION, TT, S, P, L. *Goals for the Correlation of Elementary Science and Mathematics. The Report of the Cambridge Conference on the Correlation of Science and Mathematics in the Schools*. Published for Education Development Center, Inc., by HM, 1969. viii + 208. $2.20. The outcome of a workshop-style conference held August 21st - September 8th, 1967 and organized by the Cambridge Conference on School Mathematics. The participants were thirty distinguished practitioners and teachers of mathematics, science and technology. After chapters on goals, curriculum, teacher training, and recommendations for immediate implementation, there are 25 appendices on particular topics. The report, intended as a basis for dialogue that ought to take place among specialists in many fields, has implications for mathematical education at all levels.

EDUCATION, T (14). *How to Teach Mathematics in Secondary Schools*. By Herbert Fremont (Queens College CUNY). Saunders, 1969. xv + 571. $10.50. A voluminous and eclectic combination of sketchy reviews of pre-college mathematics and pedagogical considerations.

EDUCATION, S, P, *L. *To Improve Secondary School Science and Mathematics Teaching*. (A Short History of the First Dozen Years of the National Science Foundation's Summer Institutes Program, 1954-1965). By Hillier Krieghbaum and Hugh Rawson. National Science Foundation (NSF 68-28), 41. For sale by the Superintendent of Documents, U.S. G.P.O. Washington, D.C. 20402. 30¢. A more lengthy report, *An Investment in Knowledge*, is scheduled for publication by the New York Univ. Press during 1969.

*EDUCATION, P, L. *Postgraduate Training in the United Kingdom. 4. Report on Applied Mathematics*. Prepared by the Mathematics Subcommittee of the Royal Society Committee to Examine Postgraduate Training in Science and Technology. The Royal Society, 6 Carlton House Terrace, London, S.W.1. 1968. 43. $1.00. This most interesting report contains ideas that ought to be considered by every mathematician interested in education problems at the advanced level It gives implicitly a reasonable definition of "applied mathematics" —equating it essentially with the study of fields of application involving extensive use of mathematics, and not confusing issues by interpreting it to mean a part of mathematics or a particular kind of mathematics. The foreword points out that only in the United Kingdom does applied mathematics exist as an organized discipline. Interestingly enough the report expresses little concern for the danger that mathematics may suffer from close association with specialized work in applications. On the contrary, it considers more seriously the danger that specialized work in applied mathematics will become separated from the applied field to which it is supposedly related and will develop into "pseudo-applied mathematics" lacking in both genuine mathematical significance and relevance to applications.

EDUCATION, TT. *Elementary Mathematical Structure*. By George R. Vick (Sam Houston State Coll). Merrill, 1969. xi + 244. $7.50. Whole numbers, sets, signed integers, rationals, decimal fractions, percentages, approximation, and probability.

ENGINEERING MATHEMATICS, T (13). *Mathematics for Technology. A New Approach*. By M. Bruckheimer, N. W. Gowar (both of City U, London),

and R. E. Scraton (U of Bradford). Am Elsevier, 1968. xiv + 558 $8.75. From sets and binary operations through vectors, matrices and complex numbers to limits and the beginnings of calculus (the bulk of the book), and a little probability and statistics. The "new" in the title refers to the effort to bring the "new math" to students in technical and junior colleges, under the slogan "Stop treating the technologist as a second class citizen, entitled to use mathematics but never to understand it."

ENGINEERING, P, L, *Fracture. An Advanced Treatise.* Edited by Harold Liebowitz (George Washington U). *Volume 2. Mathematical Fundamentals.* Acad Pr, 1968. xvi + 759. $35.00. The seven volume treatise "is concerned chiefly with the sudden, catastrophic failure of structures." The theory began with A. A. Griffith in the early 1920's, and was continued notably by Theodore von Karman from the 1940's. Seven chapters by as many authors and the editor covering the basic mathematical theories of the phenomena. Main tool is classical analysis.

*FOUNDATIONS, S, P, L, *Applications of Model Theory to Algebra, Analysis, and Probability.* Edited by W. A. Luxemburg (Calif. Inst. of Tech.). HR & W, 1969. vii + 307. $9.95. Non-standard analysis (the rigorous development of calculus by means of infinitely small and infinitely large numbers in the sense of Liebniz) is based on the use of model theory. This volume constitutes the proceedings of the International Symposium on the Applications of Model Theory held at the Calif. Inst. of Tech. in May 1967. A pre-requisite is some familiarity with *Non-Standard Analysis* by Abraham Robinson (North-Holland Publishing Co., 1966). (Telegraphic review March 1967, extended reviews May and November 1967).

FOUNDATIONS, P, L, *Set Theory and its Logic.* Revised Edition. By Willard Van Orman Quine (Harvard U). Harvard U Pr, 1969. xvii + 361. $7.95. The first edition (1963) extensively revised.

FUNCTIONAL ANALYSIS, P, L, *Representation Theorems on Banach Function Spaces.* By Neil E. Gretsky (U of Calif. Riverside). *Memoirs of the American Mathematical Society No. 84.* Am Math, 1968. 56. $1.50 (paper).

GENERAL, T (13), TT, *S, *Excursions into Mathematics.* By Anatole Beck, Michael N. Bleicher, and Donald W. Crowe (all of U of Wisconsin). Worth, New York, 1969. xxi + 489. $10.75. Six independent essays presupposing only modest high school mathematics and the maturity of a high school graduate. Topics are: Euler's formula for polyhedra and related topics; The search for perfect numbers; What is area?; Some exotic geometries; Games; What's in a name? (Numeration systems). The exposition is informal and includes historical comment and varied motivations. There are lively illustrations, portraits of mathematicians, and exercises. Contains much interesting mathematics and could serve for a year course with plenty of left over material for future student reading.

GENERAL, T (13), TT (ELEMENTARY), *Elementary Mathematics for Teachers.* By Charles F. Brumfiel and Eugene F. Krause (U of Michigan)., A-W, 1969. x + 436. $9.50. The guiding idea is the abstraction process, applied to the development of the number system. Last four chapters on set theory, number theory, geometry, and probability.

*GENERAL, S, P, L, *The Mathematical Sciences. A Collection of Essays.* Edited by the National Research Committee on Support of Research in the Mathematical Sciences (COSRIMS), with the collabora-

tion of George A. W. Boehm. Published for the National Academy of Sciences—National Research Council by M.I.T. Press, Cambridge, Mass, 1969. x + 271. $8.95. This collection of 22 essays, three reprints and the rest specially written, was prepared as a supplement to the COSRIMS report, *The Mathematical Sciences: A Report*, by the Committee on Support of Research in the Mathematical Sciences, National Academy of Sciences (1968). The motive of COSRIMS was to supply information on the nature of mathematical activity for the benefit of readers of their report who might not otherwise understand its significance. However, the result is of interest to all mathematicians as well as to laymen. The essays cover a very broad spectrum of mathematics and its applications, and the authors form a veritable galaxy of well informed and skillful expositors. The book is well produced and includes a brief biographical note on each author.

GENERAL, T (13; 2), TT. *Modern Principles of Mathematics*. By Robert T Craig (Herbert Lehman College CUNY). P-H, 1969. xii + 400. $8.95. Assumes three years of high school mathematics. A variety of mathematical topics organized around such notations as abstraction, specialization, generalization, functions and numbers.

GENERAL, T (13), TT, S. *Sets, Logic and Numbers*. By Clayton W. Dodge (U of Maine). Prindle, 1969. xiii + 346. $8.50. Logic, sets, Boolean Algebra, natural numbers, ..., complex numbers, etc. for the general education of future mathematics majors and, in particular, future mathematics teachers. Informal, elementary, attractive.

GENERAL, T (13; 1), TT. *Probability and Calculus: A Brief Introduction*. By J. B. Fraleigh (U of Rhode Island). A-W, 1969. vi + 250. $7.25. Assumes high school algebra and geometry. Might be suitable for training of elementary teachers or to precede a serious course in calculus.

GENERAL, T (13; 1), S. *Elementary Functions and Coordinate Geometry*. By Tze-Tsen Hu (U of Calif. Los Angeles). Markham Publishers, Chicago, 1969. xvii + 349. $7.50. First in a series planned to implement the proposal of CUPM (in their *General Curriculum in Mathematics for Colleges*) for a set of semester courses from which a multiplicity of student programs could be constructed, this book is intended to provide a text for "Mathematics 0".

GENERAL, ALGEBRA, T (13), TT, S, L. *Boolean Systems*. By Douglas Kaye (Nottingham College of Education). Longmans, London, 1968. v + 191. $3.05 (paper). An elementary treatment of three Boolean systems (switching circuits, set algebra, and propositional calculus), followed by abstract Boolean algebra and simplification methods including McCluskey's method, developed in the late 1950's.

GENERAL, T (13). *Applied Mathematics for Engineering and Science*. By Waris Shere and Gordon Love (both of Manitoba Inst. of Tech.). P-H, 1969. xiv + 672. $10.95. In the tradition of unified freshmen texts covering elementary algebra, trigonometry, analytic geometry, a little probability and statistics, and calculus of one variable.

GENERAL, *T (13), S, L. *Mathematics the Man-Made Universe. An Introduction to the Spirit of Mathematics*. 2nd edition. By Sherman K. Stein (U of Calif. Davis). W. H. Freeman, 1969. xvi + 415. $8.25. The first edition published in 1962, has been well reviewed and widely used. In this edition substantial changes are based on suggestions of users.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association representing the Sections indicated:

Allegheny Mountain	H. L. Krall, The Pennsylvania State University
Indiana	B. E. Rhoades, Indiana University
Kentucky	J. H. Wells, University of Kentucky
Metropolitan New York	Gerald Freilich, The City College (CUNY)
Nebraska	W. E. Mientka, University of Nebraska
Northern California	L. H. Lange, San Jose State College
Oklahoma-Arkansas	J. E. Scroggs, University of Arkansas
Rocky Mountain	W. N. Smith, University of Wyoming
Wisconsin	E. W. Swokowski, Marquette University

The highest percentage of voters was 55%, occurring in the Oklahoma-Arkansas Section. The Kentucky Section was the runner-up with 53%.

A. B. WILLCOX, *Executive Director*

THE 1969 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The thirtieth annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 6, 1969. This competition, which is supported by the William Lowell Putnam Intercollegiate Memorial Fund, is under the sponsorship of the Mathematical Association of America. Colleges and universities in the United States and Canada are eligible to register undergraduates in the competition.

Application blanks will be mailed about October 1 to the mathematics department chairmen of the schools on the regular mailing list and also to those who supervised the competition in 1968. If an application blank is not received by October 15, one may be secured by writing the director, James H. McKay, Department of Mathematics, Oakland University, Rochester, Michigan 48063. Your application should be mailed to the director not later than November 1, 1969. Further details are provided in the Announcement Brochure which is mailed with the registration forms.

Reports of the previous competitions, including past examination questions, may be found in the MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August-September 1947; December 1948; August-September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; January 1957; August-September (announcement of winners) and November (questions and solutions) 1957; August-September 1958, 1959; January (questions and solutions for the eighteenth, nineteenth, and twentieth competitions) 1961; August-September 1961; October 1962; August-September 1963; June-July 1964; August-September 1965, 1966, 1967, 1968.

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1969 recipients of these awards, selected by a committee consisting of Ivan Niven, Chairman; Edwin Hewitt, and D. E. Richmond, were announced by President Young at the Business Meeting of the Association on August 26, 1969, at the University of Oregon. The recipients of the Ford Awards for articles published in 1968 were the following:

Harley Flanders, A Proof of Minkowski's Inequality for Convex Curves, MONTHLY, 75 (1968), 581–593.

G. E. Forsythe, What To Do Till the Computer Scientist Comes, MONTHLY, 75 (1968), 454–462.

M. F. Neuts, Are Many 1-1 Functions on the Positive Integers Onto?, MATH. MAG., 41 (1968), 103–109.

Pierre Samuel, Unique Factorization, MONTHLY, 75 (1968), 945–952.

Hassler Whitney, The Mathematics of Physical Quantities, I and II, MONTHLY, 75 (1968), 115–138 and 227–256.

Albert Wilansky, Spectral Decomposition of Matrices for High School Students, MATH. MAG., 41 (1968), 51–59.

HENRY L. ALDER, *Secretary*

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual meeting of the Northern California Section of the MAA was held at the University of Santa Clara on February 8, 1969. Dr. H. J. Osner, Chairman of the Section, presided; Professor C. A. Hayes was Program Chairman. There were 142 persons in attendance, 108 of whom were members of the Association.

At the business meeting the following officers were elected: Chairman, Mary V. Sunseri, Stanford University; Vice-Chairman, T. H. Southard, California State College, Hayward. G. L. Alexanderson will continue as Secretary-Treasurer.

Professor George Polya was presented the Blue Ribbon Award (in the category "Mathematics and Physics") of the 1968 American Film Festival of the Educational Film Library Association for his film "Let Us Teach Guessing." The award was made by Professor Victor Klee, First Vice-President of the MAA.

Also at the business meeting Professor David Blakeslee reported on the meetings of the Board of Governors. Professor Alexanderson gave a report on the Visiting Secondary School Lecturer Program sponsored by the Section and Professor E. M. Beesley reported on the Visiting Lecturer Program for Colleges and Universities.

The following papers were presented:

1. *A Report on the Statewide Mathematics Advisory Committee*, by William Chinn, City College of San Francisco (replacing J. L. Kelley, University of California, Berkeley, who was unable to attend).
2. *Some mathematicians I have known*, by George Polya, Stanford University.
3. *Shape of the future*, by Victor Klee, University of Washington.
4. *The S.E.E.D. Program*, by William Johntz, Director of Project S.E.E.D.

G. L. ALEXANDERSON, *Secretary-Treasurer*

MARCH MEETING OF THE OKLAHOMA-ARKANSAS SECTION

The annual spring meeting of the Oklahoma-Arkansas Section of the MAA was held on March 21–22 at Arkansas State University, Jonesboro, Arkansas. There were 134 persons registered of whom 91 were members of the Association.

Mr. John Rieger presided at the business session on Friday evening. Reports were given on the activities of the Association as follows: John Rieger on the Sectional Officers Meeting in August at the University of Wisconsin, R. B. Deal on activities of the Board of Governors, and Lysle Mason on the Annual High School Mathematics Contest spon-

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19. *A General Expression for the Variance of Variance Component Estimators Using Henderson's Method 3*, by J. E. Dunn, University of Arkansas.

20. *On Ratios of Independent Integer Valued Random Variables*, by R. B. Deal, Jr., University of Oklahoma Medical Center.

H. V. HUNEKE, *Secretary-Treasurer*

MARCH MEETING OF THE SOUTHEASTERN SECTION

Winthrop College, Rock Hill, South Carolina, was host to the 48th annual meeting of the Southeastern Section of the MAA, March 28–29, 1969. Professor J. H. Wahab, Chairman of the Section, and Professor Billy Hodges, Chairman of the Mathematics Department of Winthrop College, presided at the General Sessions. Two major addresses were scheduled for the General Sessions: "Topological Methods in Analysis" by G. S. Young (President of the MAA) and "Cones of Matrices" by Emilie V. Haynsworth (Auburn University). Two special sections were devoted exclusively to presentations by students. With the cooperation of Modern Learning Aids, three films of mathematical interest were shown during the meeting: "Göttingen and New York," "The Search for Solid Ground," and "Nim and Other Oriented Graph Games."

Total registration for this meeting was 265, including 90 non-members of the Association. The following officers were elected: Chairman, T. J. Pignani (East Carolina University); Vice-Chairman, Andrew Sobczyk (Clemson University). A mail ballot for the office of Secretary-Treasurer resulted in the election of Professor Billy Bryant (Vanderbilt University) to a three-year term beginning April, 1969.

The Committee on Special Projects reported at the Business Meeting and the following motions were passed:

1. To elect a second vice-chairman whose main area of concern would be the community and junior colleges in the Southeastern Section.
2. To authorize a one dollar registration fee at future meetings of the Southeastern Section to be paid by all registrants except students.
3. To establish an award of $25 to be presented to that student in a Southeastern Section school who scores highest in the Putnam Competition.

The following papers were presented:

1. *On the asymptotic behavior of the gamma function*, by J. V. Baxley, Wake Forest University.
2. *A continuous progress plan for a course in theory of arithmetic*, by Mrs. S. D. Calkins, Winthrop College.
3. *Sequences and inversions*, by Leonard Carlitz, Duke University.
4. *The four color problem*, by R. L. Carroll, Baptist College at Charleston.
5. *Decision method for additive number theory*, by D. M. Clark, Emory University.
6. *The equivalence of perfect compactness, Heine-Borel compactness, and a limit point property*, by D. L. Cozart, Guilford College.
7. *Necessary and sufficient conditions that an operator into Orlicz spaces of Lebesgue-Bochner measurable functions be continuous*, by Joseph Diestel, West Georgia College.
8. *A simple example of an unsolvable problem*, by Trevor Evans, Emory University.
9. *Finitely and countably additive set functions in ordered linear topological spaces*, by William Hattaway, West Georgia College.
10. *A continuity property defined on sequences of sets*, by Jean E. Kelso, Guilford College.
11. *Extending mutually orthogonal partial Latin squares*, by C. C. Lindner, Emory University.
12. *Alternatives to uniform convergence*, by G. E. Parker, Guilford College.
13. *On a theorem of Alexiewicz and Orlicz*, by M. D. Roach, University of Alabama in Huntsville.
14. *Some determinants related to graph theory*, by Nancy Jo Ross, Emory University.

15. *The long line*, by D. F. Spillman, Guilford College.
16. *An iterative solution of the iteration problem*, by C. S. Sutton, The Citadel.
17. *The long line as a subset of $P(R)$*, by J. P. Thomas, Western Carolina University.
18. *On the inverse of Euler's ϕ-function*, by K. W. Wegner, Spelman and Morehouse Colleges.
19. *A proof of the general recursion theorem*, by K. E. Whipple, Georgia State College.
20. *A Fubini-type theorem for a vector valued volume related to an Orlicz space of Bochner measurable functions*, by Vernon Zander, West Georgia College.

HENRY SHARP, JR., *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The forty-ninth meeting of the Southern California Section of the MAA was held at California State College, Fullerton, on March 15, 1969. The registered attendance was 99, including 84 members of the Association. Professor C. W. Seekins, Chairman of the Section, presided at the morning and afternoon sessions.

At the business meeting Professor Vincent Harris, Chairman of the Nominating Committee, reported the election of the following officers who are to serve beginning July 1, 1969: Chairman, Dr. Edward Posner, Jet Propulsion Laboratory—Cal Tech; Vice-Chairman, Professor John Ferling, Claremont Men's College. The Chairman and Vice-Chairman will serve for one year. The following members were elected to the Program Committee for the 1970 meeting of the Section: Professor Thomas Robertson, Occidental College (Chairman); Professor Basil Gordon, UCLA; Professor Raymond Killgrove, Cal State-Los Angeles; Professor Shirley Trembley, Bakersfield College, Dr. Alexander Hurwitz, IBM.

Professor Robert Herrera, Governor of the Section, reported on the meeting of the Board of Governors held in January at New Orleans. A motion was passed activating an article in the Section By-laws permitting the Treasurer to assess Section dues. These dues will be used to provide funds for special projects within the Section.

The following program was presented:

1. *Analysis in non-Archimedean fields*, by E. G. Straus, University of California, Los Angeles.
2. *Spectral representation of self-adjoint extensions and dilations of a symmetric operator*, by R. C. Gilbert, California State College at Fullerton.
3. *On nonparametric modal intervals and estimation with applications*, by B. P. Lientz, System Development Corporation.
4. *Proper raising points in a generalization of backgammon*, by E. Keeler and J. H. Spencer, The Rand Corporation.
5. *A study of modern algebra in the undergraduate curriculum*, a panel discussion chaired by G. K. Kalisch, University of California, Irvine, together with Nelson Dinerstein, California State College at Fullerton, and Ronald Schryer, Orange Coast College.
6. *Invertibly positive linear operators on spaces of continuous functions*, by T. A. Brown, M. L. Juncosa, and V. L. Klee, The Rand Corporation.

D. H. POTTS, *Secretary-Treasurer*

APRIL MEETING OF THE NEBRASKA SECTION

The forty-fifth annual meeting of the Nebraska Section of the MAA was held on Saturday, April 26, 1969, at the Nebraska Center for Continuing Education, Lincoln, Nebraska, in conjunction with the seventy-ninth annual meeting of the Nebraska Academy of Sciences. Professor D. M. Mesner, Chairman of the Section, presided. There were seventy-five persons in attendance of whom forty were members of the Association.

The following officers were elected for 1969–1970: Chairman, Professor J. F. Wampler, Nebraska Wesleyan University; Vice-Chairman, Professor D. M. Mesner, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska. Pro-

fessor J. M. Earl was recognized as Chairman of the MAA Committee on High School Contests. Professor G. S. Young, President of the MAA, discussed the program and policies of the Association.

MAA films were shown at the beginning of the morning session. The following papers were presented:

1. *Models in the history of mathematics*, by D. W. Erbach, University of Nebraska.
2. *The $S(r, t)$ summability transform*, by S. D. Luke, Nebraska Wesleyan University.
3. *A theorem on nilpotency of the Jacobson radical*, by Ahmad Mirbagheri, University of Nebraska.
4. *Topological applications to analysis*, G. S. Young, Tulane University and Case Western Reserve University (invited lecture).
5. *The Nebraska-South Dakota Mathematics Contest*, by J. M. Earl, University of Nebraska at Omaha, and H. M. Cox, University of Nebraska.
6. *Discontinuity diagrams, by* J. A. Eidswick, University of Nebraska.
7. *On exp $A(x)$*, by K. D. Shere, University of Nebraska.
8. *Some fine points of logic in a mathematical context*, by D. J. Gross, Doane College (introduced by Mrs. Mildred Gross).
9. *On a theorem of Aliev*, by A. C. Peterson, University of Nebraska.
10. *A brief look at projective geometry*, by Steve Dondlinger, Doane College (introduced by Mrs. Mildred Gross).
11. *The classification of rings by the characteristic free modules admitted*, by Linda M. Bruning, University of Nebraska (introduced by W. G. Leavitt).
12. *Computer inspired mathematics*, by D. F. Costello, University of Nebraska (introduced by D. M. Mesner).

HENRY M. COX, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The Spring Meeting of the Upper New York State Section of the MAA was held at the University of Western Ontario on May 10, 1969. There were 61 persons in attendance, including 55 members of the Association. Professor Frank Olson, State University of New York at Fredonia, presided at the morning session and Professor John Perry, Wells College, presided at the afternoon session.

At the business meeting the following officers were elected: Chairman, Professor J. Perry, Wells College; Vice-Chairman, Professor R. Sloan, Alfred University; Secretary-Treasurer, Professor P. Schaefer, State University College at Geneseo.

Professor A. J. Coleman, Queen's University, gave the second annual Harry M. Gehman Invited Lecture, "Induced and Subduced Representation of Groups."

The following papers were presented:

1. *Toward a geometry from city streets*, by A. G. Davis, Clarkson College.
2. *On L^p, $1 < p < \infty$*, by L. T. Gardner, University of Toronto.
3. *Solution of certain differential equations using Dirichlet Series*, by R. D. Larsson, Mohawk Valley Community College.
4. *Toeplitz basic sequences in a locally convex metric space*, by D. R. Kerr, SUNY at Albany.
5. *Bounds on the magnitude of the coefficients of the cyclotomic polynomials*, by Sister Marion Beiter, Rosary Hill College.
6. *A note on multiplicative functions*, by A. Somayajulu, Canisius College.
7. *Extraction of homogeneous parts of a polynomial transformation and the pure derivative theorem*, by S. Gagola, SUNY at Buffalo (Winning paper of the Undergraduate Paper Contest).

P. SCHAEFER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-third Annual Meeting, Miami, Florida, January 24–26, 1970.

Fifty-first Summer Meeting, University of Wyoming, Laramie, Wyoming, August 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, May 2, 1970.

FLORIDA, Rollins College, Winter Park, Spring 1970.

ILLINOIS, Loyola University, Chicago, May 8–9, 1970.

INDIANA, University of Notre Dame, Notre Dame, November 15, 1969.

IOWA, Grinnell College, Grinnell, April 17, 1970.

KANSAS, Kansas State Teachers College, Emporia, March 1970.

KENTUCKY, University of Kentucky, Lexington, Spring 1970.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA US Naval Academy, Annapolis, October 18, 1969.

METROPOLITAN NEW YORK, Wagner College, Staten Island, Spring 1970.

MICHIGAN, Wayne State University, Detroit, April 4, 1970.

MISSOURI, Central Missouri State College, Warrensburg, May 2, 1970.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 24–25, 1970.

NEW JERSEY, Seton Hall University, South Orange, November 1, 1969.

NORTH CENTRAL, University of North Dakota, Grand Forks, October 18, 1969.

NORTHEASTERN, Wheaton College, Norton, Massachusetts, November 29, 1969.

NORTHERN CALIFORNIA, Diablo Valley College, Concord, February 7, 1970.

OHIO, Denison University, Granville, October 25, 1969.

OKLAHOMA-ARKANSAS, Southwestern State College, Weatherford, Oklahoma, March 1970.

PACIFIC NORTHWEST

PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Wyoming, May 8–9, 1970.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, Spring 1970.

SOUTHERN CALIFORNIA, University of California, Irvine, March 21, 1970.

SOUTHWESTERN

TEXAS, Sam Houston State College, Huntsville, April 10–11, 1970.

UPPER NEW YORK STATE, Canisius College, Buffalo, November 1, 1969.

WISCONSIN, University of Wisconsin, Waukesha, May 1970.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, Miami, Florida, January 22–25, 1970.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Ohio State University, June 22–25, 1970.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Milwaukee, Wisconsin, November 27–29, 1969.

FIBONACCI ASSOCIATION, San Jose State College, San Jose, October 18, 1969.

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami, Florida, November 10–12, 1969.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Disneyland Hotel, Anaheim, California, October 26–30, 1969.

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NOTICE TO AUTHORS

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FORMAL POWER SERIES

IVAN NIVEN, University of Oregon

1. Introduction. Our purpose is to develop a systematic theory of formal power series. Such a theory is known, or at least presumed, by many writers on mathematics, who use it to avoid questions of convergence in infinite series. What is done here is to formulate the theory on a proper logical basis and thus to reveal the absence of the convergence question. Thus "hard" analysis can be replaced by "soft" analysis in many applications.

John Riordan [4] has discussed these matters in a chapter on generating functions, but his interest is in the applications to combinatorial problems. A more abstract discussion is given by de Branges and Rovnyak [1]. Many examples of the use of formal power series could be cited from the literature; we mention only two, one by John Riordan [5] the other by David Zeitlin [6].

The scheme of the paper is as follows. The theory of formal power series is developed in Sections 3, 4, 5, 6, 7, 11, and 12. Applications to number theory and combinatorial analysis are discussed in Sections 2, 8, 9, 10, and in the last part of 11.

The paper is self-contained insofar as it pertains to the theory of formal power series. However, in the applications of this theory, especially in the application to partitions in Section 9, we do not repeat here the fundamental results needed from number theory. Thus Sections 9 and 10 may be difficult for a reader who is not too familiar with the basic theory of partitions and the sum of divisors function. This difficulty can be removed by use of the specific references given in these sections; only a few pages of fairly straightforward material are needed as background. In Section 11 on the other hand, the background material is set forth in detail because the source is not too readily available.

2. An example from algebra. To motivate the theory we begin with an illustration from algebra, to be found in Jacobson [2, p. 19]. Let q_n denote the number of ways of associating an n-product $a_1a_2a_3 \cdots a_n$ in a nonassociative system. For example $q_3=2$ because $a_1(a_2a_3)$ and $(a_1a_2)a_3$ are the only possibilities. Similarly $q_4=5$ because of the cases $a_1(a_2(a_3a_4))$, $a_1((a_2a_3)a_4)$, $(a_1a_2)(a_3a_4)$, $(a_1(a_2a_3))a_4$, $((a_1a_2)a_3)a_4$. For $n \geqq 2$ it is easy to establish the recursive formula

$$q_n = \sum_{j=1}^{n-1} q_j q_{n-j}, \tag{1}$$

by the following argument. In imposing a system of parentheses on $a_1a_2a_3 \cdots a_n$

Professor Niven has worked in number theory since his Chicago dissertation on the Waring problem under L. E. Dickson. He was a fellow at Pennsylvania, working with H. Rademacher, and since has been at Illinois, Purdue, and Oregon, interrupted by leaves twice to Berkeley. He was the MAA Hedrick lecturer in 1960 and has long been an active participant in the MAA. His six books include the Carus Monograph *Irrational Numbers* and (with H. S. Zuckerman) *Introduction to the Theory of Numbers*. *Editor.*

to make it a well-defined n-product, we can begin by writing

$$(a_1a_2 \cdots a_j)(a_{j+1}a_{j+2} \cdots a_n). \tag{2}$$

Now the number of ways of associating the product $a_1a_2 \cdots a_j$ is q_j by definition, and likewise the second factor in (2) can be associated in q_{n-j} ways. Hence (2) can be associated in q_jq_{n-j} ways, and formula (1) follows by considering the possible values for j. Now define the power series

$$f(x) = \sum_{j=1}^{\infty} q_jx^j. \tag{3}$$

Taking for granted (for the moment) the multiplication of power series, we see that for $n \geqq 2$ the coefficient of x^n in $\{f(x)\}^2$ is

$$q_1q_{n-1} + q_2q_{n-2} + q_3q_{n-3} + \cdots + q_{n-1}q_1.$$

But this is q_n by (1), and so we see that $\{f(x)\}^2 = f(x) - x$ or $f^2 - f + x = 0$.

Solving this quadratic equation for f we get

$$f(x) = f = \tfrac{1}{2}\{1 \pm (1 - 4x)^{1/2}\}. \tag{4}$$

The binomial theorem gives

$$(1 - 4x)^{1/2} = 1 + \frac{1}{2}(-4x) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}(-4x)^2 + \cdots + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2) \cdots (\frac{1}{2} - n + 1)}{n!}(-4x)^n + \cdots.$$

The coefficient of x^n here can be simplified by multiplying numerator and denominator by 2^n to give

$$\begin{aligned} \frac{(1)(-1)(-3)(-5) \cdots (-2n + 3)}{2^n \cdot n!}(-4)^n &= -\frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{n!} \cdot 2^n \\ &= -\frac{(2n - 2)!}{(n!)2^{n-1}(n - 1)!}2^n \\ &= -2\frac{(2n - 2)!}{n!(n - 1)!}. \end{aligned}$$

In view of the minus sign here we see that (4) holds with the minus sign and not the plus sign. Comparing coefficients of x^n in (4) we get the simple formula for q_n,

$$q_n = \frac{(2n - 2)!}{n!(n - 1)!}. \tag{5}$$

This analysis, however, leaves a number of questions unanswered. Why can we solve the quadratic to derive (4)? Why can we equate coefficients on the two

sides of (4) to obtain (5)? To avoid hard analysis in answering such questions, we now develop a theory of formal power series that involves no questions of convergence or divergence. At the end of Section 5 we shall return to the question of the validity of the procedure leading to formula (5).

3. Formal power series. Define α to be an infinite sequence of complex numbers

$$\alpha = [a_0, a_1, a_2, a_3, \cdots]. \tag{6}$$

By P we denote the class of all such infinite sequences α, and these are the formal power series. There are three subsets of P that play a significant role:

P_r: those sequences α all of whose components a_j are real numbers;

P_1: those sequences α with $a_0=1$;

P_0: those sequences α with $a_0=0$.

Although we have specified that the components a_j in the elements of P are complex numbers, the theory could be developed with the a_j in any integral domain.

If $\beta \in P$, say $\beta=[b_0, b_1, b_2, b_3, \cdots]$, define addition by

$$\alpha + \beta = [a_0 + b_0, a_1 + b_1, a_2 + b_2, \cdots].$$

Define multiplication by

$$\alpha\beta = \left[a_0b_0,\ a_1b_0 + a_0b_1,\ a_2b_0 + a_1b_1 + a_0b_2, \cdots, \sum_{j=0}^{n} a_jb_{n-j}, \cdots\right].$$

The definition of equality is that $\alpha=\beta$ if and only if $a_j=b_j$ for all j, i.e., $j=0, 1, 2, 3, \cdots$.

It is not difficult to establish that the set P is a commutative ring with a unit. The zero element and the unit element are

$$z = [0, 0, 0, 0, \cdots] \quad \text{and} \quad u = [1, 0, 0, 0, \cdots].$$

Given any $\alpha=[a_0, a_1, a_2, a_3, \cdots]$ the additive inverse of α is $-\alpha=[-a_0, -a_1, -a_2, -a_3, \cdots]$. The verification of the associative property of multiplication is not difficult, and it is the only property of any depth in establishing that P is a commutative ring.

Moreover, $\alpha\beta=z$ if and only if $\alpha=z$ or $\beta=z$. If $\alpha=z$ or $\beta=z$ it is obvious that $\alpha\beta=z$. To establish the converse, suppose that $\alpha\beta=z$ but $\alpha\neq z$ and $\beta\neq z$. Let j be the least nonnegative integer such that $a_j\neq 0$, and similarly let k be the least nonnegative integer such that $b_k\neq 0$. Then the component in the $(j+k+1)$-th position in $\alpha\beta$ is

$$\sum_{r=0}^{j+k} a_rb_{j+k-r} = a_jb_k \neq 0,$$

which contradicts $\alpha\beta=z$.

It follows that if $\alpha\beta=\alpha\gamma$ and $\alpha\neq z$ then $\beta=\gamma$, and P is an integral domain.

Given any α in P, there corresponds a multiplicative inverse α^{-1} if there is an element α^{-1} in P such that

$$\alpha\cdot\alpha^{-1} = \alpha^{-1}\cdot\alpha = u = [1, 0, 0, 0, \cdots].$$

THEOREM 1. *If* $\alpha = [a_0, a_1, a_2, \cdots]$, α^{-1} *exists if and only if* $a_0 \neq 0$.

Proof. Denote α^{-1} by $[c_0, c_1, c_2, \cdots]$. We see that $\alpha\,\alpha^{-1} = u$ amounts to an infinite system of equations

$$a_0c_0 = 1,\ a_1c_0 + a_0c_1 = 0, \cdots, \sum_{j=0}^{n} a_jc_{n-j} = 0.$$

These equations can be solved successively for $c_0, c_1, c_2, \cdots$ if and only if $a_0 \neq 0$.

LEMMA 2. *Let* $\beta \in P_1$, *so that* β *is of the form* $[1, b_1, b_2, b_3, \cdots]$. *Then for any positive integer* n *we see that* $\beta^n \in P_1$, *say* $\beta^n = [1, c_1, c_2, c_3, \cdots]$. *Also* $c_1 = n\,b_1$ *and for each* $k \geqq 2$ *we have* $c_k = n\,b_k + f_{n,k}\,(b_1, b_2, \cdots, b_{k-1})$ *where* $f_{n,k}$ *is an appropriate polynomial in* $b_1, b_2, \cdots, b_{k-1}$.

Proof. This result can be readily established by induction on n.

THEOREM 3. *Let* $\alpha \in P_1$, *say* $\alpha = [1, a_1, a_2, a_3, \cdots]$, *and let* n *be any positive integer. Then there is a unique* $\beta \in P_1$, *say* $\beta = [1, b_1, b_2, b_3, \cdots]$, *such that* $\beta^n = \alpha$. *Define* $\alpha^{1/n} = \beta$.

Proof. Using Lemma 2 we can solve the equations

$$nb_1 = a_1,\ nb_2 + f_{2,n}(b_1) = a_2, \cdots, nb_k + f_{k,n}(b_1, b_2, \cdots, b_{k-1}) = a_k, \cdots,$$

successively for $b_1, b_2, b_3, \cdots$.

THEOREM 4. *For any positive integer* n *and* $\alpha \in P_1$, *we have* $(\alpha^{-1})^n = (\alpha^n)^{-1}$. *Define* $\alpha^{-n} = (\alpha^n)^{-1}$ *and* $\alpha^0 = u$.

Proof. We see that $\alpha^n(\alpha^{-1})^n = \alpha\cdot\alpha \cdots \alpha\cdot\alpha^{-1}\cdot\alpha^{-1} \cdots \cdot\alpha^{-1} = u$. (Another way of establishing Theorem 4 is to observe that P_1 is a multiplicative group.)

THEOREM 5. *Let* m *and* n *be any integers,* $n > 0$. *To any* $\alpha \in P_1$ *there corresponds a unique* $\beta \in P_1$ *such that* $\alpha^m = \beta^n$, *i.e.,* $\beta = \alpha^{m/n}$.

Proof. This is a corollary of Theorem 3 with α in that theorem replaced by α^m.

4. A power series notation. Let λ denote the particular element $[0, 1, 0, 0, 0, \cdots]$ of P so that

$$\lambda^2 = [0, 0, 1, 0, 0, \cdots],\quad \lambda^3 = [0, 0, 0, 1, 0, 0, \cdots],$$

and in general λ^{n-1} is the sequence with zeros in all positions except the nth, where 1 occurs. We now introduce the notation

$$\sum_{j=0}^{\infty} a_j\lambda^j = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots \tag{7}$$

for $\alpha=[a_0, a_1, a_2, \cdots]$. What this amounts to is an agreement that a_j in (7) stands for $[a_j, 0, 0, 0, \cdots]$ and that $\lambda^0=[1, 0, 0, 0, \cdots]$. Thus we are *not* extending the integral domain P to a vector space by introducing scalar multiplication; this could be done, but all we intend by (7) is an alternative, convenient notation for the elements of P. Thus z and u can now be written simply as 0 and 1. The definitions of addition, multiplication, and equality of elements of P can be rewritten as follows. With α as in (7) and

$$\beta = [b_0, b_1, b_2, \cdots] = \sum_{j=0}^{\infty} b_j\lambda^j,$$

then

$$\alpha + \beta = \sum_{j=0}^{\infty} (a_j + b_j)\lambda^j, \quad \alpha\beta = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_k b_{j-k}\right)\lambda^j,$$

and $\alpha=\beta$ if and only if $a_j=b_j$ for all $j=0, 1, 2, 3, \cdots$.

For example, in the earlier notation we could write

$$[1, -1, 0, 0, 0, \cdots]\cdot[1, 1, 1, 1, 1, \cdots] = [1, 0, 0, 0, 0, \cdots].$$

This can now be written as $(1-\lambda)(1+\lambda+\lambda^2+\lambda^3+\cdots)=1$, or $(1-\lambda)^{-1}=1+\lambda+\lambda^2+\lambda^3+\cdots$. A general binomial theorem is established later, in Theorems 11 and 17.

THEOREM 6. *Let n be any positive integer, let $\alpha\in P_r$ and $\beta\in P_r$, so that α and β are real sequences. If n is odd, $\alpha^n=\beta^n$ implies $\alpha=\beta$. If n is even, $\alpha^n=\beta^n$ implies $\alpha=\beta$ or $\alpha=-\beta$.*

Proof. We may presume $\alpha\neq 0$ and $\beta\neq 0$. For if $\alpha=0$, for example, then $\alpha^n=0$, $\beta^n=0$ and so $\beta=0$, $\alpha=\beta$. Let ω denote the nth root of unity

$$\omega = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n).$$

Then $\alpha^n-\beta^n=0$ can be factored $\alpha^n-\beta^n=\prod_{j=1}^{n}(\alpha-\omega^j\beta)=0$. If ω^j is not real then $\alpha-\omega^j\beta\neq 0$ because α and β are real sequences with $\alpha\neq 0$ and $\beta\neq 0$. If n is odd, ω^j is real only in the case $j=n$ and hence

$$\alpha - \omega^n\beta = 0, \quad \alpha - \beta = 0, \quad \alpha = \beta.$$

If n is even, ω^j is real in the two cases $j=n$ and $j=n/2$, leading to the conclusion that $\alpha=\beta$ or $\alpha=-\beta$.

Consider an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ of elements of P, say

$$\alpha_k = \sum_{j=0}^{\infty} a_{jk}\lambda^j, \qquad k = 1, 2, 3, \cdots. \tag{8}$$

DEFINITION. *A sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ as in (8) is said to be a sequence admitting addition if corresponding to any integer $r\geq 0$ there is an integer $N=N(r)$ such that for all $n\geq N$, $a_{0n}=a_{1n}=a_{2n}=\cdots=a_{rn}=0$.*

If this condition is satisfied we also say that $\sum \alpha_j$ is an *admissible sum*, and we can write

$$\sum_{j=1}^{\infty} \alpha_j = \sum s_r \lambda^r,$$

where for each integer $r \geqq 0$ the coefficient s_r is the coefficient of λ^r in the finite sum $\alpha_1 + \alpha_2 + \cdots + \alpha_N$, i.e.,

$$s_r = a_{r1} + a_{r2} + \cdots + a_{rN}.$$

We note that s_r is the coefficient of λ^r in every finite sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ with $n \geqq N$.

LEMMA 7. *Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of elements of P admitting addition. Let $\beta_1, \beta_2, \beta_3, \cdots$ be a rearrangement of the α's in the sense that given any j there exists a unique k such that $\alpha_j = \beta_k$. Then $\beta_1, \beta_2, \beta_3, \cdots$ is also a sequence admitting addition, and*

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots = \beta_1 + \beta_2 + \beta_3 + \cdots.$$

Proof. Let r be any given nonnegative integer. For n sufficiently large the coefficient of λ^r in $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ equals the coefficient of λ^r in the finite sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$. Similarly for n sufficiently large the coefficient of λ^r in $\beta_1 + \beta_2 + \beta_3 + \cdots$ equals the coefficient of λ^r in $\beta_1 + \beta_2 + \cdots + \beta_n$. And clearly $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\beta_1 + \beta_2 + \cdots + \beta_n$ have identical terms in λ^r.

Next we get a result analogous to Lemma 7 for multiplication. Consider an infinite sequence $\gamma_1, \gamma_2, \gamma_3, \cdots$ of elements of P of the form

$$\gamma_k = \sum_{j=1}^{\infty} c_{jk} \lambda^j, \qquad k = 1, 2, 3, \cdots. \tag{9}$$

Note that the sums begin with $j = 1$. If this is a sequence admitting addition, then we say that the related sequence

$$1 + \gamma_1,\ 1 + \gamma_2,\ 1 + \gamma_3, \cdots \tag{10}$$

is a sequence admitting multiplication. Furthermore, we write

$$\prod_{k=1}^{\infty} (1 + \gamma_k) = 1 + \sum_{j=1}^{\infty} q_j \lambda^j,$$

where q_r is the coefficient of λ^r in any finite product $\prod_{k=1}^{n} (1 + \gamma_k)$ with n sufficiently large that $c_{jk} = 0$ for $1 \leqq j \leqq r$ if $k > n$. Then it is clear that we can state a result analogous to Lemma 7 as follows:

LEMMA 7a. *If* (10) *is a sequence admitting multiplication, so is any rearrangement* $1 + \delta_1, 1 + \delta_2, 1 + \delta_3, \cdots$ *of* (10), *and*

$$\prod_{k=1}^{\infty}(1+\gamma_k)=\prod_{k=1}^{\infty}(1+\delta_k).$$

5. Formal derivatives. Given any α in P, say $\alpha=\sum_{j=0}^{\infty} a_j\lambda^j$, define the derivative $D(\alpha)$ and the scalar $S(\alpha)$ by

$$D(\alpha)=\sum_{j=1}^{\infty} ja_j\lambda^{j-1},\qquad S(\alpha)=a_0. \tag{11}$$

Define $D^2(\alpha)=D(D(\alpha))$, and in general for any positive integer n, the nth derivative is $D^n(\alpha)$. Taking $D^0(\alpha)=\alpha$ for convenience, we can now write a McLaurin series expansion.

THEOREM 8. $\alpha=\sum_{n=0}^{\infty}(1/n!)\,S(D^n(\alpha))\cdot\lambda^n$.

The proof of this is quite easy.

THEOREM 9. *If $\alpha\in P$, $\beta\in P$ then $D(\alpha+\beta)=D(\alpha)+D(\beta)$ and $D(\alpha\beta)=\alpha D(\beta)+\beta D(\alpha)$, and $D(\alpha^n)=n\alpha^{n-1}D(\alpha)$ for any positive integer n. Also if α^{-1} exists then $D(\alpha^{-1})=-\alpha^{-2}D(\alpha)$ and $D(\alpha^{-n})=-n\alpha^{-n-1}D(\alpha)$.*

Proof. The formula for $D(\alpha\beta)$ can be established easily by comparing coefficients of λ^n. By using induction on n we get the formula for $D(\alpha^n)$. Next if we differentiate $\alpha\alpha^{-1}=1$ we get the formula for $D(\alpha^{-1})$. Finally, $\alpha^{-n}=(\alpha^{-1})^n$ can be used to write

$$D(\alpha^{-n})=D((\alpha^{-1})^n)=n(\alpha^{-1})^{n-1}D(\alpha^{-1})=-n\alpha^{-n-1}D(\alpha).$$

THEOREM 10. *Let $\alpha\in P_1$ so that $S(\alpha)=1$. For any rational number r, $D(\alpha^r)=r\alpha^{r-1}D(\alpha)$.*

Proof. By Theorem 5 there is a unique meaning for α^r. If $r=m/n$ where m and n are integers we can write

$$D((\alpha^r)^n)=n(\alpha^r)^{n-1}D(\alpha^r),\quad D((\alpha^r)^n)=D(\alpha^m)=m\alpha^{m-1}D(\alpha),$$

by Theorem 9. The result follows at once.

A simple version of the binomial theorem can be easily obtained from Theorems 8 and 10, as follows:

THEOREM 11. *For any rational number r and any complex number k,*

$$(1+k\lambda)^r=1+r(k\lambda)+\frac{r(r-1)}{2!}(k\lambda)^2+\cdots$$
$$+\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}(k\lambda)^n+\cdots.$$

Proof. First note that $D(1+k\lambda)^r=r(1+k\lambda)^{r-1}D(1+k\lambda)=rk(1+k\lambda)^{r-1}$, and

so by induction on n,

$$D^n(1 + k\lambda)^r = r(r - 1)(r - 2) \cdots (r - n + 1)k^n(1 + k\lambda)^{r-n}.$$

Now $(1+k\lambda)^{r-n}$ is a unique element of P_1 by Theorems 3 and 5, and so $S(1+k\lambda)^{r-n}=1$. It follows that

$$S(D^n(1 + k\lambda)^r) = r(r - 1)(r - 2) \cdots (r - n + 1)k^n.$$

Now use Theorem 8 with α replaced by $(1+k\lambda)^r$, and the result follows.

The form of the binomial theorem just established is sufficient in most applications, for example, to justify the argument given in Section 2. To see this, we replace equation (3) with this definition of α,

$$\alpha = \sum_{j=1}^{\infty} q_j\lambda^j,$$

where the q_j have the same meaning as in Section 2. Then the analysis following equation (3) leads to $\alpha^2=\alpha-\lambda$. From this we can write $4\alpha^2-4\alpha+1=1-4\lambda$, or

$$(1 - 2\alpha)^2 = ((1 - 4\lambda)^{1/2})^2.$$

By Theorem 6 it follows that $1-2\alpha=(1-4\lambda)^{1/2}$, and so by Theorem 11 we conclude that

$$\begin{aligned} &1 - 2q_1\lambda - 2q_2\lambda^2 - 2q_3\lambda^3 - \cdots \\ &\quad = 1 + \frac{1}{2}(-4\lambda) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}(-4\lambda)^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!}(-4\lambda)^3 + \cdots. \end{aligned}$$

From the definition of equality in Section 3 we can now equate the coefficients of λ^n to get equation (5).

We want to get a more general form of the binomial theorem, namely the expansion of $(1+\alpha)^r$ where $\alpha\in P_0$, so that $S(\alpha)=0$. To do this we define a formal logarithm. But first we establish one more result about derivatives.

THEOREM 12. *If $\alpha_1+\alpha_2+\alpha_3+\cdots$ is an admissible sum of elements of P in the sense of Section 4, then*

$$D(\alpha_1 + \alpha_2 + \alpha_3 + \cdots) = D(\alpha_1) + D(\alpha_2) + D(\alpha_3) + \cdots.$$

Proof. For any nonnegative integer r the coefficient of λ^r in the infinite sum $\alpha_1+\alpha_2+\alpha_3+\cdots$ equals the coefficient of λ^r in the finite sum $\alpha_1+\alpha_2+\cdots+\alpha_n$ provided $n\geqq N=N(r)$. Hence the coefficients of λ^{r-1} are equal in the equation in Theorem 12. But this holds for all r, so the result follows.

6. Logarithms and the binomial theorem. A formal logarithm is not defined for any element of P, but only for $\alpha\in P_1$, so that $S(\alpha)=1$. For any $\alpha\in P_1$, say $\alpha=1+\beta$ with $\beta\in P_0$, define

$$L(\alpha) = L(1 + \beta) = \beta - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\beta^3 - \tfrac{1}{4}\beta^4 + \cdots = \sum_{j=1}^{\infty}(-1)^{j+1}\beta^j/j,$$

noting that this is an admissible sum as in Section 4. Thus L is a formal logarithmic function from P_1 to P_0.

THEOREM 13. $D(L(\alpha))=\alpha^{-1}D(\alpha)$.

Proof. With $\alpha=1+\beta$ we use Theorem 12 to write

$$\begin{aligned} D(L(\alpha)) = D(L(1+\beta)) &= D[\beta - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\beta^3 - \tfrac{1}{4}\beta^4 + \cdots \\ &= D(\beta) + D(-\tfrac{1}{2}\beta^2) + D(\tfrac{1}{3}\beta^3) + D(-\tfrac{1}{4}\beta^4) + \cdots \\ &= D(\beta) - \beta D(\beta) + \beta^2 D(\beta) - \beta^3 D(\beta) + \cdots \\ &= D(\beta)[1 - \beta + \beta^2 - \beta^3 + \cdots] \\ &= D(\beta)\cdot(1+\beta)^{-1} = D(\alpha)\cdot\alpha^{-1}, \end{aligned}$$

because $D(\alpha)=D(\beta)$ by definition.

THEOREM 14. *If* $\alpha\in P_1$ *and* $\gamma\in P_1$ *then* $L(\alpha\gamma)=L(\alpha)+L(\gamma)$.

Proof. We use Theorems 13 and 9 to observe that

$$\begin{aligned} D(L(\alpha\gamma)) = (\alpha\gamma)^{-1}D(\alpha\gamma) &= (\alpha\gamma)^{-1}\{\alpha D(\gamma) + \gamma D(\alpha)\} \\ &= \alpha^{-1}D(\alpha) + \gamma^{-1}D(\gamma) \\ &= D(L(\alpha)) + D(L(\gamma)) \\ &= D(L(\alpha) + L(\gamma)). \end{aligned}$$

Now $L(\alpha\gamma)$ and $L(\alpha)+L(\gamma)$ are elements in P_0, and it is clear from the definition of a derivative that if $\theta_1\in P_0$ and $\theta_2\in P_0$ and $D(\theta_1)=D(\theta_2)$, then $\theta_1=\theta_2$.

THEOREM 15. *For any rational number* r, $L(\alpha^r)=rL(\alpha)$.

Proof. By definition $L(1)=0$. Then $\alpha\cdot\alpha^{-1}=1$ implies $L(\alpha)+L(\alpha^{-1})=L(\alpha\cdot\alpha^{-1})=L(1)=0$ and so $L(\alpha^{-1})=-L(\alpha)$. For any integer n we have $L(\alpha^n)=nL(\alpha)$ by induction. If $r=m/n$ where m and n are integers we see that

$$mL(\alpha) = L(\alpha^m) = L((\alpha^r)^n) = nL(\alpha^r).$$

THEOREM 16. $L(\alpha)=0$ *if and only if* $\alpha=1$. *Also if* $L(\alpha)=L(\beta)$ *then* $\alpha=\beta$.

Proof. If $L(\alpha)=0$ then $D(L(\alpha))=D(0)=0$ and so $\alpha^{-1}D(\alpha)=0$. But $\alpha^{-1}\neq 0$ and hence $D(\alpha)=0$ and $\alpha=1$.

THEOREM 17. *If* r *is rational, if* β *is an element of* P_0 *so that* $S(\beta)=0$, *then*

$$(1+\beta)^r = 1 + r\beta + \frac{r(r-1)}{2!}\beta^2 + \cdots + \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}\beta^n + \cdots. \tag{12}$$

Proof. For convenience we write

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

Let γ denote the right side of equation (12) so that

$$D(\gamma) = D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j-1},$$

$$\begin{aligned}(1+\beta)D(\gamma) &= D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j-1} + D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j}\\ &= D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j-1} + D(\beta)\sum_{j=2}^{\infty}(j-1)\binom{r}{j-1}\beta^{j-1}\\ &= D(\beta)\cdot r + D(\beta)\sum_{j=2}^{\infty}\left\{j\binom{r}{j} + (j-1)\binom{r}{j-1}\right\}\beta^{j-1}\\ &= D(\beta)\cdot r + D(\beta)\sum_{j=2}^{\infty} r\binom{r}{j-1}\beta^{j-1}\\ &= rD(\beta)\left[1+\sum_{j=1}^{\infty}\binom{r}{j}\beta^{j}\right] = r\gamma D(\beta).\end{aligned}$$

Multiplying by $\gamma^{-1}(1+\beta)^{-1}$ we get $\gamma^{-1}D(\gamma) = r(1+\beta)^{-1}D(\beta) = r(1+\beta)^{-1}D(1+\beta)$. But $D(L(\gamma)) = \gamma^{-1}D(\gamma)$ and $D(L((1+\beta)^r)) = D(rL(1+\beta)) = r(1+\beta)^{-1}D(1+\beta)$, and so $D(L(\gamma)) = D(L((1+\beta)^r))$. Since $L(\gamma)$ and $L(1+\beta)^r$ are in P_0 it follows that $L(\gamma) = L((1+\beta)^r)$, and so $\gamma = (1+\beta)^r$ by Theorem 16.

7. The exponential function. Let β be an element of P_0, so that $S(\beta)=0$. Then we define

$$E(\beta) = 1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{\beta^n}{n!},$$

so that E is a function from P_0 to P_1. Since $E(\beta)$, as defined, is an admissible sum, we can apply Theorem 12 to get

$$D(E(\beta)) = D(\beta)\left\{1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots\right\} = D(\beta)\cdot E(\beta).$$

THEOREM 18. *If $E(\beta) = E(\gamma)$ then $\beta = \gamma$.*

Proof. We observe that $D(E(\beta)) = D(E(\gamma))$, so that $D(\beta)\cdot E(\beta) = D(\gamma)\cdot E(\gamma)$. But $E(\beta) \neq 0$ so that $E(\beta)$ and $E(\gamma)$ can be cancelled giving $D(\beta) = D(\gamma)$, and hence $\beta = \gamma$.

THEOREM 19. *If $\beta \in P_0$ then $L(E(\beta)) = \beta$. If $\alpha \in P_1$ then $E(L(\alpha)) = \alpha$. Thus L and E are inverse functions, L being one-to-one from P_1 onto P_0, and E one-to-one from P_0 onto P_1.*

Proof. By Theorem 13 we see that

$$D(L(E(\beta))) = \{E(\beta)\}^{-1}\cdot D(E(\beta)) = \{E(\beta)\}^{-1}\cdot E(\beta)\cdot D(\beta) = D(\beta).$$

It follows that $L(E(\beta))=\beta$. Next, given any α in P_1 suppose that $E(L(\alpha))=\alpha_1$. Then $L(E(L(\alpha)))=L(\alpha_1)$ and so $L(\alpha)=L(\alpha_1)$. Hence $\alpha=\alpha_1$ by Theorem 16.

THEOREM 20. *Given* $\beta\in P_0$, $\gamma\in P_0$, *then* $E(\beta+\gamma)=E(\beta)\cdot E(\gamma)$.

Proof. By Theorems 14 and 19 we see that

$$L(E(\beta)\cdot E(\gamma)) = L(E(\beta)) + L(E(\gamma)) = \beta + \gamma.$$

Taking the exponential function of each side, and using Theorem 19 again, we get the result.

By Theorems 15 and 19 we see that $\alpha^r=E(rL(\alpha))$ for any $\alpha\in P_1$ and any rational r. This equation we take as the definition of α^r for any complex number r, so that such properties of exponents as $\alpha^r\cdot\alpha^s=\alpha^{r+s}$ follow at once for complex numbers r and s. Also by use of this definition we note that Theorem 10 can be extended to any complex number r; thus

$$D(\alpha^r) = D(E(rL(\alpha))) = E(rL(\alpha))\cdot D(rL(\alpha)) = \alpha^r\cdot r\alpha^{-1}D(\alpha) = r\alpha^{r-1}D(\alpha).$$

Also Theorem 15 extends to any complex r by use of Theorem 19. Finally, Theorem 17 holds for complex r; in fact the proof of this result needs no alteration for this generalization in view of the extended versions of Theorems 10 and 15 just mentioned.

8. An application to recurrence functions. For any given a, b, x_0, x_1 define a sequence $x_0, x_1, x_2, x_3, \cdots$ by the recurrence relation $x_{n+1}=ax_n+bx_{n-1}$ for $n=1, 2, 3, \cdots$. The Fibonacci sequence is the special case with $a=b=x_0=x_1=1$. The problem is to determine x_n explicitly in terms of a, b, x_0, x_1. If we define $\alpha=x_0+x_1\lambda+x_2\lambda^2+x_3\lambda^3+\cdots$ we see that

$$\alpha - a\lambda\alpha - b\lambda^2\alpha = x_0 + (x_1 - ax_0)\lambda. \tag{13}$$

If k_1 and k_2 are the roots of $k^2-ak-b=0$ we see that (13) can be written as

$$\alpha(1 - k_1\lambda)(1 - k_2\lambda) = x_0 + (x_1 - ax_0)\lambda. \tag{14}$$

CASE 1. Suppose that $k_1=k_2$. Then we see that

$$\alpha = \{x_0 + (x_1 - ax_0)\lambda\}\cdot(1 - k_1\lambda)^{-2}. \tag{15}$$

Now by Theorem 11 or Theorem 17 we have

$$(1 - k_1\lambda)^{-2} = 1 + 2k_1\lambda + 3k_1^2\lambda^2 + 4k_1^3\lambda^3 + 5k_1^4\lambda^4 + \cdots,$$

and so equating coefficients of λ^n in (15) we get

$$\begin{aligned} x_n &= x_0(n + 1)k_1^n + n(x_1 - ax_0)k_1^{n-1} \quad \text{or} \\ x_n &= nx_1k_1^{n-1} - (n - 1)x_0k_1^n. \end{aligned} \tag{16}$$

CASE 2. Suppose that $k_1 \neq k_2$. Multiplying the identity

$$k_1 - k_2 = k_1(1 - k_2\lambda) - k_2(1 - k_1\lambda)$$

by $(1-k_1\lambda)^{-1}(1-k_2\lambda)^{-1}$ we get

$$(k_1 - k_2)(1 - k_1\lambda)^{-1}(1 - k_2\lambda)^{-1} = k_1(1 - k_1\lambda)^{-1} - k_2(1 - k_2\lambda)^{-1}.$$

Multiplying this into (14) we have

$$(k_1 - k_2)\alpha = \{x_0 + (x_1 - ax_0)\lambda\}\{k_1(1 - k_1\lambda)^{-1} - k_2(1 - k_2\lambda)^{-1}\}. \tag{17}$$

Also we use $k_1(1-k_1\lambda)^{-1} = k_1 + k_1^2\lambda + k_1^3\lambda^2 + k_1^4\lambda^3 + \cdots + k_1^{n+1}\lambda^n + \cdots$. Equating coefficients of λ^n in (17) we have

$$(k_1 - k_2)x_n = x_0(k_1^{n+1} - k_2^{n+1}) + (x_1 - ax_0)(k_1^n - k_2^n),$$

or

$$x_n = \{x_0(k_1^{n+1} - k_2^{n+1}) + (x_1 - ax_0)(k_1^n - k_2^n)\}/(k_1 - k_2). \tag{18}$$

The results (16) and (18) are well known; an alternative derivation is given in [3, page 100]. An entirely different way of treating equation (13) is as follows. We can write

$$\alpha = (1 - a\lambda - b\lambda^2)^{-1}\{x_0 + (x_1 - ax_0)\lambda\}. \tag{19}$$

Now by Theorem 17 we have

$$(1 - a\lambda - b\lambda^2)^{-1} = 1 + (a\lambda + b\lambda^2) + (a\lambda + b\lambda^2)^2 + (a\lambda + b\lambda^2)^3 + \cdots.$$

The coefficient of λ^n here is

$$a^n + \binom{n-1}{1}a^{n-2}b + \binom{n-2}{2}a^{n-4}b^2 + \binom{n-3}{3}a^{n-6}b^3 + \cdots$$
$$= \sum_{j=0}^{[n/2]} \binom{n-j}{j} a^{n-2j}b^j.$$

Equating coefficients of λ^n in (19) gives therefore

$$x_n = x_0 \sum_{j=0}^{[n/2]} \binom{n-j}{j} a^{n-2j}b^j + (x_1 - ax_0) \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} a^{n-1-2j}b^j.$$

Finally, let us return to the method used for deriving (16) and (18). This method can be used with recurrence relations of higher order. Consider for example any given real (or complex) numbers x_0, x_1, x_2, a, b, c and a recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n + cx_{n-1}, \qquad n = 1, 2, 3, \cdots.$$

If we define $\alpha = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + \cdots$ we note that

$$\alpha(1 - a\lambda - b\lambda^2 - c\lambda^3) = x_0 + (x_1 - ax_0)\lambda + (x_2 - ax_1 - bx_0)\lambda^2. \tag{20}$$

If the equation $k^3-ak^2-bk-c=0$ has roots k_1, k_2, k_3 say, then (13) can be rewritten as

$$\text{(21)}\quad \alpha(1-k_1\lambda)(1-k_2\lambda)(1-k_3\lambda)=x_0+(x_1-ax_0)\lambda+(x_2-ax_1-bx_0)\lambda^2.$$

There are now three cases depending on the nature of the roots k_1, k_2, k_3: three equal roots, two equal roots, or distinct roots. The case of equal roots follows the pattern of equation (15),

$$\alpha=[x_0+(x_1-ax_0)\lambda+(x_2-ax_1-bx_0)\lambda^2]\cdot(1-k_1\lambda)^{-3}.$$

In the other two cases it is a matter of partial fraction expansions, in the sense that constants q_1, q_2, q_3, q_4, q_5, q_6 can be found so that

$$(1-k_1\lambda)^{-2}(1-k_2\lambda)^{-1}=q_1(1-k_1\lambda)^{-1}+q_2(1-k_1\lambda)^{-2}+q_3(1-k_2\lambda)^{-1},$$
$$(1-k_1\lambda)^{-1}(1-k_2\lambda)^{-1}(1-k_3\lambda)^{-1}=q_4(1-k_1\lambda)^{-1}+q_5(1-k_2\lambda)^{-1}+q_6(1-k_3\lambda)^{-1},$$

in the case of two equal roots or the case of distinct roots, respectively.

For example if $a=6$, $b=-11$, $c=6$ then we find that $k_1=1$, $k_2=2$, $k_3=3$, $q_4=\frac{1}{2}$, $q_5=-4$, $q_6=9/2$. Then (21) implies that

$$\begin{aligned}\alpha &= [x_0+(x_1-6x_0)\lambda+(x_2-6x_1+11x_0)\lambda^2]\\ &\qquad\cdot[\tfrac{1}{2}(1-\lambda)^{-1}-4(1-2\lambda)^{-1}+\tfrac{9}{2}(1-3\lambda)^{-1}],\\ x_n &= x_0(\tfrac{1}{2}-4\cdot 2^n+\tfrac{9}{2}\cdot 3^n)+(x_1-6x_0)(\tfrac{1}{2}-4\cdot 2^{n-1}+\tfrac{9}{2}\cdot 3^{n-1})\\ &\qquad+(x_2-6x_1+11x_0)(\tfrac{1}{2}-4\cdot 2^{n-2}+\tfrac{9}{2}\cdot 3^{n-2}).\end{aligned}$$

9. An application to partitions. The notation $p(n)$ represents the number of ways that a positive integer n can be written as a sum of positive integers. Two partitions are not different if they differ only in the order of their summands. As usual, we define $p(0)=1$.

Let α_j denote $1+\lambda^j+\lambda^{2j}+\lambda^{3j}+\cdots$ for every positive integer j. Then $\alpha_1, \alpha_2, \alpha_3, \cdots$ is a sequence admitting multiplication in the sense of (10) in Section 4. By the standard argument, for example in [**3**, pp. 226, 227], we have

$$\text{(22)}\qquad \alpha_1\cdot\alpha_2\cdot\alpha_3\cdot\cdots=\prod_{j=1}^{\infty}\alpha_j=\sum_{k=0}^{\infty}p(k)\lambda^k.$$

But also we see that $\alpha_j(1-\lambda^j)=1$ so that $\alpha_j=(1-\lambda^j)^{-1}$, and

$$\text{(23)}\qquad \prod_{j=1}^{\infty}\alpha_j=\prod_{j=1}^{\infty}(1-\lambda^j)^{-1}.$$

Next let $q^e(n)$ denote the number of partitions of any positive integer n into an even number of distinct summands, and similarly let $q^0(n)$ be the number of partitions of n into an odd number of distinct summands. It is customary to take $q^e(0)=1$ and $q^0(0)=0$. Then the coefficient of λ^n in the expansion of the admissible product

$$(1-\lambda)(1-\lambda^2)(1-\lambda^3)\cdots = \prod_{j=1}^{\infty}(1-\lambda^j)$$

is seen to be $q^e(n)-q^0(n)$ by a simple combinatorial argument. It follows that

$$\prod_{j=1}^{\infty}(1-\lambda^j) = \sum_{n=0}^{\infty}\{q^e(n)-q^0(n)\}\lambda^n. \tag{24}$$

By use of graphs of partitions it can be proved, cf. [3, pp. 224–226], that $q^e(n)-q^0(n)=(-1)^j$ if n is of the form $(3j^2+j)/2$ or $(3j^2-j)/2$ for some nonnegative integer j, and $q^e(n)-q^0(n)=0$ otherwise. It is easy to prove that the sets of positive integers

$$\{(3j^2+j)/2; j=1,2,3,\cdots\},\ \{(3j^2-j)/2; j=1,2,3,\cdots\}$$

are distinct, and hence (24) can be written as

$$\begin{aligned}\prod_{j=1}^{\infty}(1-\lambda^j) &= 1+\sum_{j=1}^{\infty}(-1)^j(\lambda^{(3j^2+j)/2}+\lambda^{(3j^2-j)/2}) \\ &= 1-\lambda-\lambda^2+\lambda^5+\lambda^7-\lambda^{12}-\lambda^{15}+\cdots.\end{aligned}\tag{25}$$

This with (22) and (23) implies that

$$\left\{1+\sum_{j=1}^{\infty}(-1)^j(\lambda^{(3j^2+j)/2}+\lambda^{(3j^2-j)/2})\right\}\sum p(k)\lambda^k = 1,$$

$$(1-\lambda-\lambda^2+\lambda^5+\lambda^7-\lambda^{12}-\lambda^{15}+\cdots)\sum p(k)\lambda^k = 1.$$

For any positive integer n, the coefficient of λ^n on the left side of this equation is $p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)-p(n-15)+\cdots$. Thus we have proved the following well-known result of Euler [3, p. 235].

THEOREM 21. *For any positive integers n,*

$$\begin{aligned}p(n) &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots \\ &= \sum_{j=1}^{\infty}(-1)^{j+1}\{p(n-(3j^2+j)/2)+p(n-(3j^2-j)/2)\}\end{aligned}$$

with $p(t)=0$ if $t<0$, so that the sum is finite.

It should be emphasized that the proof given here of Theorem 21 is not new. The proof above is simply the usual one formulated in terms of the "soft" analysis of formal power series.

10. An application to the sum of divisors function. For any positive integer n let $\sigma(n)$ denote the sum of the positive divisors of n; for example $\sigma(6)=1+2+3+6$. We establish a known recurrence relation [3, p. 236] for $\sigma(n)$, and again the positive integers of the form $(3k^2-k)/2$ and $(3k^2+k)/2$ play a role, namely, the positive integers 1, 2, 5, 7, 12, 15, 22, 26, $\cdots$.

THEOREM 22. *For any positive integer k,*

$$\sigma(k) - \sigma(k-1) - \sigma(k-2) + \sigma(k-5) + \sigma(k-7) - \cdots = \begin{cases} (-1)^{j+1}k \text{ if } k = (3j^2+j)/2 \text{ or } k = (3j^2-j)/2, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Define $\beta = \prod_{j=1}^{k} (1-\lambda^j)$ so that $L(\beta) = \sum_{j=1}^{k} L(1-\lambda^j)$,

$$\begin{aligned} -D(L(\beta)) = -\beta^{-1}D(\beta) &= \sum_{j=1}^{k} j(1-\lambda^j)^{-1}\lambda^{j-1} \\ &= \sum_{j=1}^{k} \{j\lambda^{j-1} + j\lambda^{2j-1} + j\lambda^{3j-1} + j\lambda^{4j-1} + \cdots\} \\ &= \sum_{n=1}^{\infty} f(n)\lambda^{n-1}, \end{aligned}$$

where $f(n)$ is seen to be the sum of all positive divisors of n that do not exceed k. Thus we have $f(n) = \sigma(n)$ if $n \leqq k$, and so we can write

$$-\beta^{-1}D(\beta) = \sum_{n=1}^{k} \sigma(n)\lambda^{n-1} + \sum_{n=k+1}^{\infty} f(n)\lambda^{n-1}. \tag{26}$$

Now equation (24) can be written with a finite product

$$\beta = \prod_{j=1}^{k} (1-\lambda^j) = \sum_{n=0}^{\infty} \{q_k^e(n) - q_k^0(n)\}\lambda^n, \tag{27}$$

where $q_k^e(n)$ denotes the number of partitions of n into an even number of distinct summands $\leqq k$, and $q_k^0(n)$ denotes the number of partitions of n into an odd number of distinct summands $\leqq k$. Define $q_k^e(0) = 1$ and $q_k^0(0) = 0$. If $n \leqq k$ we note that $q_k^e(n) = q^e(n)$ and $q_k^0(n) = q^0(n)$, so (27) can be written as

$$\beta = \sum_{n=0}^{k} \{q^e(n) - q^0(n)\}\lambda^n + \sum_{n=k+1}^{\infty} \{q_k^e(n) - q_k^0(n)\}\lambda^n. \tag{28}$$

We now equate the coefficients of λ^{k-1} in $-D(\beta)$ and in the product $\beta(-\beta^{-1}D(\beta))$. From (28) it is clear that the coefficient of λ^{k-1} in $-D(\beta)$ is

$$-k\{q^e(k) - q^0(k)\} = \begin{cases} -(-1)^j k \text{ if } k = (3j^2 \pm j)/2, \\ 0 \text{ otherwise.} \end{cases}$$

From (28) and (26) the coefficient of λ^{k-1} in $\beta(-\beta^{-1}D(\beta))$ is

$$\begin{aligned} &\sigma(k)\{q^e(0) - q^0(0)\} + \sigma(k-1)\{q^e(1) - q^0(1)\} + \sigma(k-2)\{q^e(2) - q^0(2)\} + \cdots \\ &\quad = \sigma(k) - \sigma(k-1) - \sigma(k-2) + \sigma(k-5) + \sigma(k-7) - \cdots, \end{aligned}$$

and so the theorem is proved.

11. Trigonometric functions and differential equations. We now return to the general theory of formal power series and make the definitions

$$\sin \alpha = \{E(i\alpha) - E(-i\alpha)\}/2i = \sum_{k=0}^{\infty} \{(-1)^k \alpha^{2k+1}\}/(2k+1)!$$

$$\cos \alpha = \{E(i\alpha) + E(-i\alpha)\}/2 = \sum_{k=0}^{\infty} \{(-1)^k \alpha^{2k}\}/(2k)!,$$

where α is any element in P_0. Thus $\sin \alpha$ is in P_0, but $\cos \alpha$ is in P_1, so we can define $\sec \alpha = (\cos \alpha)^{-1}$ and $\tan \alpha = (\sin \alpha)(\cos \alpha)^{-1}$. However, we cannot now define $\operatorname{cosec} \alpha$ and $\cot \alpha$, but in the next section we extend the theory, to encompass these two functions. All the rules of differentiation now apply, such as $D(\sin \alpha) = (\cos \alpha)D(\alpha)$.

The standard theory of homogeneous linear differential equations with constant coefficients is valid. For example, in the second order case, let a and b be any complex numbers, and let r_1 and r_2 be the roots of $x^2+ax+b=0$. Then a solution for ρ in P of the equation $D^2(\rho)+aD(\rho)+b\rho=0$ is

$$\rho = c_1E(r_1\lambda) + c_2E(r_2\lambda)$$

with arbitrary constants c_1 and c_2. It is easy to prove that this is the general solution if $r_1 \neq r_2$. If $r_1 = r_2$ the general solution is of course $\rho = c_1E(r_1\lambda) + c_2\lambda E(r_1\lambda)$.

We now give a brief sketch of the use of a differential equation to solve a combinatorial problem, as in André [**7**, p. 172]. Our approach differs from that of André in that we treat the differential equation in a purely formal sense, which he did not. For $n \geqq 2$ let b_n be the number of permutations $a_1, a_2, \cdots, a_n$ of $1, 2, \cdots, n$ such that $a_j > a_{j-1}$ if j is even, and $a_j < a_{j-1}$ if j is odd. Call such a permutation an E-permutation. Similarly, say that $a_1, a_2, \cdots, a_n$ is an O-permutation of $1, 2, \cdots, n$ if $a_j > a_{j-1}$ if j is odd, and $a_j < a_{j-1}$ if j is even. Note that if $a_1, a_2, \cdots, a_n$ is an O-permutation then $n+1-a_1, n+1-a_2, \cdots, n+1-a_n$ is an E-permutation, and conversely. Thus there is a one-to-one correspondence between E-permutations and O-permutations; there are b_n of each type. Define $b_0 = 1$ and $b_1 = 1$.

Next, consider the number of O-permutations with $a_1 = n$. It is not difficult to see that there are b_{n-1} of these. Also, there are no E-permutations with $a_1 = n$. Turning to permutations with $a_2 = n$, there are no O-permutations of this type. However, the number of E-permutations with $a_2 = n$ is $(n-1)b_{n-2}$, or what is the same thing $(n-1)b_1b_{n-2}$; the reason for this is that a_1 can be any element among $1, 2, \cdots, n-1$ and the rest can be set up as $a_3, a_4, \cdots, a_n$ in b_{n-2} ways. A similar argument shows that there are no E-permutations with $a_3 = n$, whereas the number of O-permutations with $a_3 = n$ is $\binom{n}{2}b_2b_{n-3}$. Thus by considering all E-permutations and all O-permutations with successively $a_1 = n$, then $a_2 = n$, then $a_3 = n, \cdots$, and finally $a_n = n$, we are led to the recurrence relation

$$2b_n = \sum_{j=0}^{n-1} \binom{n-1}{j} b_j b_{n-j-1} \quad \text{or } 2nc_n = \sum_{j=0}^{n-1} c_j c_{n-j-1},$$

where c_n is defined as $b_n/n!$ for all nonnegative integers n. Taking α to be the formal power series

$$\alpha = \sum_{n=0}^{\infty} c_n \lambda^n$$

we can readily verify that the differential equation $2D(\alpha) = \alpha^2 + 1$ holds. Now it is easy to verify from the definitions of the formal trigonometric functions that $\sin^2 \lambda + \cos^2 \lambda = 1$, $\sec^2 \lambda = 1 + \tan^2 \lambda$, $D(\tan \lambda) = \sec^2 \lambda$, $D(\sec \lambda) = \sec \lambda \tan \lambda$. Thus the unique formal solution of the differential equation is $\alpha = \tan\lambda + \sec\lambda$. (André gives the solution of the differential equation as $\alpha = \tan(\lambda/2 + \pi/4)$ which has no meaning in our formal definition of the trigonometric functions. The usual formula for $\tan(\alpha+\beta)$ in terms of $\tan \alpha$ and $\tan \beta$ is valid, but $\tan \pi/4 = 1$ cannot be established in the formal theory. In fact $\tan \pi/4$ is not even defined because $\pi/4$ is not an element of P_0, although it is an element of P.) Thus we have

$$\alpha = \sum b_n \lambda^n/n! = \tan \lambda + \sec \lambda.$$

Now the power series for $\tan \lambda$ has odd powers of λ only, with coefficients closely connected with the Bernoulli numbers [8, p. 268]. Similarly the power series for $\sec \lambda$ has even powers of λ only, with coefficients related to the Euler numbers [8, p. 269]. Thus André was able to relate the combinatorial numbers b_n to the Bernoulli numbers for odd n, and to the Euler numbers for even n. (A different approach to this problem has been given recently by R. C. Entringer [9].)

From our point of view in this paper, the important aspect of this is that André's conclusions can be drawn with only a formal use of calculus and differential equations and without any convergence questions in the use of α^2, the square of a power series, in the differential equation. The series expansions for $\tan \lambda$ and $\sec \lambda$ come from those for $\sin \lambda$ and $\cos \lambda$, and these are defined in terms of the exponential functions $E(i\lambda)$ and $E(-i\lambda)$. The formal structure carries the entire argument, with no need for the classical infinitesimal calculus. Of course, such relations as $\sin^2 \lambda + \cos^2 \lambda = 1$ have meaning only in terms of formal power series in this context and not in terms of the geometry of right-angled triangles.

12. Extension to a field. Since the set of formal power series P is a commutative integral domain, it can be imbedded in a field P^* in the classical manner by use of pairs of elements, cf. [2, pp. 87–92]. This construction is very well known in the extension of the integers to the rational numbers. Thus P^* is the field of all pairs (α, β) with $\alpha \in P$, $\beta \in P$ and $\beta \neq 0$. Addition and multiplication are defined by

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1\beta_2 + \alpha_2\beta_1, \beta_1\beta_2),$$

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2).$$

Two elements (α_1, β_1) and (α_2, β_2) are said to be equal if and only if $\alpha_1\beta_2 = \alpha_2\beta_1$.

If $\beta = 1$ we agree to write α for $(\alpha, \beta) = (\alpha, 1)$, so that P is a subset of P^*. Similarly we agree to write

$$\left(\sum_{j=0}^{\infty} a_j\lambda^j, \lambda^r\right) \quad \text{as} \quad \sum_{j=0}^{\infty} a_j\lambda^{j-r}, \tag{29}$$

where r is a positive integer. We prove in Theorem 23 that every element of P can be written in this way, so that P^* can be thought of as the class of Laurent power series expansions, with a finite number of negative exponents allowed.

To do this we first define the degree of α for any α in P, $\alpha \neq 0$. If $\alpha = \Sigma a_j\lambda^j$ then the degree of α, written $\deg(\alpha)$, is the subscript of the first nonzero coefficient in the sequence of coefficients $a_0, a_1, a_2, \cdots$. If α_1 and α_2 are nonzero elements of P it follows that $\deg(\alpha_1\alpha_2) = \deg(\alpha_1) + \deg(\alpha_2)$. This definition is extended to P^* as follows: if $(\alpha, \beta) \in P^*$ with $\alpha \neq 0$ then $\deg(\alpha, \beta) = \deg(\alpha) - \deg(\beta)$. Degree is well-defined, because if $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$ then $\alpha_1\beta_2 = \alpha_2\beta_1$ and so we have

$$\deg(\alpha_1) + \deg(\beta_2) = \deg(\alpha_2) + \deg(\beta_1),$$
$$\deg(\alpha_1) - \deg(\beta_1) = \deg(\alpha_2) - \deg(\beta_2).$$

Next for any (α, β) in P^* with $\alpha \neq 0$, let $\deg(\alpha) = m$, $\deg(\beta) = n$ so that $\deg(\alpha, \beta) = m - n$. Then we see that $\beta = \lambda^n\beta_1$ where β_1 has degree 0, so that β_1 has an inverse. It follows that $(\alpha, \beta) = (\alpha, \lambda^n\beta_1) = (\alpha\beta_1^{-1}, \lambda^n)$. Now $\alpha\beta_1^{-1}$ has degree m, so it can be written in the form

$$\alpha\beta_1^{-1} = \sum_{j=m}^{\infty} a_j\lambda^j, \qquad a_m \neq 0.$$

Thus we have

$$(\alpha, \beta) = \sum_{j=m}^{\infty} a_j\lambda^{j-n}, \qquad a_m \neq 0, \tag{30}$$

by virtue of (29).

THEOREM 23. *The representation* (30) *of any nonzero element* (α, β) *of* P^* *is unique.*

Proof. Suppose that (α, β) can also be written as

$$(\alpha, \beta) = \sum_{j=h}^{\infty} c_j\lambda^{j-n}, \qquad c_h \neq 0.$$

By the invariance of degree under different representations we see that $m - n = h - n$ and $m = h$. Also we have

$$(\alpha, \beta) = \left(\sum_{j=m} a_j\lambda^j, \lambda^n\right) = \left(\sum_{j=m}^{\infty} c_j\lambda^j, \lambda^n\right),$$

and so by the definition of equality in P^*,

$$\sum_{j=m}^{\infty} a_j\lambda^{j+n} = \sum_{j=m}^{\infty} c_j\lambda^{j+n}.$$

The theorem follows by the definition of equality in P.

In the preceding section we saw that the trigonometric functions $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, and $\sec \alpha$ could be defined for any element α of P, but not $\operatorname{cosec} \alpha$ and $\cot \alpha$. If $\alpha \neq 0$ we can define the latter two functions from P to P^*; thus

$$\operatorname{cosec} \alpha = (1, \sin \alpha), \quad \cot \alpha = (\cos \alpha, \sin \alpha).$$

A simple calculation shows that

$$\operatorname{cosec} \lambda = \lambda^{-1} + (\lambda/6) + (7\lambda^3/360) + \cdots.$$

Finally, we note that the theory of formal power series, developed here in analogy to power series in a single variable, can be extended in a similar way to the multiple variable case.

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AN ELEMENTARY SOLUTION OF THE BRACHISTOCHRONE PROBLEM

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1. Introduction. The main purpose of this article is to present a solution of the brachistochrone problem which is elementary in the sense that students completing calculus should be able to follow it. This is done in Sections 2 through 4. This result is embedded in more general results in [1] which include the isoperimetric inequality in the plane and the minimal surfaces of revolution. It seems worthwhile to discuss this special case (the brachistochrone problem) *ab initio* in order to show more clearly the simplicity of the method and to make the result accessible to a mathematically unsophisticated audience. Nevertheless, the author believes that the novelty of the method will interest experts as well as beginners.

Section 5 shows how the results of Section 4 can be extended from smooth curves to rectifiable curves. It should be added that calculus students are not expected to follow this section because it is not self-contained as it refers to theorems involving, for example, the concept of absolute continuity.

2. The brachistochrone problem. The Swiss mathematician John Bernoulli published in 1696 a rather arrogant challenge to his fellow mathematicians to solve a problem which he had solved and which he considered very beautiful and very difficult. The reader may be interested to read John Bernoulli's colorful remarks about this problem in [2, pp. 644–655]. The solvers of this problem included John's brother James Bernoulli, as well as Newton, Leibniz, and l'Hospital.

The problem, called the brachistochrone problem, is the following:

Given two points A and B in three dimensional space find, among all smooth curves with endpoints A and B, the curve such that a bead which slides without friction along the curve under the influence of gravity will travel from the one point to the other in the least possible time.

This problem is important because it led to the systematic consideration of similar problems. The new discipline which developed thereby is called the *calculus of variations*.

Other solutions of the brachistochrone problem, known to the writer, either omit consideration of a certain difficult point, or else dispose of that difficulty using rather elaborate machinery. The difficulty is the question of whether or not the problem has any solution at all. The student in freshman calculus should be aware of this sort of difficulty in the maximum-minimum problems which he encounters. Necessary conditions for extrema are easy, but the usual sufficient conditions only establish *relative* maxima and minima. The solution for the brachistochrone problem presented here disposes of this difficulty in a very elementary way. (In fact we shall see that the problem has a solution if and only if a certain first order differential equation has a solution. The ques-

tion of existence of solutions of such differential equations is generally considered to be more elementary than the existence problems in the calculus of variations.)

3. The inequality. The following inequality, the simplest case of Cauchy's inequality, is crucial in solving the problem. For any real numbers a, b, c, d we have

$$ac + bd \leqq (a^2 + b^2)^{1/2}(c^2 + d^2)^{1/2}, \tag{3.1}$$

and equality holds if and only if one has $ad = bc$, i.e., if and only if a and b are proportional to c and d respectively. (A straightforward computation shows that (3.1) holds because of the obvious inequality

$$(ad - bc)^2 \geqq 0, \tag{3.2}$$

and that equality holds in (3.2) if and only if equality holds in (3.1).)

4. The solution. Returning to the problem described in Section 2, one can show that it is sufficient to consider curves which lie in a vertical plane through A and B. We omit this detail.

We make the following assumption for the sake of simplicity. *We assume that the points A and B lie in a horizontal plane.*

Let us suppose that the curve lies in the x-y plane, and that gravity acts in the negative y-direction. We suppose that the points A and B are $(x_1, 0)$ and $(x_2, 0)$ respectively with $x_1 < x_2$. Let Γ denote a smooth curve with endpoints A and B, contained in the half plane $y \leqq 0$. That Γ is smooth means that Γ has a parametric representation $x = \phi(t)$, $y = \psi(t) \leqq 0$, $0 \leqq t \leqq T$, where ϕ and ψ are continuously differentiable on the closed interval $[0, T]$. We will use the notation $\dot{x} = (d/dt)\phi(t)$ and $\dot{y} = (d/dt)\psi(t)$. Using the fact that the sum of potential and kinetic energy must remain constant, one sees that the speed of the bead is equal to $\sqrt{-2gy}$, where g is the gravitational constant. Using the fact that the differential of arc length is $(\dot{x}^2 + \dot{y}^2)^{1/2}dt$, we see that the total time for the bead to complete its journey is proportional to

$$\int_0^T \left(\frac{\dot{x}^2 + \dot{y}^2}{-y}\right)^{1/2} dt. \tag{4.1}$$

(This integral may be infinite.)

Let $-M$ denote the value of y at the lowest point on the curve. Now use (3.1) with

$$a = |\dot{x}|, \quad b = |\dot{y}|, \quad c = M^{-1/2}, \quad d = \left(\frac{1}{-y} - \frac{1}{M}\right)^{1/2}, \tag{4.2}$$

and integrate to obtain

$$\int_0^T \left(\frac{\dot{x}^2 + \dot{y}^2}{-y}\right)^{1/2} dt \geqq M^{-1/2} \int_0^T |\dot{x}|\, dt + \int_0^T \left(\frac{1}{-y} - \frac{1}{M}\right)^{1/2} |\dot{y}|\, dt. \tag{4.3}$$

Noting (3.2) one sees that equality holds in (4.3) if and only if the first order differential equation $M^{-(1/2)}|\dot{y}| = ((1/-y)-(1/M))^{1/2}|\dot{x}|$ is satisfied.

For the first integral on the right hand side of (4.3) we have

$$\int_0^T |\dot{x}|\,dt \geqq x_2 - x_1, \tag{4.4}$$

with equality if and only if x is a nondecreasing function of t.

Suppose that t_0 $(0 \leqq t_0 \leqq T)$ is a number such that $\psi(t_0) = -M$. Then for the last integral in (4.3) we have

$$\begin{aligned}\int_0^T \left(\frac{1}{-y} - \frac{1}{M}\right)^{1/2} |\dot{y}|\,dt &\geqq \int_0^{t_0} \left(\frac{1}{-y} - \frac{1}{M}\right)^{1/2} (-\dot{y})dt \\ &+ \int_{t_0}^T \left(\frac{1}{-y} - \frac{1}{M}\right)^{1/2} \dot{y}dt = 2\int_0^M \left(\frac{1}{\eta} - \frac{1}{M}\right)^{1/2} d\eta = \pi M^{1/2}.\end{aligned} \tag{4.5}$$

(The integrals from 0 to t_0 and from t_0 to T have both been transformed by the substitution $\eta = -\psi(t)$.)

The inequality in (4.5) becomes equality if and only if $\dot{y}$ is nonpositive in $(0, t_0)$ and nonnegative in (t_0, T). That is, equality holds if and only if y is nonincreasing to the left of t_0 and nondecreasing to the right of t_0.

Putting together (4.3), (4.4) and (4.5), we have

$$\int_0^T \left(\frac{\dot{x}^2 + \dot{y}^2}{-y}\right)^{1/2} dt \geqq M^{-1/2}(x_2 - x_1) + \pi M^{1/2}. \tag{4.6}$$

Putting together the various conditions for equality we see that we have equality in (4.6) if and only if the following conditions hold:

$$\begin{aligned} &\dot{x} \geqq 0 && (0 \leqq t \leqq T) \\ &\dot{y} = -((M/-y) - 1)^{1/2}\dot{x} && (0 \leqq t \leqq t_0) \\ &\dot{y} = ((M/-y) - 1)^{1/2}\dot{x} && (t_0 \leqq t \leqq T). \end{aligned} \tag{4.7}$$

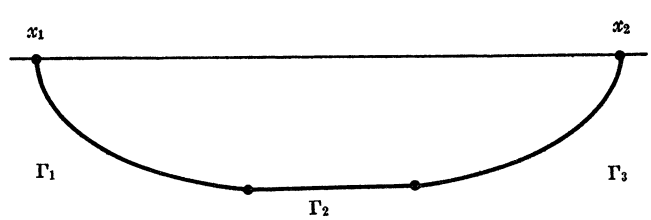

FIG. 1

It is not difficult to see that these conditions are fulfilled if and only if $x_2 - x_1 \geqq M\pi$ and Γ consists of three subarcs, Γ_1, Γ_2, and Γ_3, (see Figure 1) where Γ_1 and Γ_3 are each half-arches of a cycloid and Γ_2 is a segment of the line $y = -M$.

The subarc Γ_2 may consist of a single point, in which case Γ is exactly one arch of a cycloid. We have the following explicit parametric representations:

$$\Gamma_1:\ \begin{aligned} x &= x_1 + \tfrac{1}{2}M(\tau - \sin\tau) \\ y &= \tfrac{1}{2}M(\cos\tau - 1) \qquad 0 \leqq \tau \leqq \pi. \end{aligned} \tag{4.8}$$

$$\Gamma_3:\ \begin{aligned} x &= x_2 - \pi M + \tfrac{1}{2}M(\tau - \sin\tau) \\ y &= \tfrac{1}{2}M(\cos\tau - 1) \qquad \pi \leqq \tau \leqq 2\pi. \end{aligned} \tag{4.9}$$

$$\Gamma_2:\ \begin{aligned} &y = -M \\ &x_1 + \tfrac{1}{2}M\pi \leqq x \leqq x_2 - \tfrac{1}{2}M\pi. \end{aligned} \tag{4.10}$$

A consequence of the foregoing is that if the numbers x_1, x_2 and M are chosen so that

$$x_2 - x_1 \geqq \pi M, \tag{4.11}$$

then the curve Γ of minimum time, subject to the additional constraint that the lowest point on Γ has ordinate equal to $-M$, is exactly the curve described in (4.8), (4.9) and (4.10).

We wish to have a result like (4.6) which is independent of M, since M does not occur in the statement of the problem which we wish to solve. Proceeding from the right hand side of (4.6) we have

$$\begin{aligned} M^{-1/2}(x_2 - x_1) + \pi M^{1/2} &= \pi^{1/2}(x_2 - x_1)^{1/2}\left(\left(\frac{\pi M}{x_2 - x_1}\right)^{1/2} + \left(\frac{x_2 - x_1}{\pi M}\right)^{1/2}\right) \\ &\geqq 2\pi^{1/2}(x_2 - x_1)^{1/2} \end{aligned} \tag{4.12}$$

since the sum of a number and its reciprocal is not less than 2. Equality holds if and only if

$$\pi M = x_2 - x_1. \tag{4.13}$$

Finally we have from (4.6) and (4.12)

$$\int_0^T \left(\frac{\dot{x}^2 + \dot{y}^2}{-y}\right)^{1/2} dt \geqq 2\pi^{1/2}(x_2 - x_1)^{1/2} \tag{4.14}$$

with equality if and only if Γ is the curve of Fig. 1 with Γ_2 consisting of a single point, i.e., if and only if Γ is the cycloid

$$\begin{aligned} x &= x_1 + \frac{1}{2\pi}(x_2 - x_1)(\tau - \sin\tau) \\ y &= \frac{1}{2\pi}(x_2 - x_1)(\cos\tau - 1) \qquad 0 \leqq \tau \leqq 2\pi. \end{aligned} \tag{4.15}$$

Thus we have shown that this cycloid solves the brachistochrone problem.

5. Extension to rectifiable curves. It will be shown in this section that the cycloid (4.15) is still optimum if one allows the rectifiable curves, instead of simply the smooth curves, to compete for the minimum time. The method of proof will be the same as that used in Section 4, but certain steps require special justification.

Many difficulties disappear if the parameter t, which appears in the previous section, is taken to be *arc-length*. In that case it is well known (see [3] p. 17) that the functions $x=\phi(t)$ and $y=\psi(t)$ are Lipschitzian and therefore *absolutely continuous*, and $\dot{x}^2+\dot{y}^2=1$ holds almost everywhere. The integral (4.1) is again proportional to the time. (This claim could not be made if ϕ and ψ were not both absolutely continuous.)

The argument of Section 4 may now be read with the understanding that the derivatives which appear are defined almost everywhere, and the differential equations are to hold almost everywhere. A delicate point occurs in (4.5) where integration by substitution is used. Without loss of generality, we may restrict our consideration to those curves Γ for which the total time (4.1) is finite, since we are seeking curves which minimize the total time. This implies that the integrands in (4.5)

$$\left(\frac{1}{-y}-\frac{1}{M}\right)^{1/2}(-\dot{y}) \quad \text{and} \quad \left(\frac{1}{-y}-\frac{1}{M}\right)^{1/2}\dot{y}$$

are integrable on the intervals $[0, t_0]$ and $[t_0, T]$, respectively. Now we use Theorem 7, part 2, of [4], p. 223, to show that the integration by substitution in (4.5) is justified.

We find that equality holds in (4.6) if and only if conditions (4.7) hold a.e. Assuming that conditions (4.7) hold, one can separate variables and integrate, using integration by substitution again, to show that the solution curve, unique apart from the length of the line segment Γ_2, is the curve in Fig. 1. Since, according to conditions (4.7), $y=\psi(t)$ is monotone in each of the intervals $[0, t_0]$ and $[t_0, T]$, this integration by substitution may be justified by Theorem 7, part 3 of [4] p. 223.

The remainder of the argument follows without difficulty.

The author is indebted to Professor Harley Flanders for his suggestions, particularly for pointing out certain difficulties in the original version which led to the material of Section 5 in this revision.

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FORMAC MEETS PAPPUS

SOME OBSERVATIONS ON ELEMENTARY ANALYTIC GEOMETRY BY COMPUTER

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1. Introduction. One of the truly great advances in mathematics was the algebraization of geometry via the notion of a coordinate system. The broad outlines of this program were indicated in the "Discours de la méthode" of René Descartes (1637) while the essential features were grasped, although not made explicit, by Pierre de Fermat. There is no doubt that Descartes regarded his invention as a universal method, and he wrote that it removed geometry as much from its previous condition as the orations of Cicero were removed from simple ABC's.

The method of Descartes is frequently regarded by students and by teachers as a "machine" into which one feeds the hypotheses of certain geometric situations and which is guaranteed to "grind out" the desired conclusions given sufficient patience on the part of the problem solver. However, it is no denigration of Descartes to assert what also has long been known: that many elementary situations give rise to impossibly long and tedious algebraic computations, and hence the universal method which replaces brains by brawn founders upon the rock of limited human patience and endurance. Ways around are then sought; these include clever coordinate systems, special transformations, determinants, other methods of abridged notation, special devices, constructions, tricks, etc., etc. Several of these devices have subsequently become of prime importance in their own right.

The object of the present paper is to describe what happens when these difficulties are deliberately met broadside and overcome by making use of the symbolic manipulation possibilities of electronic computers. The problem to which we have applied Descartes' method is a classic theorem of Pappus. The language in which we tackled the problem was FORMAC, and the machine was an IBM 360/50 at Brown University with 256 K (K = 1,024) bytes of core storage. As of Summer 1968, this is considered to be medium-sized storage.

Phillip Davis received his Harvard PhD in 1960 under Ralph Boas. He has taught at Harvard, MIT, American University, Maryland, and his present university, Brown. He has had extensive industrial and government experience including five years as Chief, Numerical Analysis Section, National Bureau of Standards; also he was a Guggenheim Fellow in 1956–57. His extensive work in numerical analysis and applied mathematics includes the books *Lore of Large Numbers* (1961), *Interpolation and Approximation* (1963), *Mathematics of Matrices* (1964), *Approximate Numerical Integration* (with P. Rabinowitz, 1967), and *3.1416 and All That* (with W. Chinn, 1969). Professor Davis received the 1960 Award in Mathematics of the Washington Academy of Sciences and the MAA Chauvenet Prize in 1963.

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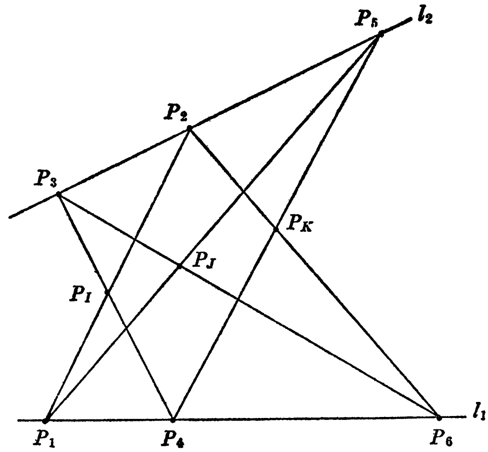

FIG. 1

2. The Theorem of Pappus. Pappus of Alexandria (c. 320 A.D.) was one of the last significant mathematicians of antiquity. There are a number of theorems which bear his name, but the one we have in mind is as follows.

Let l_1 and l_2 be two straight lines in the plane. On l_1 take three points P_1, P_4, P_6 arbitrarily and on l_2 take three points P_3, P_2, P_5 arbitrarily. Now connect up the points in the criss-cross fashion indicated in the figure. Let the points of intersection of the three criss-crosses be designated by P_I, P_J, P_K, respectively. Then, P_I, P_J, and P_K are collinear.

This beautiful theorem (see Figure 1), it turns out, is basic to certain investigations in the foundations of projective geometry. (If Pappus' theorem holds in a projective plane, then the plane is isomorphic to a projective plane over a field.) The interested reader can find information on ancient methods of proof (see p. 289 in [7]). A modern analytic proof can be found on p. 81 of [4].

The program of the present paper is to assign coordinates to $P_1, \cdots, P_6$, to solve for the intersections P_I, P_J, P_K in terms of those coordinates and then simply to verify by algebra that the points P_I, P_J, P_K are, in fact, collinear.

3. Details of the Method. We shall assign general (letter) coordinates to the points. We shall try insofar as possible not to take advantage of the projective group nor of the group of rigid motions or dilations. This, following a remark of Professor Ulf Grenander, can be described as "the method of artificial stupidity" and is to be contrasted with current studies in Computer Science called "artificial intelligence." We did not wholly succeed in this. For reasons explained later we used a rigid motion to place the configuration in a simple position. We shall employ the usual rectangular coordinates, although, effectively, we will be using homogeneous coordinates in that we have arranged our computations so that no divisions occur.

The first formula we need to work out is the point of intersection of a simple criss-cross (Figure 2).

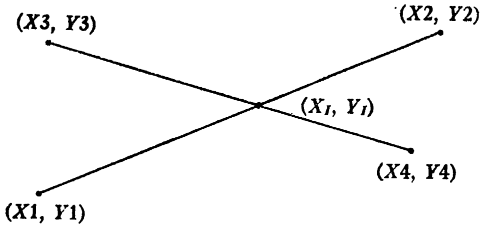

FIG. 2

Let two lines be determined by (X1, Y1), (X2, Y2) and (X3, Y3), (X4, Y4). The point of intersection is given by

$$X_I = \mathrm{NI/DI}, \qquad Y_I = \mathrm{MI/DI}, \tag{3.1}$$

where

$$\begin{aligned} \mathrm{NI} &= \mathrm{Y2\,X3\,X1 - Y4\,X3\,X1 - X4\,Y2\,X1 + X4\,Y3\,X1 - X3\,X2\,Y1} \\ &\quad \mathrm{+ X4\,X2\,Y1 + Y4\,X3\,X2 - X4\,Y3\,X2}, \\ \mathrm{MI} &= \mathrm{Y3\,Y2\,X1 - Y4\,Y2\,X1 - Y3\,X2\,Y1 + Y4\,X2\,Y1 - Y4\,X3\,Y1} \\ &\quad \mathrm{+ X4\,Y3\,Y1 + Y4\,Y2\,X3 - X4\,Y3\,Y2}, \\ \mathrm{DI} &= \mathrm{Y3\,X1 - Y4\,X1 - X3\,Y1 + X4\,Y1 - Y3\,X2 + Y4\,X2 + Y2\,X3} \\ &\quad \mathrm{- X4\,Y2}. \end{aligned} \tag{3.2*}$$

Similar formulas pertain to the points P_J and P_K.

Formulas listed with an asterisk (*) were derived by the computer. The interested reader is invited to check them by whatever means he has at his disposal.

Notice that each coordinate X_I, Y_I is the ratio of two sums of 8 monomials in the variables X1, Y1, etc.

The second formula we need is the condition that three points P_I: (X_I, Y_I), P_J: (X_J, Y_J), P_K: (X_K, Y_K) be collinear. This condition is

$$\begin{vmatrix} X_I & Y_I & 1 \\ X_J & Y_J & 1 \\ X_K & Y_K & 1 \end{vmatrix} = 0. \tag{3.3}$$

In this paper, determinants are only employed as a shorthand for their brute expansions.

We can use (3.1) to rewrite this as

$$\begin{vmatrix} \mathrm{NI/DI} & \mathrm{MI/DI} & 1 \\ \mathrm{NJ/DJ} & \mathrm{MJ/DJ} & 1 \\ \mathrm{NK/DK} & \mathrm{MK/DK} & 1 \end{vmatrix} = 0, \tag{3.4}$$

or as

$$\frac{1}{(\mathrm{DI})(\mathrm{DJ})(\mathrm{DK})}\begin{vmatrix} \mathrm{NI} & \mathrm{MI} & \mathrm{DI} \\ \mathrm{NJ} & \mathrm{MJ} & \mathrm{DJ} \\ \mathrm{NK} & \mathrm{MK} & \mathrm{DK} \end{vmatrix} = 0. \tag{3.5}$$

Call the determinant part of (3.5) DE. Thus, DE is the sum of six terms of the form (NI)(MJ)(DK), etc. Each of the NI, MJ, etc. is the sum of eight monomials in the Xi, Yi, and so (NI)(MJ)(DK) will consist of (at most) $8\times8\times8=512$ monomials. The determinant DE will consist of (before possible reductions) $6\times512=3{,}072$ monomials. (To put this figure in some perspective, recall that the complete expansion of an $n\times n$ determinant consists of $n!$ terms.) It now should be clear why a broadside attack on Pappus' Theorem is tedious.

4. Machine Proof of Pappus' Theorem. FORMAC is a computing system that provides the capability of doing nonnumerical manipulation as well as numerical calculation. The interested reader may consult references [**8**] and [**9**] for details. The system consists of a preprocessor program and the PL/I compiler. The preprocessor translates the FORMAC program into a PL/I program which in turn calls various FORMAC routines at execution time. The PL/I compiler is that program which translates the PL/I language into machine language.

Numerical calculation can be done either in floating point arithmetic or in rational arithmetic. For example, $2\times(3/10)$ can either be computed as .6 or as the rational number 3/5. Some of the algebraic capabilities of FORMAC are expansion of products of sums, substitution of one expression for another, symbolic differentiation, and automatic simplification. Simplification of symbolic expressions by computer is by no means a trivial task. It requires explicit programming of such simple transformations as $x^1\rightarrow x$, $y+0\rightarrow y$, $xy-yx\rightarrow 0$. In addition, the program must run through every expanded algebraic expression and combine like terms. A special version of FORMAC would be required to deal with noncommutative multiplication.

The nonnumeric features of FORMAC that were most essential in the Pappus program were expansion of products and automatic simplification of products. This simplification is crucial to the economy of memory space. The computer form of these expansions required a considerable amount of core storage. In our memory of 256 K bytes, the Pappus program itself required only 35 statements or about 11,000 bytes. The FORMAC system required about 134,000 bytes leaving approximately 117,000 bytes for the algebraic paper work. In the IBM 360, one byte will hold one symbolic (i.e., alphabetic or numeric) character.

A FORMAC program was written which accepted the symbolic coordinates of the points $P_1, \cdots, P_6$ as input and criss-crossed them in the following order: P_1P_2 with P_3P_4, P_1P_5 with P_3P_6, P_4P_5 with P_2P_6. Call the three points of intersection P_I, P_J, P_K the *Pappus points* for $P_1, \cdots, P_6$. The program then computed the determinant DE in terms of the symbolic coordinates of $P_1, \cdots, P_6$.

We now assume that the lines l_1 and l_2 are parametrized as follows:

$$(4.1) \qquad l_1: \begin{cases} x = t \\ y = 0 \end{cases} \qquad l_2: \begin{cases} x = ct \\ y = at + b. \end{cases}$$

The input to our Pappus program was therefore

$$(4.2) \qquad \begin{array}{ll} \mathrm{X1} = \mathrm{T1} & \mathrm{Y1} = 0 \\ \mathrm{X2} = \mathrm{CT2} & \mathrm{Y2} = \mathrm{B} + \mathrm{AT2} \\ \mathrm{X3} = 0 & \mathrm{Y3} = \mathrm{B} \\ \mathrm{X4} = \mathrm{T4} & \mathrm{Y4} = 0 \\ \mathrm{X5} = \mathrm{CT3} & \mathrm{Y5} = \mathrm{B} + \mathrm{AT3} \\ \mathrm{X6} = \mathrm{T6} & \mathrm{Y6} = 0. \end{array}$$

The above simplifications were adopted after it was found that using six general points on two arbitrary lines caused core space to be exceeded. The output (after 4.52 minutes of execution time which included compile and preprocessor time) was

$$(4.3)^* \qquad \mathrm{DE} = 0.$$

As a curiosity, we have reproduced in (4.4)* one of the six terms in the determinant DE. It should be observed that special selection of the coordinates and other cancellations and simplifications have reduced the number of monomials to 41 from a possible 512.

$$(4.4)^* \qquad \begin{aligned} \mathrm{P1} = {} & \mathrm{B^3\ A\ T2\ C\ T1\ T6\ T3\ T4} + \mathrm{B^2\ A^2\ T2^2\ C\ T1\ T6\ T3\ T4} \\ & + 2\ \mathrm{B^2\ A^2\ T2\ T1^2\ T6\ T3\ T4} - \mathrm{B^4\ T2\ C^2\ T6\ T3\ T4} \\ & + \mathrm{B^3\ A\ T2^2\ C^2\ T6\ T3\ T4} + \mathrm{B^4\ T2\ C^2\ T1\ T3\ T4} \\ & - \mathrm{B^3\ A\ T2^2\ C^2\ T1\ T3\ T4} - 2\ \mathrm{B^2\ A^2\ T2\ T1\ T6^2\ T3\ T4} \\ & - 2\mathrm{B^3\ A\ T2\ C\ T6^2\ T3\ T4} + \mathrm{B^3\ A\ T2\ C\ T1^2\ T3\ T4} \\ & - \mathrm{B^2\ A^2\ T2^2\ C\ T1^2\ T3\ T4} + \mathrm{B^4\ T2\ C\ T1\ T6\ T4} \\ & + \mathrm{B^3\ A\ T2^2\ C\ T1\ T6\ T4} + \mathrm{B^3\ A\ T2\ T1^2\ T6\ T4} + \mathrm{B^4\ T2^2\ C^2\ T6\ T4} \\ & - \mathrm{B^4\ T2^2\ C^2\ T1\ T4} + \mathrm{B\ A^3\ T2\ T1^2\ T6\ T3^2\ T4} \\ & - \mathrm{B^3\ A\ T2\ C^2\ T6\ T3^2\ T4} + \mathrm{B^3\ A\ T2\ C^2\ T1\ T3^2\ T4} \\ & - \mathrm{B\ A^3\ T2\ T1\ T6^2\ T3^2\ T4} - \mathrm{B^2\ A^2\ T2\ C\ T6^2\ T3^2\ T4} \\ & + \mathrm{B^2\ A^2\ T2\ C\ T1^2\ T3^2\ T4} - \mathrm{B^3\ A\ T2\ T1\ T6^2\ T4} - \mathrm{B^4\ T2\ C\ T6^2\ T4} \\ & - \mathrm{B^3\ A\ T2^2\ C\ T1^2\ T4} + \mathrm{B^2\ A^2\ T2\ T1\ T6\ T3\ T4^2} \\ & + \mathrm{B\ A^3\ T2^2\ T1\ T6\ T3\ T4^2} + \mathrm{B^3\ A\ T2\ C\ T6\ T3\ T4^2} \\ & + \mathrm{B^2\ A^2\ T2^2\ C\ T6\ T3\ T4^2} - \mathrm{B^3\ A\ T2\ C\ T1\ T3\ T4^2} \\ & - \mathrm{B^2\ A^2\ T2^2\ C\ T1\ T3\ T4^2} - \mathrm{B^2\ A^2\ T2\ T1^2\ T3\ T4^2} \\ & - \mathrm{B\ A^3\ T2^2\ T1^2\ T3\ T4^2} + \mathrm{B^3\ A\ T2\ T1\ T6\ T4^2} \\ & + \mathrm{B^2\ A^2\ T2^2\ T1\ T6\ T4^2} + \mathrm{B^4\ T2\ C\ T6\ T4^2} \\ & + \mathrm{B^3\ A\ T2^2\ C\ T6\ T4^2} - \mathrm{B^4\ T2\ C\ T1\ T4^2} - \mathrm{B^3\ A\ T2^2\ C\ T1\ T4^2} \\ & - \mathrm{B^3\ A\ T2\ T1^2\ T4^2} - \mathrm{B^2\ A^2\ T2^2\ T1^2\ T4^2}. \end{aligned}$$

5. Pascal's Theorem. The theorem of Pappus is a special case of the more general theorem of Blaise Pascal: *if a hexagon is inscribed in a conic then the intersections of opposite sides of the hexagon are collinear.* This theorem was discovered in 1640 when Pascal was 16. An immense literature has grown up around this so-called "mystic hexagram." For example, six given points will (in some order) determine sixty different hexagrams. If these points lie on a conic, sixty Pascal lines will be determined. These lines fall into twenty groups of three, each group passing through a common point. These twenty points lie by fours on fifteen lines, three of the lines going through each point. See, e.g., G. Salmon, Appendix. Pascal was himself reputed to have derived four hundred other theorems from his theorem.

A broadside attack on Pascal by FORMAC might go like this: Parametrize the conic in some way, e.g., take the ellipse $x=a\cos t$, $y=b\sin t$ and then take the six points P_i on the ellipse as $x_i=a\cos t_i$, $y_i=b\sin t_i$, $i=1, 2, \cdots, 6$. Now use the program to form the Pappus points for P_i and then form DE. FORMAC has the capability of dealing with sin and cos symbolically, and can be instructed to reduce by using $\sin^2 x=1-\cos^2 x$.

A second possibility is to use $y=(b/a)\sqrt{a^2-x^2}$ and take P_i: (x_i, y_i) where $y_i=(b/a)\sqrt{a^2-x_i^2}$. FORMAC also has fractional power capabilities.

Neither of these approaches succeeded with our 256K memory. (In Summer, 1968, the storage of the Brown computer was increased to 512 K. This was still insufficient.) The message "no more free list space available" was received before the second intersection point PK was computed. An indirect machine approach to Pascal will be indicated shortly.

6. New Geometrical Theorems by Machine. By leaving a little slack in the situation, one can come up with new theorems or generalizations of old theorems. For example, let us *not* require that $P_1, \cdots, P_6$ lie on two straight lines but compute, quite generally, DE and DI, DJ, DK for arbitrary positions of $P_1, \cdots, P_6$. Since DE/DIDJDK$=2$ times the signed area of the triangle $P_IP_JP_K$, we can obtain a complete formula for this area and hence the possibility of deriving theorems. The following theorem was obtained after an inspection of the machine print out.

THEOREM.* *Let $P_1, \cdots, P_6$ be six points in the plane and let P_I, P_J, P_K be their three Pappus points in the order previously adopted. Consider $P_1, \cdots, P_5$ to be fixed while P_6 is variable. The locus of points P_6 such that the signed area σ of the Pappus triangle $P_IP_JP_K$ is a constant is a conic. If $\sigma=0$ then the conic passes through $P_1, \cdots, P_5$. As σ varies, the conic varies in a pencil of conics.*

Proof. The analytic condition for the constancy of the Pappus area is

$$\text{DE/DIDJDK} = \sigma = \text{constant}, \quad \text{or} \tag{6.1}$$

$$\text{DE} - \sigma\text{DIDJDK} = 0. \tag{6.2}$$

We used the following input to the Pappus program: (a simple rigid motion

does it)

$$X1 = 0,\ Y1 = 0;\quad X2 = F,\ Y2 = G;\quad X3 = P,\ Y3 = E;$$

$$X4 = A,\ Y4 = 0;\quad X5 = H,\ Y5 = K;\quad X6 = B,\ Y6 = C.$$

Single letter variables are preferable to subscripted variables insofar as the storage requirements are less. Now we have

$$\begin{aligned}
\mathrm{DE} = &- E^2 F^2 K H A C + P^2 G^2 K H A C + E^2 F^2 B K A C\\
&- P^2 G^2 B K A C - E^2 G B H^2 A C + E G^2 B H^2 A C\\
&+ E^2 G F H^2 A C - E P G^2 H^2 A C + P^2 G B K^2 A C\\
&- E F^2 B K^2 A C - P^2 G F K^2 A C + E P F^2 K^2 A C\\
&+ E^2 F K H A^2 C - P G^2 K H A^2 C + E^2 G B H A^2 C\\
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&- E G F H^2 A C^2 + E P G H^2 A C^2 - E P F^2 K A C^2\\
&- E F K H A^2 C^2 + P G K H A^2 C^2 + E G F H A^2 C^2\\
&- E P G H A^2 C^2 - P G F K A^2 C^2 + E P F K A^2 C^2\\
&- E^2 G B K H A^2 + E G^2 B K H A^2 + E^2 G F B K A^2\\
&- E P G^2 B K A^2 - E G F B K^2 A^2 + E P G B K^2 A^2
\end{aligned} \tag{6.3}^*$$

$$\begin{aligned}
\mathrm{DIDJDK} = &\,2 G B K H A C - G F K H A C - P G K H A C - E G F H A C\\
&- E^2 F H A C + E P G H A C + P G^2 H A C - E F B K A C\\
&+ P G B K A C + E P F K A C - P^2 G K A C + E G H^2 A C\\
&+ G^2 H^2 A C + 2 E F B K H C - 2 P G B K H C + P G F K H C\\
&- E P F K H C + P^2 G K H C - E F^2 K H C - E G H A^2 C\\
&- G^2 H A^2 C - G B K A^2 C + P G K A^2 C + E G F H^2 C\\
&+ E^2 F H^2 C - E P G H^2 C - P G^2 H^2 C - E G B K H A\\
&- G^2 B K H A + E G F K H A + P G^2 K H A + E^2 G F H A\\
&- E P G^2 H A + E G F B K A - P G^2 B K A - E P G F K A\\
&+ P^2 G^2 K A - E G^2 H^2 A + G F B K^2 A + P G B K^2 A\\
&- P G F K^2 A - G B^2 K^2 A - E G F B K H\\
&- E^2 F B K H + E P G B K H + P G^2 B K H + E^2 F^2 K H\\
&- P^2 G^2 K H + E F H A C^2 - P G H A C^2 - G H^2 A C^2\\
&+ G H A^2 C^2 - E F H^2 C^2 + P G H^2 C^2 + E G^2 H A^2 + G^2 B K A^2\\
&- P G^2 K A^2 - E^2 G F H^2 + E P G^2 H^2 - P G F B K^2\\
&+ E P F B K^2 - P^2 G B K^2 + E F^2 B K^2 + P^2 G F K^2\\
&- E F B^2 K^2 + P G B^2 K^2 - E P F^2 K^2.
\end{aligned} \tag{6.4}^*$$

Now note that for fixed $P_1, \cdots, P_5$ and variable P_6, DE$-\sigma$DIDJDK is a linear combination of two quadratic forms in the coordinates of P_6.

This theorem was derived after an inspection of a machine print out and this

process can be described as *computer assisted theorem derivation.*

Notice that the quadratic form DE is such that when two points P_i, P_j coincide, the form reduces to 0. Therefore, DE = 0 represents the condition that P_i $i=1, \cdots, 6$ lie on a conic. But DE is the computed collinearity determinant of the three Pappus points. Hence, this computation demonstrates plainly that collinearity of the Pappus points is equivalent to the six original points P_i lying on a conic. Thus we have a form of Pascal's theorem. DE is, of course, the 6×6 determinant

$$\text{DE} = -\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \text{F}^2 & \text{G}^2 & \text{FG} & \text{F} & \text{G} & 1 \\ \text{P}^2 & \text{E}^2 & \text{PE} & \text{P} & \text{E} & 1 \\ \text{A}^2 & 0 & 0 & \text{A} & 0 & 1 \\ \text{H}^2 & \text{K}^2 & \text{HK} & \text{H} & \text{K} & 1 \\ \text{B}^2 & \text{C}^2 & \text{BC} & \text{B} & \text{C} & 1 \end{vmatrix}. \tag{6.5}$$

We consider yet another problem: what are the conditions that the Pappus points P_I, P_J, P_K form a right angled triangle with right angle at P_J? The conditions are

$$\frac{(\text{MJ}/\text{DJ}) - (\text{MI}/\text{DI})}{(\text{NJ}/\text{DJ}) - (\text{NI}/\text{DI})} = -\frac{(\text{NJ}/\text{DJ}) - (\text{NK}/\text{DK})}{(\text{MJ}/\text{DJ}) - (\text{MK}/\text{DK})} \tag{6.6}$$

or

$$\begin{aligned} S \equiv (\text{MJ}\cdot\text{DI} - \text{DJ}\cdot\text{MI})(\text{MJ}\cdot\text{DK} - \text{DJ}\cdot\text{MK}) \\ + (\text{NJ}\cdot\text{DI} - \text{DJ}\cdot\text{NI})(\text{NJ}\cdot\text{DK} - \text{NK}\cdot\text{DJ}) = 0. \end{aligned} \tag{6.7}$$

From (3.2), the number of monomials implicit in the left hand side of (6.7) is $2\cdot(8\cdot 8+8\cdot 8)(8\cdot 8+8\cdot 8) = 2^{15} = 32768$, which again indicates the enormous build-up of the formal algebra corresponding to very simple geometrical operations. This kind of storage requirement may saturate memories of moderate size, and some simplifications may be in order. Again, we take P_1 at the origin and P_4 on the x-axis by means of the input

$$\text{X1} = 0, \text{Y1} = 0; \text{X2} = \text{F}, \text{Y2} = \text{G}; \text{X3} = \text{D}, \text{Y3} = \text{E}; \text{X4} = \text{A}, \text{Y4} = 0;$$
$$\text{X5} = \text{H}, \text{Y5} = \text{K}; \text{X6} = \text{B}, \text{Y6} = \text{C}.$$

The introduction of the three zeros will reduce the Pappus points as follows: NI, MI = 1 monomial, DI = 3 monomials; NJ, MJ = 2 monomials; DJ, MK = 4 monomials; NK, DK = 6 monomials.

The number of monomials implicit in the left hand side of (6.7) is therefore reduced to

$$(2\cdot 3 + 4\cdot 1)(2\cdot 6 + 4\cdot 4) + (2\cdot 3 + 4\cdot 1)(2\cdot 6 + 6\cdot 4) = 640.$$

Even with a computer at one's disposal, transformations and shorthand notations may therefore be sought to reduce storage requirements and to interpret the output. The race against the $n!$ buildup of determinants cannot be won by the computer alone operating in the crude mode outlined.

The final computation output (combined and simplified) was

(6.8)* S = approximately 300 monomials each of degree 10.

The first three monomials listed were

$$-\mathrm{E}^2\mathrm{DF}^2\mathrm{BKHAC}+\mathrm{D}^3\mathrm{G}^2\mathrm{BKHAC}+\mathrm{E}^2\mathrm{F}^2\mathrm{B}^2\mathrm{KHAC}-\cdots.$$

A (human) scan of the output for S yields the following computer assisted theorem.

THEOREM.* *Let $P_1, \cdots, P_6$ be six points in the plane and P_I, P_J, P_K be their Pappus points. Let $P_1, \cdots, P_5$ be fixed while P_6 is variable. The locus of points P_6 such that P_IP_J is perpendicular to P_JP_K is a cubic curve.*

7. What Constitutes a Proof in Mathematics? The reader who is not used to thinking about mathematics in terms of machine work may object to the claim that the printout

$$\mathrm{DE}=0$$

constitutes a proof of Pappus' Theorem. What if the programming was erroneous? What if the initial data were false? What if there was a machine malfunction? What if the programmer, in a moment of pique, simply programmed the computer to type out $\mathrm{DE}=0$, and let it go at that?

These are certainly valid objections. Similar objections, however, can be raised with conventional proofs. One aspect of a mathematical proof is that it consists of a finite string of symbols which must be recognized one by one and processed either by a person or by a machine or by both. Now symbols must have physical traces on paper, in the brain, or elsewhere, and cannot be reproduced and recognized with perfect fidelity. Human processing is subject to such things as fatigue, limited knowledge or memory, and to the psychological desire to force a particular result to "come out."

The splicing together of several theorems may cause difficulty. A colleague tells the following story. He received a paper for refereeing which was written by a competent mathematician. The conclusion of the paper seemed to our colleague to be intuitively erroneous. He therefore checked the details of the proof. The details seemed to be in order. He was forced to conclude, therefore, that there was probably an error in one of the theorems used in the paper, but not proved in the paper. The author's references led him to a theorem in a well-known book in probability theory. In this book, the cited theorem was printed erroneously. The words 'closed set' and 'open set' had inadvertently been interchanged.

This sort of story is unfortunately common. A former editor of the *Math-*

nal goals of these subjects. However, if the characteristic means of the computer can supersede these goals—whether the means are to be found in computer languages or in combinatorial power or in heuristic power is not clear—then a genuinely new subject of historical significance can emerge. The Descartes of computer geometry must point an identifying finger firmly at these means.

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ON NEWTON'S INEQUALITY FOR REAL POLYNOMIALS

J. N. WHITELEY, University of Melbourne

1. Introduction. The result of Newton on the coefficients of real polynomials with all roots real, namely

$$\frac{A_r^2}{(n-r+1)} \geqq \frac{A_{r-1}A_{r+1}}{(n-r)},$$

where the polynomial is given in the form

$$P_n(z) = \sum_{r=0}^{n} \frac{A_r}{r!} z^{n-r}$$

is well known. There is an elegant and short proof of this, depending on Rolle's Theorem, in [1]. What is not generally known however, and not as easy to prove (although the ideas involved are still completely elementary), is that there exists a dual to Newton's Inequality. In it the zeros of the real polynomial must

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is well known. There is an elegant and short proof of this, depending on Rolle's Theorem, in [1]. What is not generally known however, and not as easy to prove (although the ideas involved are still completely elementary), is that there exists a dual to Newton's Inequality. In it the zeros of the real polynomial must

lie on a line at right angles to the real axis and within a sector of angle $\pi/2$; then the inequality in the result runs the other way.

Although this result is not difficult, it borders on a theory of considerable difficulty, namely the behaviour of the coefficients of a real polynomial as its zeros move off the real line in one way or another. The famous (but nowadays little-known) paper of Sylvester [5] (see also [6]) and the attendant result of Van Vleck [4] must be mentioned here. The problem of finding a class of polynomials for which the number of variations in sign in Newton's Rule of Signs (or in Sylvester's generalization) is maximal, remains open.

The reader should be acquainted with the elementary results in [3]. Without a knowledge of these I probably should not have attempted the present problem.

2. Preliminary results.

LEMMA 1. *Let* $z_1, \cdots, z_n$ *be the zeros of*

$$f(z) = z^n + \frac{A_1}{1!} z^{n-1} + \frac{A_2}{2!} z^{n-2} + \cdots + \frac{A_n}{n!}. \tag{2.1}$$

Then

$$\frac{1}{n(n-1)} \sum_{i>j} (z_i - z_j)^2 = \frac{A_1^2}{n} - \frac{A_2}{n-1}.$$

Proof. By elementary symmetric functions

$$\begin{aligned} \frac{A_1^2}{n} - \frac{A_2}{n-1} &= \frac{1}{n}\left(\sum z_i\right)^2 - \frac{2}{n-1} \sum_{i>j} z_i z_j \\ &= \frac{1}{n(n-1)}\left[(n-1) \sum z_i^2 - 2 \sum_{i>j} z_i z_j\right] \\ &= \frac{1}{n(n-1)} \sum_{i>j} (z_i - z_j)^2. \end{aligned}$$

COROLLARY. *If* $z_1, \cdots, z_n$ *are the zeros of a polynomial of degree* n *and* $\theta_1, \cdots, \theta_{n-1}$ *are the zeros of its derivative, then*

$$\sum_{i>j} (z_i - z_j)^2 = \frac{n^2}{(n-1)(n-2)} \sum_{i>j} (\theta_i - \theta_j)^2. \tag{2.2}$$

Proof. We differentiate (2.1):

$$\frac{1}{n} f'(z) = z^{n-1} + \frac{n-1}{n} A_1 z^{n-2} + \frac{n-2}{2n} A_2 z^{n-3} + \cdots.$$

By Lemma 1, applied to this polynomial,

$$\frac{1}{(n-1)(n-2)}\sum_{i>j}(\theta_i-\theta_j)^2=\frac{1}{n-1}\left(\frac{n-1}{n}A_1\right)^2-\frac{1}{n-2}\left(\frac{n-2}{n}A_2\right)$$
$$=\frac{n-1}{n}\left(\frac{A_1^2}{n}-\frac{A_2}{n-1}\right).$$

The result follows from Lemma 1.

Note. This is a property like the well-known Center of Gravity property of the roots of an equation (see [2]). It can be shown that

$$\sum_{i>j}(z_i-z_j)^2=n\sum_{i=1}^{n}(z_i-g)^2,$$

where $g=(z_1+\cdots+z_n)/n$, and this form gives a simpler value to the constant in (2.1). It also can be shown that this function is a directional mean, i.e. it has the direction of a line on which "the sum of the squares of the projections of the vectors (z_i-z_j) or the vectors (z_i-g) is maximal."

LEMMA 2. *Suppose all zeros z_i are negative so that $A_i>0$ $(i=1,\cdots,n)$. Set $A_r=\prod_{i=1}^{r}a_i$ $(r=1,\cdots,n)$, $A_0=1$. Then Newton's Inequality for* (2.1) *is equivalent to*

$$\frac{a_r}{n-r+1}\geqq\frac{a_{r+1}}{n-r}\qquad(r=1,\cdots,n-1).\tag{2.3}$$

Let the numbers $b_s^{(j)}$, where $0\leqq j\leqq n-2$, $1\leqq s\leqq n-j$, correspond to the polynomial $w^{n-j}f^{(j)}(1/w)$, obtained by differentiating $f(z)$ of (2.1) *j times and replacing z by $w=1/z$. Then*

$$\frac{b_s^{(j)}}{s}=\frac{n-s-j+1}{a_{n-s-j+1}}>0,\tag{2.4}$$

and (2.3) *is equivalent to*

$$\frac{b_1^{(j)}}{n-j}\geqq\frac{b_2^{(j)}}{n-j-1}\qquad(j=0,1,\cdots,n-2).\tag{2.5}$$

LEMMA 3. *Let $w_1,\cdots,w_n$ be the zeros of a real polynomial of degree n. Suppose these zeros lie on the circle $|w|=1$ and that each lies on the left half-plane $R(w)\leqq 0$. Then*

$$\sum_{i>j}(w_i-w_j)^2\leqq 0.$$

Proof. Write $w_1=-\cos\theta+i\sin\theta$, $w_2=-\cos\eta+i\sin\eta$, $0\leqq(\theta,\eta)\leqq\pi/2$, so that four typical roots are w_1, $\bar{w}_1$, w_2, $\bar{w}_2$. These contribute to the sum

$$x^n + A_1 x^{n-1} y + \frac{A_2}{2!} x^{n-2} y^2 + \cdots + \frac{A_n}{n!} y^n \tag{3.3}$$

by repeated differentiation with respect to x or y will have two nonreal zeros (unless all zeros of (3.1) are equal).

It is worthwhile when trying to understand these results, to plot a rough graph of a_r vs. r, for many of these inequalities admit simple graphical interpretations. This is particularly true of Newton's Inequality and its Dual.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

J. H. McKAY, Oakland University

The following results of the twenty-ninth William Lowell Putnam Mathematical Competition held on December 7, 1968, have been determined in accordance with the regulations governing the Competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of **Massachusetts Institute of Technology,** Cambridge, Massachusetts. The members of the team were Gerald S. Gras, Don Coppersmith, and Jeffrey C. Lagarias; to each of these a prize of one hundred dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the **University of Waterloo,** Waterloo, Ontario. The members of the team were Lee James, William Cunningham, and Alan Adamson; to each of these a prize of seventy-five dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the **University of California at Los Angeles,** Los Angeles, California. The members of the team were Michael Klass, Chris Landauer, and Martin J. Cohen; to each of these a prize of fifty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of **Michigan State University,** East Lansing, Michigan. The

$$x^n + A_1 x^{n-1} y + \frac{A_2}{2!} x^{n-2} y^2 + \cdots + \frac{A_n}{n!} y^n \tag{3.3}$$

by repeated differentiation with respect to x or y will have two nonreal zeros (unless all zeros of (3.1) are equal).

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The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of the **University of Kansas,** Lawrence, Kansas. The members of the team were Walter R. Stromquist, William D. Homer, and Douglas A. Hensley; to each of these a prize of fifty dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are **Don Coppersmith,** Massachusetts Institute of Technology; **Gerald A. Edgar,** University of California, Santa Barbara; **Gerald S. Gras,** Massachusetts Institute of Technology; **Dean G. Huffman,** Yale University; and **Neal I. Koblitz,** Harvard University. Each of these has been designated as Putnam Fellows by the Mathematical Association of America and is awarded a prize of two hundred and fifty dollars.

The seven persons ranking second highest in the examination, named in alphabetical order, are *Allen J. Beadle,* Michigan State University; *Stephen Gagola*, State University of New York at Buffalo; *Jerrold W. Grossman*, Stanford University; *Michael E. Grost*, Michigan State University; *Dennis Hejhal*, University of Chicago; *Robert L. Scott*, University of Michigan; and *Lansing J. Sloan*, South Dakota School of Mines and Technology. To each of these a prize of one hundred dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: *Brown University*, the members of the team were Kenneth A. Ribet, Douglas M. Lublin, and Stephen M. Zucker; *University of California at Berkeley*, the members of the team were Peter Montgomery, Arthur Mirin, and George Evans; *McGill University*, the members of the team were Peter Doubilet, Stephen Tanny, and David Edwards; *Stanford University*, the members of the team were Gordon E. Gullahorn, Jerrold W. Grossman, and Sudhir Aggarwal; *Yale University*, the members of the team were Gregg Zuckerman, Dean G. Huffman, and Ethan Kra.

Honorable mention is given to the following twenty-three individuals, named in alphabetical order: Alan Adamson, *University of Waterloo*; Rich Arratia, *Massachusetts Institute of Technology*; Alan Russell Beale, *Rice University*; Jacob Bergmann, *Dartmouth College*; J. Lawrence Carter, *Dartmouth College*; David A. Cox, *Rice University*; Peter Doubilet, *McGill University*; William Dwyer, *Boston College*; David Edwards, *McGill University*; Irwin Gaines, *Harvard University*; Douglas A. Hensley, *University of Kansas*; William Hibbard, *University of Wisconsin at Madison*; Lee James, *University of Waterloo*; John Keary, *Massachusetts Institute of Technology*; Henry King, *Brown University*; Jeffrey C. Lagarias, *Massachusetts Institute of Technology*; Gordon D. Phillips, *University of Calgary*; Russell D. Meredith, *University of Santa Clara*; Peter Montgomery, *University of California at Berkeley*; James A. Reeds, *University of Michigan*; Walter R. Stromquist, *University of Kansas*; Robert E. Tarjan, *California Institute of Technology*; and Steven Winker, *Massachusetts Institute of Technology.*

The other individuals who were ranked in the top one hundred, arranged by college, are: Robert David Skeel, *University of Alberta;* Douglas M. Lublin and Steven M. Zucker, *Brown University;* Robert J. Epp, *University of British Columbia;* Kenneth Gordon Logan, *University of Calgary;* John L. Davis and Arthur A. Mirin, *University of California, Berkeley;* Martin J. Cohen, Michael J. Klass, and Christopher A. Landaver, *University of California, Los Angeles;* Michael A. Amling and Mark L. Bartelt, *University of California, Santa Barbara;* Thomas R. Davis, *California Institute of Technology;* Robert J. Walcott, *Calvin College;* Frederick W. Call, *Carnegie-Mellon University;* David S. Fried, Kiyoshi Igusa, and Robert B. Israel, *University of Chicago;* David A. Bassein, *City College of New York;* Russell K. Rew, *University of Colorado;* Louis H. Rowen, *Columbia University;* Steven O. Hobbs and Robert B. Lumbert, *Dartmouth College;* Jack R. Pace, *Emory University;* Alexander F. Hunter, *Florida Presbyterian College;* Seaton D. Purdom and Roy L. Smith, *Georgia Institute of Technology;* Daniel E. Frohardt, *Grinnell College;* Michael C. Bix, David W. Collins, Nicholas Littestone, Mark A. Mostow, Gerald I. Myerson, Mark I. Schwimmer, Daniel B. Shapiro, and George Sicherman, *Harvard University;* James L. Hlavka, *Harvey Mudd College;* Erick Douglas Bedford and Daniel E. Putnam, *University of Illinois;* George F. Cornelius, *Illinois Institute of Technology;* Eric J. Isaackson, *Indiana University;* David A. Brubaker, *Lebanon Valley College;* James J. Callahan, *Manhattan College;* Walter L. Griffith and Eric Schechter, *Massachusetts Institute of Technology;* Joseph L. Dunn, *New York University at Washington Square;* William G. Fleissner, *Oberlin College;* Peter Malcolmson, *Princeton University;* Richard L. Enison, *Pratt Institute;* Richard E. Crandall, *Reed College;* Roland T. Smith, Jr., *Rice University;* Thomas J. Marlowe, Jr., *Seton Hall University;* Stephen Allan Reid, *Simon Fraser University;* Gordon E. Gullahorn, *Stanford University;* Mary E. Kramer and Benjamin J. Kuipers, *Swarthmore College;* Edward Bierstone and David J. Oakden, *University of Toronto;* Dean G. Hoffman, *Union College;* Robert J. Kimble, *United States Naval Academy;* Stephen C. Helmreich, *Valparaiso University;* William H. Cunningham, *University of Waterloo;* David Callin Mitchell, *University of Western Ontario;* Frederick J. Bruch, *University of Wisconsin;* Lance W. Jayne and Gregg J. Zuckerman, *Yale University;* David S. Lawrence, *York University.*

One thousand three hundred ninety-eight students from two hundred fifty-three colleges and universities participated in the examination on December 7, 1968.

A listing of the top five hundred contestants may be obtained from the Director. The list, which includes addresses and expected dates of graduation, may be helpful to departments of mathematics in selecting graduate students.

The Questions Committee, consisting of N. D. Kazarinoff (chairman), Leo Moser, and Albert Wilansky, prepared the problems (listed below) for the competition.

PROBLEMS. PART A

A-1. Prove

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx.$$

A-2. Given integers a, b, e, c, d, and f with $ad \neq bc$, and given a real number $\epsilon > 0$, show that there exist rational numbers r and s for which

$$0 < |ra + sb - e| < \epsilon,$$
$$0 < |rc + sd - f| < \epsilon.$$

A-3. Prove that a list can be made of all the subsets of a finite set in such a way that (i) the empty set is first in the list, (ii) each subset occurs exactly once, (iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

A-4. Given n points on the sphere $\{(x, y, z): x^2+y^2+z^2=1\}$, demonstrate that the sum of the squares of the distances between them does not exceed n^2.

A-5. Let V be the collection of all quadratic polynomials P with real coefficients such that $|P(x)|\leqq 1$ for all x on the closed interval $[0, 1]$. Determine

$$\sup\{\,|P'(0)|: P\in V\}.$$

A-6. Determine all polynomials of the form $\sum_0^n a_i x^{n-i}$ with $a_i=\pm 1 (0\leqq i\leqq n, 1\leqq n<\infty)$ such that each has only real zeros.

PART B

B-1. The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:
(i) $P(x^\circ=70^\circ)$, the probability that the temperature in Chicago is 70°,
(ii) $P(y^\circ=70^\circ)$, and
(iii) $P(\max(x^\circ, y^\circ)=70^\circ)$.
Determine $P(\min(x^\circ, y^\circ)=70^\circ)$.

B-2. A is a subset of a finite group G (with group operation called multiplication), and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A.

B-3. Assume that a 60° angle cannot be trisected with ruler and compass alone. Prove that if n is a positive multiple of 3, then no angle of $360/n$ degrees can be trisected with ruler and compass alone.

B-4. Show that if f is real-valued and continuous on $(-\infty, \infty)$ and $\int_{-\infty}^{\infty} f(x)dx$ exists, then

$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)dx.$$

B-5. Let p be a prime number. Let J be the set of all 2×2 matrices $\binom{a\,b}{c\,d}$ whose entries are chosen from $\{0, 1, 2, \cdots, p-1\}$ and satisfy the conditions $a+d\equiv 1 \pmod p$, $ad-bc\equiv 0 \pmod p$. Determine how many members J has.

B-6. A set of real numbers is called compact if it is closed and bounded. Show that there does not exist a sequence $\{K_n\}_{n=0}^{\infty}$ of compact sets of rational numbers such that each compact set of rationals is contained in at least one K_n.

SOLUTIONS. PART A

The number in parentheses, immediately following the problem number, is the number of participants who received a score of 8, 9 or 10 (10 is maximum possible) on a problem. In the case of A-1, A-2, B-1 and B-2, this applies to all 1398 participants. For the other problems, the count applies only to the 794 qualifiers.

A-1 (641) The standard approach, from elementary calculus, applies. By division, rewrite the integrand as a polynomial plus a rational function with numerator of degree less than 2. The solution follows easily.

A-2 (331) The easy solution is obtained by selecting a rational number ρ with $0<\rho<\epsilon$ and solving the linear system

$$ar + bs = e + \rho$$

$$cr + ds = f + \rho.$$

The solution for r and s exist, since $ad \neq bc$, and are rational numbers which satisfy the given inequalities.

Comment. Several students sought a point, with rational coordinates, near the intersection of the two lines $ax+by=e$ and $cx+dy=f$. This was a common but awkward approach and sometimes the student failed to show that the point was on neither of the two lines.

A-3 (257) The proof is by induction. For a singleton set $\{1\}$ the list is $\varnothing$, $\{1\}$. Thus the result is true for singleton sets. Suppose the result is true for all sets with $n-1$ members. Let $S=\{1, 2, 3, \cdots, n\}$ and $T=\{1, 2, 3, \cdots, n-1\}$. Let $T_0, T_1, \cdots, T_t$ $(t=2^{n-1}-1)$ be the list of subsets of T satisfying the requirements. Then the desired list of subsets of S are $S_0, S_1, \cdots, S_s$ $(s=2^{s-1})$ where $S_i=T_i$, for $0 \leqq i < t$, and $S_t=T_t \cup \{n\}$, $S_{t+1}=T_{t-1} \cup \{n\}, \cdots, S=\{n\}$.

Comments: This problem is equivalent to finding a Hamiltonian circuit on an n-cube.

David Bloom observes that S_k is obtained from S_{k-1} by adding or deleting $j+1$, where 2^j is the highest power of 2 which divides k.

A-4 (29) The n points can be represented by vectors v_i $(i=1, \cdots, n)$ with $|v_i|=1$. Expanding "the sum of the squares of the distances between them" in the case $n=3$ suggests the following general identities:

$$\begin{aligned}
\sum_{1 \leq i<j \leq n} |v_i - v_j|^2 &= \sum_{1 \leq i<j \leq n} (v_i - v_j)\cdot(v_i - v_j) \\
&= (n-1) \sum_{1 \leq i \leq n} v_i \cdot v_i - 2 \sum_{1 \leq i<j \leq n} v_i \cdot v_j \\
&= n \sum_{1 \leq i \leq n} v_i \cdot v_i - \left[\sum_{1 \leq i \leq n} v_i \cdot v_i + 2 \sum_{1 \leq i<j \leq n} v_i \cdot v_j \right] \\
&= n^2 - \left(\sum_{1 \leq i \leq n} v_i \right) \cdot \left(\sum_{1 \leq i \leq n} v_i \right).
\end{aligned}$$

Thus the result follows and, in addition, equality exists if and only if $\sum_{1 \leq i \leq n} v_i = 0$.

A-5 (11) Let $f(x)=ax^2+bx+c$ be an arbitrary quadratic polynomial. Then $f(0)=c, f(\frac{1}{2})=\frac{1}{4}a+\frac{1}{2}b+c$, and $f(1)=a+b+c. f'(0)=b=4f(\frac{1}{2})-3f(0)-f(1)$. Using the given conditions, $|P'(0)| \leqq 4|P(\frac{1}{2})|+3|P(0)|+|P(1)| \leqq 8$. Furthermore, $P(x)=8x^2-8x+1$ satisfies the given conditions and has $|P'(0)|=8$.

Comment: The above solution is due to W. G. Hammerle.

A-6 (0) The desired polynomials with $a_0=-1$ are the negative of those with $a_0=1$, so consider $a_0=1$. The sum of the squares of the zeros of $x^n+a_1x^{n-1}+ \cdots +a_n$ is $a_1^2-2a_2$. The product of the squares of these zeros is a_n^2. If all the zeros are real, we can apply the arithmetic-geometric mean inequality to obtain

$$\frac{a_1^2 - 2a_2}{n} \geqq (a_n^2)^{1/n},$$

with equality only if the zeros are numerically equal. In our case this inequality

becomes $(1\pm 2)/n \geqq 1$ or $n \leqq 3$. Note that $n>1$ implies $a_2=-1$ and $n=3$ implies all zeros are ± 1. Thus the list of polynomials is:

$$\pm(x-1), \qquad \pm(x+1), \qquad \pm(x^2+x-1), \qquad \pm(x^2-x-1),$$
$$\pm(x^3+x^2-x-1), \qquad \pm(x^3-x^2-x+1).$$

SOLUTIONS. PART B

B-1 (216) Denote the four events $x^\circ=70^\circ$, $y^\circ=70^\circ$, $\max(x^\circ, y^\circ)=70^\circ$, $\min(x^\circ, y^\circ)=70^\circ$ by A, B, C, D, respectively. Then $A\cup B=C\cup D$, and $A\cap B=C\cap D$. Hence $P(A)+P(B)=P(A\cup B)+P(A\cap B)=P(C\cup D)+P(C\cap D)=P(C)+P(D)$ and $P(\min(x^\circ, y^\circ)=70^\circ)=P(x^\circ=70^\circ)+P(y^\circ=70^\circ)-P(\max(x^\circ, y^\circ)=70^\circ)$.

Comments: There were numerous approaches used on this problem, and not all were valid. Many students had the correct formula for $P(\min(x^\circ, y^\circ)=70^\circ)$, but assumed independence of x and y in their derivation. Only minor credit was given for the correct formula in such cases.

B-2 (261) Let g be any element of G. The set $\{ga^{-1} \mid a\in A\}$ has the same number of elements as A. If these two sets are disjoint, their union would contain more elements than G. Thus there exist a_1, $a_2\in A$ such that $a_1=ga_2^{-1}$ and $g=a_1a_2$.

Alternate Solution: Let G have n elements, A have m elements, and consider the multiplication table of G. An element g in G must appear exactly once in each row and column of the multiplication table. It appears at most $2(n-m)$ times outside the table for A and n times in the table for G. Thus it appears at least $n-2(n-m)=2m-n$ times in the table for A, and we are given that $2m>n$.

B-3 (0) We need to make use of the following facts about fields and constructibility: (1) If Q is the field of rational numbers, the degree of Q extended by $\cos(360^\circ/k)$, where k is a positive integer, is $\phi(k)$, where ϕ is the Euler function. (2) If K, L, M are fields with $K\subset L\subset M$ and $[L:K]<\infty$, $[M:L]<\infty$ then $[M:K]=[M:L]\cdot[L:K]$. (3) Given $\cos(360^\circ/k)$, then $\cos(360^\circ/3k)$ is constructible if and only if

$$\left[Q\left(\cos\frac{360^\circ}{3k}\right):Q\left(\cos\frac{360^\circ}{k}\right)\right]$$

is a power of 2.

Consequently, $[Q(\cos 360^\circ/3k):Q(\cos 360^\circ/k)]\cdot\phi(k)=\phi(3k)$. Now

$$\phi(3^a) = 3^{a-1}\cdot 2 = \begin{cases} 3\phi(3^{a-1}), & \text{if } a>1 \\ 2, & \text{if } a=1 \end{cases}$$

and, by the multiplicative property of the Euler function,

$$\phi(3k) = \begin{cases} 3\phi(k), & \text{if } 3\mid k \\ 2\phi(k), & \text{if } 3\nmid k. \end{cases}$$

Therefore an angle of size $360^\circ/k$ is trisectible if and only if $3\nmid k$.

Comment: A common erroneous approach was the following: "If $360°/3k$ is trisectible, then we can construct $40°/k$. Repeat this angle k times to obtain $40°$ and subtract from $60°$ to obtain $20°$. But since angle of $60°$ is not trisectible, $20°$ is not constructible." The error in this argument is that one is given the angle $360°/3k$ in order to attempt the trisection and with the aid of this additional figure perhaps $60°$ is trisectible (e.g. the case $k=6$).

B-4 (13) The graph of $y=x-1/x$ suggests splitting the integral into the form

$$\int_{-\infty}^{\infty} f(x-1/x)dx = \lim_{a\to-\infty}\int_a^{-1} f(x-1/x)dx + \lim_{b\to 0^-}\int_{-1}^{b} f(x-1/x)dx + \lim_{c\to 0^+}\int_c^1 f(x-1/x)dx + \lim_{d\to\infty}\int_1^d f(x-1/x)dx$$

and making the change of variables $x=\frac{1}{2}[y-\sqrt{y^2+4}]$, in the first two integrals, and the change of variables $x=\frac{1}{2}[y+\sqrt{y^2+4}]$, in the second two integrals. Since both of these functions of y have continuous first derivatives on the intervals involved, the change of variables is valid. After the changes of variable, we have four improper integrals. The convergence of each of these integrals is established by a corollary of the Dirichlet Test (Advanced Calculus, R. C. Buck, McGraw-Hill, p. 143). Thus it is permissable to rewrite the first and third of these improper integrals as a single integral by adding the integrands, since they have the same limits from $-\infty$ to 0. The result is $\int_{-\infty}^{0} f(y)dy$. Likewise, the other two integrals combine to give $\int_0^\infty f(y)dy$. In this combining, there is a canceling of a term involving $y/\sqrt{y^2+4}$ because it appears once with a plus sign and once with a minus sign. We have shown both the convergence of $\int_{-\infty}^{\infty} f(x-1/x)dx$ and the desired equality.

B-5 (183) If $a=0$ then $d=1$, and if $a=1$ then $d=0$. In either case $bc=0$ and b or c is 0, while the other is arbitrary. There are $2p-1$ distinct solutions to $bc=0$ and thus the case $a=0$ or $a=1$ accounts for a total of $4p-2$ solutions. If $a\neq 0$ or 1, then d is uniquely determined and $bc\equiv ad\not\equiv 0 \pmod p$ implies that for each $b\neq 0$, there is a unique c, since the integers mod p form a field. Hence for each a in this case, there are $p-1$ solutions. The total number of solutions is $4p-2+(p-2)(p-1)=p^2+p$.

B-6 (95) Let $\{K_n\}$ be any sequence of compact sets of rational numbers. For each n, there is a rational $r_n\notin K_n$, with $0\leq r_n<1/n$. Otherwise, it would be that K_n contained all rationals in $[0, 1/n]$, and hence some irrationals (since K_n is closed). Let $S=\{0, r_1, r_2, \cdots\}$. Then S is compact and not included in any K_n.

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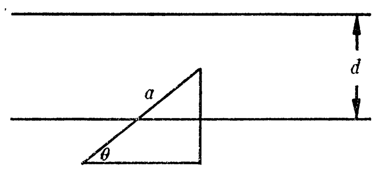
BUFFON'S NOODLE PROBLEM

J. F. RAMALEY, Bowling Green State University, Ohio

In 1733, Georges Louis Leclerc, Comte de Buffon, considered the following problem: Given a needle of length a and an infinite grid of parallel lines with common distance d between them, what is the probability $P(E)$ that a needle, tossed at the grid randomly, will cross one of the parallel lines?



If $a \leqq d$ and we choose a sector $\Delta\theta$ as the direction in which the needle is to fall, we see that $P(\theta \in \Delta\theta) = (\Delta\theta)/\pi$ and $P(\text{cross} \mid \theta \in \Delta\theta) = (a \sin \theta)/d$. Thus $P(E)$ may be calculated using the formula for conditional probability $P(AH) = P(A \mid H)P(H)$ and integrating θ from 0 to π as follows:

$$P(E) = \int_0^\pi \frac{a \sin \theta d\theta}{\pi d} = (a/\pi d) \int_0^\pi \sin \theta d\theta = 2a/\pi d.$$

If $d < a$, the calculations are rougher, but

$$P(E) = (2a/\pi d)(1 - \cos \alpha) + (\pi - 2\alpha)/\pi \quad \text{where} \quad \alpha = \arcsin (d/a),$$

which actually coincides with the earlier result if $d \geqq a$.

Gnedenko [1, p. 43] generalized the problem first to n-sided convex polygons with diameter less than d, and then to convex closed curves with diameter less than d by considering such curves as limits of inscribed polygons, giving the formula $P(\text{cross}) = a/\pi d$. The requirement on the diameter was given to insure a probability $P \leqq 1$, for in the case of a circle with diameter greater than d, the probability of a cross is always 1 although the formula gives a value greater than 1. The convexity requirement on the polygons allowed one to assert that, with probability one, the polygon crosses a line if and only if exactly two sides of the polygon cross the line. A computational proof was then given to find the probability of this latter event.

In order to drop these assumptions on the curve (closed, convex, and with restricted diameter), we generalize the problem in another direction by asking, "How many lines might we expect the needle to cross?" To do this we shall say that there are exactly n *line-crossings* (or just *crossings*) on a given toss if there

Gnedenko's result then follows since, under his hypothesis, the event E that the curve intersects a line of the grid would occur, with probability one, if and only if there occur exactly two line-crossings. Hence $P(E)$ is exactly one-half the expected number of line-crossings, $P(E)=(1/2)(2a/\pi d)=a/\pi d$, where a is the perimeter of the curve.

We remark that this result may be obtained geometrically by taking the "grid's point of view" and considering the experiment as one of tossing the grid onto a noodle fixed in the plane. The expected number of crosses is independent of the position of the noodle, and in fact one could change the noodle's position after each throw.

Reference

1. B. V. Gnedenko, The Theory of Probability, 2nd ed., Chelsea, New York, 1963.

AN APPLICATION OF TWO ESTIMATES FOR e

R. B. DARST, Purdue University and University of California at Riverside

A. F. Sololev recently obtained the following formula for the jump at a point $a\in(0, 1)$ of a nondecreasing function g defined on $[0, 1]$:

$$(S)\ g(a^+) - g(a^-) = \inf_{n=1,2,\cdots} \inf_{k=1,\cdots,n} \int_0^1 \frac{x^k(1-x)^{n-k}}{a^k(1-a)^{n-k}}\, dg.$$

Sololev came to g via a Hausdorff moment problem and obtained his formula in that setting. The purpose of this note is to give an elementary verification of his formula based on two estimates for e which arise in a beginning calculus class:

$$u(x) = \left(1+\frac{1}{x}\right)^x \text{ increases to } e$$

and

$$v(x) = \left(1+\frac{1}{x}\right)^{x+1} \text{ decreases to } e$$

as x increases from 1.

Because elementary properties of the Riemann-Stieltjes integral tell us that the right side of (S) is greater than or equal to the left, it is the reverse inequality to which we shall direct our attention. Let f be defined on $[0,1]$ by $f(x)=x^k(1-x)^{n-k}$, where n and k remain to be chosen. Then f obtains its max at k/n, and a simple computation gives

$$\frac{f\{(k-1)/n\}}{f(k/n)} = \frac{u(n-k)}{v(k-1)} \quad\text{and}\quad \frac{f\{(k+1)/n\}}{f(k/n)} = \frac{u(k)}{v(n-k-1)}.$$

Moreover, if each of n, k, and $n-k$ is large and each of a and $x\in[(k-1)/n,$

$(k+1)/n]$, then $f(x)/f(a)$ is close to one. Let m be a positive integer satisfying $m\cdot\min\{a, 1-a\}>3$. Let t be a positive integer (to be specified) and let $n=mt$. Let k satisfy $|(k/n)-a| = \min\{|(i/n)-a|; i=1, 2, \cdots\}$. Then each of k and $n-k$ is greater than $2t$, and hence,

$$\frac{f[(k-t)/n]}{f(k/n)} = \left(\frac{k-t}{k}\right)^k\left(1+\frac{t}{n-k}\right)^{n-k} = \left[\frac{u\{(n-k)/t\}}{v\{(k-t)/t\}}\right]^t < \left[\frac{u(m)}{e}\right]^t$$

and

$$\frac{f\{(k+t)/n\}}{f(k/n)} = \left(\frac{k+t}{k}\right)^k\left(\frac{n-k-t}{n-k}\right)^{n-k}$$
$$= \frac{[u(k/t)]^t}{[1+(t/(n-k-t))]^{n-k}} < \left[\frac{u(m)}{v\{(n-k-t)/t\}}\right]^t < \left[\frac{u(m)}{e}\right]^t.$$

Let $\epsilon>0$. Choose m so that $g(a+2/m)-g(a-2/m)<g(a^+)-g(a^-)+\epsilon$; next, choose t such that $[u(m)/e]^t<\epsilon$ and $f(x)/f(a)<1+\epsilon$ on $[(k-1)/n, (k+1)/n]$. Then

$$\int_0^1 \frac{f(x)}{f(a)}\,dg < (1+\epsilon)[g(a^+)-g(a^-)] + \epsilon(1+\epsilon) + \frac{\epsilon}{f(a)}[g(1)-g(0)],$$

and (S) is verified.

The corresponding formulas (and strategies for their verification) for the jumps of g at the endpoints of $[0, 1]$ should be obvious to the reader.

Reference

1. A. F. Sololev, On H. Hamburger's problem of moments, Studies Contemporary Problems Constructive Theory of Functions, (Proc. Second All-Union Conf., Baku, 1962) 616–619. Izv. Akad. Nauk. Azerbaidzan. SSR., Baku, 1965.

REDUCIBILITY OF POLYNOMIALS OF ODD DEGREE (II)

D. B. Lloyd, District of Columbia College

In a recent issue of this journal [5], the author presented a sufficient condition for reducibility of the general polynomial of odd degree with rational integer coefficients. The present paper is a companion thereof in that it provides another such criterion, but entails less computation in its use. For this purpose we present and prove the key

THEOREM I. *Let $f(x)=x^n+c_1x^{n-1}+\cdots+c_n$ be an irreducible monic polynomial in $Z[x]$ of odd degree $n>1$. Let α and β be roots of $f(x)=0$. Then $\alpha\beta$ is irrational.*

Proof. Assume $\alpha\beta=r\in Q$, the field of rationals. If θ is *any* root of $f(x)=0$,

$(k+1)/n]$, then $f(x)/f(a)$ is close to one. Let m be a positive integer satisfying $m\cdot\min\{a, 1-a\}>3$. Let t be a positive integer (to be specified) and let $n=mt$. Let k satisfy $|(k/n)-a| = \min\{|(i/n)-a|; i=1, 2, \cdots\}$. Then each of k and $n-k$ is greater than $2t$, and hence,

$$\frac{f[(k-t)/n]}{f(k/n)} = \left(\frac{k-t}{k}\right)^k\left(1+\frac{t}{n-k}\right)^{n-k} = \left[\frac{u\{(n-k)/t\}}{v\{(k-t)/t\}}\right]^t < \left[\frac{u(m)}{e}\right]^t$$

and

$$\frac{f\{(k+t)/n\}}{f(k/n)} = \left(\frac{k+t}{k}\right)^k\left(\frac{n-k-t}{n-k}\right)^{n-k}$$
$$= \frac{[u(k/t)]^t}{[1+(t/(n-k-t))]^{n-k}} < \left[\frac{u(m)}{v\{(n-k-t)/t\}}\right]^t < \left[\frac{u(m)}{e}\right]^t.$$

Let $\epsilon>0$. Choose m so that $g(a+2/m)-g(a-2/m)<g(a^+)-g(a^-)+\epsilon$; next, choose t such that $[u(m)/e]^t<\epsilon$ and $f(x)/f(a)<1+\epsilon$ on $[(k-1)/n, (k+1)/n]$. Then

$$\int_0^1 \frac{f(x)}{f(a)}\,dg < (1+\epsilon)[g(a^+)-g(a^-)] + \epsilon(1+\epsilon) + \frac{\epsilon}{f(a)}[g(1)-g(0)],$$

and (S) is verified.

The corresponding formulas (and strategies for their verification) for the jumps of g at the endpoints of $[0, 1]$ should be obvious to the reader.

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REDUCIBILITY OF POLYNOMIALS OF ODD DEGREE (II)

D. B. Lloyd, District of Columbia College

In a recent issue of this journal [5], the author presented a sufficient condition for reducibility of the general polynomial of odd degree with rational integer coefficients. The present paper is a companion thereof in that it provides another such criterion, but entails less computation in its use. For this purpose we present and prove the key

THEOREM I. *Let $f(x)=x^n+c_1x^{n-1}+\cdots+c_n$ be an irreducible monic polynomial in $Z[x]$ of odd degree $n>1$. Let α and β be roots of $f(x)=0$. Then $\alpha\beta$ is irrational.*

Proof. Assume $\alpha\beta=r\in Q$, the field of rationals. If θ is *any* root of $f(x)=0$,

$$f = (x^2 + mx + n)(x^3 + px^2 + qx + r).$$

Substituting $n = -2$, multiplying out, and equating coefficients of like powers, we solve the resulting system of equations to find

$$f = (x^3 + x + 6)(x^2 - 2).$$

As the converse of Theorem I is not in general true for $n > 5$, many reducible polynomials are not detected by this method. For such cases, the author's method (see [3] and [4]) using finite fields can be used. For example, the nonic

$$x^9 - 2x^7 + 7x^6 - x^5 + 9x^3 - 13x^2 + 26x - 15$$

factors into three irreducible cubics

$$(x^3 + x - 1)(x^3 - x + 3)(x^3 - 2x + 5),$$

as easily found by the finite field method. Yet the S and P functions for this case have no rational roots and thus fail to reveal the reducibility.

For some polynomials, P may have a rational root while S does not, and vice-versa; e.g., for $x^6 - 6x^3 + 8$, the P function has the rational root $\theta_1\theta_2 = \sqrt[3]{2}\sqrt[3]{4} = 2$, and S has none. Whereas, for the rational sextic $\Pi_{i=1}^{6}(x - \theta_i)$, where the six roots are $1 \pm \rho$, $\omega \pm \omega^2\rho$, $\omega^2 \pm \omega\rho$, with $\rho^3 = 3$ and ω a primitive cube root of unity, P has no rational root, while S has the root 2. Clearly, this distinction between P and S holds for polynomials of both even and odd degree.

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AN EXTREMUM PROPERTY OF CONVEX FUNCTIONS

M. J. Miles, Environmental Science Services Administration, Boulder, Colorado

Let S be a closed and bounded convex subset of R^n, real n-dimensional Euclidean space, and consider the convex functions $f(x)$ defined there.

It is observed that these functions, together with their supporting hyperplanes, possess an extremum property that, surprisingly, depends only on S. Loosely speaking, the "closest" support hyperplane to any such function occurs at the common point $\bar{x}$, the centroid of S.

THEOREM 1. *Let S be a closed and bounded convex subset of R^n whose centroid is $\bar{x}$. If $f(x)$ is a convex function and $h(x, \xi)$ is a support hyperplane at ξ, then the function*

$$\phi(\xi) = \int_S [f(x) - h(x, \xi)]dx$$

has a minimum at $\xi = \bar{x}$.

Proof. Two properties of a support hyperplane to f at ξ are

$$h(\xi, \xi) = f(\xi) \quad \text{and} \quad h(x, \xi) \leqq f(x).$$

If σ is the positive volume of S, integration shows that

$$\phi(\xi) = \int_S f(x)dx - \sigma h(\bar{x}, \xi). \tag{1}$$

By the first of the two properties,

$$\phi(\bar{x}) = \int_S f(x)dx - \sigma f(\bar{x}), \tag{2}$$

and by the second, equations (1) and (2) reveal that $\phi(\xi) \geqq \phi(\bar{x})$; that is, the minimum value of ϕ is at $\bar{x}$.

An inspection of the proof yields the following generalization:

THEOREM 2. *Let S be a closed and bounded subset of R^n that contains its centroid $\bar{x}$. Let $f(x)$ be a real-valued function that is integrable on S and possesses a lower support hyperplane, $h(x, \xi)$, at least at $\xi = \bar{x}$. Then $\phi(\xi)$ has a minimum at $\bar{x}$.*

ON REPRESENTING A SQUARE AS THE SUM OF THREE SQUARES

OWEN FRASER, Los Angeles Valley College, and BASIL GORDON, University of California, Los Angeles

In his book on number theory [**3**; 194], Nagell says "it follows from Lebesgue's identity

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac + 2bd)^2 + (2ad - 2bc)^2$$

that every integral square may be written as the sum of three integral squares." Since there is always the trivial representation $n^2 = n^2 + 0^2 + 0^2$, he presumably means to imply that there is also a representation $n^2 = x^2 + y^2 + z^2$ with $xyz \neq 0$. This, however, is not always the case; the complete answer is as follows:

THEOREM. *If n is a positive integer, then the equation $n^2 = x^2 + y^2 + z^2$ has a solution in positive integers x, y, z if and only if n is not of the form 2^k or $2^k \cdot 5$.*

This theorem was stated without proof by Hurwitz [**2**; 751], who also gave a

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This theorem was stated without proof by Hurwitz [**2**; 751], who also gave a

formula for the number of solutions. It appears that his proof was probably based on the theory of elliptic functions. Here we shall give a completely elementary proof.

Proof. It is trivial to verify the unsolvability of our equation for $n=1$ and $n=5$. Suppose that $k\geqq 1$, and that unsolvability has already been established for $n=2^{k-1}$ and $2^{k-1}\cdot 5$. Let n be either 2^k or $2^k\cdot 5$, and suppose that $n^2=x^2+y^2+z^2$, where $xyz\neq 0$. Since n is even, either two or none of the numbers x, y, z are odd. In the first case $x^2+y^2+z^2\equiv 2 \pmod 4$, contradicting the fact that $n^2\equiv 0 \pmod 4$. In the second case, the equation can be divided by 4, giving $(n/2)^2=(x/2)^2+(y/2)^2+(z/2)^2$. This contradicts the induction hypothesis, completing the proof of unsolvability for $n=2^k$ and $n=2^k\cdot 5$.

Now suppose n has neither of these forms. Then n is divisible either by 25, or by some prime $p\neq 2, 5$. Clearly if $n=ab$, and if $a^2=x^2+y^2+z^2$ with $xyz\neq 0$, then $n^2=(bx)^2+(by)^2+(bz)^2$ with $(bx)(by)(bz)\neq 0$. Hence our theorem will be proved if we can show that each of the numbers 25^2 and p^2 ($p\neq 2, 5$) is the sum of three positive squares. We have $25^2=12^2+15^2+16^2$, so we can suppose from now on that $n=p$, where p is a prime $\neq 2, 5$. By Lagrange's Theorem we can write $p=a^2+b^2+c^2+d^2$, and Lebesgue's identity then gives

$$p^2 = (a^2+b^2-c^2-d^2)^2 + (2ac+2bd)^2 + (2ad-2bc)^2. \tag{1}$$

Consider first the case where $p\equiv 3 \pmod 4$. Here it is known [1; 299] that there is no representation of p^2 as the sum of two positive squares. Hence the number of positive terms on the right hand side of (1) is either 1 or 3. The first term is clearly odd, so it suffices to show that the second and third terms cannot both vanish. If $ac+bd=ad-bc=0$, then

$$(ac+bd)^2 + (ad-bc)^2 = (a^2+b^2)(c^2+d^2) = 0.$$

Hence either $a=b=0$ or $c=d=0$. In either case, the formula $p=a^2+b^2+c^2+d^2$ leads to a representation of p as a sum of two squares, which is well known to be impossible, since $p\equiv 3 \pmod 4$.

If $p\equiv 1 \pmod 4$, we can write $p=a^2+c^2$, and Lebesgue's identity reduces to $p^2=(a^2-c^2)^2+(2ac)^2$. Clearly both terms on the right are positive. Since $p\neq 5$, we have $p^2\equiv 1$ or $4 \pmod 5$. Every square is congruent to 0, 1, or 4 (mod 5); from this it is easily seen by enumeration of cases that whenever $p^2=u^2+v^2$, we must have either $u\equiv 0 \pmod 5$ or $v\equiv 0 \pmod 5$. Suppose for definiteness that $v=5w$. Then $p^2=u^2+25w^2=u^2+(3w)^2+(4w)^2$. Since we found a representation $p^2=u^2+v^2$ with $uv\neq 0$, this leads to a representation $p^2=x^2+y^2+z^2$ with $xyz=u\cdot(3v/5)(4v/5)\neq 0$. The proof of our theorem is now complete.

The second author was sponsored in part by NSF Grant GP-5497.

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As an example of what can be done using a little machinery in conjunction with Theorem 1, we offer the following:

COROLLARY 3. *Let G be any group with order $2^t \cdot 3$. Then G has a normal subgroup of order 2^t or 2^{t-1}.*

Proof. Sylow's theorem tells us that G has a subgroup H with $|H| = 2^t$; we set $m=3$, $n=2^t$, and apply Theorem 1. Here, $(n, (m-1)!) = (2^t, 2!) = 2$, so that $[H: K] \mid 2$; hence, $|K| = 2^t$ or $|K| = 2^{t-1}$, and our conclusion follows.

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

ARE ALMOST COMMUTING MATRICES NEAR COMMUTING MATRICES?

PETER ROSENTHAL, University of Toronto

Let $\mathcal{H}$ be a finite-dimensional complex vector space and let $\mathcal{B}(\mathcal{H})$ denote the collection of all linear transformations on $\mathcal{H}$. Let $\|\ \|$ be a norm on $\mathcal{B}(\mathcal{H})$; perhaps the first norm that should be considered is the Hilbert-Schmidt norm; i.e., if the linear transformation A has matrix $\{a_{ij}\}$ relative to some orthonormal basis of $\mathcal{H}$ then

$$\|A\| = (\sum |a_{ij}|^2)^{1/2}.$$

The following problem is apparently open: if A and B "almost commute" must there exist "nearby" matrices A' and B' which commute? More precisely, does there exist a function f mapping the positive reals into the positive reals, and with $\lim_{t\to 0^+} f(t) = 0$, such that whenever $\epsilon > 0$ and $A, B \in \mathcal{B}(\mathcal{H})$ with $\|AB - BA\| < \epsilon$, then there exist $A', B' \in \mathcal{B}(\mathcal{H})$ such that

$$\|A - A'\| < f(\epsilon), \qquad \|B - B'\| < f(\epsilon),$$

and $A'B' = B'A'$? Such an f would probably depend upon dim $\mathcal{H}$. Alternatively, the problem could be changed to take into account the norms of A and B (e.g., perhaps one could make the additional assumption that $\|A\| = \|B\| = 1$).

As an example of what can be done using a little machinery in conjunction with Theorem 1, we offer the following:

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$$\|A - A'\| < f(\epsilon), \qquad \|B - B'\| < f(\epsilon),$$

and $A'B' = B'A'$? Such an f would probably depend upon dim $\mathcal{H}$. Alternatively, the problem could be changed to take into account the norms of A and B (e.g., perhaps one could make the additional assumption that $\|A\| = \|B\| = 1$).

Of course if such an f exists for one norm on $\mathcal{B}(\mathcal{H})$ then one exists for all norms on $\mathcal{B}(\mathcal{H})$, since all norms on a finite-dimensional vector space are equivalent. However, the form of f would depend upon the norm used, and the relation between f and dim $\mathcal{H}$ would also depend upon the norm. The growth of f with dim $\mathcal{H}$ would be important for applications to the study of compact operators on infinite-dimensional spaces. For example, the existence of such an f might imply (if f didn't grow too fast with dim $\mathcal{H}$) the existence of common invariant subspaces for certain classes of compact operators.

It is easily seen that such an f exists for dim $\mathcal{H}=2$. It is also fairly easy to show that there is an f that works for all pairs (A, B) of matrices such that at least one of A and B is normal (i.e., is unitarily equivalent to a diagonal matrix).

ARE EQUIDECOMPOSABLE PLANE CONVEX SETS CONVEX EQUIDECOMPOSABLE?

G. T. SALLEE, University of California, Davis

We shall say that the plane sets A and B are *equidecomposable* if A is the union of disjoint sets $A_1, \cdots, A_n$, and B is the union of disjoint sets $B_1, \cdots, B_n$ such that A_i is congruent to B_i for all i. The A_i are termed *factors* of the equidecomposition. X and Y are *congruent* to each other if there exists a one-one distance-preserving map of the plane onto itself taking X onto Y. If each factor is a convex set, we say that A and B are *convex equidecomposable.*

The central problem of this paper may now be stated.

PROBLEM 1. *Suppose A and B are compact, convex sets in the plane which are equidecomposable. Are A and B convex equidecomposable?*

This problem was suggested by an old one which Tarski posed many years ago [4].

PROBLEM 2. *Are a circle and a square of the same area equidecomposable?*

One possible approach to solving either of the problems would be to restrict the nature of the factors in some way. For example, if A and B are plane sets and each factor is either the interior of a simple, closed Jordan curve or part of such a curve, we say that A and B are *scissors equidecomposable* (see [2]). For a large class of sets, scissors equidecomposable sets are completely characterized. A plane set is termed *elementary* if it is compact, convex, and its boundary consists of a finite number of arcs, each of which is either strictly convex or a straight line segment.

THEOREM 1 (Dubins-Hirsch-Karush). *Suppose A and B are elementary plane sets. Then A and B are scissors equidecomposable if and only if they have the same area and their respective boundaries are scissors equidecomposable.*

We may use this result to establish the following:

THEOREM 2. *Let A and B be elementary plane sets. Then A and B are convex equidecomposable if and only if they are scissors equidecomposable.*

Proof. The necessity of the condition is obvious. To prove sufficiency, we observe that if A and B are scissors equidecomposable we may partition the boundary of A into a finite number of disjoint arcs $I_1, \cdots, I_n$ which can be rearranged to form the boundary of B. Let I_k' denote the set on boundary of B congruent to I_k. Let $J_k[J_k']$ denote the convex hull of the relative interior of $I_k[I_k']$. Clearly J_k is congruent to J_k' for $1 \leqq k \leqq n$. Moreover $A \sim \bigcup J_k$ and $B \sim \bigcup J_k'$ are closed polygons of equal area, and it is well known that these are convex equidecomposable ([1], p. 260). The result follows.

Note that a solution to Problem 1 in the affirmative would give a generalization of Theorem 1 to all compact, convex plane sets.

This problem is essentially the only one of its type that remains unsolved. The one-dimensional analogue is trivial, and any similar assertion in three or more dimensions is easily seen to be false by considering the Banach-Tarski Paradox [1]. Likewise the analogous conjecture permitting a countable number of factors is false since all two d-dimensional bodies $(1 \leqq d)$ with interior are known to be countably equidecomposable [1]. A good summary of many related questions may be found in [3].

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

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LIMITING DISTRIBUTIONS FOR SAMPLE QUANTILES

D. S. MOORE, Purdue University

Let $X_1 < X_2 < \cdots < X_n$ be the order statistics from a random sample of size n from an absolutely continuous distribution with distribution function $F(x)$. For $0 < \alpha < 1$, $X_{[n\alpha]+1}$ is the *sample α-quantile*, where $[n\alpha]$ is the greatest integer

This observation is the key to the proof. For since $F(x)$ is continuous, $p_{in} \to p_i$ as $n \to \infty$, where $p_1 = F(x_\alpha) = \alpha$ and $p_2 = \beta - \alpha$. It may be easily checked that the usual characteristic function proof of asymptotic normality for multinomial random variables is not affected by substitution of a convergent sequence of cell probabilities. Therefore, any two of the random variables $Q_{in} = \sqrt{n}(n_i/n - p_{in})$, $i = 1, 2, 3$, are asymptotically normal with means 0 and covariance

$$\sigma_{ii} = p_i(1 - p_i); \quad \sigma_{ij} = - p_i p_j, \qquad i \neq j. \tag{2}$$

The proof is now straightforward. Taylor's Theorem gives

$$F(x_\alpha + u/\sqrt{n}) = \alpha + f(x_\alpha)u/\sqrt{n} + o(1/\sqrt{n})$$

and a similar expression for $F(x_\beta + v/\sqrt{n})$. (1) thus becomes

$$F_n(u, v) = P[Q_{1n} \geqq - f(x_\alpha)u + o(1), \qquad Q_{1n} + Q_{2n} \geqq - f(x_\beta)v + o(1)].$$

The result of the theorem follows from the known joint asymptotic distribution of the Q_{in}. For since $Q_{1n} + Q_{2n} + Q_{3n} = 0$,

$$F_n(u, v) = P\left[- \frac{Q_{1n}}{f(x_\alpha)} \leqq u + o(1), \frac{Q_{3n}}{f(x_\beta)} \leqq v + o(1) \right] \to P[Z_1 \leqq u, Z_2 \leqq v],$$

where (Z_1, Z_2) have the limiting law of $(-Q_{1n}/f(x_\alpha), Q_{3n}/f(x_\beta))$, which may be trivially calculated to be the stated distribution.

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NOTE ON SIMPSON'S RULE

ANON, Erewhon-upon-Wabash

We derive the remainder in Simpson's Rule by using an integral formula.

Suppose the interval of integration is $[-h, h]$, centered at the origin, and the given integrand $F(x)$ satisfies

(i) F has a continuous third derivative on $[-h, h]$,

(ii) the restrictions of F''' to the intervals $[-h, 0]$ and $[0, h]$ are differentiable, and the derivatives F^{iv} are integrable (but do not necessarily satisfy $F^{iv}(0-) = F^{iv}(0+)$).

Let $P_2(x)$ be the quadratic interpolating $F(x)$ at $-h$, 0, h. The error in

Simpson's Rule is obtained by integrating the difference $f(x)=F(x)-P_2(x)$. This difference satisfies (i) and (ii), and in addition $f(-h)=f(0)=f(h)=0$. The following formula is proved by integration by parts:

$$\int_{-h}^{0}(x+h)^3(3x-h)f^{iv}(x)dx+\int_{0}^{h}(x-h)^3(3x+h)f^{iv}(x)dx = 72\int_{-h}^{h}f(x)dx.$$

Now assume $F(x)$ satisfies (i), (ii), and

(iii) $|F^{iv}(x)|\leqq M$ on $[-h, h]$.

Then $f(x)$ also satisfies (iii) and the error can be estimated:

$$72\left|\int_{-h}^{h}f(x)dx\right| \leqq M\int_{-h}^{0}(x+h)^3(h-3x)dx+M\int_{0}^{h}(h-x)^3(3x+h)dx$$

$$= 2M\int_{0}^{h}(h-x)^3(3x+h)dx = \tfrac{4}{5}Mh^5.$$

Finally,

$$\left|\int_{-h}^{h}F(x)dx-\int_{-h}^{h}P_2(x)dx\right| \leqq \tfrac{1}{90}Mh^5.$$

Reference

John Todd, Introduction to the Constructive Theory of Functions, Academic Press, New York, 1963, p. 122.

FUNCTIONS WITH CLOSED GRAPHS

P. E. Long, University of Arkansas

For two topological spaces X and Y and any function $f: X\rightarrow Y$, the subset $\{(x, f(x))\}$ of the product space $X\times Y$ is called the *graph* of f and is denoted by $G(f)$. This note investigates conditions for $G(f)$ to be a closed subset of $X\times Y$.

Example 1 shows the well-known fact that even continuous functions may not have closed graphs. Example 2 shows the equally well-known fact that a function having closed graph need not be continuous.

Example 1. Let $X=\{0, 1\}$ with topology $T=\{X, \varnothing, \{0\}\}$. The identity function $i: X\rightarrow X$ is continuous but $G(i)=\{(0, 0), (1, 1)\}$ is not closed in $X\times X$.

In view of Theorems 1 and 4, note that the identity function on any space X will have closed graph if and only if X is T_2.

Example 2. Let X be the closed unit interval and Y the nonnegative reals, where both X and Y have the subspace topology induced from the reals. Let $f: X\rightarrow Y$ be defined as $f(x)=1/x$ for $x\neq 0$ and $f(0)=0$. Then $G(f)$ is closed in $X\times Y$, but f is not continuous at $x=0$.

In the theorems which follow no restrictions on the spaces or functions are assumed unless explicitly stated.

Simpson's Rule is obtained by integrating the difference $f(x)=F(x)-P_2(x)$. This difference satisfies (i) and (ii), and in addition $f(-h)=f(0)=f(h)=0$. The following formula is proved by integration by parts:

$$\int_{-h}^{0}(x+h)^3(3x-h)f^{iv}(x)dx+\int_{0}^{h}(x-h)^3(3x+h)f^{iv}(x)dx = 72\int_{-h}^{h}f(x)dx.$$

Now assume $F(x)$ satisfies (i), (ii), and

(iii) $|F^{iv}(x)|\leqq M$ on $[-h,h]$.

Then $f(x)$ also satisfies (iii) and the error can be estimated:

$$72\left|\int_{-h}^{h}f(x)dx\right| \leqq M\int_{-h}^{0}(x+h)^3(h-3x)dx+M\int_{0}^{h}(h-x)^3(3x+h)dx$$

$$= 2M\int_{0}^{h}(h-x)^3(3x+h)dx = \tfrac{4}{5}Mh^5.$$

Finally,

$$\left|\int_{-h}^{h}F(x)dx-\int_{-h}^{h}P_2(x)dx\right| \leqq \tfrac{1}{90}Mh^5.$$

Reference

John Todd, Introduction to the Constructive Theory of Functions, Academic Press, New York, 1963, p. 122.

FUNCTIONS WITH CLOSED GRAPHS

P. E. LONG, University of Arkansas

For two topological spaces X and Y and any function $f: X\to Y$, the subset $\{(x,f(x))\}$ of the product space $X\times Y$ is called the *graph* of f and is denoted by $G(f)$. This note investigates conditions for $G(f)$ to be a closed subset of $X\times Y$.

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In the theorems which follow no restrictions on the spaces or functions are assumed unless explicitly stated.

LEMMA. *Let $f: X \to Y$ be given. Then $G(f)$ is closed if and only if for each $x \in X$ and $y \in Y$, where $y \neq f(x)$, there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V = \varnothing$.*

Proof. If the condition holds and $(x, y) \notin G(f)$, then $U \times V$ is an open set in $X \times Y$ containing (x, y) such that $(U \times V) \cap G(f) = \varnothing$, which implies $G(f)$ is closed. If $G(f)$ is closed and $(x, y) \notin G(f)$, then $y \neq f(x)$, so there exists a basic open set of the form $U \times V$ where U and V are open containing x and y, respectively, such that $(U \times V) \cap G(f) = \varnothing$ and therefore $f(U) \cap V = \varnothing$.

Theorem 1 may be found in [**1**, Theorem 1.5(3) page 140].

THEOREM 1. *Let $f: X \to Y$ be continuous where X is arbitrary and Y is T_2. Then $G(f)$ is closed.*

THEOREM 2. *Let $f: X \to Y$ be any function where Y is countably compact and X is first countable. If $G(f)$ is closed, then f is continuous.*

Proof. Suppose f is not continuous. Then there exists an open set $V \subset Y$ such that $f^{-1}(V)$ is not open in X. Therefore $f^{-1}(V)$ contains a point $x \in X$ such that x is a limit point of $X \backslash f^{-1}(V)$. Since X is first countable, there exists a sequence $\{x_n\}$, with $x_n \in X \backslash f^{-1}(V)$, that converges to x. In the countably compact space Y, the set $\{f(x_n)\}$ has an accumulation point $y \notin V$. Then $(x, y) \notin G(f)$, but (x, y) is a limit point of $G(f)$ since any open set in $X \times Y$ containing (x, y) clearly contains points of the form $(x, f(x))$. This contradiction to the hypothesis that $G(f)$ is closed implies f is continuous.

Theorem 2 is known for the case where Y is compact and X is arbitrary, and may be found in [**3**, exercise 16.10 page 130]. Example 2 shows that Theorem 2 is not true for Y locally compact and X compact.

Theorem 3 is actually a corollary to Lemma 1 [**2**, page 2].

THEOREM 3. *Let $f: X \to Y$ be any surjection with $G(f)$ closed. Then Y is T_1.*

Proof. Let y and w be distinct points in Y. Then there exists an $x \in X$ such that $f(x) = w$. Thus $(x, y) \notin G(f)$, so by the Lemma there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V = \varnothing$. Therefore $w \notin V$ and Y is T_1.

THEOREM 4. *Let $f: X \to Y$ be any open surjection with $G(f)$ closed. Then Y is T_2.*

Proof. Let y and w be distinct points in Y. Then there are distinct points x and z in X such that $f(x) = y$ and $f(z) = w$. Since $(x, w) \notin G(f)$ and $G(f)$ is closed, there exist open sets U and V containing x and w, respectively, such that $f(U) \cap V = \varnothing$; but $f(U)$ is open and contains y. Consequently, Y is T_2.

THEOREM 5. *Let $f: X \to Y$ be injective with $G(f)$ closed. Then X is T_1.*

Proof. Let x and z be distinct points in X. Then $f(x) \neq f(z)$, so there exist open sets U and V containing x and $f(z)$, respectively, such that $f(U) \cap V = \varnothing$. Thus $z \notin U$ implying X is T_1.

THEOREM 6. *If $f: X \to Y$ is bijective with closed graph, then both X and Y are T_1.*

Proof. Theorems 3 and 5.

THEOREM 7. *Let $f: X \to Y$ be injective and continuous with $G(f)$ closed. Then X is T_2.*

Proof. By hypothesis f^{-1} is an open surjection from $f(X)$ onto X. Furthermore $G(f) \cong G(f^{-1})$, hence f^{-1} has a closed graph in $Y \times X$. By Theorem 4, X is T_2.

THEOREM 8. *Let $f: X \to Y$ be a homeomorphism of X onto Y having $G(f)$ closed. Then both X and Y are T_2.*

Proof. Theorems 4 and 7.

THEOREM 9. *Let $f: X \to Y$ be injective, open, connected, and having $G(f)$ closed. Then if X is locally connected, X is T_2.*

Proof. Let x and z be distinct points of X. Then $f(x) \neq f(z)$, so there exists an open connected set U containing x and an open set V containing $f(z)$ such that $f(U) \cap V = \varnothing$. Since $f(U)$ is open, $z \notin \overline{U}$. For otherwise $U \cup \{z\}$ is connected, so that $f(U \cup \{z\}) = f(U) \cup \{f(z)\}$ is connected, since f is connected. This is an impossibility.

Example 3 shows that a function satisfying the hypothesis of Theorem 9 need not be continuous.

Example 3. Let X be the reals with standard topology. Let Y be the graph of $g(x) = \sin 1/x$, $x \neq 0$, $g(0) = 0$ with the subspace topology induced from the standard topology of the plane. Define $f: X \to Y$ by $f(x) = (x, g(x))$. Then f is clearly injective, open, and connected, and the Lemma shows $G(f)$ is closed. However, f is not continuous at $x = 0$.

References

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2. T. Husain, Almost Continuous Functions, Prace Mat., 10(1966) 1–7.
3. Wolfgang J. Thron, Topological Structures, Holt, Rinehart, Winston, New York, 1966.

Evolution of Pure Mathematics.

"We should not hide from ourselves that the golden age of mathematics has passed. It rests at the head of the classification of the sciences and keeps a place of honour in the scholastic programs, but the respect in which it is held resembles a little that which is attached to something dead . . . more than ever, in fact, mathematics wanders at random. . . . No overall plan relates the researches undertaken from twenty different points of view, and the treatises themselves are often merely the juxtaposition of detached chapters. . . . Will we not see the end of this anarchy? May one not hope, in fact, that the taste for synthetic studies and the exercise of philosophical thinking might aid the mathematician to know better the object and the goal of his researches?" P. Boutroux in *Scientia*, 1909.

MATHEMATICAL EDUCATION

EDITED BY J. G. HARVEY AND M. W. POWNALL

Material for this Department should be sent to either of the editors: J. G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706; M. W. Pownall, Department of Mathematics, Colgate University, Hamilton, NY 13346.

MATHEMATICS TRAINING OF SECONDARY MATHEMATICS TEACHERS

R. E. REYS, R. D. KERR and J. W. ALSPAUGH, University of Missouri

Mathematics departments in colleges and universities are deeply concerned about providing adequate training for mathematics teachers. In particular, the Committee on Undergraduate Program in Mathematics, the major group influencing the modernization of collegiate mathematics, has recommended minimal mathematics requirements for various levels of mathematics teachers (1, 2). Furthermore, the National Science Foundation has provided in-service opportunities for mathematics teachers to obtain additional training.

In light of the massive effort to improve the preparation of secondary mathematics teachers, this study was undertaken. The purposes were two-fold: (1) to identify selected academic characteristics of secondary mathematics teachers, such as amount of college level mathematics; (2) to ascertain the influence of NSF institutes as an agent of change in secondary school mathematics programs.

The following results are based upon data collected from selected junior and senior high school mathematics teachers in Missouri. The population list for this study was the *1967–68 Missouri School Directory* which identified nearly 2200 mathematics teachers in Missouri public schools. Initially, 250 mathematics teachers were randomly selected from this list. Only one mathematics teacher from a given school was contacted. The original mailing of the questionnaire, together with a follow-up letter in some cases and a personal telephone call in a few others, resulted in a return of 98 per cent of the questionnaires. Twelve questionnaires were returned but not completed. Hence, the following analysis is based upon data from 233 mathematics teachers in Missouri.

Many suburban school districts in Missouri are able to pay their teaching faculty \$500 to \$1,500 more per year than urban and rural school districts (3). It has been suggested that a migration of qualified staff to suburban school districts has occurred at the expense of rural and urban schools. Consequently, an effort was made in this study to identify some academic characteristics of mathematics teachers in rural, suburban and urban areas. Each school district represented in the sample was placed in one of three groups according to the following criteria: An urban school district was one located in any metropolitan area with a population exceeding 35,000. A suburban school district was one located in a municipality adjacent to an urban area or within six miles of an urban area. School districts not classified as urban or suburban were considered rural.

This criterion resulted in the following distribution of the 233 mathematics teachers in the sample: 144 from rural school districts, 49 from suburban school districts and 40 from urban school districts. The number of teachers in the sample from rural, suburban, and urban schools was found to be approximately proportional to the total number of rural, suburban, and urban mathematics teachers in the state.

One section of the questionnaire included items related to the mathematics training of the teachers selected. It was found that approximately 14 different undergraduate majors were represented including sociology, business, English, music, and engineering. Nearly 29 per cent of the teachers held a masters degree. The Master of Science in Education, the Master of Arts in Mathematics, and the Master of Arts in Mathematics Education were held by most of these mathematics teachers. Several other graduate academic majors were also represented including both music and English.

The percentages of mathematics teachers in rural, suburban, and urban school districts with masters degrees were 19%, 55%, and 41% respectively. A chi square test of these three groups on the basis of possession or non-possession of masters degrees revealed significant differences beyond the .01 level. No significant differences were observed between the urban and suburban categories. However, there was a significant difference between the number of

TABLE I

Distribution of Total Semester Hours in Mathematics Completed by 233 Mathematics Teachers in Missouri

Semester Hours	*Per cent of Teachers*			
	Rural $N=144$	*Suburban* $N=49$	*Urban* $N=40$	*All Teachers* $N=233$
5–9	.70	0	0	.85
10–14	2.82	0	2.50	2.14
15–19	10.56	4.08	0	7.36
20–24	12.68	6.12	10.00	10.78
25–29	12.68	6.12	12.50	11.22
30–34	19.02	20.44	17.50	19.00
35–39	19.72	10.20	15.00	16.84
40–44	4.93	14.28	7.50	7.35
45–49	9.15	4.08	12.50	8.64
50–54	2.82	8.16	10.00	5.19
55–59	2.82	6.12	2.50	3.40
60–64	.70	8.16	2.50	2.56
65–69	0	2.04	0	.42
70–74	.70	2.04	0	.85
75–79	0	4.08	5.00	1.70
over 79	.70	4.08	2.50	1.70
Mean	32.25	44.04	39.80	36.06

teachers in rural school districts with masters degrees and teachers in urban and suburban districts.

Table I reports a distribution of the total semester hours in mathematics earned by mathematics teachers in the three geographic areas in Missouri.

TABLE II

One Way ANOV of Total Semester Hours in Mathematics of 233 Mathematics Teachers from Rural, Suburban, and Urban School Districts

Source	*df*	*SS*	*MS*	*F*
Among	2	5,745.577	2,872.789	13.76*
Within	230	48,012.187	208.748	
Total	232	53,757.764		

* Significant beyond .01 level.

The teachers in rural, suburban, and urban school districts had mean semester hours in mathematics of 32.25, 44.04, and 39.80 respectively. Table II summarizes a one way analysis of variance which led to rejection of the hypothesis that the total mean number of semester hours in mathematics of teachers from rural, suburban, and urban school districts were not significantly different at the .01 level. Using the same level of significance, Scheffe's multiple comparisons were made to locate significant differences between these groups. Although teachers in suburban and urban school districts were not significantly different in total hours completed in mathematics, both of these groups had significantly greater mathematics training than teachers in rural school districts.

Another section of the questionnaire was related to participation in National Science Foundation institutes by secondary school mathematics teachers in Missouri. It was found that 35 per cent of these teachers had attended at least one institute. The number of teachers selected from rural, suburban, and urban schools was found to be approximately proportional to the total number of rural, suburban, and urban mathematics teachers in the state. Thirteen per cent of the teachers who had attended an NSF institute attended an academic year institute, whereas 87 per cent attended one or more summer institutes. Furthermore, it was found that slightly more than 57 per cent of the teachers involved attended an institute within the state, whereas nearly 43 per cent attended a National Science Foundation institute held outside of Missouri.

The institute participants were asked to rate the effectiveness of the institute in promoting change. The results are reported in Table III. From Table III it is apparent that NSF Institutes have promoted considerable improvement in upgrading the academic mathematics background of high school mathematics teachers. This is evidenced by the fact that 99 per cent of the mathematics teachers who had attended an institute reported at least some improvement in their mathematics background, with the majority identifying this change as

institute training. The academic training in mathematics of a sizeable portion of these high school teachers still leaves much to be desired. Serious attention by mathematics faculties needs to be given to devising ways to provide for continuing mathematics preparation of the secondary teachers.

The research reported herein was financed by a grant from the Research Council of the University of Missouri-Columbia.

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1. Committee on Undergraduate Programs in Mathematics, Recommendations of the Mathematical Association of America for the training of mathematics teachers, this MONTHLY, 67 (1960).

2. D. A. Johnson and Robert Rahtz, The New Mathematics in our Schools, Macmillan, New York, 1966.

3. Salary Schedules Missouri Public Schools: 1967–68, School and Community, December, 1967.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before January 31, 1970. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

E 2190. *Proposed by Harry Pollard, Purdue University*

Show that if m and n are positive integers, the smaller of the numbers $\sqrt[n]{m}$ and $\sqrt[m]{n}$ cannot exceed $\sqrt[3]{3}$.

E 2191. *Proposed by M. J. Zerger, Umpqua Community College, Roseburg, Oregon*

Find the solution set of $x^{(x+1)}=(x+1)^x$ in the reals.

E 2192. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

It is easy to show that

$$\sum_p \frac{1}{p^2}+\sum_p \frac{1}{p^3}+\sum_p \frac{1}{p^4}+\cdots<1,$$

where the summation runs over all primes in each term of the above series. Determine a better upper bound.

E 2193. *Proposed by D. J. Simanaitis, Case Western Reserve University*

For A, B, and C, three points in the Euclidean plane, define B to be *weakly-between* A and C if and only if angle $ABC \geqq 120°$. Determine the minimal number of points required to insure the existence of at least one such weak-betweenness relation.

E 2194. *Proposed by Marion B. Smith, University of Wisconsin, Baraboo*

Do there exist nonzero integers a, b, c, d such that $a^2+b^2=c^2-d^2$ and $ab=cd$?

E 2195. *Proposed by T. J. Bruggeman, Xavier University, Cincinnati, Ohio*

Which polynomials of the form $\sum_{i=1}^{n} x^{a_i}$ are divisible by $\sum_{i=1}^{m} x^{i-1}$? That is, find necessary and sufficient conditions for a_i, $i=1, 2, \cdots, n$.

SOLUTIONS OF ELEMENTARY PROBLEMS

All Triangles Generate Right Triangles

E 2124 [1968, 899]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Construct on the sides BC, CA, AB of a triangle ABC, exteriorly, the squares $BCDE$, $ACFG$, $BAHK$ and build parallelograms $FCDQ$, $EBKP$. Show that APQ is an isosceles right triangle.

Solution by W. E. Buker, Pittsburgh Public Schools. Assign coordinates $A(0, 0)$; $B(a, 0)$; $C(b, c)$. Then find by inspection the coordinates $F(b-c, b+c)$, $D(b+c, c+a-b)$, $Q(b, a+c)$, $K(a, -a)$, $E(a+c, a-b)$, $P(a+c, -b)$. Since AQ and AP have equal lengths and are perpendicular, the theorem follows.

Also solved by forty other readers.

Note. It follows at once that if parallelogram $HAGR$ is constructed, then BQR and CRP are also isosceles right triangles.

A. W. Walker points out that an extensive investigation of triangles bordered by squares is found in a paper by Musselman, this MONTHLY, 43 (1936), 539–548.

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x	$\log u_1$	$\log u_2$	$\log u_3$	$\log u_4$	$\log u_5$
.409	.976	.976			
.442	.999	.990	.990		
.458	1.010		.993	.993	
.466	1.015			.995	.995

Also solved by the proposer.

Polynomials without Integral Zeros

E 2126 [1968, 1007]. *Proposed by Erwin Just, Bronx Community College, New York*

If n is an integer greater than 2, and k is a nonzero integer, prove that $\sum_{r=0}^{n} (rk+1)x^{n-r}$ has no integral zeros.

Solution by L. E. Mattics, University of South Alabama. Suppose that for some $n>2$ and $k\neq 0$ the polynomial $\sum_{r=0}^{n} (rk+1)x^{n-r}$ has an integral zero $x=a$. Then

$$a^n + a^{n-1} + \cdots + 1 = -k(a^{n-1} + 2a^{n-2} + \cdots + n),$$

and since $k\neq 0$ we see that $a\neq -1$. A further manipulation yields

$$n + 1 = (-a - k + 1)(a^{n-1} + 2a^{n-2} + \cdots + n),$$

from which we see that $a^{n-1}+2a^{n-2}+ \cdots +n$ divides $n+1$. By inspection we have that $a\leqq -2$. But a simple induction proof shows that

$$\left| b^{n-1} + 2b^{n-2} + \cdots + n \right| > n + 1$$

for integers $n\geqq 3$ and $b\leqq -2$ except for the cases $b=-2$, $n=3$ and $b=-2$, $n=4$. A calculation shows that in these cases $b^{n-1}+2b^{n-2}+ \cdots +n$ does not divide $n+1$. Thus a cannot be an integer.

Also solved by William Fox, E. F. Schmeidel, and the proposer.

A Four Point Condition in Three Space

E 2127 [1968, 1007]. *Proposed by John Wilker, University of Toronto*

Let A, B, C, and D be any four points in Euclidean 3-space. If every sphere or plane through A and B meets every sphere or plane through C and D, what can be said about the points?

Solution by Anders Bager, Hjørring, Denmark. The following two conditions will be proved equivalent:

(a) Every sphere or plane through A and B meets every sphere or plane through C and D.

(b) A, B, C and D are on the same circle or line, and the pairs (A, B) and (C, D) separate each other.

Proof. If AB and CD are skew or parallel and distinct, then (b) is not fulfilled. Neither is (a), as there exist a plane through A and B and a plane through C and D which do not meet. From now on, we consider only points A, B, C, and D in the same plane α. Clearly, the spheres and planes in (a) henceforth can be replaced by circles and lines in α.

Let A, B, C and D be on the same line. Clearly, (a) is fulfilled if and only if (A, B) and (C, D) separate each other, that is, if (b) is fulfilled. From now on we assume that the line AB cuts the line CD.

If segments AB and CD do not meet, clearly neither (a) nor (b) is fulfilled. From now on we assume that the segments AB and CD do meet, and we put a circle c through A, B and C.

Assume D is not on c, that is, part of (b) is not fulfilled. It is easily seen that we may pass a circle d through A and B such that C and D are either both inside or both outside d. As some circle through C and D will not meet d, (a) is not fulfilled. Then D is on c.

In view of our earlier assumptions, this is equivalent to saying that (b) is fulfilled. Let x be a line or circle through A and B, distinct from c. Then C and D are in different ones of the two domains bounded by x. Hence, every arc or segment connecting C with D must meet x. This concludes the proof.

Also solved by W. D. Bouwsma, Michael Goldberg, Norman Miller, and the proposer.

Note. The problem is related to Putnam problem B-6, this MONTHLY, 73 (1966) 728.

A Triangle Inequality with Many Solutions

E 2128 [1968, 1007]. *Proposed by Jovan Vukmirović, Belgrade, Yugoslavia*

In the triangle ABC let M be any point on side BC. Prove that

$$(AM-AC)\cdot BC \leqq (AB-AC)\cdot MC.$$

I. *Solution by Leon Bankoff, Los Angeles, California*. The desired result is obtained by considering AM and BC the diagonals of the degenerate quadrilateral $ABMC$ and applying Ptolemy's theorem, whence

$$AM\cdot BC \leqq MC\cdot AB + AC(BC - MC)$$

or

$$(AM - AC)\cdot BC \leqq (AB - AC)\cdot MC.$$

Equality occurs when A, B, M, C are concyclic, that is, when M coincides with B or C.

This problem and its solution were published as a Quickie (Q 272 and A 272) in the *Mathematics Magazine*, January 1961, pp. 181, 184.

II. *Solution by M. G. Greening, University of New South Wales, Australia*. Set $\boldsymbol{CM}=h\boldsymbol{CB}$; then $0\leqq h\leqq 1$. $\boldsymbol{AM}=h\boldsymbol{AB}+(1-h)\boldsymbol{AC}$. Then

$$|\boldsymbol{BC}|\,\{h|\boldsymbol{AB}| + (1-h)|\boldsymbol{AC}|\} \geqq |\boldsymbol{BC}|\,|\boldsymbol{AM}|,$$

so

$$h|BC|\{|AB| - |AC|\} \geqq |BC|\cdot|AM| - |BC|\cdot|AC|,$$

giving the stated inequality.

III. *Solution by M. A. Bershad, Bureau of the Census, Washington, D. C.* Using the analytic notation, $C(0, 0)$, $B(a, 0)$, $A(k, h)$, and $M(xa, 0)$ with $0<x<1$, and defining $f(x)=\{(k-xa)^2+h^2\}^{1/2}$, the problem is to prove that

$$a[f(x) - f(0)] \leqq ax[f(1) - f(0)]$$

or that $f(x)\leqq xf(1)+(1-x)f(0)$. Since

$$\frac{d^2f(x)}{dx^2} = \frac{a^2b^2}{[f(x)]^3} > 0,$$

$f(x)$ is convex so that the inequality holds.

Also solved by Anders Bager (Denmark), W. D. Bouwsma, M. A. Ettrick, D. S. George, Michael Goldberg, D. C. Kay, J. D. E. Konhauser, J. R. Kuttler & N. Rubinstein, Graham Lord, Alice P. Meyer, Norman Miller, E. F. Schmeichel, and the proposer.

Fermat-type Triples in Modular Systems

E 2129 [1968, 1007]. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha, and H. E. Chrestenson, Reed College*

Let J_k denote the integers modulo k. We seek three nonzero elements x, y, z of J_k such that $x^n+y^n=z^n$ for all positive integers n. Show that if k is not a power of a prime, then J_k contains such a triple of elements. For what other values of k do these triples exist?

Solution by C. V. Heuer, Concordia College. Such a triple exists in J_k if and only if k is composite and $k\neq 4$. Indeed, if $k\neq 4$ is composite and p is a prime dividing k, then

$$\left(p+\frac{k}{p}\right)^n \equiv p^n + \left(\frac{k}{p}\right)^n \pmod{k}$$

for all positive integers n. So $x=p$, $y=k/p$, $z=p+k/p$ is a suitable triple.

One easily checks that there is no suitable triple satisfying $x^2+y^2=z^2$ in J_8 or J_4. If p is an odd prime and x, y, and z are elements of J_p such that $x+y=z$, then $x^2+y^2=z^2$ requires that $p|2xy$ and hence that either $x=0$ or $y=0$.

Also solved by Anders Bager (Denmark), W. D. Bouwsma, Robert Gilmer, M. G. Greening (Australia), Erwin Just, D. C. B. Marsh, Norman Miller, Margaret S. Ogelsby, Stephen Pierce, Ira Rosenholtz, Ross Schipper, E. H. Schmeichel, and (partially) by the proposers.

Octahedron with Vertices on Six Planes

E 2130 [1968, 1008]. *Proposed by J. M. Quoniam, Saint-Etienne, France*

Show how to construct a regular octahedron whose vertices lie on six parallel

so

$$h|BC|\{|AB| - |AC|\} \geqq |BC|\cdot|AM| - |BC|\cdot|AC|,$$

giving the stated inequality.

III. *Solution by M. A. Bershad, Bureau of the Census, Washington, D. C.* Using the analytic notation, $C(0, 0)$, $B(a, 0)$, $A(k, h)$, and $M(xa, 0)$ with $0 < x < 1$, and defining $f(x) = \{(k-xa)^2+h^2\}^{1/2}$, the problem is to prove that

$$a[f(x) - f(0)] \leqq ax[f(1) - f(0)]$$

or that $f(x) \leqq xf(1)+(1-x)f(0)$. Since

$$\frac{d^2f(x)}{dx^2} = \frac{a^2b^2}{[f(x)]^3} > 0,$$

$f(x)$ is convex so that the inequality holds.

Also solved by Anders Bager (Denmark), W. D. Bouwsma, M. A. Ettrick, D. S. George, Michael Goldberg, D. C. Kay, J. D. E. Konhauser, J. R. Kuttler & N. Rubinstein, Graham Lord, Alice P. Meyer, Norman Miller, E. F. Schmeichel, and the proposer.

Fermat-type Triples in Modular Systems

E 2129 [1968, 1007]. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha, and H. E. Chrestenson, Reed College*

Let J_k denote the integers modulo k. We seek three nonzero elements x, y, z of J_k such that $x^n+y^n=z^n$ for all positive integers n. Show that if k is not a power of a prime, then J_k contains such a triple of elements. For what other values of k do these triples exist?

Solution by C. V. Heuer, Concordia College. Such a triple exists in J_k if and only if k is composite and $k\neq 4$. Indeed, if $k\neq 4$ is composite and p is a prime dividing k, then

$$\left(p+\frac{k}{p}\right)^n \equiv p^n + \left(\frac{k}{p}\right)^n \pmod k$$

for all positive integers n. So $x=p$, $y=k/p$, $z=p+k/p$ is a suitable triple.

One easily checks that there is no suitable triple satisfying $x^2+y^2=z^2$ in J_2 or J_4. If p is an odd prime and x, y, and z are elements of J_p such that $x+y=z$, then $x^2+y^2=z^2$ requires that $p|2xy$ and hence that either $x=0$ or $y=0$.

Also solved by Anders Bager (Denmark), W. D. Bouwsma, Robert Gilmer, M. G. Greening (Australia), Erwin Just, D. C. B. Marsh, Norman Miller, Margaret S. Ogelsby, Stephen Pierce, Ira Rosenholtz, Ross Schipper, E. H. Schmeichel, and (partially) by the proposers.

Octahedron with Vertices on Six Planes

E 2130 [1968, 1008]. *Proposed by J. M. Quoniam, Saint-Etienne, France*

Show how to construct a regular octahedron whose vertices lie on six parallel

e, and the angle ϕ. With these, the regular octahedron can be constructed on the given set of parallel planes.

Since ϕ cannot be less than $\alpha = \text{arc sin } 1/\sqrt{3} \approx 35°16'$, solutions violating this condition arise from impossible sets of planes. If $\phi = \alpha$, then there are only two planes in each of which three vertices lie. If $\theta = 0$, then there are only four planes on two of which two vertices lie.

Note. Other special cases which reduce the number of distinct planes are $\phi = 90°$, $\theta = 45°$, and $\phi = \text{arc tan } (\sin (\theta + 45°))$. The conditions given in Solution I are implicit in Solution II.

A Homeomorphism Condition

E 2131 [1968, 1008]. *Proposed by R. A. Struble, North Carolina State University at Raleigh*

If $f: R^n \to R^n$ is bijective and maps connected sets onto connected sets and nonconnected sets onto nonconnected sets, then f is a homeomorphism.

Solution by G. P. Speck and F. Prokop, Bradley University. Consider a nonempty closed connected set S in R^n. If $f(S)$ is not closed, then there exists a point p such that $p \notin S$ and $f(p) \in \overline{f(S)}$, the closure of $f(S)$. Now the set $\{p\} \cup S$ is nonconnected, but since $f(S)$ is connected and $f(S) \subset f(\{p\} \cup S) \subset \overline{f(S)}$ we know that $f(\{p\} \cup S)$ is connected, a contradiction. Thus, f carries any closed connected set onto a closed set, from which it follows that f is an open map by considering the closed connected complement of any open spherical neighborhood in R^n. Now it is easily seen that f^{-1}, the inverse of f, satisfies all of the hypotheses that f satisfies. Therefore f^{-1} is open, as is f, and it follows that f is a homeomorphism.

Also solved by Ralph Bennett, John Comiskey, W. F. Fox, R. V. Fuller, Douglas Lind (England), L. E. Mattics, W. G. McArthur, D. Roddick, E. F. Schmeichel, and the proposer.

Triangular Numbers Generate the Rationals

E 2132 [1968, 1008]. *Proposed by Gregory Wulczyn, Bucknell University*

Let r, s be positive integers, $(r, s) = 1$, rs not a square. Then there are infinitely many pairs of triangular numbers, $T_a = \frac{1}{2}a(a+1)$, $T_b = \frac{1}{2}b(b+1)$ such that $T_a/T_b = s/r$.

Solution by G. E. Engebretsen, Fort Ord, California. Rewrite the given equation $r(2a+1)^2 - s(2b+1)^2 = r - s$. This has the solution $a = b = 0$. Write $x^2 - rsy^2 = 1$, which always has a smallest solution $x_0^2 - rsy_0^2 = 1$, since rs is not a square (H. Davenport, *The Higher Arithmetic* (London), 1952). Infinitely many integral solutions to the original equation are given by

$$(\sqrt{r} + \sqrt{s})(x_0 + \sqrt{rs}\, y_0)^n = x_n\sqrt{r} + y_n\sqrt{s}, \qquad 2a + 1 = x_n, \qquad 2b + 1 = y_n;$$

$$(\sqrt{r} - \sqrt{s})(x_0 + \sqrt{rs}\, y_0)^n = j_n\sqrt{r} + k_n\sqrt{s}, \qquad 2a + 1 = j_n, \qquad 2b + 1 = k_n,$$

unless rs is even and x_0, y_0 are both odd, in which case only even values of n give integral a, b. The two sets of solutions are distinct unless $r - s = 1$.

Also solved by Michael Goldberg, M. G. Greening (Australia), J. S. Vigder, and the proposer.

Goldberg finds the problem in L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 33 and reference to solution by A. Cunningham and R. W. D. Christie in Math. Quest. Educ. Times, vol. 74 (1901), pp. 87–88.

A System of Linear Equations

E 2134 [1968, 1113]. *Proposed by Harley Flanders, Purdue University*

Find the rank of the system of linear equations over a field of characteristic zero:

$$\begin{cases} x_{ijkl} + x_{jkil} + x_{ijlk} + x_{jlik} = 0 \\ x_{ijkl} + x_{jkil} + x_{jikl} + x_{ikjl} = 0. \end{cases}$$

The indices run independently from 1 to n.

Solution by the proposer. The system is equivalent to

$$\begin{cases} y_{ijkl} + y_{ijlk} = 0 \\ y_{ijkl} + y_{jikl} = 0, \end{cases}$$

where $y_{ijkl}=(x_{ijkl}+x_{jkil})$, or $x_{ijkl}=\frac{1}{2}(y_{ijkl}-y_{jkil}+y_{kijl})$. There are n^4 unknowns y_{ijkl}. We may specify arbitrarily those for which $i<j$ and $k<l$; the others are uniquely determined. Thus the nullity is $[n(n-1)/2]^2$, and the rank is

$$n^4 - [n(n-1)/2]^2 = n^2(n+1)(3n-1)/4.$$

Signs in an Ellipse

E 2135 [1968, 1113]. *Proposed by F. A. Butter, Jr., California State College at Long Beach*

Two radii (distinct or not) of lengths r_1, r_2 with respective slopes m_1, m_2 are drawn from the center of an ellipse $b^2x^2+a^2y^2=a^2b^2$, $0<b<a$, to the open quadrant for which $x>0, y>0$. Prove that (r_1r_2-ab) and $(ba^{-1}-m_1m_2)$ are alike in sign.

Solution by W. G. Wild, Wisconsin State University, Stevens Point. With all terms positive, $r_1r_2-ab>0$ iff $r_1^2r_2^2-a^2b^2>0$, $ba^{-1}-m_1m_2>0$ iff $b^2a^{-2}-m_1^2m_2^2>0$. Now, from

$$r^2 = x^2 + y^2, \qquad m^2 = y^2/x^2, \qquad x^2b^2 + y^2a^2 = a^2b^2,$$

eliminate x^2 and y^2 to get $r^2=a^2b^2(1+m^2)/(b^2+a^2m^2)$. It follows that

$$r_1^2r_2^2 - a^2b^2 = \left[\frac{(a^2-b^2)a^4b^2}{(b^2+a^2m_1^2)(b^2+a^2m_2^2)}\right](b^2a^{-2} - m_1^2m_2^2),$$

but the bracketed expression is positive by the conditions of the problem. Hence the left side of the equation bears the sign of the expression in parentheses.

Also solved by Anders Bager (Denmark), A. P. Boblett, L. D. Black, W. D. Bouwsma, D. M. Danvers, S. L. Feigenbaum, W. F. Fox, Michael Goldberg, M. G. Greening (Australia), Cornelius

Groenewoud, N. V. Klesyzewski, Lew Kowarski, D. C. B. Marsh, Alice P. Meyer, Norman Miller, J. C. Molluzzo, S. Smith, Charles Wexler, and the proposer.

Sums of Powers of Integers

E 2136 [1968, 1113]. *Proposed by A. Inselberg and B. Dimsdale, IBM Los Angeles Scientific Center*

Let

$$S_r = \sum_{k=1}^{n} k^r.$$

It is well known that $S_3 = S_1^2$. Are there other values of p, q, u, v such that $S_p^u = S_q^v$ for all n?

Solution by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. We will consider p, q, u, v to be any real numbers. By the Euler-MacLaurin expansion, the first few terms of the asymptotic expansion of S_r for $r > -1$ is given by

$$S_r \sim \frac{n^{r+1}}{r+1} + \frac{1}{2} n^r + \frac{rn^{r-1}}{12} + \cdots.$$

Thus,

$$S_p^u \sim \left(\frac{n^{p+1}}{p+1}\right)^u \left\{1 + \frac{p+1}{2n} + \frac{p(p+1)}{12n^2} + \cdots\right\}^u,$$

$$S_q^v \sim \left(\frac{n^{q+1}}{q+1}\right)^v \left\{1 + \frac{q+1}{2n} + \frac{q(q+1)}{12n^2} + \cdots\right\}^v.$$

Since $S_p^u = S_q^v$ for all n, we must have by equating the first few terms of the expansion that

$$(p+1)^u = (q+1)^v, \tag{1}$$

$$(p+1)u = (q+1)v, \tag{2}$$

$$\frac{u(u-1)(p+1)^2}{8} + \frac{up(p+1)}{12} = \frac{v(v-1)(q+1)^2}{8} + \frac{vq(q+1)}{12} \tag{3}$$

(assuming that there are at least three terms in both expansions). It now follows from (2) and (3) that $p = q$ and then that $u = v$. If the expansion of S_p has less than three terms, then $p = 0$ or $p = 1$. In this case the term $up(p+1)/12$ does not appear in (3). This then leads to $q = 3p$. Thus for $p = 1$, either $q = 1$ or $q = 3$, and for $p = 0$, $q = 0$.

We now consider the cases $p, q \leqq 1$. The case $p = -1 \neq q$ is ruled out since here $S_p \sim \ln n$ and S_q is not. For $p, q < -1$, we have

$$S_p^u \sim \left\{\zeta(-p) - \frac{n^{p+1}}{p+1} + \frac{n^p}{2} - \cdots\right\}^u.$$

On comparison of the first two terms of the expansion for S_p^u with that for S_q^v, we must have $p=q$. Thus the only solution is the known identity $S_3=S_1^2$.

Also solved by Anders Bager (Denmark), M. A. Bershad, D. M. Bloom, W. J. Blundon, H. M. Edgar, Michael Goldberg, M. G. Greening (Australia), Emil Grosswald, J. C. Hickman, John Ivie, Erwin Just, J. D. E. Konhauser, Dan Marcus, D. C. B. Marsh, Stephen Pierce, Charles Wexler, W. G. Wild, and the proposers.

Ivie and many others note that the solution is well known and has been published. See D. Allison, *A note on sums of powers of integers*, this MONTHLY, 1961, p. 272. A related result is developed in S. Cavior, *A theorem on power sums*, in the April 1968 Fibonacci Quarterly, pp. 157–161. He considers the more general problem of finding polynomials

$$f(x) = \sum_{i=0}^{r} a_i x^i, \qquad g(x) = \sum_{i=0}^{s} b_i x^i$$

over the real field such that

$$\{f(1) + \cdots + f(n)\}^p = \{g(1) + \cdots + g(n)\}^q$$

for positive integral r, p, s, q. For this condition to hold, it is shown that the only monic solutions occur when $p=2$, $q=1$, and

$$f(x) = a + x, \qquad g(x) = x^3 + 3ax^2 + (2a^2 - a)x - a^2,$$

where a is an arbitrary real constant. (For $a=0$, this is the result of the present problem.) Cavior also considers the problem of finding non-monic polynomials f and g for arbitrary p and q, and proves a general theorem.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before January 31, 1970. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

The (*) means the proposer or editors did not supply a solution.

5689. *Proposed by Frank DeMeyer, Colorado State University*

Let G be a finite group with cyclic center Z and assume G/Z is abelian. Show $G/Z=H\times H$ (direct product) for some abelian group H.

5690. *Proposed by Irving Kaplansky, University of Chicago*

Let R be an associative ring. Suppose $a\in R$ is nilpotent and $b\in R$ is a right zero-divisor. Suppose a and b commute. Prove there exists a $c\neq 0$ in R such that $ac=bc=0$.

5691. *Proposed by John Bond and Harold Reiter, Clemson (S. C.) University*

Does there exist a discrete subset of the real line (usual topology) which has an uncountable closure?

5692. *Proposed by Olga Taussky, California Institute of Technology*

Let F be a real closed field and A a finite matrix with elements in F. Show explicitly that the characteristic roots of $A'A$ (A' the transposed matrix) are sums of squares in F.

5693.* *Proposed by S. Abhyankar, Purdue University*

Show that the curve $x=t+t^5$, $y=t^3$, $z=t^4$ is the intersection of three surfaces. Is it the intersection of two algebraic surfaces, either set-theoretically or ideal-theoretically?

5694. *Proposed by Jürg Rätz and Albert Wilansky, Lehigh University*

As a subspace of R (the reals) with the half-open interval topology (Kelley, *General Topology*, p. 59), Q (the rationals) is first countable, hence second countable, hence metrizable. Exhibit an explicit metric for this topology.

SOLUTIONS OF ADVANCED PROBLEMS

An Ordinary Differential Equation

5415 [1966, 783]. *Proposed by R. D. Driver, the Sandia Corporation*

The ordinary differential equation

$$x'(t) = 1 - g(x(t)) + g(t) \text{ for } t > 0 \text{ with } x(0) = 0$$

has a solution $x(t)=t$. Is the solution unique (locally) if $g(t)$ is continuous for $t \geqq 0$?

Partial solution by the proposer and R. J. Thompson and D. W. Sasser, the Sandia Corporation. There is no such $g(t)$ for which there exists a solution $x(t)<t$ for $t>0$ with $x'(t)$ nonincreasing.

Assume (for contradiction) that there is such a solution, $x(t)$. Choose any $t_0>0$ such that $x(t)>0$ for $0<t\leqq t_0$, and define $t_i=x(t_{i-1})$ for $i=1, 2, \cdots$, then, $0<t_i<t_{i-1}\leqq t_0$, and therefore $\lim_{i\to\infty} t_i=0$.

From the differential equation we obtain $g(t_i)=g(t_{i-1})+1-x'(t_{i-1})$. Therefore $g(t_i)=g(t_0)+\sum_{j=1}^{i} [1-x'(t_{j-1})]$. Letting $i\to\infty$ yields

$$g(0) = g(t_0) + \sum_{j=1}^{\infty} [1 - x'(t_{j-1})].$$

We shall now show that this series diverges.

From the monotonicity of $x'(t)$,

$$1 - x'(t_{j-1}) \geqq 1 - x(t_{j-1})/t_{j-1} = (t_{j-1} - t_j)/t_{j-1}.$$

Thus, for any $k\geqq 1$,

$$\sum_{j=k}^{\infty} [1 - x'(t_{j-1})] \geqq \sum_{j=k}^{\infty} \frac{t_{j-1} - t_j}{t_{k-1}} = 1.$$

A similar argument holds if $x(t)>t$ for $t>0$ and $x'(t)$ is nondecreasing. Here one works with $t_i=x^{-1}(t_{i-1})$.

A related problem, which has been studied, is that of finding a continuous solution, $g(t)$, of the functional equation

$$g(f(t)) - g(t) = F(t),$$

where f and F are given continuous functions with $f(t)$ strictly increasing, $f(t)>t$ for $t>0$, and $f(0)=0$, $F(0)=0$. It is known, for example, [cf. J. Kordylewski and M. Kuczma, Ann. Polon. Math. X (1961) 41–48] that a solution exists if $f(t)$, $F(t)\in C^1$ and $f'(0)\neq 1$. This last requirement is violated in the proposed problem.

Disjoint Arcs in the Plane

5629 [1968, 1017]. *Proposed by M. W. Hudgins, University of Iowa*

Does there exist in the Euclidean plane a family of disjoint open arcs containing, for each point P distinct from the origin, one joining P to O?

Solution by J. B. Linder, University of North Carolina at Charlotte. The answer is in the negative, for suppose such a family F exists. For each point P let $\overline{P}$ denote the arc of F from (0, 0) to P. For each arc A of F let Af be a homeomorphism from the closed interval [0, 1] whose range is the closure of A such that $Af(0)=(0, 0)$. For each arc A of F the closed subarc of A from $Af(1/4)$ to $Af(3/4)$ and the closure of the subarc of $\overline{Af(1/2)}$ from $\overline{Af(1/2)}f(7/8)$ to $Af(1/2)$, (the common endpoint being $Af(1/2)$ for these arcs) form a triod. (A triod is the union of three arcs having a common endpoint and otherwise not intersecting.) Hence there is a triod associated with each arc of F, and the family of all such triods is uncountable and pairwise disjoint. R. L. Moore (Proc. N.A.S., vol. 14 (1928), 86) has shown that no such family of triods exists.

Similar argument shows that there exists no uncountable family of disjoint open arcs each of which contains an endpoint of another.

Also solved by G. R. MacLane, W. J. Roth, and the proposer.

A Simple Inequality

5630 [1968, 1018]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove the inequality $1+r^\lambda \leqq (1+r^2)^{\lambda/2}$ $(0\leqq r\leqq 1;\lambda\geqq 2)$, and determine positive numbers a, b, c such that the inequality $1+r^a=(1+r^b)^c$ $(0\leqq r\leqq 1)$ holds true.

Solution by G. A. Heuer, Concordia College. Necessary and sufficient conditions are that $c\geqq 1$ and $a\geqq b$. For sufficiency, note that $0\leqq r\leqq 1$ implies $1+r^a \leqq 1+r^b\leqq (1+r^b)^c$.

For $r=1$ we have $2\leqq 2^c$, so $c\geqq 1$ is necessary. By L'Hospital's rule,

$$\lim_{x\to 0^+} \frac{\log(1+x^b)^c}{\log(1+x^a)} = \frac{bc}{a} \lim_{x\to 0^+} x^{b-a},$$

which cannot be less than 1, and so $a \geqq b$ is necessary.

The first part of the problem has $a=\lambda$, $b=2\leqq\lambda$, $c=\lambda/2$.

Also solved by Marcia Ascher, Leon Bankoff, L. E. Clarke (England), J. H. Halton, Richard Gisselquist, D. A. Hejhal, D. A. Herrero, H. P. Jagani, M. S. Klamkin, Archan Kumer De (India), J. R. Kuttler & Nathan Rubinstein, M. L. Laplaza (Puerto Rico), R. L. McFarland, J. C. Molluzzo, R. A. Moore, C. B. A. Peck, and J. D. Riley.

A Functional Identity

5631 [1968, 1018]. *Proposed by A. D. Ziebur, State University of New York at Binghamton*

Suppose f is continuous in a neighborhood of the point $(0, c)$ in R^2, and is such that $f(t, x)$ is nondecreasing in x. Define the *Picard transformation* P by means of the equation

$$P\phi(t) = c + \int_0^t f(s, \phi(s))ds.$$

Show that if $P^2\phi(t)=\phi(t)$ for each t in some interval $[0, a)$, then $P\phi(t)=\phi(t)$ for $t\in[0, a)$.

Solution by R. J. Driscoll, Loyola University. Let $\psi=(P\phi-\phi)^2$. The relations

$$P\phi(t) = c + \int_0^t f(s, \phi(s))ds$$

$$\phi(t) = P^2\phi(t) = c + \int_0^t f(s, P\phi(s))ds$$

imply that ϕ and $P\phi$ are differentiable on $[0, a)$ and satisfy

$$(P\phi - \phi)'(t) = f(t, \phi(t)) - f(t, P\phi(t)),$$

so that ψ satisfies $\psi(0)=0$, $\psi'(t)\leqq 0$ on $[0, a)$; and this implies that $\psi(t)=0$ on $[0, a)$.

Also solved by D. R. Anderson & J. H. George, W. G. Dotson, Jr., Richard Gisselquist, D. A. Hejhal, O. P. Lossers (Netherlands), L. F. Meyers, and the proposer.

Fixed Point Homeomorphisms

5632 [1968, 1018]. *Proposed by D. K. Cohoon, W. Lafayette, Indiana*

Show that the line E^1 is the only one of the Euclidean spaces $E^n (n\geqq 1)$ which does not admit a fixed-point-free homeomorphism f with the property that the iterate f^m has fixed points for some positive integer m.

Solution by Dan Marcus, Harvard University. For $n\geqq 2$, set $f(x_1, x_2, \cdots, x_n) = (x_1+|x_2|-1, -x_2, x_3, \cdots, x_n)$. Then f is a fixed-point-free homeomorphism on E^n. Any point with $|x_2|=1$ is left fixed by f^2.

If, on the other hand, g is any fixed-point-free homeomorphism on E^1, then

$g(x)-x$ has constant sign by the Intermediate Value Theorem. Thus no iterate of g has a fixed point.

Also solved by Ralph Bennett, Julio Cano, R. A. Christiansen, D. A. Hejhal, Dennis Henkel, Ellen Hertz, L. F. Meyers, Ira Rosenholtz, B. L. Schwartz, J. B. Wilker & H. Radjavi (Iran), and the proposer.

Free Groups on Two Generators

5633 [1968, 1018]. *Proposed by George Shapiro, Harvard University*

Let $G(x, y)$ designate the free group on two generators x and y, and let $G(x)$ designate the subgroup generated by x. Does there exist a sub-semigroup S of $G(x, y)$ such that the set-theoretic union $S\cup S^{-1}=G(x, y)$ and $S\cap S^{-1}=G(x)$, where $S^{-1}=\{x^{-1}: x\in S\}$?

Solution by S. D. Promislow, University of British Columbia. We shall develop some more general results.

DEFINITION. Let G be any group. We shall say that (H, R) is an o-pair in G if H and R are sub-semigroups of G such that

(i) $R\cap R^{-1}=\phi$, (ii) $(R\cup R^{-1})\cup H=G$,
(iii) $(R\cup R^{-1})\cap H=\phi$.

$\cup$ and $\cap$ refer to set-theoretic union and intersection, respectively. The terminology is suggested by connections with order. For any o-pair (H, R) we can define a partial order on the set (G/H) by $xH\leqq yH$ if $x^{-1}y\in R$. This satisfies $xH\leqq yH$ implies $zxH\leqq zyH$. Conversely, for any such partial ordering on G/H, $R=\{r: H\leqq rH, r\notin H\}$ is a semigroup such that (H, R) is an o-pair.

H is necessarily a subgroup, since if $x\in H$, and $x^{-1}\notin H$, then $x^{-1}\in R\cup R^{-1}$ by (ii), so $x\in(R\cup R^{-1})$, which is a contradiction to (iii).

We consider ϕ as a sub-semigroup of G, so that (G, ϕ) is an o-pair in G. Any other o-pair will be called *proper*.

$$\left.\begin{aligned} H\cdot R &= R\cdot H = R \\ H\cdot R^{-1} &= R^{-1}\cdot H = R^{-1}\end{aligned}\right\} \textit{for any o-pair } (H, R). \tag{1}$$

Proof. Let $h\in H$, $r\in R$. Then

$$\begin{aligned} &\text{if } hr\in H, && r=(h^{-1})hr\in H, && \text{a contradiction to (iii),}\\ &\text{if } hr\in R^{-1}, && h=(hr)r^{-1}\in R^{-1}, && \text{a contradiction to (iii).}\end{aligned}$$

So by (ii), $hr\in R$. The remaining statements are similarly proved.

(2) *Let (K, Q) be an o-pair in H and let (H, R) be an o-pair in G. Then $(K, R\cup Q)$ is an o-pair in G.*

Proof. Since $Q\subseteq H$, $R\cup Q$ is a sub-semigroup of G by (1), and conditions (i), (ii) and (iii) are easily verified.

For a given group G we can partially order the set of o-pairs in G by $(H_1, R_1)\leqq(H_2, R_2)$ if $H_1\supseteq H_2$ and $R_1\subseteq R_2$. Given any increasing family $\{(H_\alpha, R_\alpha)\}$ of o-pairs in G, it is easily verified that (H, R) is an o-pair in G, where $H=\bigcap_\alpha H_\alpha$,

$R=\bigcup_\alpha R_\alpha$. Therefore we can apply Zorn's lemma and the fact that (G, ϕ) is an o-pair in G to deduce

(3) *Given any subgroup F of a group G, there exists an o-pair (H, R) in G, maximal with respect to the property that $F\subseteq H$.*

(4) *If (H, R) is an o-pair in G, and if α is any homomorphism from a group G' into G, then $(\alpha^{-1}(H), \alpha^{-1}(R))$ is an o-pair in G'.*

(5) *Let F be any free group and let K be any subgroup of F which has finite rank strictly less than the rank of F. Then there exists a proper o-pair (H, R) in F, with $K\subseteq H$.*

Proof. Let $x_1, x_2, \cdots, x_n$ be a set of free generators of K, let $\{\psi_\alpha\}, \alpha\in I$, be a set of free generators of F, and let $\mathcal{R}$ denote the additive group of real numbers and $\mathcal{R}^+$ denote the semigroup of positive numbers. Any real-valued function f on the set $\{\psi_\alpha\}$ determines by freeness a homomorphism $\hat{f}$ from F into $\mathcal{R}$. Since $|I|>n$, we can find $(n+1)$ linearly independent real-valued functions, $f_1, f_2, \cdots, f_{n+1}$ on $\{\psi_\alpha\}$.

Let $a_{ij}=\hat{f}_j(x_i)$ and solve the system of n equations in the $(n+1)$ unknowns, $r_1, r_2, \cdots, r_{n+1}$,

$$\sum_{j=1}^{n+1} r_j a_{ij} = 0, \qquad i = 1, 2, \cdots, n.$$

Let $f=\sum_{j=1}^{n+1} r_j f_j$, a function on $\{\psi_\alpha\}$. Since $(0, \mathcal{R}^+)$ is an o-pair in $\mathcal{R}$, by (4) $(\hat{f}^{-1}(0), \hat{f}^{-1}(\mathcal{R}^+))$ is an o-pair in F and it is proper since f is not the zero function because of the linear independence of the f_i. By the definition of f, $\hat{f}(x_i)=0$ for $i=1, 2, \cdots, n$, and so $K\subseteq\hat{f}^{-1}(0)$.

(6) *Let F be any free group, and let K be any cyclic subgroup of F. Then there exists a sub-semigroup R of G such that (K, R) is an o-pair in G.*

Proof. By (3), there exists an o-pair (H, R) in F, such that $K\subseteq H$, and which is maximal with respect to this property. If K is strictly contained in H, then K has rank 1, which is strictly less than the rank of H. We can apply (5) to H and K, to get (K', Q), a proper o-pair in H with $K\subseteq K'$. From (2), $(K', R\cup Q)$ is an o-pair in F. Since $Q\neq\phi$, $(H, R)\leqq(K', R\cup Q)$, but not equal, contradicting the maximality of (H, R). Therefore, we conclude that $K=H$.

From (6) we get an affirmative answer to the given problem. Here $F=G(x, y)$, $K=G(x)$. We take $S=K\cup R$. Then S is a sub-semigroup of F by (1), and satisfies the conditions of the problem.

Also solved by the proposer.

Sums of Consecutive Integers and Consecutive Squares

5634 [1968, 1018]. *Proposed by H. E. Thomas, Jr., University of Michigan*

Find all integral solutions (n, r) of $\sum_{i=1}^{n} i = \sum_{i=1}^{r} i^2$.

Partial solution by C. L. Sabharwal, Saint Louis University. We are led directly to the equation

$$3(n^2 + n) = r(r + 1)(2r + 1).$$

On testing various values of r we find that the following four sets (n, r) satisfy:

$$(1, 1), \qquad (10, 5), \qquad (13, 6), \qquad (645, 85).$$

NOTE. The proposer has reduced this challenging problem to finding solutions of the Diophantine equation $y^2=x^3-9x+81$ in odd integers divisible by 3. Although some solutions of this equation are known they have not led to additional solutions of the original question.

Isometries in Non-Compact Spaces

5635 [1968, 1018]. *Proposed by Otto Plaat, University of San Francisco*

Let S be a metric space such that (1) closed spheres in S are compact, and (2) given points p, q in S there is an isometry T of S such that $T(p)=q$. Prove that every isometry of S (into S) is onto S. (This yields a non-algebraic proof that the isometries of E^n are onto.)

Solution by P. R. Meyer, Hunter College, C.U.N.Y. If f is an isometry of S then f maps each closed sphere with center x onto the closed sphere of the same radius about $f(x)$. This follows from a well-known fact about compact metric spaces (see, e.g., J. L. Kelley, *General Topology*, Exercise 5 D, p. 162) because there is an isometry of S which carries $f(x)$ to x. Thus, if range f is a proper subset of S, we can obtain a contradiction by choosing a point x and a sufficiently large radius so that the closed sphere about $f(x)$ meets $S-$range f.

Also solved by P. R. Chernoff, W. G. Dotson, Jr., R. J. Driscoll, W. F. Fox, R. V. Fuller, Ellen Hertz, M. L. Laplaza (Puerto Rico), L. E. Mattics, M. D. Mavinkurve (India), Dan Putnam, Robert Silber, Garrett Van Meter & John Connett, P. van der Steen (Netherlands), J. B. Wilker & H. Radjavi (Iran), and the proposer.

On Operators A, B: $BA=\omega AB$

5637 [1968, 1125]. *Proposed by P. M. Cohn, University of London, England*

Let ω be a primitive nth root of unity. Given two linear operators A, B satisfying the commutation rule $BA=\omega AB$, show that

$$(A+B)^n = A^n + B^n.$$

Solution by E. P. Del Norte, University of Texas, El Paso. Let

$$(A+B)^{n-1} = \sum_{j=0}^{n-1} p_j^{n-1}(\omega) A^j B^{n-1-j},$$

where $p_j^{n-1}(x)$ is a polynomial over the integers. Left multiplication by $(A+B)$ leads to the equations

$$p_j^n(x) = p_{j-1}^{n-1}(x) + x^j p_j^{n-1}(x), \qquad 0 < j < n,$$

while right multiplication by $(A+B)$ yields

$$p_j^n(x) = x^{n-j} p_{j-1}^{n-1}(x) + p_j^{n-1}(x), \qquad 0 < j < n.$$

Eliminating $p_j^{n-1}(x)$, we have

$$p_j^n(x) = \frac{x^n - 1}{x^j - 1} p_{j-1}^{n-1}(x), \qquad 0 < j < n,$$

and therefore $p_j^n(\omega) = 0$ for $0<j<n$. It now follows that $(A+B)^n = A^n + B^n$.

Also solved by Joel Anderson, David Ballew, D. Borwein, D. Ž. Djoković, D. T. Graves, M. G. Greening (Australia), D. A. Herrero, M. S. Klamkin, J. R. Kuttler, Douglas Lind (England), Roger Lyndon, M. D. Mavinkurve (India), Arnold Singer, J. B. Skinner, D. M. Smiley & M. F. Smiley, Stoddard Smith, Jr., Joel Spencer, E. C. Thompson, H. Thornton (England), E. W. Trost (Switzerland), and the proposer.

REVIEWS

EDITED BY KENNETH O. MAY

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.*

C *Multilinear Algebra*. By W. H. Greub. Grundlehren der Mathematischen Wissenschaften, vol. 136. Springer-Verlag, New York, 1967. x+224 pp. $8.00. (Telegraphic Review, March, 1968.)

This book is a valuable and important addition to the limited literature on multilinear algebra. Aside from the Bourbaki vol. VII, Chapter III, *Algèbre Multilinéaire*, there are few books that are devoted exclusively to this subject. There are a number of textbooks and monographs, however, which include substantial sections on tensor and Grassmann algebras. Some of these are: *Fundamental Concepts of Algebra*, C. Chevalley, Academic Press, Inc., 1956; *Elements of Modern Algebra*, S. T. Hu, Holden Day, 1965; *Algebra*, S. Lang, Addison-Wesley, 1965; *Algebra*, S. MacLane and G. Birkhoff, Macmillan, 1967; *Fundamental Structures of Algebra*, G. Mostow, J. Sampson, J. P. Meyer, McGraw-Hill, 1963; *The Construction and Study of Certain Important Algebras*, C. Chevalley, The Mathematical Society of Japan, 1955; *Lectures in Abstract Algebra* vol. II, N. Jacobson, Van Nostrand, 1953; *Advanced Calculus*, H. Nickerson, D. Spencer, N. Steenrod, Van Nostrand, 1959. We have not mentioned any of the recent differential geometry books that usually contain chapters on tensor and exterior algebras. Some of the books in this very partial list are intended for the basic graduate courses and confirm this reviewer's opinion concerning the importance of multilinear algebra.

Greub starts by defining the tensor product of a finite collection of not nec-

essarily finite dimensional spaces by means of the usual universal factorization property. The eight chapters of the book then cover: tensor algebras, exterior algebras, mixed exterior algebras, symmetric algebras, and multilinear functions. There are a large number of interesting sections devoted to such topics as differential algebras, ideals in the exterior algebra, derivations and antiderivations, polynomial algebras.

The book was used by the reviewer as a text for a graduate course in multilinear algebra. The average preparation of the students was about a one-year standard graduate course in algebra. A rather extensive list of specific corrections were found during the course. There are also a number of places in which proofs can be shortened or improved. However, because of a lack of space the details of these comments will not be reproduced here. However, I might mention one item which appears in several places.

An error on p. 15 states that if $\phi: E\times F\rightarrow G$ is bilinear and induces the linear map $f: E\otimes F\rightarrow G$ then $\ker f=N_1(\phi)\otimes F+E\otimes N_1(\phi)$ where $N_1(\phi)$ and $N_2(\phi)$ are the left and right null spaces of ϕ. This is not true, e.g., take $E=F=V_2(C)$, $\phi(x, y)=x_1y_1+x_2y_2$. If $x=(1, i)$ then $f(x\otimes x)=\phi(x, x)=0$ but $x\notin N_1(\phi)$ i.e., $\phi(x, V_2(C))\neq 0$. What is true is that $\ker f\supset N_1(\phi)\otimes F+E\otimes N_2(F)$. It would have been useful if this chapter tied up the notion of the rank of a matrix with the "irreducible" length of a tensor in $E\otimes F$. The properties of Kronecker products of matrices should also have been included here, together with the important but infrequently treated elementary divisor theory for these items.

The perfect book on any subject has yet to be written. However, Greub's effort certainly deserves credit. This subject is notationally ghastly and so ridden with canonical isomorphisms that one occasionally gets lost. Greub has done a good job of organization here, and we would certainly use this book again as a text.

MARVIN MARCUS, University of California, Santa Barbara

Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. By A. Hurwitz, R. Courant and H. Röhrl. Springer-Verlag, New York, 1966. 706 pp. DM 49.00.

This is the "fourth enlarged and improved edition" of a book first published in 1922 as Volume 3 of the Springer yellow series. The book is divided into four parts. The first part, by Hurwitz, is an introduction to complex analysis based on power series ("according to Weierstrass"). The second part, also by Hurwitz, is a beautiful and very readable introduction to elliptic functions. These first two parts seem to be a photolithographic reprint from the third edition—including a couple of misprints. In places they seem old-fashioned or unclear (e.g., "variable" on page 17, definition of function on page 43). The third part by Courant, comprises almost half the book; it has been reset and slightly revised. Here we are given complex analysis "according to Riemann." There is some duplication with the first part. In the section on conformal mappings potential theory is used. The definition of the Riemann surface of a function could be clearer. It is also very disturbing to find a step in a proof simply left

out (page 442)! The fourth part is an appendix (new in this edition), by Röhrl, which treats further problems of conformal mapping and the existence of functions on Riemann surfaces. Holomorphic vector bundles are introduced and used.

In some sections of the book it is hard to pick out the theorems, mainly because of the typography. There are no problems and few examples. This is not a book for the beginner, but someone who already has some familiarity with complex analysis will find much clever and skillful mathematics here. The teacher will also find much that he can use with profit.

C. ANAGNOSTAKIS, University of Connecticut

Tensor Analysis on Manifolds. By Richard L. Bishop and Samuel I. Goldberg. Macmillan, New York, 1968. viii+280 pp. $11.95. (Telegraphic Review, May 1968.)

In spite of its ever-increasing importance in mathematics and the physical sciences, manifold theory has yet to find a secure position in the undergraduate curriculum. The present authors, by emphasizing tensor calculus on manifolds, place the subject as the direct successor to multivariable calculus. Their book is elementary in the best sense: (1) the prerequisites are minimal (calculus through advanced calculus—including a smattering of differential equations and linear algebra); (2) the exposition is efficiently organized and its pace is unhurried; (3) examples and problems abound. This book should put an end to the still prevalent misconception that tensor calculus is the science of rearranging indices; the following partial outline will indicate the broad range of topics covered.

Chapter 1 is a straightforward account of differentiable manifolds and mappings and their linear approximation by tangent spaces and differential maps. Chapter 2 is an excellent exposition of the tensor algebra of a vector space. Its first third develops (from the very beginning) the necessary linear algebra. Tensors are then deftly defined as multilinear functionals on products of the vector space and its dual. It is made quite clear that tensors appear in many different guises ("interpretations"), and the classical definition of tensor is derived by introducing coordinates and observing the resulting transformation laws. (Classical notation is used frequently throughout the rest of the book.) Chapter 3 deals with vector fields, the notion of integral curve leading to the Lie derivative, thence to the bracket operation on vector fields (geometrically well motivated), and finally to the Frobenius Theorem. For the latter the available space is wisely devoted to explanation and example rather than proof. Integration theory is presented in Chapter 4. Differential forms are already available as skew-symmetric covariant tensor fields. The exterior derivative is defined, axiomatically characterized, and related to gradient, curl, divergence, and laplacian. Forms are integrated over cubical singular chains, and emphasis is not on the formalism of homology theory but on such practical matters as fitting cubical chains onto variously shaped regions. A proof of Stokes' Theorem and a look at partial differential equations complete the main part of the book,

which turns now to applications. Chapter 5 on Riemannian (and semi-Riemannian) geometry applies variational methods to the arclength and energy of a curve. Geodesics, defined as self-parallel curves, are shown equivalently to be critical points of the energy function. There is a technically fine but perhaps too detailed treatment of covariant derivatives (the level of sophistication is rising rapidly now), and curvature is introduced. A final brief chapter utilizes an impressive amount of previous work to give a modern account of Hamiltonian mechanics.

This is a first-rate book and deserves to be widely read.

BARRETT O'NEILL, University of California, Los Angeles

Foundations of Real Numbers. By Claude W. Burrill. McGraw-Hill, New York, 1967. 163 pp. $6.95.

This enjoyable, well-written little book presents a concise development of the real numbers from the foundations of set theory. The first five chapters deal with sets, the natural numbers, and integers. The system of real numbers is then defined directly in terms of the integers without first introducing the rationals. Since this definition is based on the decimal representation, the student should find this approach natural and easy to grasp. Using the isomorphism of complete ordered fields, he then shows that the usual Dedekind and Cantor definitions are equivalent to this. A short appendix indicates how the rational number system can be constructed from the system of integers.

The development is carefully motivated, with illustrative examples, and provides a clear exposition of the subject. The book is suitable as a text for mature undergraduate or beginning graduate students, or as collateral reading and independent study.

GERALDINE A. COON, Goucher College

An Introduction to Fluid Dynamics. By G. K. Batchelor. Cambridge Univ. Press, 1967. xviii+615 pp. $14.50. (Telegraphic Review, June, 1968.)

There are few existing books which expound theoretical fluid mechanics from a modern viewpoint in a form accessible to the student meeting the subject for the first time. With this in mind Professor Batchelor has written a text aimed at filling such a need and focuses his attention on the most important and central area of fluid mechanics, namely, the motion of a uniform incompressible viscous fluid. In accordance with the progress of fluid mechanics during the past fifty years physical reasoning is employed to explain the principles of the subject, to develop the flow structures and to analyze the manner in which vorticity is diffused and convected in the numerous flow models taken up for discussion. Throughout, the mathematical treatment is straightforward and requires a familiarity with methods of potential theory, vector integral calculus and some knowledge of tensor notation. The book is intended for final year honors course students as a basic text and will be useful also to graduate students and teachers who wish for an up-to-date account of well-known material.

K. B. RANGER, University of Toronto

A Comparative Study of Programming Languages. By Bryan Higman. American Elsevier, New York, 1967. 164 pp. $8.50. (Telegraphic Review, May, 1968.)

This is a useful little book. Its modest size plus its eminently readable style result in a book which can be read and appreciated by mathematicians and others who are not computer specialists. It is not, however, as its title might suggest, nearly as complete or ambitious as the computer specialist might hope.

The first seven chapters introduce some basic concepts and terminology and include a very brief but useful treatment of the lambda calculus, recursion, formal language structure and the like. However, comparatively little of the apparatus introduced is actually used in the remainder of the book. Chapters 8 through 13 offer brief introductions of features of macro languages, assembly languages, FORTRAN, COBOL, ALGOL-60, CPL, and PL/I. The treatment can not really be considered as comparative with the exception of Chapter 13 which provides an especially good and comparative analysis of CPL and PL/I. None of the languages is treated in any depth, but a reasonable introduction is given to most of the languages covered. Chapter 8 on macro languages is quite weak and might best be omitted unless one is willing to turn to the original source for details and clarification. Chapter 12 on list processing languages also leaves an impression which, in this reviewer's opinion, is not correct, namely that "the day for list processing languages is over . . .".

In summary, this book provides a very useful sketch of some of the basic concepts of programming linguistics and a brief survey of a number of programming languages in a style which makes it very accessible to the nonspecialists.

T. E. CHEATHAM, JR., Lexington, Massachusetts

Matrices and Linear Algebra. By Hans Schneider and George Phillip Barker. Holt, Rinehart and Winston, New York, 1968. viii+385 pp. $7.95. (Telegraphic Review, Aug. 1968.)

The text appears to be an excellent introduction to matrices and linear transformations. With aims of making the material usable in the early undergraduate years and to both mathematics majors and others, matrix concepts are emphasized initially as being somewhat less abstract. However, linear transformations are introduced early in the text, and a later chapter illustrates how both viewpoints can be utilized in establishing various theorems.

An attempt is made to motivate many of the theorems and definitions, and most of the theory is well-illustrated with concrete examples. There is a wide variety of problems, both simple and difficult, and answers are provided.

The number of slips and misprints does not seem to be excessive for a new text—perhaps one per ten pages.

Chapter headings are: The Algebra of Matrices, Linear Equations, Vector Spaces, Determinants, Linear Transformations, Eigenvalues and Eigenvectors, Inner Product Spaces, and Applications to Differential Equations. There should be adequate material for a one semester course, or perhaps a two quarters course.

A. B. FARNELL, Colorado State University

Real Analysis: An Introduction. By A. J. White. Addison-Wesley, Reading, Mass., 1968. vii+244 pp. \$8.75. (Telegraphic Review, August 1968.)

This is a very useful book in one variable analysis. It is clearly and intelligently written in modern style and notation. The selection of material is good and approximately one fourth of the book is devoted to interesting project-type problems. Metric spaces are taken up in Chapter 2 and play a continuing role in the development, especially in the problems. The important theorems in the calculus, whose proofs are commonly omitted in first courses, are nicely handled in Chapters 4 and 5. Differential equations are briefly discussed in Chapter 7 with Picard's Theorem presented in the context of contraction mappings. The other chapters deal with real numbers, functions, and infinite series. As the reader progresses through the book he not only learns to think of functions in the classical way but also naturally thinks of them as elements of function spaces or other suitable algebraic structures. The book is admirably suited to a first course in analysis taught at the junior-senior level. It could be used even earlier by an able student. The book is primarily a text, though I think it would also be welcome in many college libraries.

J. B. ROBERTS, Reed College

Integration. By A. C. Zaanen. North-Holland Publishing Company, Amsterdam, and Wiley, New York, 1967. 604 pp. \$16.75. (Telegraphic Review, June 1968.)

This is a substantially rewritten and enlarged edition of the author's *An Introduction to the Theory of Integration* (1958). As such it retains the distinctive features of that book. Chief among these is the definition of a measure as a nonnegative function on a semiring of sets, satisfying certain conditions which are shown to imply monotonicity and σ-additivity and, conversely, to be implied by them. This approach is modern in the sense that the concept of semiring appeared relatively recently (ca. 1950), and it has certain advantages: (1) The collection of cells (half-open intervals $(a, b]$) in R_k is a semiring, so that Lebesgue and Lebesgue-Stieltjes measures can be introduced more easily, and (2) the extension procedure from a measure on a semiring, to an outer measure on all subsets, to a measure on the σ-algebra of measurable sets, can be exploited to extend an elementary integral (a nonnegative linear functional I on a vector-lattice L of real functions, satisfying $I(f_n) \downarrow 0$ whenever $f_n \downarrow 0$) to a Daniell integral. The device is to form a semiring $\bar{\Gamma}$ of subsets of $X \times R_1^+$ from the "ordinate sets" of differences $f-g$ with $f, g \in L$, and to define a measure on $\bar{\Gamma}$ in terms of I. The resulting extended measure is then used to define the extended integral. It is not obvious that this results in an overall economy of effort, since the linearity properties of the Daniell integral seem somewhat harder to prove in this setting, but the idea is interesting and shows, in the author's words, "that the integral of a nonnegative function has something to do with the measure . . . of the ordinate set of the function" (i.e., with the "area under the curve" in the case where $X = R_1$).

The very distinctiveness of Zaanen's approach is one reason for the reviewer's feeling that the book is extremely valuable as a reference for those with a special interest in integration theory but less so as a text for the basic graduate course in Real Variable. Another is that in Section 17 there is a discussion of ways to extend the elementary integral defined for step functions relative to a measure on a ring of sets (by extending the measure first and then defining the integral in the "usual" way, or by extending the elementary integral to a Daniell integral). The two approaches are brought together and shown to coincide in Theorem 9 of that section, which the author rightly believes is "one of the most important" in the book. But in the course of this discussion the word "measurable" is used, with various prefixes, in enough different ways so as to leave the beginner somewhat confused.

The scope of the book is substantial. It includes all that is basic to a thorough graduate course of one year (unless integration on locally compact spaces is basic: the only topological space mentioned is R_k), and much beyond. Notable among the more advanced topics are: a very complete discussion of the Radon-Nikodym Theorem for non-σ-finite spaces, a discussion of the Bochner integral, chapters on the Fourier transform (mostly for functions on R_1) and ergodic theory, and a chapter on normed Köthe spaces (of which the L_p-spaces are a special case). The classical Lebesgue integral is not slighted, being the subject of sections or chapters devoted to integration by parts and change of variable formulas, the gamma function, and the above-mentioned chapter on Fourier transforms, as well as numerous exercises.

The number of exercises is greatly enlarged, and they are arranged in groups which bear the name of their general topic. Many subjects not developed in the text are outlined in the exercises, and the book contains (at the end) 100 pages of brief solutions for exercises. This feature contributes to the value of the book as a reference or for self-study.

C. W. AUSTIN, California State College at Long Beach

An Introduction to Analysis. By Wilson M. Zaring. Macmillan, New York, 1967. xi+364 pp. $9.95. (Telegraphic Review, December 1967.)

The preface to this book begins with the declaration that "This text was designed specifically for the prospective teacher of the calculus." Whatever that may imply, the intent is to provide the reader with a careful development of the mathematical foundations of the elements of the theory of functions of one real variable from a set-theoretic viewpoint with "Emphasis . . . placed upon the continuity of thought, the motivation of ideas, and the clarity of exposition rather than rigor *per se*." Part II, which is four-fifths of the whole and follows a preliminary account of logic and set theory, was "developed over a period of several years as a two-semester course in introductory real analysis taught to the participants in Academic Year Institutes at the University of Illinois." The author says that Landau's *Calculus* was the major source of his ideas for Part II. Proofs of central theorems are given in considerable detail. An involved argu-

ment is usually prefaced by an attractive preliminary "proof procedure." The proofs of many results are left to the reader, sometimes preceded by an informative sketch. The exercises are commonly extensions of the theory. Some important sections are identified by the author as being possible of omission from an abbreviated course without loss of logical continuity (e.g., Heine-Borel Theorem, axiom of choice, Schröder (Cantor)-Bernstein Theorem, van der Waerden's classic example). Another of these is an account of the conditions for the existence of an integral which the author identifies as one of his "favorite sections"; he would no doubt feel happier with it if Lebesgue's name had not been consistently misspelled. Preceding a rather lengthy proof of his "rule," l'Hospital is selected for a historical recognition denied others. For a bibliography the reader must be satisfied with the listing of ten well-regarded texts as "references" for Part I and eleven others (four by one author) for Part II, together with another in a footnote. Some transitional or otherwise expository passages (for example, that introducing the concepts of limit and continuity) are not expressed with the felicity that one is led to expect from the purely technical presentations.

S. G. HACKER, Washington State University

C *Calculus for Students of Business and Management.* By Bevan K. Youse and Ashford W. Stalnaker. International Textbook Company, Scranton, Pa,. 1967. viii+271 pp. $7.50.

This book was designed as a brief introduction to calculus for business students with limited backgrounds in mathematics. The selection of topics is good. In particular the early treatment of calculus of several variables is desirable for business problems. This is an advantage over most calculus texts which could be used for this audience.

Unfortunately the treatment of topics is not as good as the selection. The mathematics is adequate for the intended level of student, but it appears to be written by a mathematician largely illiterate in business who has asked a business expert to supply examples which were inserted at appropriate points in a mathematics text. The reviewer (a mathematician) used the book with a group of graduate business students. A number of times the better students were able to supply more appropriate terminology for the mathematical ideas presented. In the reviewer's opinion it would be more helpful if the chapter on differential equations emphasized more of the geometric aspects of solutions rather than the mechanics of finding them. The exercises were poorly checked. Problem 15 on page 4 is inconsistent; it requires -46 people in one set. In the maximum and minimum problems in several variables after explaining the significance of positive and negative values for the second derivative criterion, the first exercise has the value zero for which no explanation was given. While it is a mediocre book, it will continue to be used since it has very little competition.

KENNETH LOEWEN, University of Oklahoma

Elementary Linear Algebra. By L. H. Lange. Wiley, New York, 1968. xiii+380 pp. $9.50 (Telegraphic Review, Aug./Sept. 1968.)

The author's prefatory notes indicate clearly how the text can be used in different situations. His description of a one-semester use is very helpful. The style is very good—careful, but with a pleasant degree of informality. The treatment of vector spaces and matrices is restricted for the most part to vector spaces and matrices over the reals. Concepts are introduced so as to appeal to the reader's experience with the real number field and with analytic geometry. For example, Euclidean n-space is presented after a short recollection of the dot product in two and three-space. The material on rank, solutions of linear systems, linear transformations, similarity, eigenvalues is well illustrated with examples accessible to the typical beginning undergraduate.

Some of the exercises are routine; others are more difficult and will help the student to develop his skill in making proofs. In addition to the material intended for a one-semester course, there are introductory and readable sections on groups and other systems, determinants, unitary space, and linear programming. An instructor who wishes to order differently the content of such a course would benefit by reading the present text and having it available for his students. The book may have a special appeal for instructors who are teaching vector spaces for the first time. Also, the book may be useful in courses designed to upgrade teachers of high school mathematics.

J. R. WESSON, Vanderbilt University

Topics in Geometry. By Howard Levi. Complementary Series, Vol. 11. Prindle, Weber and Schmidt, Boston, Mass., 1968. viii+104 pp. $2.95 (paper). (Telegraphic Review, August 1968.)

This collection of essays is based on lectures presented by the author to audiences ranging from high school to an advanced college level. As the author points out, this explains an unevenness in the level of exposition.

Chapter 1 reconciles the notion of a plane congruence considered as a distance preserving mapping with that of a mapping which preserves shape and size. Properties of reflections are studied in Chapter 2, and it is shown that any plane congruence is the composition of at most three line reflections. The approach to these matters, as with most topics in the book, is mainly coordinate oriented. Chapter 3 consists of a few qualitative remarks on elliptic geometry. Inversions are discussed in Chapter 5 and used in Chapter 7 to give a development of hyperbolic geometry (using the Poincaré circular model) based on motions considered as products of hyperbolic line reflections. Later chapters treat length and area in hyperbolic geometry and give a nice development of affine geometry. The book closes with a readable sketch of how one introduces coordinates in a synthetically presented geometry.

This is not intended to be a textbook, although there are exercises at the ends of some chapters. The book would provide good supplementary reading for a college course which touches on these topics.

G. D. CHAKERIAN, University of California at Davis

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G. D. CHAKERIAN, University of California at Davis

TELEGRAPHIC REVIEWS

Telegraphic reviews are intended to give prompt notice of books with sufficient information to assist in deciding whether to order an examination copy or suggest library purchase. Possible uses are coded as follows: T = textbook, S = supplementary student reading, P = professional reading, TT = teacher training, L = library purchase, 13 to 18 = freshman to second graduate year level, 1 to 4 = one to four semesters. An asterisk is used for emphasis. Books covering standard high school material are called "remedial." All textbooks are examined carefully, and mention is made of noteworthy features that are not evident from the title and coding. Publishers are indicated by the standard abbreviations used in *Books in Print* (which gives full names and addresses).

ALGEBRA, T(14-16). *An Introduction to Abstract Algebra.* By Dennis B. Ames. Intl. Textbk, 1969. 378 pp. $10. Groups, vector spaces, structure of groups, rings, factorization and ideals, modules, algebras, field theory, Galois theory, homological algebra, and elementary structure theory of rings, presented at a gradually increasing level of sophistication.

ALGEBRA, T(16), S, L. *Introduction to Commutative Algebra.* By M. F. Atiyah and I. G. Macdonald, A-W, 1969. 137 pp. $7.50. To follow a first course in algebra and preceed one in homological algebra. Focuses on commutative rings, prime ideals.

ALGEBRA, P, L. *Studies on Abelian Groups.* Edited by B. Charles. Springer-Verlag, 1968. 365 pp. $10.50. Twenty three papers given at a symposium held at Montpellier University in 1967. Emphasis is on structure, utilizing homological and topological methods.

ALGEBRA, *T(14-15; 1-2). *Linear Algebra: An Introductory Approach.* 2nd ed. By Charles W. Curtis. Allyn, 1968. 262 pp. $8.95. Major changes make the beginning more elementary and introduce abstract ideas earlier.

ALGEBRA, T(17-18), P, *L. *Universal Algebra.* By George Gratzer. Van Nostrand, 1968. 384 pp. $12.50. The author defines universal algebra as "the study of finitary operations on a set" whose purpose is "finding and developing those properties which such diverse algebras as rings, fields, Boolean algebras, lattices and groups may have in common." He intends to "give a systematic treatment of the most important results in the field." Exercises, a brief historical note, a scholarly 29 page *bibliography* by the author and Catherine M. Gratzer.

ALGEBRA, P, L. *Structure and Representations of Jordan Algebras.* By Nathan Jacobson. AMS Colloq. Pub. 39. Am Math, 1968. 563 pp. $10.80. "...a comprehensive account of the structure and representation theory of Jordan algebras over a field of characteristics not two." The subject stems from the efforts of P. Jordan, von Neumann and E. P. Wigner to reformulate quantum mechanics in terms of the Jordan products (AB+BA)/2. *Bibliography.*

ALGEBRA, P(17), L. *Completely 0-Simple Semigroups.* By Kenneth M. Kapp and Hans Schneider. W.A.Benjamin, 1969. 110 pp. $12.50 (cloth) $3.95 (paper). Lecture notes to "introduce...basic tools of...the algebraic theory of semigroups and...some recent results..."

ALGEBRA, TT(13; 1). *Basic Algebraic Concepts.* By F. Lynwood Wren and John W. Lindsay. McGraw, 1969. 387 pp. $8.95. For future elementary and junior high school teachers with minimal high school mathematics, but a semester college course along the lines of the senior author's *Basic Mathematical Concepts* (McGraw-Hill, 1965). Intended to meet the Level I recommendations of CUPM on number systems and algebra.

ANALYSIS, *T(15), P, L. *Advanced Calculus.* By Harold M. Edwards. HM, 1969. 523 pp. $10.50. Not just another advanced calculus, but a fresh treatment of the most important topics through the use of differential forms. Central theme: calculus of several variables. Major topics: convergence, the algebra of forms, the implicit function theorem, fundamental theorems of calculus. Included are Lagrange multipliers, Stokes' theorem, integrability conditions for partial differential equations, the Lebesgue integral, etc. The treatment is informal and the author urges flexible reading as opposed to the old insistence on mastering every detail before moving forward.

ANALYSIS, T(17-18), P, *L. *Geometric Measure Theory.* By Herbert Federer. Springer-Verlag, 1969. 690 pp. $29.50. An impressive comprehensive treatise with historical references and *bibliography.* Assumes some set theory, topology, linear algebra and commutative ring theory but deals with required topics in multi-linear algebra, analysis, differential geometry and algebraic topology.

ANALYSIS, T(15-16), S, P, L. *Introduction to Spectral Theory in Hilbert Space.* By Gilbert Helmberg. North-Holland, 1969. Distributed by Wiley, New York. 359 pp. $19.50. "...to make the reader familiar with everything needed in order to understand, believe, and apply the spectral theorem for self-adjoint operators (not necessarily bounded) in Hilbert space." Detailed exposition assuming only classical analysis.

ANALYSIS, NUMERICAL METHODS, P, L. *The Method of Quasi-Reversibility. Applications to Partial Differential Equations.* By R. Lattès and J.-L. Lions. Translated from the French edition and edited by Richard Bellman. Am Elsevier, 1969. 408 pp. $20. Numerical solutions of intrinsically unstable equations ("improperly posed problems" in the sense of Hadamard). The translator speaks of the "peculiar snobbery" of the "aberration" by which "pure" and "applied" have been used to bifurcate mathematical activity in the 20th century. He points out the basic instability of this departure from the central mathematical tradition, mentioning the curiosity of the young generation of mathematicians about problems "posed by the outside world," the universality of mathematics ("mathematics is too vital for the monastery; it will not stay cloistered") and the impact of the digital computer on algorithmic feasibility ("The game has become unbelievably more interesting."). *Bibliography.*

ANALYSIS, P, L. *Topology on Spaces of Holomorphic Mappings.* By Leopoldo Nachbin. *Ergebnisse 47.* Springer-Verlag, 1969. 66 pp. $4.50. For specialists interested in "a natural method of endowing certain vector spaces of holomorphic mappings with locally convex topology."

ANALYSIS, T(17), P. *Analysis on Real and Complex Manifolds.* By Raghavan Narasimhan. Masson, Paris and North-Holland, Amsterdam, 1968. 251 pp. $14. Differentiable functions in real n-space, manifolds, linear elliptic differential operators. Assumes Bourbaki

on multilinear algebra and general topology.

ANALYSIS, S(18), P, L. *Perturbation Theory of Eigenvalue Problems.* By Franz Rellich. Assisted by J. Berkowitz. Gordon, 1969. 138 pp. $9.25 (cloth), $6.25 (paper). Notes from lectures given by the late Professor Rellich at the New York University in 1953. Theme: interplay of abstract operator theory with significant applications.

ANALYSIS, T(17), P, L. *The Approximation of Functions. Vol. 2. Nonlinear and Multivariate Theory.* By John R. Rice. A-W, 1969. 347 pp. $16.75. The first volume, entitled *Linear Theory*, appeared in 1964. This one contains chapters seven through thirteen. *Bibliography.*

ANALYSIS, P. *Denjoy Integration in Abstract Spaces.* By Donald W. Solomon. *Memoirs of AMS 85.* Am Math, 1969. 69 pp. $1.90 (paper).

ANALYSIS, PHYSICS, P. *Generalized Feynman Amplitudes.* By Eugene R. Speer. *Annals of Math. Studies 62.* Princeton U Pr, 1969. 120 pp. $3.50 (paper).

*ANALYSIS, APPLICATIONS S(17), P, L. *Lectures on the Calculus of Variations and Optimal Control Theory.* By L. C. Young. Saunders, 1969. 342 pp. $15. The first 212 pages deal with the calculus of variations from a modern point of view, especially utilizing the ideas growing out of the concept of generalized curves and surfaces introduced by the author in the thirties. The rest of the book is on the theory of optimal control. The highly personal style, in the tradition of the authors parents, W. H. and G. C. Young, involves talking intelligently about the material and reaching for a broad audience while holding the interest of the specialist.

ANALYTIC GEOMETRY, T(13; 1-2), *Contemporary Analytic Geometry.* By Thomas L. Wade and Howard E. Taylor. McGraw, 1969. 337 pp. $8.50. The first word of the title refers to the authors' intention to include analytic geometry in the "updating and modernizing of mathematics," especially by using logic, set theoretic concepts, and the properties of the real number system.

APPLICATIONS, T(13-14), S, *L. *Mathematics for Science and Engineering. 2nd ed.* By Philip L. Alger. McGraw, 1969. 384 pp. $9.75. Starting with arithmetic, then continuing from elementary to advanced topics, "substantially all the mathematics taught in undergraduate college courses and used by practising engineers is made available in a single volume." The first edition of 1957 was inspired by and based on the three editions (1911-1917) of the famous *Engineering Mathematics* by C. P. Steinmetz. This edition adds some newer topics.

APPLICATIONS, SOCIAL SCIENCES, S, P, *L. *Mathematics of the Decision Sciences. Pt. 2.* Edited by George B. Dantzig and Arthur F. Veinott, Jr. Am Math, 1968. 450 pp. $17.20. Volume 1 (Tel. Rev. June 1969) included papers on linear programming, pivot theory and quadratic programs, convex polyhedra and integer programs, conbinatorics, nonlinear programming. This volume contains papers on control theory, mathematical economics, dynamic programming, applied probability and statistics, mathematical psychology and linguistics, and computer science.

APPLICATIONS, PROBABILITY, T(15), S, P, L. *Introduction to Queueing Theory.* By B. V. Gnedenko and I. N. Kovalenko. Translated by R. Kondor. Translation edited by D. Louvish. Israel Program for Scientific Translations, Jerusalem 1968. Distributed by Davey, Conn. 290 pp. $10. No exercises. The Russians call this topic "the theory of mass service."

APPLICATIONS, T(13, 1), S. *Mathematics in Architecture.* By Mario Salvadori. P-H, 1968. 187 pp. $7.95. Traditional elementary mathematics through elementary calculus for those who are "afraid of mathematics, who would like to learn how to use it for purposes of architecture and who do not want to spend more than a few weeks...". Good, but in addition to these trivial bits of mathematical technology, the architect needs some acquaintance with the big ideas of mathematics that relate it to art and spacial conceptions, symmetry for example.

BIOLOGY, S(15), P, L. *Some Mathematical Problems in Biology.* By Murray Gerstenhaber, Egbert R. Leigh, Richard C. Lewontin, and Theodosios Pavlidis. Proceedings of the First Symposium on Mathematical Biology sponsored by the AMS and SIAM in 1966. Am Math, 1968. 122 pp. $6.10. The first essay begins with a history of mathematical ecology, which stems from the work of Lotka and Volterra.

BUSINESS MATH, T(13), S. *Mathematics for Decision Making. A Programmed Basic Text. Vol. 1: Linear Mathematics. Vol. 2: Calculus.* By E. Wainwright Martin, Jr. Irwin, 1969. 688 pp. $10. 472 pp. $10. Programmed presentation (linear sequences of easy questions and answers, conventional exposition, outlines, problems, and tests) of the most important topics for future business and government decision makers.

CALCULUS, T(13; 1). *Calculus and Analytic Geometry with Applications.* By Robert Breusch, with additional material by C. Stanley Ogilvy. Prindle, 1969. 282 pp. $8.50. The senior author wrote the first version for a combined physics-calculus course for freshmen. The "additional material" consists of applications to the social sciences.

CALCULUS, T(13, 1). *Introduction to Calculus and Analytic Geometry. Vol. 2.* By Robert Breusch. Prindle, 1969. 330 pp. $6.95. For the second semester of calculus, reaching to multiple integrals and differential equations.

CALCULUS, PROBLEM BOOK. *Problems in Mathematical Analysis.* Edited by B. Demidovich. Translated from the Russian by G. Yankovsky. Gordon, 1968? 496 pp. $29.50. An overpriced collection of routine problems on traditional calculus.

CALCULUS, T(13-14). *Analytic Geometry and the Calculus. 2nd ed.* By A. W. Goodman. Macmillan, 1969. 841 pp. $12.95. The author believes that rigor should be tempered by the needs of the student. There is a nice example of an incomplete, but understandable, statement of a theorem compared with completely rigorous, but almost unreadable, formulation.

CALCULUS, T(13; 2; SOCIAL, BIOLOGICAL SCIENCES). *Introduction to Calculus.* By Vincent O. McBrien. Appleton, 1969. 313 pp. $7.95. Assumes only three years high school mathematics and has chapters on coordinate geometry and circular functions.

CALCULUS, T(13; 1), S. *A Preliminary Course in Analysis.* By R. M. F. Moss and G. T. Roberts. Chapman Hall, 1968. Distributed by B & N New York. 225 pp. $5.75 (paper). Differentiation is based on continuity, and the concept of limit is postponed until sequences. Exercises.

CALCULUS, T(13; 2-3). *Calculus with Analytic Geometry.* By Paul K. Rees and Fred W. Sparks. McGraw, 1969. 629 pp. $10.95.

CALCULUS, T(13-14). *Calculus with Analytic Geometry.* By William K. Smith. Macmillan, 1969. 926 pp. $12.95.

CALCULUS, T(13). *Calculus, with Analytic Geometry. Functions of one Variable.* By Angus E. Taylor and Charles J. A. Halberg, Jr. P-H, 1969. 950 pp. $12.95. Emphasis is on problem solving. A second volume is planned.

CALCULUS, T(13-14; 2-3). *Calculus for the Social and Natural Sciences.* By Bevan K. Youse and Ashford W. Stalnaker. Intl Textbk, 1969. 490 pp. $8.50. Chapter on probability.

COMPLEX ANALYSIS, L. *Complex Numbers and Elementary Complex Functions.* By F. M. Hawkins and J. Q. Hawkins. Gordon, 1968. 153 pp. $12.80. A leisurely but rigorous development of complex numbers and elementary functions, not including the calculus of complex functions. But for the ridiculous price, it might be a useful supplement in analysis courses.

COMPLEX ANALYSIS, T(16-17). *Elements of Complex Analysis.* By John D. Depree and Charles C. Oehring. A-W, 1969. 399 pp. $10.95. Last four chapters are on geometric function theory, harmonic functions, entire functions, analytic continuation.

COMPUTER, T(14-15), S, P, L. *The Art of Computer Programming. Vol. 2: Seminumerical Algorithms.* By Donald E. Knuth. A-W, 1969. 635 pp. $18.50. Continues the encyclopaedic, but lively and readable, effort of volume one (Tel. Rev. Oct. 1968). Two chapters (random numbers, arithmetic), about 650 graded exercises, and three appendices: MIX, tables of numerical quantities, index to notations. The author says that he has found the chapters suitable for texts in elementary probability and number theory.

COMPUTERS, T. *Machine, Assembly, and Systems Programming for the IBM 360.* By William H. Payne. Har-Row, 1969. 332 pp. $5.95 (paper). For a second semester course on programming.

COMPUTER, S, P, *L. *Programming Languages: History and Fundamentals.* By Jean E. Sammet. P-H, 1969. 815 pp. $18. ($13.50 text). A comprehensive reference work describing over 100 computer languages with examples, extensive selected *bibliography*, and several indexes.

COMPUTERS, P, L. *Advances in Information Systems Science. Vol. 1.* Edited by Julius T. Tou. Plenum Pub, 1969. 318 pp. $14. This first in a planned series of annual review volumes contains five papers: *Theory of Algorithms and Discrete Processors* by V. M. Glushkov and A. A. Letichevskii. *Programming Languages* by Alfonso Carracciolo di Forino. *Formula Manipulation—The User's Point of View* by M. E. Engeli. *Engineering Principles of Pattern Recognition* by Julius T. Tou. *Learning Control Systems* by K. S. Fu.

COMPUTERS, T, S, P, L. *Programming Languages, Information Structures, and Machine Organization.* By Peter Wegner. McGraw, 1968. 421 pp. $10.95. On the basis of the unifying concept of information structures many topics are considered.

DIFFERENCE EQUATIONS, T(16-17). *Linear Difference Equations.* By Kenneth S. Miller. W. A. Benjamin, 1968. 115 pp. $12.50 (cloth) $4.95 (paper). The locale is vector spaces and the tool matrix equations.

DIFFERENCE EQUATIONS, T(14-15; 1-2). *Elementary Differential Equations and Boundary Value Problems. 2nd ed.* By William E. Boyce and

Richard C. DiPrima. Wiley, 1969. 547 pp. $10.95. Major revisions: the use of matrices in dealing with systems (preceded by an introduction for those who have not studied linear algebra), a chapter on stability theory.

DIFFERENTIAL EQUATIONS, T(18), P, L. *Abstract Methods in Partial Differential Equations.* By Robert W. Carroll. Har-Row, 1969. 383 pp. $14.95. "...textbook and guide to the literature about certain aspects of PDE". Main topics: elliptic theory, evolution equations, and global analysis. Exercises, *bibliography.*

DIFFERENTIAL EQUATIONS, P, L. *Linear and Quasilinear Equations of Parabolic Type.* By O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Translations of Math. Mon. 23.* Am Math, 1968. 659 pp. $34.20. Solvability of boundary value problems and the connection between the smoothness of solutions and of the known functions.

DIFFERENTIAL EQUATIONS, APPLICATIONS, T(17-18), P, L. *Differential and Integral Inequalities. Theory and Applications. Vol. I: Ordinary Differential Equations.* By V. Lakshmikantham and S. Leela. Academic, 1969. 399 pp. $18.50. Research monograph, guide to the literature, and textbook. This volume contains four untitled chapters on ordinary differential equations and one on Volterra integral equations. The second volume will deal with time lag, partial differential equations, differential equations in abstract spaces, and complex differential equations. *Bibliography.*

DIFFERENTIAL GEOMETRY, T(17-18), S, P. *Symmetric Spaces.* By Ottmar Loos. Vol. I: General Theory. Vol. II: Compact Spaces and Classification. W. A. Benjamin, 1969. 198 pp. 190 pp. Each volume $12.50 (cloth), $3.95 (paper). Notes from a graduate course on topics in advanced differential geometry given at Minnesota in 1967-1968. *Bibliographies* and indexes but no problems.

DIFFERENTIAL GEOMETRY T(15-16), P, L. *Differential Geometry.* By J. J. Stoker. Wiley, 1969. 425 pp. $14.95. Presupposing only elementary linear algebra and calculus, the author intends rather thorough treatment of differential geometry from a contemporary point of view.

EDUCATION, *P, *TT, *L. *An Outline of Piaget's Developmental Psychology for Students and Teachers.* By Ruth M. Beard. Basic, 1969. 155 pp. $4.95. Jean Piaget has told us more than anyone else about the way children see mathematical concepts.

EDUCATION *P, TT. *Freedom to Learn. An Active Learning Approach to Mathematics.* By Edith E. Biggs and James R. MacLean. A-W, 1969. 205 pp. $6.95. This very interesting book describes teaching procedures based on the suggestion of Z. P. Dienes "that we shift the emphasis from teaching to learning, from our experience to the childrens', in fact from our world to their world." Fine photographs and illustrations.

EDUCATION, TT, S. *Laboratory Manual for Elementary Mathematics.* By William M. Fitzgerald, David P. Bellamy, Paul H. Boonstra, John W. Jones, and William J. Oosse. Prindle, 1969. 157 pp. $3.95 (paper) Developed at Michigan State University for use in weekly labs associated with a required course for future elementary teachers, the manual has 16 units covering a wide variety of topics. (See *The Arithmetic Teacher,* Oct. 1968).

EDUCATION, COMPUTERS, S(TT), P, L. *Computer-Assisted Instruction and the Teaching of Mathematics. Proceedings of a National Conference*

on Computer-Assisted Instruction Conducted at Pennsylvania State University, September 24-26, 1968. NCTM, Washington, D.C. 1969. 158 pp. $2. (paper). The editor, R. T. Heimer, describes it as "an up-to-date compendium of thoughts on computer-assisted instruction as expressed by a collection of the most experienced and knowledgeable people to be found."

EDUCATION, NEW ABSTRACTING JOURNAL, P, *L. *Investigations in Mathematics Education. A Journal of Abstracts and Annotations.* SMSG, Stanford Univ. Vol. I, 1969. The intention is to give abstracts and commentary on all research in the field of mathematical education, and listings with brief annotations of many other related publications. The first issue, described as "but a first small approximation," contains 16 abstracts with comment. Since *Mathematical Reviews* ignores mathematical education almost completely, this journal fills an important gap in the field of mathematical communication and scholarship. (There are no subscriptions. Requests for inclusion on the distribution list should be addressed to SMSG, Cedar Hall, Stanford, Calif., 94305).

*EDUCATION, S, P, *L. *Journal of Undergraduate Mathematics.* Published by the Department of Mathematics, Guilford College Greensboro, N.C. 27410. Vol. I, No. I, March, 1969. "...to provide an outlet for significant research in mathematics done by undergraduates and ...a source of topics for such research. Undergraduate research papers...should be the original work of students, but need not be original to mathematics. Proposals...are welcomed from any source." This first issue, in an attractive format with 52 pages, contains 6 undergraduate articles and several research proposals.

EDUCATION, COMPUTERS, TT, S, P. *Problem-Solving with the Computer.* By Edwin R. Sage. Entelek, 1969. 255 pp. $3.95. Designed for use in the high school where the computer is used as a classroom auxiliary tool for calculation. Suggestive of possible college uses.

ENGINEERING, P. *Skeletal Structures. Matrix Methods of Linear Structural Analysis Using Influence Coefficients.* By C. M. Bommer and D. A. Symonds. Printed in Hungary by Egyetemi Printing House, 1968. Distributed by Gordon and Breach. 106 pp. $9.50. Many detailed examples.

FOUNDATIONS, *P, *L. *Formalized Recursive Functionals and Formalized Realizability.* Memoirs AMS. 89, Am Math, 1969. 106 pp. $2.60.

FOUNDATIONS, T(16-17). *Introduction to Set Theory.* By J. Donald Monk. McGraw, 1969. 202 pp. $10.95. "...self contained introduction to all the set theory needed by most mathematicians." Axiomatic but based on intuitive logic. The main thrust is on the theory of cardinals.

FOUNDATIONS, T(13), S(13), TT. *Sets—Relations—Functions. 2nd ed.* By Samuel Selby and Leonard Sweet. McGraw, 1969. 392 pp. $6.95 (cloth), $4.95 (paper). The last chapter of 67 pages describes various structures.

*GENERAL, S, P, L. *Mathematics: Its Content, Methods, and Meaning.* Edited by A. D. Alexsandrov, A. N. Kolmogorov, and M. A. Lavrent'ev. Translated by S. H. Gould, K. A. Hirsch and T. Bartha. M.I.T. Pr, 1969. Three volumes. Vol. I. 387 pp. Vol. II. 405 pp. Vol. III. 383 pp. $10 (three paperback volumes boxed). The original hard cover edition, whose price is $30, was published by the American Mathematical Society in 1963 (reprinted 1965). Though perhaps a

little old fashioned and inclined to over-emphasize manipulations and calculations, this is one of the best surveys of mathematics ever written.

GENERAL, S, P, L. *1969 Britannica Book of the Year.* Encyclopaedia Britannica, Chicago, 1969. 896 pp. $6.95. Irving Kaplansky reviews selected dramatic achievements in 1968. He discusses the discovery of six new finite simple groups (giving a table of the fourteen known non-classical finite simple groups) and mentions progress with respect to the four colour problem, game theory, homological algebra, and functional analysis.

GENERAL, P, L. *First Employment of Ph.D's in the Mathematical Sciences: 1966-67, 1967-68, 1968-69.* A survey by the Conference Board of the Mathematical Sciences. April, 1969. Copies on request from the CBMS. 37 pp. Gives the Ph.D. origins of people hired by each employing institution as well as overall data on input and output of new Ph.D's by states for the academic years ending 1967 through 1969.

GENERAL, T(13; 1-2). *A Survey of Finite Mathematics.* By Marvin Marcus. HM, 1969. 496 pp. $9.50. Aimed at the course now often entitled "finite mathematics", devotes about a third to logic, sets, etc... through probability, a third to linear algebra and a third to convexity and applications to linear programming, game theory and Markov chains.

GENERAL, S(13). *Modern Algebra.* By Kaj L. Nielsen. College Outline Series. B & N, 1969. 288 pp. $1.75. Keyed to 27 freshman texts on algebra and general mathematics.

GENETICS, T, S, P, *L. *Population Genetics.* By W. J. Ewens. Methuen, London, 1968. Distributed by Barnes and Noble. 158 pp. $5. A treatment of the mathematical theory of population genetics by a mathematician. Begins with the Hardy-Weinberg law (Yes, G. H. Hardy, 1908, *Science,* Vol. 28, 49-50). *Bibliography.*

GEOMETRY, TT, T(13), S, *P, *L. *Geometry in a Modern Setting.* By Gustave Choquet. Hermann, Paris and HM, 1969. 142 pp. $7.50. The word "modern" means here algebraic, axiomatic, and Bourbakian.

GEOMETRY, GRAPHICS, S(13), P, *L. *Four-Dimensional Space.* By Ludwig Eckhart. Translated by Arthur L. Bigelow and Steve M. Slaby. U Ind Pr, 1968. 90 pp. $6.75. Presupposing only high school mathematics, the author uses graphics (descriptive geometry) to give a representation of four-dimensional space.

GEOMETRY, T(13; 1-2), TT. *An Intuitive Approach to Elementary Geometry.* By Beauregard Stubblefield. Brooks-Cole, 1969. 265 pp. $7.95. An informal, imaginative, elementary exposition in line with the Level 1 recommendations of CUPM, but useful also for remedial work. Topics include non-euclidean geometry, topological notions, measure, vectors, and axiomatics.

GEOMETRY, T(13), TT. *Elementary Geometry for College.* By Charles W. Tryon. HB & W, 1969. 304 pp. $7.95. Plane and solid synthetic Euclidean geometry based on modern concepts, including the number line.

GEOMETRY, TT. *Modern Coordinate Geometry.* A Wesleyan experimental curricular study. Supported by NSF. HM, 1969. 460 pp. $6.40. An effort to fuse algebra and geometry by means of a few powerful axioms and to present the results in a course suitable for high

school honors classes.

GEOMETRY, T(13), *S, L. *Geometric Transformations II.* By I. M. Yaglom. Translated from the Russian by Allen Shields. *New Mathematical Library 21.* Random, 1969. 197 pp. $1.95 (paper). Part I (*New Mathematical Library 8,* 1962) was devoted to isometries. This volume deals with similarities. Part III will be on affine and projective transformations.

GRAPHING, *S(13). *Functions and Graphs.* By I. M. Gelfand, E. G. Glagoleva, and E. E. Shnol. Translated from the 2nd Russian edition by Thomas Walsh and Randell Magee. M.I.T Pr, 1969. 110 pp. $6. (cloth), $1.95 (paper). This is volume 2 of the *Library of School Mathematics,* translated under a grant from the NSF by the Survey of East European Mathematical Literature, directed by Izaak Wirszup at the University of Chicago.

GROUPS, P. *Theory of Finite Groups. A Symposium.* Edited by Richard Brauer and Chih-Han Sah. W. A. Benjamin, 1969. 279 pp. $12.50. Three parts: Characterizations of old and new simple groups, representation theory, miscellaneous topics.

GROUP THEORY, APPLICATIONS, S, P, L. *How to use Groups.* By J. W. Leech, and D. J. Newman. Methuen, London, 1969. Distributed by Barnes and Noble (USA), Methuen, Toronto (Canada). 133 pp. $5.25 (cloth), $3.50 (paper). By physicists primarily for physicists, for independent study or use in advanced undergraduate courses. "Group theory...it's virtue is not that it leads to new results but that it minimizes calculation...by incorporating from the beginning the symmetry characteristics of the system..."

GROUPS, PHYSICS, S(17), P, L. *Group Theory and Its Applications.* Edited by Ernest M. Loebl. Academic, 1968. 722 pp. $19.50. Fifteen authors from five countries. Interesting titles, e.g., "Projective Representation of the Poincare Group in Quaternionic Hilbert Space."

GROUPS, P, *L. *Matrix Representations of Groups.* By Morris Newman. Applied Math. Ser. 60. Nat. Bur. Stan. USGPO, Washington, D.C. 60¢. A handy reference and "simple but complete exposition."

HILBERT SPACE, OPERATORS. *Linear Operators in Hilbert Space.* By J. L. Soule. Gordon, 1968. 40 pp. $3.50 (paper). Intended as a supplementary introduction. No problems, index, novelty. Reviewed by C. R. Putnam, *American Scientist,* Spring, 1969, p. 81A.

HISTORY, RUMANIA, *L. *Istoria Matematicii în România.* By George St. Andonie. 3 volumes. Editura Stiintifica, Bucarest, 1965-1967. 414, 470, 515 pp. A careful, authoritative work covering from the earliest times to 1966 with bibliographies, biographies, portraits and thorough indexes. If all countries had such histories, we would have a much better picture of mathematics.

HISTORY, S, P, *L. *The Origins of the Infinitesimal Calculus.* By Margaret E. Baron. Pergamon, 1969. 312 pp. $13. A detailed documented history from Greek times to Newton and Leibniz, who are discussed in a brief epilogue. *Bibliography.*

HISTORY, T(14), TT. *An Introduction to the History of Mathematics.* 3rd ed. By Howard Eves. HR & W, 1969. 479 pp. $9.95. Only 51 pages deal with mathematics since 1700 and the statement that history prior to 1700 covers "elementary mathematics in the form that we have it today" is no longer true, although it was approximately correct

when the first edition was published (1953).

HISTORY, P, *L. *Collected Papers of G. H. Hardy, Including joint Papers with J. E. Littlewood and Others.* Edited by a committee appointed by the London Math. Soc. Vol. II. Oxford U Pr, 1967. 702 pp. $16.75. This second of seven volumes (Vol. I. Teleg. Rev. May 1967) contains papers on multiplicative number theory (including the zeta function,) other number theory, and inequalities.

HISTORY, P, *L. *The Mathematical Papers of Isaac Newton. Vol. III. 1670-1673.* Edited by D. T. Whiteside with the assistance in publication of M. A. Hoskin and A. Prag. Cambridge U Pr, 1969. 613 pp. $32.50. Continuing at the high standard of scholarship and printing established in the first two volumes (Teleg. Rev. Nov. 1967, Dec. 1968), this volume contains Newton's elaborate tract on infinite series and fluxions, including a hitherto unpublished appendix, papers on integration of algebraic functions, short texts dealing with geometry and harmonic motion, mathematical excerpts from his notes on light and the theory of lenses, and an appendix summarizing mathematical highlights in his correspondence of the time. (See the eloquent review by D. J. Struik in *Science* 8 Aug. 1969, p. 578.)

HISTORY, S, P, *L. *The History of the Abacus.* By J. M. Pullan. Praeger, 1969. 140 pp. $4.95. This first book on the Abacus since the one by F. P. Barnard in 1916 is based in part on archaeological findings, medieval written sources, and early arithmetic textbooks. Beautifully illustrated. Extensive *bibliography*.

LINEAR ALGEBRA, T(13-14; 1). *Applied Linear Algebra.* By Ben Noble. P-H, 1969. 539 pp. $9.95. Matrix algebra, finite dimensional vector spaces, eigenvalues with emphasis on applications, numerical aspects, concrete motivation. The author is known for his *Applications of Undergraduate Mathematics in Engineering* (Macmillan 1967). A candidate for the semester devoted to linear algebra in the calculus sequence.

LINGUISTICS, P, L. *Mathematical Linguistics in Eastern Europe.* By Ferenc Kiefer. Am Elsevier, 1968. 188 pp. $12.50. A "critical review of East European contributions."

LOGIC, T(13; 1). *The Grammar of Mathematics.* By Lincoln K. Durst. A-W, 1969. 178 pp. $6.95. An informal treatment of logic and proof intended to provide a bridge to mathematics courses in which proof is important.

LOGIC, T(14). *An Introduction to Mathematical Logic.* By Gerson B. Robison. P-H, 1969. 223 pp. $5.95. For math majors. Concentration on first-order predicate calculus and proof theory.

MATRIX THEORY, APPLICATIONS, T(14-15), S, P, L. *Theory of Matrices.* By Peter Lancaster. Academic, 1969. 328 pp. $11. Primarily for applications in engineering or science. Besides the usual introduction, includes topics unusual at this level: functions of matrices, norms of vectors and matrices, perturbation theory and bounds for eigenvalues, direct products, solutions of matrix equations, stability problems, non-negative matrices.

*NUMBER THEORY, S(16-18), P, L. *Studies in Number Theory.* Edited by W. J. LeVeque. *Studies in Mathematics Vol. 6.* Published by MAA. Distributed by Prentice-Hall, 1969. 212 pp. $6. (Members of MAA, one copy at $3 from the Washington Office). Intended to "illustrate the remarkable breadth both of the subject itself and of the array of other mathematical theories that have been successfully brought

to bear on arithmetical questions," this volume contains the following five papers: *A Brief Survey of Diophantine Equations* by W. J. LeVeque. *Diophantine Equations, p-adic Methods* by D. J. Lewis. *Diophantint Decision Problems* by Julia Robinson. *Computer Technology Applied to the Theory of Numbers.* by D. H. Lehmer. *Asymptotic Distribution of Beurling's Generalized Prime Numbers* by P. T. Bateman and H. G. Diamond. All papers are self-contained in the sense that they require only rudimentary acquaintance with number theoretic ideas and advanced undergraduate mathematics. Each is followed by a bibliography and suggestions for further exploration. The volume is a worthy addition to a series that have become a most useful source for those who desire to "keep with it".

NUMERICAL ANALYSIS, APPROXIMATION, T(15-16; 1), S. *An Introduction to the Approximation of Functions.* By Theodore J. Rivlin. Blaisdell, 1969. 158 pp. $7.50. Approximation of continuous functions by functions that depend on a finite number of parameters. Prerequisites: linear algebra and advanced calculus. Informal and elementary. Chapters are uniform approximation, least-squares approximation, least-first-power approximation, polynomial and spline interpolation, and approximation and interpolation by rational functions.

OPERATIONS RESEARCH, T(16), P. *System Analysis Techniques.* By Ralph Deutsch. P-H, 1969. 488 pp. $13.50. Introductory text and reference works for engineers. *Bibliography* (132 items).

OPERATIONS RESEARCH, S, P, L. *Boolean Methods in Operations Research and Related Areas.* By Peter L. Hammer. Preface by Richard Bellman. Springer-Verlag, 1968. 344 pp. $11.50. The first comprehensive monograph using as its main tool pseudo-Boolean functions (real valued function of bivalent variables). Knowledge of Boolean algebra not assumed. New results. *Bibliography.*

OPERATIONS RESEARCH, P, *L. *The Proceedings of the Fourth International Conference on Operational Research.* Edited by David B. Hertz and Jacques Melese. Publications in Operations Research of the Op. Res. Soc. of America 14. Wiley, 1966, (actually published in 1969) 1128 pp. $24.95. The conference, organized by the International Federation of Operational Research Societies, was held in Boston in 1966.

OPTIMIZATION, P, L. *Mathematics of Adaptive Control Processes.* By Sidney J. Yakowitz. Am Elsevier, 1969. 173 pp. $11. The author brings together information theory, sequential decision theory, and dynamic programming to present "a rigorous, unified, and inclusive systems theory for multi-stage decision processes". Only general probability theory assumed. The name goes back to Richard Bellman's work on *Two-armed Bandits.*

PHYSICS, T(16), P, L. *Basic Equations and Special Functions of Mathematical Physics.* By V. Ya. Arsenin. Trans. by S. Chomet. Transl. editor: S. Doniach. Am Elsevier, 1968. 361 pp. $13.50. Formulas and methods.

PROBABILITY, *T(13). *The Elements of Probability.* By Simeon M. Berman. A-W, 1969. 237 pp. $6.50. Requires only high school algebra. Since "the student benefits more from a few profundities than from a lot of trivia," the author presents in an elementary way the many ideas that are usually postponed to advanced courses. Coin tossing is the main model.

PROBABILITY, S(17), P, L. *Theory of Random Functions.* By Blanc-Lapierre and R. Fortet. Translated from French by J. Gani. Two volumes. Gordon, 1965, 1967. 464 pp. $29.50 ($14.50 to prof.) 344 pp. $19.50 ($9.50 to prof.). Although there are now a number of books in the field, this ten year old treatise is notable for its interplay of theory with physical applications.

PROBABILITY, T(15). *Probability and Stochastic Processes: With a View Toward Applications.* By Leo Breiman. HM, 1969. 336 pp. $9.50. Assumes two years of calculus. Emphasis is on translating a physical situation into a probability model and on "why and how" and "how to apply" rather than proof. Topics include the continuous time Markov processes, vector independence, multivariate normal distribution, stationary time series. The book contains some informal proofs, but unfortunately the author also uses the word "proof" to describe verification by special cases or numerical calculation (See page 102).

PROBABILITY, T(13; 1 OR 16; 1). *Introduction to Mathematical Probability Theory.* By Martin Eisen. P-H, 1969. 556 pp. $12.95. The first three chapters are hitched to those "with no prerequisites other than mathematical ability." The next two require some calculus. From chapter six, advanced calculus is prerequisite, and topics are measure theory, integration, distributions, characteristic functions, independent random variables, and limit theory.

PROBABILITY, STATISTICS, APPLICATIONS; T(AFTER CALCULUS), P, L. *Mathematical Methods of Reliability Theory.* By B. V. Gnedenko, Yu. K. Belyayev, and A. D. Solovyev. Trans. by Scripta Technica. Transl. editor Richard E. Barlow. Academic, 1969. 517 pp. $24.50. Broad coverage. *Bibliographies.* For practicing engineers, statisticians and graduate students in the field.

PROBABILITY, T(14; 1). *Probability Theory.* By Henry E. Kyburg, Jr., P-H, 1969. 304 pp. $10.95. Designed "primarily for the future critical consumer of statistics" with some calculus, this book ties probability with measure concepts. Three chapters on statistics. Presentation is limited to subjective approaches, especially the Bayesian and Fisherian.

PROBABILITY, STATISTICS, T(14-15; 1-2). *Introduction to Probability Theory and Statistical Inference.* By Harold J. Larson. Wiley, 1969. 398 pp. $10.95. "...a more rigorous (but not more difficult) introduction...than is commonly available..." Topics include estimation, tests of hypotheses, Bayesian methods, least squares, regression theory.

PROBABILITY, NUMERICAL ANALYSIS, P, *L. *Stochastic Approximation.* By M. T. Wasan. Cambridge U Pr, 1969. 212 pp. $9.50. A rigorous treatment (plus examples of applications) that draws together a relatively new field. Prerequisites are summarized in three appendices, which presume some knowledge of probability theory and numerical analysis.

PROGRAMMING, S(16-17), P, L. *Dynamic Programming.* By D. J. White. Oliver & Boyd, Edinburgh, 1969, and Holden-Day, San Francisco. 187 pp. $10.50. Assumes knowledge of DP and concentrates on analysis of pros and cons.

PROGRAMMING, T(16-17), P, L. *Nonlinear Programming: A Unified Approach.* By Willard I. Zangwill. P-H, 1969. 372 pp. $12.50. Assumes advanced calculus, linear algebra, and introduction to linear programming. Included are the most important topics of current inter-

est, latest research results of the author and others, a variety of applications, many exercises developing topics not included, and a 15 page *bibliography*.

RELATIVITY, P, L, *Einstein Spaces*. By A. Z. Petrov. Translated by R. F. Kelleher. Translation edited by J. Woodrow. Pergamon, 1969. 424 pp. $12. An exposition of the mathematical basis of the general relativistic theory of gravitation, using invariant methods. (An Einstein space is one whose Ricci curvature tensor is proportional to its metric tensor). Prerequisite only calculus. The first chapter is on basic tensor analysis. *Bibliography* (509 items in chronological order).

REMEDIAL, T(13; 1), S(13), *Contemporary Algebra and Trigonometry*. By Walter A. Albrecht, and Francis J. Mueller. Dickenson, 1969. 149 pp. $8.95. For students with very weak background.

REMEDIAL, *Trigonometry*. By Edward B. Anders. Merrill, 1969. 332 pp. $7.50.

REMEDIAL, *Essentials of Trigonometry*. By E. Allan Davis and Jean J. Pedersen. Prindle, 1969. 247 pp. $7.50. Traditional topics with emphasis on functions and only 8 pages on triangle solution.

REMEDIAL, *College Arithmetic*. By Steven J. Bryant, Leon Nower, Daniel Saltz. Glencoe, 1969. 374 pp. $7.50. Uses number line, Newton's method for square root extraction.

REMEDIAL, *Elementary Algebra*. By Steven J. Bryant, Leon Nower, Daniel Saltz. Glencoe, 1969. 312 pp. $6.50. Topics include: lines and linear systems, functions and their graphs, families of curves, proportion and variation, sets and functions. The preface ends with the logically interesting quotation (given with approval but without indication of source): "mathematics is either good mathematics, or it is not mathematics at all."

REMEDIAL, *Intermediate Algebra*. By Steven J. Bryant, Leon Nower, Daniel Saltz. Glencoe, 1968. 351 pp. $6.95. Topics include real number system (nothing on the logical structures and no definition or discussion of the nature of real numbers), matrices, set theory, probability, linear programming.

REMEDIAL, *Basic Technical Mathematics*. By Thomas C. Crooks and Harry L. Hancock. Macmillan, 1969. 480 pp. $8.95. To eighth grade level with no mention of modern mathematical technology.

REMEDIAL, *Algebra*. By Frank J. Fleming. HB & W, 1969. 500 pp. $8.95. What used to be called "intermediate algebra," beginning with the real numbers and "first degree open sentences" and ending with rational expressions, systems of equations and inequalities, complex numbers and vectors, matrices, exponential and logarithmic functions, sequences.

REMEDIAL, *Geometry and its Methods*. By John N. Fujii. Wiley, 1969. 379 pp. $8.95. Plane and solid classical Euclidean geometry for students with a year of elementary high school algebra.

REMEDIAL, *Elementary Algebra*. By Alan R. Hoffer and Gary L. Musser. Prindle, 1969. 294 pp. $6.95.

REMEDIAL, *S(13), *Trigonometry. A Programmed Text*. By Mervin L. Keedy and Marvin L. Bittinger. HR & W, 1969. 270 pp. $5.95. Mainly analytic trigonometry, using conventional exposition and exercises as

well as programmed sequences. Used as a supplement or for independent study, it could save much teaching time.

REMEDIAL. *Elementary College Arithmetic.* By David A. Ledbetter. Goodyear, 1969. 279 pp. $7.95. From sets through the usual arithmetic topics to final chapters on measurements and real numbers, using the number line and algebraic concepts.

REMEDIAL, T(13; 1). *Algebra: An Intermediate Approach.* By Florence M. Lovaglia, Merritt A. Elmore and Donald Conway. Har-Row, 1969. $8.95. From sets, logic, and real numbers as a field to complex numbers, polynomials, exponential and logarithmic functions, matrix algebra, determinants and sequences.

REMEDIAL, T(13). *Elementary Functions.* By Thomas K. Maddox and Lawrence H. Davis. P-H, 1969. 280 pp. $8.95. Two years of high school mathematics assumed. Coordinates, the function concept (a function is a rule together with a domain and a range), linear, absolute value, quadratic, exponential, periodic (mostly trigonometric) functions, composition, inverse, restrictions and extensions.

REMEDIAL. *Beginning Algebra.* By John H. Minnick and Raymond C. Strauss. P-H, 1969. 389 pp. $7.95. To quadratics through such topics as "first degree sentences", "expressions with two variables".

REMEDIAL, S. *Elements of Algebra. A Worktext. Revised Edition.* By Jon M. Plachy and Orason L. Brinker. Prindle, 1969. 240 pp. $4.95 (paper). Brief expositions followed by tear-out work sheets. Manipulations only. To quadratics and inequalities.

REMEDIAL. *Mathematics. An Introduction.* By Charles N. Podraza, Larry L. Blevins, Arlys W. Hanson and Harry C. Prall. Goodyear, 1969. 265 pp. $8.50. Arithmetic, assuming no previous algebra and starting with the set concept, followed by a short introduction to informal geometry.

REMEDIAL, T, *S. *Fundamentals of Trigonometry.* By Earl W. Swokowski. Prindle, 1969. 219 pp. $7.50. *A Programmed Supplement to Fundamentals of Trigonometry.* By Roy A. Dobyns. Prindle, 1969. 172 pp. $2.50.

REMEDIAL, T, *S. *Arithmetic: A First Course in Mathematics.* By Margaret F. Willerding. Prindle, 1969. 250 pp. $6.95. Accompanied by a partially programmed workbook entitled *Arithmetic Worktext.* (297 pp. $1.95). The usual topics of elementary arithmetic ending with percent and measurement.

SET THEORY, P, L. *Théorie Axiomatique des Ensembles.* By Jean-Louis Krivine. Presses Univ. Paris, 1969. 120 pp. $2. An introduction to the results of P. Cohen on the independence of the axioms of choice and the continuum hypotheses as well as to more recent work, this little treatise assumes previous acquaintance with set theory and logic. Uses Zermello-Fraenkel axioms and model theory.

SOCIAL SCIENCES, T(13). *Mathematics for the Social and Behavioral Sciences. Probability, Calculus and Statistics.* By Bernard R. Gelbaum and James G. March. Saunders, 1969. 349 pp. $8.75. Since students of the social and behavioral sciences must know a substantial amount of mathematics, it is desirable that competent mathematicians participate, as in this case, in the writing of appropriate special materials. This volume, presupposing only intermediate high school algebra, starts with sample spaces, basic probability, count-

ing, conditional probability and random variables. Then it takes up limits, differentiation, integration before returning to countable sample spaces, continuous random variables, estimation, and distribution of estimates. A companion volume is projected to cover linear algebra, difference and differential equations, special problems. The book draws on the recommendation of the Committee on the Undergraduate Program of the Mathematical Association of America (not the American Mathematical Society as is stated in the preface) and the experience in teaching a two year sequence. Style is informal with numerous "experiments," examples, and exercises.

SOCIAL SCIENCES, S, L. *Mathematical Thinking in Behavioral Sciences. Readings from the Scientific American.* Edited by David M. Messick W. H. Freeman, 1968. 231 pp. $10 (cloth), $4.95 (paper). Fine expositions with introductions, biographies of the authors, additional bibliography and an index, should be very useful as a supplement in courses in the behavioral sciences, in general education courses in mathematics, in mathematics courses for social scientists, and for the mathematics student who wants a broader picture than he can get in specialized courses.

STATISTICS, T(13; 1), L. *Basic Statistics.* By David Blackwell. McGraw, 1969. 148 pp. $5.50. An "intuitive, informal, concrete, decision-theoretic, and Bayesian" introduction for students with varied interests and very modest background (hardly more than arithmetic, a bit of algebra and graphing). Brevity is achieved by limiting the exposition rather than by reducing the number of topics, worked examples, or problems. The distinction of the author should tempt the expert to find out how he treats particular topics.

STATISTICS, T(13). *Statistical Methods for Decision Making.* By William A. Chance. Irwin, 1969. 453 pp. $9.50. For future executives. Emphasis is on applications "rather than the theoretical or mathematical aspects of the methods."

STATISTICS, T(13; 2). *Statistics: Methods and Analyses.* By Lincoln L. Chao. McGraw, 1969. 523 pp. $11.50. "Although essentially not mathematical," the text goes from sets, relations, and basic probability to such topics as F-distributions, correlation, time series, Bayesian decision theory and non-parametric methods.

STATISTICS. *Statistics for Experimentalists.* By B. E. Cooper. Pergamon, 1969. 345 pp. $9. A handbook assuming little mathematics and arranged according to type of experiment.

STATISTICS, T(13; 1-2). *Introduction to Statistical Analysis.* By Wilfrid J. Dixon and Frank J. Massey. McGraw, 1969. 646 pp. $10.50. This well-seasoned (first edition 1951, second 1957) comprehensive text for students with minimal mathematical competence is here revised and updated in many details.

STATISTICS, REFERENCE, P, *L. *Systems of Frequency Curves.* By William Palin Elderton and Norman Lloyd Johnson. Cambridge U Pr, 1969. 216 pp. $10. A substantial revision and updating of the classical *Frequency Curves and Correlation* first published by the senior author in 1906 and in three later editions. The material on correlation has been replaced by new topics.

STATISTICS, T(13; 1). *General Statistics.* By Audrey Haber and Richard P. Runyon. A-W, 1969. 364 pp. $7.50. For non-mathematicians with varying interests. Calculus not assumed. Nearly half on descriptive statistics, but the inferential part includes analysis

of variance and non-parametric methods.

STATISTICS, T(13; 1). *Business Decision Theory*. By Paul Jedamus and Robert Frame. McGraw, 1969. 300 pp. $9.95. Classical and Bayesian inference at the very elementary pre-calculus level.

STATISTICS, T. *Fundamentals of Statistics*. By H. Mulholland and C. R. Jones. Plenum Pub, 1968. 201 pp. $5.95. Fisherian statistics without benefit of Neyman, Pearson, etc.

STATISTICS, P, *L. *A Dictionary and Bibliography of Discrete Distributions*. By Ganapati P. Patil and Sharadchandra W. Joshi. With a foreword by C. Radhakrishna Rao. Published for the International Statistical Institute by Hafner, 1968. 280 pp. $23.95. Describes 115 distributions and lists 3022 titles with full bibliographic information, review location, indexed and coded.

STATISTICS *S, *P, *L. *Statistical Tables*. By F. James Rohlf and Robert R. Sokal. W. H. Freeman, 1969. 264 pp. $7.50 (cloth), $2.75 (paper). Thirty-three tables, mostly computer generated and all reproduced from printout, chosen for the needs of students and researchers in the biological, social and earth sciences, and on the assumption that desk calculators are available. Section on interpolation. Individual explanation and instructions for each table. Up-to-dateness is suggested by the inclusion of tables for computations of square and cubed roots by a desk calculator instead of the tables of the roots themselves. Modest price!

STATISTICS, T(13). *Mathematics in Management. The language of Sets, Statistics and Variables*. By P. Rosenstiehl and J. Mothes. Preface by P. Massé. Translated by A. Silvey. North-Holland, 1968. 408 pp. $12.60. Sets, probability and some statistics with applications, written originally for French college students headed for careers in business and public affairs with the goal of achieving both rigor and close links with applications.

STATISTICS, T(13). *Measuring Uncertainty. An Elementary Introduction to Bayesian Statistics*. By Samuel A. Schmitt. A-W, 1969. 400 pp. $9.75. "It just seemed high time that someone stirred the Bayesian pot on an elementary level so that practitioners, rather than theorists, could start discussions and supply feedback to one another." The word "frequency" does not appear in the index. Probability is an expression of strength of knowledge or belief.

STATISTICS, T(13-14). *Introductory Statistics*. By Thomas H. Wonnacott and Ronald J. Wonnacott. Wiley, 1969. 415 pp. $9.95. An economist and a mathematical statistician have here teamed up to write a mathematical exposition of modern sophisticated statistics at a level between the most elementary treatments and the standard post-calculus texts. Calculus is not essential, but it is used, and presumably the students are expected to have a very solid high-school background or some collegiate mathematics. Essential probability is included. The treatment is classical but there is a discussion of Bayesian methods and decision theory in general (with strong emphasis on regression theory).

STOCHASTIC PROCESSES, T(17-18), P. *Analytical Treatment of One-Dimensional Markov Processes*. By Petr Mandl. Springer-Verlag, 1968. *Grundlehren der Math. Wiss. 151*. 212 pp. $9. Presupposes substantial background in algebra (e.g. semi-groups), functional analysis and probability.

SYSTEMS, T(17-18), S, P, L. *Topics in Mathematical System Theory.* By R. E. Kalman, P. L. Falb, and M. A. Arbib. McGraw, 1969. 372 pp. $16.50. After a general introduction come four independent essays. Topics include elementary control theory, regulators of linear plants, optimal control theory, control system design, automata theory, decomposition for finite automata, algebraic theory of linear systems. The intention is not complete or systematic coverage but rather a stimulating survey, useful to anyone who wishes to find out about and/or work in this field that brings together algebra and analysis in interesting ways.

TOPOLOGY, P, *L. *Topological Papers of Eduard Cech.* Academia, Prague, 1968. 514 pp. All 31 of Cech's topological papers, a few in the original German, the rest translated into French or English. Portrait, biography, *bibliography.*

TOPOLOGY, T(16-17; 1). *Differentiable Manifolds.* By S. T. Hu. HR & W, 1969. 192 pp. $11.50. Chapters are on differentiable manifolds, differential forms, Riemannian manifolds, and de Rham's theorem. Might be used for part of a course in modern differential geometry or in algebraic topology, possibly in troika with Hu's *Holomogy Theory* (1966) and *Cohomology Theory* (1968).

TOPOLOGY, T(16-17), P, L. *Modern General Topology.* By Jun-iti Nagata. *Bibliotheca Mathematica 7.* North-Holland, 1968. Distributed by Wiley, New York. 361 pp. $14.75. To make the book suitable as a text as well as an advanced reference the author has avoided "too much attention to the more abstract spaces" and "abstract description" and chosen "rather more popular than novel" methods. The intention is to select "only the most significant results" from "as many aspects as possible." The author says that the exercises are easy, but he recommends only the first one hundred pages for undergraduates and has inserted some sections later in the book to appeal to the expert. *Bibliography* (11 pages).

TIME SERIES, T(17), S, P, L. *Spectral Analysis and its Applications.* By Gwilym M. Jenkins and Donald G. Watts. Holden-Day, 1968. 543 pp. $18.75. Spectral analysis of time series for engineers and scientists not expert in statistics.

VARIATIONAL THEORY, T(16-17), S, P. *Calculus of Variations with Applications.* By George M. Ewing. Norton, 1969. 355 pp. $10. An introduction, assuming only advanced calculus or introductory real analysis. From classical calculus of variations to recent problems and methods, with increasing demands on student knowledge and sophistication.

VECTORS, APPLICATIONS, T(13-14). *A Course in Vector Analysis.* By L. G. Chambers. Chapman Hall, London, 1969. In the U.S. Barnes and Noble. 238 p. $7.25. In Canada from Methuen, Toronto. $7.65. Algebra and calculus of vectors in three dimensions leading up to line, surface and volume integrals, the divergence theorems, and miscellaneous topics related to vector functions, diadics, vector spaces, tensors, and matrices. Pointed toward physicists and engineers and emphasizing precisely those topics that are underemphasized in "modern" texts in linear algebra and analysis.

VECTOR ANALYSIS, S(14), T(14). *Vector Analysis.* By N. M. Queen. McGraw, 1967. 87 pp. $2.50 (paper). A "concise but self-contained account" presupposing elementary calculus and some knowledge of matrices and determinants, with stress on transformation theory.

1962 and program director for the Undergraduate Research Participation Program since 1966, has been appointed program director for the new program.

A brochure containing suggestions for submission of proposals is available; institutions eligible to submit proposals under the program are four-year colleges and universities that have, or are actively planning, elementary- or secondary-school teacher programs in the sciences.

WAUKESHA MATHEMATICAL SOCIETY

The Spring 1969 issue of DELTA is out containing fascinating articles by Professors Carlitz (Duke University), Wilansky (Lehigh University), Amir-Moéz (Texas Technological College), Maxwell (Queen's College, England), Lightstone (Queen's University, Canada), Starke and a problem section. Yearly subscription is $1.00. Order your copies from Waukesha Mathematical Society, University of Wisconsin, Waukesha, Wisconsin 53186.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

LIFE MEMBERSHIPS

So that members of the Association may be encouraged to maintain their standing in our organization after their retirement, the Board of Governors at its meeting on August 24, 1969, at the University of Oregon has voted to establish Life Memberships on the following terms:

Any member of the Association may become a Life Member after he has passed the age of 60 by paying a lump sum of $150. He may become a "Patron Life Member" by paying a lump sum of $300 or more. Each Life Member will be entitled to all the privileges of membership, including a subscription to the MONTHLY. The names of Patron Life Members will be published at appropriate times in the MONTHLY except for those who wish to remain anonymous.

HENRY L. ALDER, *Secretary*

NEW TYPES OF MEMBERSHIP

In order to finance the growing activities of the MAA in a time of inflation and decreasing government support, the Board of Governors at its meeting on August 24, 1969, at the University of Oregon has voted to establish three types of contributing memberships in the Association, so that there are now the following types of membership:

	Annual Dues
Ordinary Member	$ 10
Contributing Member	25
Sponsor	50
Patron	100 or more

These types of membership are intended to encourage additional support of the activities of the Association by those members who are financially able to do so. An opportunity to subscribe to one of these memberships will be contained in the annual statement of dues. The privileges of these new types of membership will be the same as those for ordinary members. The names of sponsors and patrons will be published at appropriate times in the MONTHLY, except for those who wish to remain anonymous.

HENRY L. ALDER, *Secretary*

APRIL MEETING OF THE IOWA SECTION

The 56th regular meeting of the Iowa Section of the MAA was held at the University of Northern Iowa, Cedar Falls, on April 18, 1969. Chairman J. C. Friedell presided. Total attendance was 92, including 47 members of the Association.

A business meeting opened the afternoon session. A motion that the Iowa Section pay for membership in the Association for each Iowa student ranking in the top 100 in the W. L. Putnam Mathematical Competition, was made and passed.

The following officers were elected: Chairman, Elsie Muller, Morningside College, Sioux City; Vice-Chairman, Timothy Robertson, University of Iowa, Iowa City; Secretary-Treasurer, B. E. Gillam, Drake University, Des Moines.

As part of the program two films were shown. "Inversion", a 13 minute film, written by Dan Pedoe and produced by the College Geometry Project at the University of Minnesota, was used to begin the morning session. "Dihedral Kaleidoscopes", a 12 minute film written by H. S. M. Coxeter and by the College Geometry Project, ended the afternoon program.

The following papers completed the program:

Intuitive vs. rigoristic approach to teaching calculus, Panel discussion: Fred Lott, Michael Millar, and John Longnecker of Cedar Falls.

Undergraduate research in mathematics—Purposes and sources, by D. H. Pilgrim, Decorah.

Some thoughts on the teaching of algebra, abstract and linear, by Sister Cathleen Real, Davenport.

Are we only concerned about conveying information? by D. V. Meyer, Pella.

Integrating the computer into the collegiate mathematics curriculum, by G. P. Weeg, Iowa City (invited address).

The University of Iowa's NSF Conference in Computer Science for Secondary Teachers, by Marilyn J. Zweng, Iowa City.

On uses of computers in teaching statistics, by J. W. L. Cole, Iowa City.

Representations of C^*-algebras, by R. S. Doran, Cedar Falls.

B. E. Gillam, *Secretary-Treasurer*

APRIL MEETING OF THE NORTH CENTRAL SECTION

The spring meeting of the North Central Section (formerly Minnesota Section) of the MAA met at the College of St. Catherine, St. Paul, Minnesota, on April 25 and 26, 1969. There were 117 persons registered for the meetings including 91 members. The following officers were elected at the business meeting: Chairman, Professor E. O. Nelson, University of North Dakota; Chairman-Elect, Professor Alfred Aeppli, University of Minnesota; Secretary-Treasurer, Professor Warren Thomsen, Moorhead State College; Members at Large (Executive Committee), Professor K. E. Dubbert, Rochester Junior College; Professor H. C. Finlayson, University of Manitoba.

Professor George Minty, University of Indiana, gave the invited address on Friday evening, "Kerszbraun's Theorem and Extension of Hölder-Continuous Functions". Professor Daniel Pedoe gave the invited address on Saturday morning, "Geometrical Thinking".

Other papers presented included:

1. *On convergence and divergence of a trigonometric series*, by O. E. Stanaitis, St. Olaf College.
2. *On the maximum modulus of certain lacunary power series*, by W. D. Serbyn, University of Minnesota.

3. *Idiot's delight. An interesting probability problem*, by R. B. Kirchner, Carleton College.

4. *A hierarchy of nested countable sets of real numbers: some remarks on a sequence proposed by Charles S. Pierce*, by T. J. Doyle, St. John's University.

5. *An ideal operation*, by L. C. Larson, St. Olaf College.

6. *Real additive functions*, by Larry Hansen, Lakehead University.

7. *Topology in terms of connectedness*, by George Brauer, University of Minnesota, and Edwin Buchman, University of California.

8. *An application of the Vietoris-Begle mapping theorem*, by C. F. Blakemore, Mankato State College.

W. J. THOMSEN, *Secretary*

APRIL MEETING OF THE OHIO SECTION

The fifty-third annual meeting of the Ohio Section of the MAA was held at Stouffer's University Inn, Columbus, Ohio, on Friday and Saturday, April 25–26, 1969. Professor Arnold Ross, Chairman of the Section, presided at the business meeting and one program session and Professor Daniel Finkbeiner, Past Chairman, and Professor James Smith, Chairman-Elect, presided at other program sessions. Two hundred seventy-one persons registered in attendance including one hundred seventy-five members of the Association.

The following officers were elected: Chairman, Professor James Smith, Muskingum College; Chairman-Elect, Professor Bernard Yozwiak, Youngstown State University; Secretary-Treasurer, Foster Brooks, Kent State University; Program Committee: Professor J. F. Leetch, Bowling Green State University, Chairman; Professor R. G. Laatsch, Miami University and Professor R. H. Rolwing, University of Cincinnati.

The following program was presented:

1. *COSRIMS: Findings and recommendations of the graduate panel*, by R. P. Boas, Northwestern University.

2. *COSRIMS: Findings and recommendations of the undergraduate panel*, by H. O. Pollak, Bell Telephone Laboratories.

3. *COSRIMS: Problems of implementation of recommendations*, by P. J. Hilton, Cornell University.

4. *Graduate training in mathematics: Some dilemmas of a practicing educator*, by Arnold Ross, The Ohio State University (Chairman's address).

5. *Some new developments in the theory of finite simple groups*, by Zvonimir Janko, Institute for Advanced Study and the Ohio State University (invited speaker).

6. *Geometric interpretations of a certain contour integral*, by B. F. Plybon, Miami University.

7. *Two algorithms for determining the formula:* $\sum_{i=1}^{n} i^k$, by L. D. Rodabaugh, University of Akron.

8. *Quasi-projective modules and semisimple rings*, by Ann Koehler, Western College for Women.

9. *Linear operators on differentiable manifolds and tensor spaces*, by F. R. Davis, Kent State University.

10. *Intrinsic functions on semisimple algebras*, by C. J. Long, The College of Steubenville.

Papers 6 through 10 were presented at a joint session of the Ohio Section, MAA, with Section L—Mathematical Sciences of the Ohio Academy of Science.

FOSTER BROOKS, *Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the MAA was held on the campus of Texarkana College, Texarkana, Texas on April 18–19, 1969. There were 204 persons registered, of which 132 were members of the Association.

3. *Idiot's delight. An interesting probability problem*, by R. B. Kirchner, Carleton College.

4. *A hierarchy of nested countable sets of real numbers: some remarks on a sequence proposed by Charles S. Pierce*, by T. J. Doyle, St. John's University.

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29. *Residue classes and string figures*, by A. R. Amir-Moez, Texas Technological College.

30. *The interface between mathematics and statistics*, by P. D. Minton, Southern Methodist University.

J. C. BRADFORD, *Secretary-Treasurer*

MAY MEETING OF THE KENTUCKY SECTION

The fifty-second annual meeting of the Kentucky Section was held at Morehead State University, Morehead, Kentucky, on May 3, 1969, Chairman Billy R. Nail presiding. Seventy persons registered at the meeting, including 48 members of the Association.

At the business meeting these officers were elected: Chairman: Dr. Carl Langenhop, University of Kentucky; Secretary-Treasurer: Dr. A. S. Howard, Eastern Kentucky University; Contest Chairman: Dr. James Simpson, University of Kentucky.

Dr. Gail Young, President of the Association, gave the invited address on *Some Topological Aspects in Analysis.*

In the morning there were two parallel program sessions. One of these, presided over by Prof. L. C. Cooper of Morehead State University, concerned itself with school mathematics. It was addressed by Mr. Russell Boyd, Mathematics Consultant with the State Department of Education, Mrs. Peggy Prater, Chairman, Mathematics Department, Bath County High School, and Dr. Martha Sudduth, University of Kentucky.

At the other session the following papers were presented:

1. *A generalization of a theorem of Kirszbraun on the intersection of convex sets*, by J. H. Wells, University of Kentucky.

2. *On extending the domain of a complex-valued function satisfying a Lipschitz condition*, by T. M. Jenkins, University of Louisville.

3. *Elementary proofs of theorems on weak convergence—linear operators approach*, by Z. Govindarajulu, University of Kentucky.

4. *The intersection projection of orthogonal projections in finite dimensional spaces*, by C. E. Langenhop, University of Kentucky.

5. *The functions quasi-periodic and the nonhomogeneous linear differential equation*, by F. G. Scorsone, Eastern Kentucky University.

6. *Hamiltonian form of the geodesic equations*, by L. E. Bragg, University of Kentucky.

W. H. SPRAGENS, *Secretary-Treasurer*

MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held on May 10, 1969 at the Indianapolis Campus of Purdue University. There were 70 persons in attendance, including 57 members of the Association.

The group was welcomed by Dr. R. C. Sanborn, Assistant Dean for Academic Affairs at the Indianapolis Campus. Professor B. E. Rhoades, Chairman of the Section, presided.

The following program was presented at the morning session:

1. *Bending of a cylindrical shell under a discontinuous load*, by A. K. Naghdi, Purdue University Indianapolis Campus.

2. *Some recent developments in Boolean geometry*, by H. J. Ludwig, Ball State University.

3. *On subsemigroups of groups*, by J. E. Kuczowski, Purdue University Indianapolis Campus.

4. *Qualitative properties of satellite orbits*, by P. C. Loh, Purdue University Indianapolis Campus.

5. *Primal decomposition of ideals in noncommutative rings*, by C. M. Murphy, Purdue University Calumet Campus.

At the business meeting in the afternoon, the Secretary-Treasurer reported that Mr. Stephen Helmreich of Valparaiso University and Mr. Eric Isaacson of Indiana University had each been awarded a one-year membership in the Association in recognition of their achievement in the 29th Putnam Mathematical Competition. Officers for 1969–70 were elected as follows: Chairman, Professor N. B. Haaser, University of Notre Dame; Vice-Chairman, Professor W. C. Swift, Wabash College; Secretary-Treasurer, Professor M. J. Mansfield, Purdue University at Fort Wayne.

Following the business meeting Professor G. S. Young, President of the MAA, addressed the group on "Topology and Analysis."

M. J. MANSFIELD, *Secretary-Treasurer*

MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The fifty-second annual meeting of the Rocky Mountain Section of the MAA was held at the University of Colorado, Boulder, Colorado, on May 9 and 10, 1969. There were 153 persons registered for the meeting, including Professor F. M. Stein of Colorado State University, Sectional Governor, and Professor J. W. Bebernes of the University of Colorado, Section Chairman. The invited address was delivered by Professor V. L. Klee, Jr., of the University of Washington, who spoke on "Shapes of the Future—Unsolved Geometric Problems for Science and Technology." Professor W. E. Briggs, Dean of the College of Arts and Science of the University of Colorado, welcomed the Section at the banquet on Friday evening.

At the business meeting, the Report of the Nominating Committee recommending that the By-Laws of the Section be amended to provide for the election of a Second Vice-Chairman to look after the interests of the junior colleges, was approved. Professor T. D. Cavanagh, Contest Chairman of the Section, reported that 8610 students from 141 high schools participated in the 1968 MAA mathematics contest. Professor Robert McKelvey reported for the High School Lecturer Program inaugurated last year, and his recommendation that the program be continued was approved. The present committee, consisting of Professor Robert McKelvey, University of Colorado, Chairman, Professor W. R. Scott, University of Utah, and Professor Verne Varineau, University of Wyoming, was reappointed to continue the administration of this program.

The following officers were elected: Chairman, Ray Hanna, University of Wyoming, Laramie, Wyoming; First Vice-Chairman, George Stratopoulos, Weber State College, Ogden, Utah; Second Vice-Chairman, James Davis, Mesa Junior College, Grand Junction, Colorado; Secretary-Treasurer, D. J. Sterling, Colorado College, Colorado Springs.

The following papers were read at the meeting:

1. *Construction of projective ideals*, by D. W. Ballew, South Dakota School of Mines and Technology.
2. *A relationship of perfect fields to compact classes*, by A. J. Boes, Colorado School of Mines.
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12. *The recent discovery of infinitesimal analysis*, by Gary Meisters, University of Colorado.
13. *The continuum hypothesis and the axiom of choice—remarks on the practical relevance of recent independence results*, by Donald Monk, University of Colorado.
14. *Some matric equations over $GF(q)$*, by A. D. Porter, University of Wyoming.
15. *Quasi symmetric functions*, by T. J. Reed, University of Colorado.
16. *Lagrange's version of Lagrange's theorem on groups, 1771*, by R. L. Roth, University of Colorado.
17. *The number of solutions of polynomial equations*, by L. E. Shader, University of Wyoming.
18. *CSU closed circuit television mathematics courses*, by F. M. Stein and R. H. Niemann, Colorado State University.
19. *Applicability of mathematics*, by Stanislaw Ulam, University of Colorado.

C. R. WYLIE, *Secretary-Treasurer*

MAY MEETING OF THE WISCONSIN SECTION

The annual meeting of the Wisconsin Section of the MAA was held at Wisconsin State University-Oshkosh, on May 2 and 3, 1969. Chairman J. A. Raab, Wisconsin State University-Oshkosh, presided. Approximately 110 persons attended.

After registration on Friday afternoon, May 2, the following papers were presented:

1. *Matrix norms composed from norms of submatrices of a partitioned matrix*, by N. E. Nirschl, St. Norbert College, West DePere.
2. *Pseudo quasi metric space*, by Y. W. Kim, Wisconsin State University, Eau Claire.
3. *Generalized characteristic exponents of linear homogeneous systems of differential equations*, by H. S. Gunderson, Wisconsin State University, Oshkosh.
4. *How important is the normal distribution?* by D. Lund, Wisconsin State University, Eau Claire.

A banquet was held Friday evening. After the banquet, several films were shown including "Challenge in the Classroom: The Methods of R. L. Moore."

The Saturday morning session was devoted to a short business meeting and the presentation of three more papers. During the business meeting, Dr. Marshall Wick, Wisconsin State University, Eau Claire, was elected Chairman; Mr. Warren White, University of Wisconsin Extension Center-Sheboygan, was elected Vice-Chairman; and Dr. R. W. Christensen, Wisconsin State University-La Crosse, was elected Secretary-Treasurer. The remainder of the morning session was devoted to the presentation of the following papers:

1. *Nearly antiflexible division algebras*, by L. W. Davis, Wisconsin State University, White water.
2. *Not solving differential equations*, by F. Brauer, University of Wisconsin, Madison.
3. *Commutative subrings of the ring of $n\times n$ matrices over a finite field*, by C. B. Henneken, Marquette University, Milwaukee.

After a luncheon break, a panel consisting of F. Brauer, University of Wisconsin, Madison, J. Lakin, Wisconsin State University-Oshkosh, L. Wahlstrom, Wisconsin State University, Eau Claire, and E. Wilde, Beloit College, presented a discussion of "Geometry and the Undergraduate Program." The Saturday afternoon session concluded with a presentation of the following papers:

1. *CL products of CL spaces*, by C. C. Braunschweiger, Marquette University, Milwaukee.
2. *Complex number algebra as a simple case of Heavyside operational calculus*, by D. Moore, University of Wisconsin, Green Bay.

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CALENDAR OF FUTURE MEETINGS

Fifty-third Annual Meeting, Miami, Florida, January 24–26, 1970.

Fifty-first Summer Meeting, University of Wyoming, Laramie, August 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, May 2, 1970.

FLORIDA, Rollins College, Winter Park, March 20–21, 1970.

ILLINOIS, Loyola University, Chicago, May 8–9, 1970.

INDIANA, University of Notre Dame, Notre Dame, November 15, 1969.

IOWA, Grinnell College, Grinnell, April 17, 1970.

KANSAS, Kansas State Teachers College, Emporia, March 1970.

KENTUCKY, University of Kentucky, Lexington, Spring 1970.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Wagner College, Staten Island, Spring 1970.

MICHIGAN, Wayne State University, Detroit, April 4, 1970.

MISSOURI, Central Missouri State College, Warrensburg, May 2, 1970.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 24–25, 1970.

NEW JERSEY, Seton Hall University, South Orange, November 1, 1969.

NORTH CENTRAL

NORTHEASTERN, Wheaton College, Norton, Massachusetts, November 29, 1969.

NORTHERN CALIFORNIA, Diablo Valley College, Concord, February 7, 1970.

OHIO, Bowling Green State University, Bowling Green, Spring 1970.

OKLAHOMA-ARKANSAS, Southwestern State College, Weatherford, Oklahoma, March 1970.

PACIFIC NORTHWEST

PHILADELPHIA, Swarthmore College, Swarthmore, November 22, 1969.

ROCKY MOUNTAIN, University of Wyoming, Laramie, May 8–9, 1970.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, Spring 1970.

SOUTHERN CALIFORNIA, University of California, Irvine, March 21, 1970.

SOUTHWESTERN, University of Texas at El Paso, March 27–28, 1970.

TEXAS, Sam Houston State College, Huntsville, April 10–11, 1970.

UPPER NEW YORK STATE, Canisius College, Buffalo, November 1, 1969.

WISCONSIN, University of Wisconsin, Waukesha, May 1970.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, Miami, Florida, January 22–25, 1970.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Ohio State University, June 22–25, 1970.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Milwaukee, Wisconsin, November 27–29, 1969.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami, Florida, November 10–12, 1969.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

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GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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VOLUME 76 NUMBER 9

CONTENTS

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NOVEMBER 1969

NOTICE TO AUTHORS

Articles should be typewritten on good quality paper, triple spaced with wide margins. Submit the original and a duplicate; keep a complete copy for protection against loss. Please follow the format used in current issues of the MONTHLY. The manuscript must make the author's intent clear to the printer. A paper accepted for publication by the editor but not acceptable to the printer will be returned to the author for retyping. Please use the American Mathematical Society *Manual for Authors of Mathematical Papers* as your guide and please conform to correct English usage. A *Manual for Monthly Authors* is available on request.

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WEAK AND STRONG INDUCTION

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Introduction. There are two familiar forms of the principle of mathematical induction. The standard form (WI(P) of Section 1 below) allows one to derive the conclusion $P(n)$ for all (natural numbers) n, from the hypotheses: (1) $P(1)$ and (2) for all n, $P(n)$ implies $P(n')$, where n' is the successor of n. This is often called *ordinary* or *weak* induction to distinguish it from the second principle (SI(P) below) called *strong* or *course of values* induction, which differs in that (2) is replaced by the weaker (2'): for all n, if $P(m)$ is true for all $m \leqq n$, then $P(n')$.

REMARK. (2') is weaker than (2) because it gets the same conclusion from a stronger hypothesis; the second principle is stronger than the first because it gets the same conclusion as the first from the weaker hypothesis (2').

A well-known argument shows that the greater strength of the second principle is only apparent, for the second principle for a given P can be proved by applying the first principle to a different property P^* (where $P^*(n)$ is defined as: $P(m)$ is true for all $m \leqq n$). So the statement that weak induction holds for all properties P implies the statement that strong induction holds for all properties P. But what is the situation if one considers these two principles WI(P) and SI(P) for the same P? One certainly does find it convenient to use strong induction SI(P), e.g., in proving unique decomposition into prime factors, or that each non-null set of natural numbers has a least element. And one feels in such cases that unless one does change as above to a more complicated property this use is inevitable, that WI(P) applied to the same P would not enable one to reach the desired conclusion. [The expression $P^*(n)$: $(Am)(m \leqq n \rightarrow P(m))$ mentioned above is *a priori* more complicated than P in that it involves an additional quantifier, (Am).]

The object of this note is to consider to what extent this feeling is justified. We shall see that within what is probably the most natural logical framework, this intuitive feeling is sometimes (but not always) correct; in some such cases WI(P) does not imply SI(P). [Of course for "most" P, WI(P) does imply SI(P), e.g., if $(An)P(n)$ is provable by weak induction on P, or is provable without using induction at all, or if the hypothesis of SI(P) is refutable, or if P is *monotonic* in the sense that $P(n) \rightarrow P(m)$ for $m < n$ (this property is possessed by P^*). What we are interested in are the rather few (see Section 7) cases where there is a natural proof by strong induction on P, but apparently not by weak induction on P.]

On the other hand it turns out that for one or two well-known methods of defining the order relation on the natural numbers, WI(P) always implies SI(P). The analysis provides simple exercises in the use of nonstandard models for fragments of arithmetic.

The reader may be interested in L. Henkin's elegant paper, *On mathematical induction*, this MONTHLY, 67 (1960) 323–338. The difference in approach is

that Henkin considers allowing free use of the axiom of induction and dropping the other axioms, whereas we consider what happens when you restrict the axiom of induction, but add other axioms.

1. Precise statement of the two principles. We shall use (but mainly as abbreviations; we shall not hesitate to use English when it is shorter than symbolism) the following symbols: & (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (is equivalent to), $\neg$ (not), (Ax) (for all x), (Ex) (there exists x), $\in$ (belongs to), $\notin$ (does not belong to). "Number" means natural number throughout. As variables for individuals (numbers) we use $x, y, z, m, n, p, \cdots$, as variables for properties of individuals or propositional functions, $P, Q, R, P_1, \cdots$, as variables for sets of individuals, $X, Y, Z, W, X_1, \cdots$. Thus the weak and strong forms of the principle of induction referred to above will be written:

$$\mathrm{WI}(P)\colon [P(1) \,\&\, (An)(P(n)\rightarrow P(n'))]\rightarrow (An)P(n),$$

$$\mathrm{SI}(P)\colon [P(1) \,\&\, (An)((Am)(m\leqq n\rightarrow P(m))\rightarrow P(n')]\rightarrow (An)P(n).$$

There is also a well-known variant of the latter, viz., $\mathrm{SI}^*(P)\colon (An[(Am)(m<n\rightarrow P(m))\rightarrow P(n)]\rightarrow (An)P(n)$. The equivalence, $\mathrm{SI}(P)\leftrightarrow \mathrm{SI}^*(P)$ of these two forms follows from simple properties of $<$, e.g., $n \not< 1$, $m<n' \leftrightarrow m\leqq n, n\neq 1 \rightarrow (Em)(n=m')$.

The elementary argument given above shows

$$(1.1) \qquad (AP)\mathrm{WI}(P) \rightarrow (AP)\mathrm{SI}(P).$$

The question we are interested in is whether for certain fixed P,

$$(1.2) \qquad \mathrm{WI}(P) \rightarrow \mathrm{SI}(P)$$

is provable.

[REMARK. For a strong counterexample one would require a little more than the unprovability of $\mathrm{WI}(P)\rightarrow \mathrm{SI}(P)$, namely the result $(An)P(n)$ should be provable by strong induction but not by weak, i.e., the hypothesis of $\mathrm{SI}(P)$ should be provable, but not that of $\mathrm{WI}(P)$. The examples below do in fact have this property.]

This may be expected to depend on what particular logical development of arithmetic is being adopted, e.g., on how $\leqq$ and the other relations and operations such as $+$ or $\cdot$ which may occur in P are defined, and on what are the other axioms, apart from the induction principle itself (which is the subject of examination). We proceed to consider various possibilities.

2. The naive approach: a particular example. I use the word naive here not in any deprecatory sense, but to describe what I imagine would be the first approach of a mathematician who considers it meaningful to ask whether a certain theorem which can be proved by strong induction on P can also be proved by weak induction on P, but is unwilling to go further than necessary

into the logical background. Let us take a specific example, say the theorem about decomposition into prime factors. Here we define, as usual

2.1. DEFINITION. p is a *prime*$\leftrightarrow p\neq 1$ & $(Ay)(Az)(yz=p\rightarrow y=1\vee z=1)$. The theorem then is

(2.2) $(An)(n\neq 1\rightarrow n$ *is a product of primes*).

[REMARK. As usual "n is a product of primes" is an abbreviation for: there exists a sequence $p_1, \cdots, p_k$ of numbers which are prime and such that $n=p_1 \cdots p_k$. If we don't want to presuppose the associative law, we should write, say, $n=p_1(p_2(\cdots p_k)\cdots)$.]

The natural way to try and prove (2.2) is by strong induction on the property P_0 defined by:

2.3. DEFINITION. $P_0(m)\leftrightarrow[n\neq 1\rightarrow n$ is a product of primes]. In other words we try and prove

$$\mathrm{SI}(P_0)\rightarrow(2.2). \tag{2.4}$$

[REMARK. An alternative interpretation of "$(An)P_0(n)$ is provable by strong induction" would be to equate this to "the hypothesis of $\mathrm{SI}(P_0)$, i.e., $P_0(1)$ & $(An)((Am)(m\leqq n\rightarrow P_0(m)\rightarrow P_0(n'))$, is provable." This is easily seen to be equivalent to the version "$\mathrm{SI}(P_0)\rightarrow(An)P_0(n)$", which is our (2.4).]

It is easy to see that this can be done if we have already established as theorems enough of the elementary properties of order and multiplication. [The following are certainly enough: $n\geqq 1$, $m<n'\leftrightarrow m\leqq n$, $n>n_1\rightarrow mn>mn_1$, $n1=n$, $m(np)=(mn)p$.]

Now can (2.2) be proved by weak induction on P_0, i.e. can we prove (2.5)?

$$\mathrm{WI}(P_0)\rightarrow(2.2). \tag{2.5}$$

Intuitively one feels the answer is no! In proving (2.4) we find $n'=yz$ for some y and z, where $1<y$, $z\leqq n$, and use the induction hypothesis to tell us that y and z are products of primes. Clearly y, $z\neq n$ so the hypothesis $P_0(n)$ of the weak induction is no use to us. But all that this shows is that we can't think of a natural way of using $\mathrm{WI}(P_0)$ to prove Theorem (2.2). If (2.2) could be proved without using induction at all, then of course (2.5) would be trivially provable; so would $\mathrm{SI}(P_0)$, and hence so would

$$\mathrm{WI}(P_0)\rightarrow\mathrm{SI}(P_0). \tag{2.6}$$

In a sophisticated approach to elementary arithmetic this could happen, for example, if one had already proved the decomposition theorem for Euclidean rings, or had already proved it for principal ideal rings and proved that the ring of integers is a principal ideal ring. In proving either of these, however, one would already have used strong induction, either directly or via the least number principle, to prove a more general result than (2.2). But this does show we can't expect a unique answer to our question without being more precise about the other axioms we are allowed to use. This is not surprising, because

the question we are asking is a fairly delicate one; for we know that (2.2) can be proved by weak induction applied to a slightly different property P^*. This doesn't mean that the question is quite uninteresting; if one accepts outright the Peano Axioms including the full principle of induction $(AP)\mathrm{WI}(P)$, then one is studying number theory, one of the richest and most interesting branches of mathematics, but one of the narrowest, for the system of natural numbers is unique to within isomorphism. It is the induction principle which is responsible for this uniqueness and also for the different feeling which number theory has compared with algebra. So it is of some interest to develop arithmetic step by step from the Peano Axioms, stopping after each use of the induction principle to see what can be proved from the theorems one has so far obtained "purely algebraically," i.e., without further use of the induction principle. This is indeed the standard route of many textbooks on the number systems, and part of its attraction is that the algebraic systems which arise along the way, or are suggested by these, include many of the basic structures of algebra.

Returning to our own little sheeptrack meandering off this broad highway, we see that perhaps the question we ought to be asking is: "Can (2.5) be proved from all the simple properties of order, addition, and all the properties of multiplication simpler than, and normally proved before (2.2)?" This still leaves much to be desired in precision, but we can answer it satisfactorily since we can produce a nonstandard model of arithmetic in which (2.5) and (2.2) are false, but which satisfies all the properties of order, addition, and multiplication which could reasonably be considered as prior to (2.2). For the sake of brevity let us make the following definition:

(2.8). DEFINITION. *ALGAR* denotes the conjunction of the axioms

$$x \neq 1, \quad x' = y' \rightarrow x = y, \quad x \neq 1 \rightarrow (Ey)(x = y'),$$

the axioms giving the inductive definitions of $+$ (viz.: $x+1=x'$, $x+y'=(x+y)'$) and $\bullet$, the commutative, associative, and distributive laws for $+$ and $\bullet$, the order axioms for $<$, and

$$x < y' \leftrightarrow x \leqq y, \quad x < y \leftrightarrow (Ez)(x + z = y),$$
$$x < y \rightarrow (x + z < y + z \;\&\; xz < yz).$$

As the name is intended to suggest, these (not independent) axioms consist of all the axioms needed to develop the algebraic part of arithmetic as far as it is taken in most treatments of the number systems. Any model $\boldsymbol{M}$ for them can be embedded in a ring $\boldsymbol{R}$ by the usual construction; this ring $\boldsymbol{R}$ will be commutative and discretely ordered (no element between 0 and 1). Conversely if we take the set of elements $\geqq 1$ of any discretely ordered commutative ring $\boldsymbol{R}$ we get a model for *ALGAR*. In the standard case, $\boldsymbol{R}$ corresponds to the ring of integers; the remainder of the usual construction of the rationals and reals can be paralleled, for $\boldsymbol{R}$ is an integral domain which has a quotient field $\boldsymbol{Q}$, and this can be embedded in a complete (in the sense that every fundamental sequence

converges) field C whose ordering is an extension of that of R. The difference from the standard theory is that if M is nonstandard (i.e., does not consist of the natural numbers with the usual definitions of 1, $+$, $\bullet$, $\leqq$), then Q and C are not archimedean ordered, and C is not order complete. Since these latter properties are equivalent to the principle of induction for all properties, there is no point in the present context in considering adding them.

Before proving the prime decomposition theorem, one would probably have defined $x|y$ as $(Ez)(xz=y)$, and proved (using induction) some elementary number theoretic facts about the existence of quotient and remainder and of the g.c.d. of two numbers and its expressibility as a linear combination of them, and possibly (although this is unnecessary) the weaker result that each number has a prime factor (2.91-4 below). However, even with these, weak induction on P_0 is not adequate to prove that every number other than 1 is a product of primes, for:

(2.9). THEOREM. *There is a model* $\mathfrak{M}=\langle M, ', +, \bullet, \leqq, 1\rangle$ *for ALGAR and for* (2.91-4) *below which fails to satisfy* (2.2), (2.5), *and* (2.6).

(2.91) $$x > y \rightarrow [y \mid x \vee (Eq, r)(x = qy + r \,\&\, r < y)].$$

(2.92) $$(Ed)[d \mid x \,\&\, d \mid y \,\&\, (Az)(z \mid x \,\&\, z \mid y \rightarrow z \mid d)].$$

(2.93.) *The* d *of* (2.92) (*it is easily seen to be unique*) *is expressible in the form* $d+ax=by$ *or* $d+by=ax$ *for some* a *and* b.

(2.94) $$(An)(n \neq 1 \rightarrow (Ep)(p \textit{ is prime and } p \mid n)).$$

Proof. Take M to be the set of elements which are $\geqq 1$ in the ring R of polynomials in an indeterminate t with rational coefficients, the constant term being an integer, i.e., of expressions

$$a = a_p t^p + a_{p-1} t^{p-1} + \cdots + a_1 t + a_0,$$

where p is a nonnegative integer, $a_1, \cdots, a_p$ are rationals, a_0 is an integer, and $a_p \neq 0$ unless $p=0$. Define $'$, $+$, $\bullet$, 1 on M in the obvious way and define $\leqq$ by saying that $a>0$ if and only if $a_p>0$ (and, of course, $x \geqq y \leftrightarrow x-y \geqq 0$), in other words, making t infinitely large. It is easy to check that the resulting structure $\mathfrak{M}=\langle M, ', +, \bullet, \leqq, 1\rangle$ satisfies *ALGAR* (i.e., that R is a discretely ordered commutative ring).

To prove (2.91-4) in $\mathfrak{M}$, it is easier to work in the ring R. What we have to show there is that, for all x, y,

(i) $y>0 \rightarrow (Eq, r)(x=qy+r \,\&\, 0 \leqq r<y)$,
(ii) *the ideal* (x, y) *generated by* x, y *is principal,*
(iii) *if* $x \neq \pm 1$, *then* x *has a prime factor.*

It is convenient to consider also R', the ring of all polynomials in t with rational coefficients. Now (i) is true in R'. For by the division algorithm we can write

$$x(t) = q(t)y(t) + r(t) \quad \text{with } \deg r(t) < \deg y(t).$$

This yields $r(t)<y(t)$ and $-r(t)<y(t)$, so if $r(t)>0$ we already have the desired result; if $r(t)<0$ we get it by replacing $q(t)$ by $q(t)-1$ and $r(t)$ by $r(t)+y(t)$. Now if $x, y \in R$, use this to write $x=qy+r$ with $0\leqq r<y$ and q, r in R'. Take q_1 in R such that $q_1\leqq q<q_1+1$. We get $x=q_1y+r_1$, where $r_1\in R$ (being $=x-q_1y$) and, since $r_1=r+(q-q_1)y$, we have $0\leqq r_1<2y$, so either q_1 and r_1 already satisfy (i), or q_1+1 and r_1-y do. (Another way of proving (i) is to take the Laurent series which is the formal quotient x/y, extend the ordering to these series in the obvious way, and take a q in R such that $q\leqq x/y<q+1$.)

In R' (ii) is also true, i.e. for $x, y\in R'$ there exist $d, a, b, f, g\in R'$ such that $x=df$, $y=dg$, and $d=ax+by$. Now suppose $x, y\in R$. For each element u of R' there exists a positive integer k such that $ku\in R$. Let m_1, m_2, n_1, n_2 be positive integers such that $m_1f, m_2g, n_1a, n_2b\in R$. If we put $d_1=d/m_1m_2$ and $n=m_1m_2n_1n_2$, we have:

$$x = f_1d_1, \quad y = g_1d_1, \quad nd_1 = a_1x + b_1y,$$

where $a_1, b_1, f_1, g_1\in R$. Each element of the ideal (x, y) generated by x, y in R is of the form $\alpha x+\beta y$, where $\alpha, \beta\in R$, i.e. of the form $(\alpha f_1+\beta g_1)d_1=\gamma d_1$, where $\gamma\in R$. Also $nd_1\in(x, y)$ and the only positive elements of R which are $\leqq n$ are $1, 2, \cdots, n$, so if we take the least positive integer n_0 such that $n_0d_1\in(x, y)$, we have in n_0d_1 a least positive element of (x, y). Using (i), the usual argument shows that $(x, y)=(n_0\, d_1)$. (The whole point of this argument is to establish the existence of a least positive element of (x, y); e.g., the ideal $(t, t/2, t/3, \cdots)$ does not have such and is not principal.)

To see that (iii) holds, let

$$f = f(t) = f_pt^p + \cdots + f_1t + f_0$$

be an element of R. If $f_0=0$, we can write $f=2\cdot f/2$, where $2, f/2\in R$ and 2 is a prime. If not, and if $p>0$ (if $p=0$ the result is obvious), write in R'

$$f(t) = a(t)b(t),$$

where $a(t)$ is irreducible in R', of degree>0, and has nonzero constant term a_0. If we put $a_1(t)=a(t)/a_0$ and $b_1(t)=a_0b(t)$ we have

$$f(t) = a_1(t)b_1(t),$$

where $a_1(t)$ has constant term 1 and $b_1(t)$ has constant term f_0, so both $a_1(t)$, $b_1(t)\in R$. Since the only factorisations of $a_1(t)$ in R' are

$$a_1(t) = r(a_1(t)/r)$$

with rational r, the only factorisations in R are the trivial ones $(\pm1)\cdot(\pm a_1(t))$; i.e., $a_1(t)$ is prime.

(2.2) is false in $\mathfrak{M}$, for the element t is not a product of primes. Indeed the only factorisations of t are

$$t = n_1n_2 \cdots n_k\cdot(t/n_1 \cdots n_k),$$

and $t/n_1 \cdots n_k$ is not prime. Also (2.5) fails in $\mathfrak{M}$, for (2.2) is false and $\mathrm{WI}(P_0)$ is true, since the premise $(An)(P_0(n) \to P_0(n'))$ is false. The latter is the case because $P_0(t)$ is false, yet $P_0(t-1)$ is true; for $t-1$ is, as above, a prime. Since (2.6) and (2.4) (which is true) imply (2.5), it follows that (2.6) also fails in $\mathfrak{M}$.

Since (2.92) and (2.93) enable one to prove that every number which is expressible as a product of primes is uniquely expressible, the same remarks apply to the proof of this by strong and weak induction.

Theorem (2.9) may also be regarded as justifying the common belief that it is often necessary, when making proofs by induction, to strengthen the hypothesis. For we have shown that $(An)P_0(n)$ cannot be proved by weak induction on P_0, but can be proved by strong induction on P_0, hence by weak induction on $P_0^*(n)$: $(Am)(m \leq n \to P_0(m))$. In this case P_0^* is only "locally" stronger than P_0; although $P_0(n) \to P_0^*(n)$ is not provable (being false in the model $\mathfrak{M}$ for $n = t+1$), $(An)P_0(n) \to (An)P_0^*(n)$ obviously is. One can also give an example where the "stronger" hypothesis is genuinely so, i.e. where $(An)P_1(n)$ cannot be proved by weak induction on P_1, $(An)P_2(n)$ can be proved by weak induction on P_2, $P_2(n) \to P_1(n)$, but $(An)P_1(n) \to (An)P_2(n)$ is not provable. Namely, take for $P_1(n)$ the statement $n \neq 1 \to (Ep)$ (p is prime & $p \mid n$) and take $P_2(n)$ to be $P_0^*(n)$. That $P_2(n) \to P_0(n) \to P_1(n)$ is clear; also we have seen (2.94) that $(An)P_1(n)$ is true in $\mathfrak{M}$ but $(An)P_2(n)$ is false, so $(An)P_1(n) \not\to (An)P_2(n)$. Furthermore $(An)P_0^*(n)$ can be proved by weak induction on P_0^*, but $(An)P_1(n)$ cannot be proved by weak induction on P_1 (from axioms *ALGAR*, 2.91-3). This can be shown by considering a model $\mathfrak{M}_1$ differing from $\mathfrak{M}$ in that the polynomials in t are replaced by fractional polynomials in t, i.e. polynomials in $t^{1/q}$ where q is allowed to vary over all positive integers. On the same lines as above, but with a little more effort, one can show that $\mathfrak{M}_1$ satisfies *ALGAR*, (2.91-3) and $\mathrm{WI}(P_1)$, but not $(An)P_1(n)$. It turns out that in $\mathfrak{M}_1$, $t-2 = 2(\frac{1}{2}t-1)$ has the prime factor 2, but $t-1$ is not prime and has no prime factor (e.g., it is equal to $(t^{1/2}-1)(t^{1/2}+1)$ but $(t^{1/2}+1)$ is divisible by $t^{1/6}+1$, etc.) Actually there is, for $P_1(n)$, an amusing short argument which shows that as soon as we have enough algebraic axioms to prove that each number is either even or odd, then $(An)P_1(n)$ can only be proved trivially by weak induction on P_1, i.e., only when it can be proved without using induction at all. For, since 2 is prime, $P_1(2n)$ is true, so if $P_1(n) \to P_1(n')$ is true, then $P_1(2n+1)$ is true. So if each number is of the form $2n$ or $2n+1$, then $P_1(n)$ is true for all n. But this is an accidental argument due to a special property of P_1; it does not work, e.g., for the P_0 above.

REMARK. P_1 provides an example of a property which is expressible in first order logic in terms of $+$, $\bullet$, and such that $(An)P_1(n)$ is provable by strong, but not by weak induction on P_1. However, P_0 above is not first order since it involves the notion of the sequence of prime factors.

3. A counterexample—the well ordering theorem. Another familiar example of the use of strong induction is in the proof that $\leqq$ well orders the natural numbers. We introduce three definitions:

$$W \text{ has a least element} \leftrightarrow (En)(n \in W \ \& \ (Am)(m < n \rightarrow m \notin W)),$$
$$P_W(n) \leftrightarrow (n \in W \rightarrow W \text{ has a least element}),$$
$$P_1(n) \leftrightarrow (AW)P_W(n).$$

Then the usual proof is by strong induction on the property P_1, viz. $P_1(1)$ is true (using $n \not< 1$); if $n' \in W$ either n' is a least element of W or (using $m < n' \rightarrow m \leqq n$) there exists $m \leqq n$ such that $m \in W$, so if $P_1(m')$ is true for all $m \leqq n$, then $P_1(n')$. [If we use the second form $\mathrm{SI}^*(P_1)$ of strong induction, then no assumptions at all about the $\leqq$ relation are needed.] In this case, the result can be proved by weak induction on P_1 by a slight change in the last step of the proof. If there exists $m \leqq n$ such that $m \in W$, then consider $W \cup \{n\}$. This has a least element by the (weak) induction hypothesis. If $W \cup \{n\} = W$, then W has a least element; if not then $n \notin W$ so there is an element $m < n$ which belongs to W, hence to $W \cup \{n\}$. Hence the least element of $W \cup \{n\}$ is not n, so it is a member of, and the least element of, W.

Of course we have made use here of the "for all W" in $P_1(n)$ by using a different W; the first argument can be used for fixed W to prove, by strong induction on P_W, that if W is nonnull it has a least element. The second argument cannot. Indeed the question, for fixed W, whether

$$\mathrm{WI}(P_W) \rightarrow (An)P_W(n), \tag{3.1$_W$}$$

takes us back to our original question in its full generality, for it can be shown fairly easily that if P is any property, then

$$\mathrm{WI}(P) \rightarrow \mathrm{SI}^*(P)$$

if and only if (3.1_W) holds for the set $W = \{n \mid \neg P(n)\}$. Hence (see Introduction) the same is true of $\mathrm{WI}(P) \rightarrow \mathrm{SI}(P)$, as soon as we can prove

$$n \not< 1, \quad m < n' \leftrightarrow m \leqq n, \quad n \neq 1 \rightarrow (Em)(n = m').$$

The use here of a different W is exactly parallel to the original proof that $(AP)\mathrm{WI}(P) \rightarrow (AP)\mathrm{SI}(P)$, by using weak induction on P^*, for

$$P_W^*(n) \leftrightarrow P_{W \cup \{n\}}(n).$$

The result to be proved by induction here is exceptional in that it is equivalent to the full induction principle $(AP)\mathrm{WI}(P)$.

4. An alternative development, where weak induction is strong. The attitude we have taken above is to regard $\leqq$, $+$, $\cdot$, etc. as undefined or primitive concepts determined only by the axioms and theorems already proved. These usually include the equations which provide the recursive definitions of the concepts, and may also include implicit definitions of some in terms of others, e.g., $x < y \leftrightarrow (Ez)(x + z = y)$. But there is a common way of developing properties of the natural numbers, where one takes 1 and $'$ as the sole primitive concepts, and defines $\leqq$ in terms of the notion of descendant, i.e. defines the set

D_x of descendants of x as the intersection of all subsets of N (the set of natural numbers), which have x as an element and are closed under successor:

(4.1.) DEFINITION.

$$D_x = \{y \mid (AW)[W \subseteq N \ \&\ x \in W \ \&\ (Az)(z \in W \to z' \in W) \to y \in W]\},$$

then puts

(4.2.) DEFINITION. $x \leqq y \leftrightarrow y \in D_x$.

Finally, general theorems on the existence and uniqueness of functions satisfying simple forms of recursive definition are proved and, using these, addition and multiplication are defined.

What can we say about the relation between weak and strong induction if we accept (4.2) as a definition of $\leqq$ (or, which amounts to the same thing, if we have a system in which (4.2) is an axiom or theorem)? Let us start with the minimal system of any real interest, where we take as axioms the Peano Axioms (leaving out, of course, the last one which is the principle of induction). Two of them merely give us the constant 1 and the uniqueness of n'. The other two are:

$$x' \neq 1, \quad x' = y' \to x = y. \tag{4.3}$$

From Definition (4.2) and Axiom (4.3) follow easily, without use of induction:

$$n \leqq n, \tag{4.41}$$

$$m \leqq n \ \&\ n \leqq p \to m \leqq p, \tag{4.42}$$

$$n \not< 1, \tag{4.43}$$

$$m < n' \to m \leqq n, \tag{4.44}$$

$$n \leqq n'. \tag{4.45}$$

[REMARK. The situation is so delicate in (4.43) that I should say we are defining $x<y$ as $x \leqq y$ & $x \neq y$; the alternative, $x \leqq y$ & $\neg y \leqq x$ would give a stronger relation and hence make SI* even weaker.]

The models for Axiom (4.3) consist of the standard model $\langle N, 1, ' \rangle$ together with any number of copies of it, any number of copies of the integers, and any number of cycles $x, x', x'', \cdots, x^{(k)} = x$. We are now considering properties P which involve 1, $'$ as the sole nonlogical constants. We have, for all such P,

$$\mathrm{WI}(P) \to \mathrm{SI}^*(P). \tag{4.5}$$

(We give the proofs of (4.5–4.7) at the end of this section.) In this context, weak induction is actually stronger than this form of strong induction, i.e.,

$$(AP)(\mathrm{SI}^*(P) \to \mathrm{WI}(P)) \text{ is false in some model for (4.3).} \tag{4.51}$$

The result that $(AP)(\mathrm{SI}^*(P) \to \mathrm{WI}(P))$ is of course easily provable when enough further properties of $\leqq$ are established; all one needs is $n < n'$(which in the presence of (4.45) is equivalent to $n \neq n'$) and $n \neq 1 \to (Em)(n = m')$.

The original form of strong induction, SI(P) is here stronger than both SI$^*(P)$ and WI(P), for one easily sees

$$\text{(4.52)} \qquad \mathrm{SI}(P) \to \mathrm{SI}^*(P),$$

$$\text{(4.53)} \qquad \mathrm{SI}(P) \to \mathrm{WI}(P).$$

(4.52) needs only (4.43), (4.44); and (4.53) needs only (4.41). Also (proof below):

(4.54) $(AP)(\mathrm{WI}(P) \to \mathrm{SI}(P))$ *is false in some model of* (4.3).

(4.55) $(AP)(\mathrm{SI}^*(P) \to \mathrm{SI}(P))$ *is false in some model of* (4.3).

The next step is to add

$$\text{(4.6)} \qquad n \neq 1 \to (Em)(n = m')$$

either as a new axiom or, more naturally, as a theorem proved by induction on the hypothesis (4.6) (WI or SI works, SI* doesn't). In the presence of (4.6) (which cuts out copies of $\boldsymbol{N}$) we have

$$\text{(4.7)} \qquad \mathrm{WI}(P) \to \mathrm{SI}(P).$$

With (4.53) this shows WI(P) and SI(P) equivalent; but they are still stronger than SI$^*(P)$, for it will be shown that results corresponding to (4.51) (and (4.55)) still hold.

Let us now add

$$\text{(4.8)} \qquad n \neq n'.$$

This is provable by WI or SI on the hypothesis (4.8) (but not by SI* as we show below in the proof of (4.51)). It merely cuts out unicycles. At last, as mentioned above, we do have SI$^*(P) \to$WI(P), hence, by (4.52) and (4.7), all forms of induction are equivalent:

$$\mathrm{WI}(P) \leftrightarrow \mathrm{SI}(P) \leftrightarrow \mathrm{SI}^*(P).$$

But this equality of (useful) status is short-lived, for they are all reduced to triviality as soon as one has proved

$$\text{(4.91)} \qquad m \leqq n \vee n \leqq m,$$

or

$$\text{(4.92)} \qquad 1 \leqq n.$$

For either of these is equivalent to the full principle of induction (AP)WI(P)! Indeed the situation described above might never arise, since one of (4.91), (4.92) might well be the first theorem to be proved—(4.92) can be proved at once either by WI or SI (but not SI*) on the property $1 \leqq n$.

Proofs of above results.

Proof of (4.5.) Let H be the hypothesis of SI$^*(P)$ i.e.:

$$H\colon (An)[(Am)(m < n \to P(m)) \to P(n)].$$

It is clearly sufficient to show that $H\to(P(n)\to P(n'))$. So let us assume H and prove $P(n)\to P(n')$. The idea of the proof is simple; let $\mathfrak{M}=\langle M, ', 1\rangle$ be any model for Axioms (4.3). This consists of (a structure isomorphic to) the standard model $\mathfrak{N}=\langle N, ', 1\rangle$, together with any number of copies of it, any number of copies of the integers I, and any number of cycles. Hypothesis H implies that $P(n)$ is true for all n in N and its copies. So $P(n')$, and hence $P(n)\to P(n')$ are true for all these n. But the mapping which sends n into itself for these n and sends $n\to n'$ for all other n, (i.e., the n in the cycles or copies of I) is an automorphism of $\mathfrak{M}$, hence $P(n)\to P(n')$ is true for all n in the cycles and copies of I as well. (See Remark below proof.) In detail, define:

$$I(x),\quad x \text{ is an } \textit{initial element}, \leftrightarrow (Ay)(y' \neq x)$$

$$FB(x),\quad x \text{ is } \textit{finitely based}, \leftrightarrow (Ey)(y \leqq x \,\&\, I(y))$$

$$\begin{cases} \phi(x) = x & \text{if } x \text{ is finitely based} \\ \phi(x) = x' & \text{otherwise.} \end{cases}$$

Now ϕ is clearly an automorphism of $\mathfrak{M}$, i.e., ϕ is one-one and $\phi(1)=1$, $\phi(x')=\phi(x)'$. (In the proof you use (4.3), (4.41), (4.45) and show $FB(x)\leftrightarrow FB(x')$.) Hence

$$P(n) \leftrightarrow P(\phi(n)).$$

So $\neg FB(n)\to(P(n)\to P(n'))$, and it remains to show (using hypothesis H) that

$$FB(n) \to (P(n) \to P(n')).$$

Actually we shall show $FB(n)\to P(n)$; since $FB(n)\to FB(n')$ this gives $FB(n)\to P(n')$, hence $FB(n)\to(P(n)\to P(n'))$. To do this, put

$$P_1(n) \leftrightarrow (Am)(m < n \to P(m)).$$

Then since $m<n'\to m\leqq n$ (4.44) we have, by H,

$$P_1(n) \to P_1(n').$$

Also

$$I(n_0) \to \neg m < n_0 \to P_1(n).$$

So the set of n's for which $P_1(n)$ holds contains all initial elements and is closed under successor. Hence, from the definition (4.2) of $\leqq$, it contains all finitely based elements. But, by H, $P_1(n)\to P(n)$, hence $FB(n)\to P(n)$.

REMARK. Most textbooks on algebra are rather vague about exactly what sort of properties *are* preserved under isomorphism. Many come close to circularity—"all algebraic properties are preserved under isomorphism; algebra is the study of those properties preserved under isomorphism." Others talk about properties which "make no reference to the individual properties of the elements." Bourbaki is the only book I know which makes a serious attempt to delimit "algebraic property." Of course all first order properties are preserved,

a fact proved in many books on logic and universal algebra. For the higher order properties, one has to be careful to avoid statements which amount to identifying elements of the structure with particular sets. Some sort of type-theory or Zermelo-Fraenkel set-theory with individuals, with the individual variables ranging over the elements of the structure, seems the appropriate logical language. I assume that such a language is used here; all the properties (e.g. $P_1(n)$ of Section 3) mentioned above are clearly preserved under isomorphism.

Proof of (4.51). What we show is that even in the presence of (4.6) as well as (4.3) there is a P and a model $\mathfrak{M}$ such that $\mathrm{SI}^*(P)\rightarrow\mathrm{WI}(P)$ is false in $\mathfrak{M}$. Take $P(n)$ to be $n\neq n'$. Then the hypotheses of $\mathrm{WI}(P)$ are satisfied, viz.: $1\neq 1'$ (by (4.3)) $n\neq n'\rightarrow n'\neq n''$ (by (4.3)). So we only need to produce a model in which $(An)P(n)$ is false, but $\mathrm{SI}^*(P)$ is true, i.e., in which the hypothesis of $\mathrm{SI}^*(P)$ is false. Take the natural model $\mathfrak{N}$ together with an element a such that $a=a'$. The hypothesis of $\mathrm{SI}^*(P)$,

$$(Am)(m < n \rightarrow m \neq m') \rightarrow n \neq n'$$

is then false for $n=a$.

Proof of (4.52), (4.53)—easy.

Proof of (4.54). We want P such that $\mathrm{WI}(P)\rightarrow\mathrm{SI}(P)$ is false in some model for (4.3). Put

$$P(n) \leftrightarrow n \geqq 1 \vee (Ey)(n = y' \ \& \ (Az)(y \neq z')).$$

Then the hypotheses of $\mathrm{SI}(P)$ are satisfied, for $P(1)$ is true by (4.41); if we assume $P(m)$ is true for all $m\leqq n$, then, since $n\leqq n$ (4.41), we have $P(n)$. If $n\geqq 1$ then, by (4.42), (4.45), $n'\geqq 1$ and $P(n')$ follows. If not, then $n=y'$, where $(Az)(y\neq z')$. By (4.45), $y\leqq n$ so by hypothesis $P(y)$ is true; but here the second alternative is false, so $y\geqq 1$, hence $y'\geqq 1$, i.e., $n\geqq 1$ in this case also, and as above $P(n')$ holds.

So all we want is a model in which the hypothesis of $\mathrm{WI}(P)$ is false (and hence also $(An)P(n)$ is false). Take $\mathfrak{N}$ together with a copy of it consisting of elements $a, a', a'', \cdots$, all distinct. Then $P(n)\rightarrow P(n')$ is false for $n=a'$.

Proof of (4.55). We want to produce a P and a model $\mathfrak{M}$ for (4.3) and (4.6) such that $\mathrm{SI}^*(P)\rightarrow\mathrm{SI}(P)$ is false in $\mathfrak{M}$. The example used for (4.51) will do by (4.53).

Proof of (4.7). Take the proof of (4.5) above replacing H by

$$P(1) \ \& \ (An)((Am)(m \leqq n \rightarrow P(m)) \rightarrow P(n')),$$

and replacing $P_1(n)$ by $P_2(n)$ defined by

$$P_2(n) \leftrightarrow (Am)(m \leqq n \rightarrow P(m)).$$

The only change needed in the proof is the justification of $I(n_0)\rightarrow P_2(n_0)$. This is because, by (4.6), $I(n_0)\rightarrow n_0=1$.

5. Another alternative development. Instead of defining $\leqq$ first it is quite common to start by showing the existence of a unique function $+$ satisfying

$$x + 1 = x', \tag{5.1}$$

$$x + y' = (x + y)', \tag{5.2}$$

and then to define

$$x < y \leftrightarrow (Ez)(x + z = y). \tag{5.3}$$

This is the route followed by E. Landau in *Foundations of Analysis*. The existence of a unique function $+$ satisfying (5.1) and (5.2) is easily proved by (weak) induction; indeed, as L. Henkin showed in the paper referred to above, the proof does not need the other Peano Axioms (4.3) at all.

Let us examine the relation between weak and strong induction here. First let us note that the theorem on the uniqueness of $+$ is, together with the Peano Axioms (4.3), very nearly equivalent to the whole induction axiom. For let $\mathfrak{M} = \langle M, ', 1\rangle$ be any model satisfying (4.3) and this theorem. By (4.3) it contains a natural part isomorphic to the standard model $\mathfrak{N} = \langle N, ', 1\rangle$. For convenience suppose M contains N and refer to the elements of N as standard elements. (5.1) and (5.2) force $x+y=x^{(y)}$ $(=x''\cdots'(y \text{ times}))$ for standard y, but it is easy to find apparently different ways of completing the definition for non-standard y. It is easy to check that

$$\left\{\begin{array}{ll} x + y = x^{(y)} & \text{for standard } y \\ x + y = y & \text{for non-standard } y \end{array}\right\} \quad \left\{\begin{array}{ll} x + y = x^{(y)} & \text{for standard } y \\ x + y = y' & \text{for non-standard } y \end{array}\right\}$$

both satisfy (5.1) and (5.2)—all you need to show is that if y is non-standard, so is y'. So if $+$ is unique, we must have $y = y'$ for all non-standard y. Also there can be at most one such y; if there are two, y_1 and y_2, then $x+y=y_1$ for all non-standard y, and $x+y=y_2$ for all non-standard y both satisfy (5.1) and (5.2) and are different if $y_1 \neq y_2$. So the only possible non-standard model left is $\mathfrak{N}$ together with a single element ω satisfying $\omega' = \omega$. This model does satisfy the uniqueness condition, since $x+\omega=\omega$ is forced.

So as soon as you have proved $x \neq x'$ (Landau actually proves this first), or that $<$ as defined by (5.3) is an ordering, or the right cancellation law for $+$, you have the full induction axiom. Even without this, i.e., on the basis of Peano Axioms (4.3), uniqueness of $+$ and definition (5.3) alone, all our forms of induction on any property P are equivalent.

For all three hypotheses:

$$P(1) \;\&\; P(n) \rightarrow P(n'),$$

$$P(1) \;\&\; (Am)(m \leqq n \rightarrow P(m)) \rightarrow P(n'),$$

$$(Am)(m < n \rightarrow P(m)) \rightarrow P(n),$$

are, when postulated for all standard n, equivalent to each other and to $P(n)$ being true for all standard n. Also they are all true (or$\leftrightarrow P(1)$) for $n=\omega$ (since

$\omega=\omega'$, $\omega\leqq\omega$, and $\omega<\omega$). So

$$\mathrm{WI}(P) \leftrightarrow \mathrm{SI}(P) \leftrightarrow \mathrm{SI}^*(P).$$

But the situation is still delicate; there is a difference here if we start the natural numbers from 0 instead of 1; define $x\leqq y\leftrightarrow(Ez)(x+z=y)$ (or, starting from 1, as $(Ez)(x+z=y')$, but this is unnatural) and $x<y\leftrightarrow x\leqq y\,\&\,x\neq y$. Then we have

$$\mathrm{WI}(P) \leftrightarrow \mathrm{SI}(P) \rightarrow \mathrm{SI}^*(P),$$

but not (until we have $n\neq n'$ or one of the alternatives above) $\mathrm{SI}^*(P)\rightarrow\mathrm{SI}(P)$. For it is easily checked using this model that if $P(n)$ is $n\neq n'$ then $(An)P(n)$ can be proved by $\mathrm{SI}(P)$ but not by $\mathrm{SI}^*(P)$.

If one takes $+$ as a primitive relation satisfying (5.1) and (5.2) (but does not postulate uniqueness), then, as soon as one has proved commutativity, associativity, and $n\neq n'$, it is easy to show that postulating the uniqueness of a multiplication satisfying

$$x\cdot 1 = x, \qquad x\cdot y' = x\cdot y + x,$$

implies the full induction axiom.

6. Conclusions. Is weak induction really weaker than strong? Despite the evidence to the contrary given, for two very standard developments of arithmetic, in the last two sections, the reasonable answer must surely be yes! If one is working in a system containing some theorem (like $1\leqq n$ or the uniqueness of $+$ together with $n\neq n'$) which is equivalent to the full induction axiom, then in practice one uses this (if one realizes its power at all) via the much more convenient standard forms of induction (or the least number principle). It is reasonable to ask whether a result which can be established by strong induction can be established by weak induction on the same property, provided one is thinking of omitting the theorem which is itself equivalent to the full induction theorem, or omitting the definitions which make it so. In other words, the original viewpoint of Section 2 is, in my opinion, the most natural one to take; to regard all the relations and operations introduced at the time of use of induction as primitive terms satisfying as axioms all the theorems and definitions previously established, with the exception of higher order ones (definition of $\leqq$ by descendant, uniqueness of $+$) which in practice are not used directly in proofs.

[REMARK. The referee has suggested it is worth pointing out that the essential difference between Section 2 and Sections 3, 4, and 5 is that the properties considered in the latter sections are essentially 2nd order while those of Section 2 are essentially 1st order. When the 2nd order properties, together with a few basic axioms are assumed, the distinction between weak and strong induction breaks down.]

7. Why is weak induction usually strong enough? The examples where $\mathrm{WI}(P)\rightarrow\mathrm{SI}(P)$ fails which one thinks of naturally are relatively complicated.

Are there any simpler ones? Let us take the attitude of Sections 2 and 6 and look for some. At a very early stage they can be found. Indeed the familiar proof of the basic result

$$(AP)\mathrm{WI}(P) \rightarrow (AP)\mathrm{SI}(P)$$

uses $m<n' \rightarrow m \leqq n$, $n \leqq n$, $n \nless 1$ and is not true in the presence of the first two only, as the model 1, 2, 3 $\cdots$ with $m \leqq n$ for all m, n shows. Even

$$\mathrm{SI}(P) \rightarrow \mathrm{WI}(P)$$

requires $n \leqq n$ (take $P(n)$ to be $n \leqq n$ and consider the model consisting of $\mathfrak{N}$ plus $\omega = \omega'$ with standard $\leqq$ relation on all standard $n \leqq \omega$ but $\omega \nless \omega$). [The need for $n \leqq n$ in these last two cases can be avoided by rewriting the hypothesis of $\mathrm{SI}(P)$ in the form

$$P(1) \,\&\, [P(n) \,\&\, (Am)(m < n \rightarrow P(m)) \rightarrow P(n')].$$

Also the proof of

$$\mathrm{SI}(P) \rightarrow \mathrm{SI}^*(P)$$

uses $m<n' \rightarrow m \leqq n$ and $n \nless 1$ (to show the former is not enough take $P(n)$ as $n \nless 1$ and use the model with $\mathfrak{N}$ and $\omega = \omega' <$all members of N).

But these are prenatal examples; the usual starting point would be the Peano Axioms (4.3) and the recursive definition of $<$.

[REMARK. Or of $\leqq$ by $m \leqq n' \leftrightarrow m = n' \vee m \leqq n$, $n \leqq 1 \leftrightarrow n = 1$. This is not equivalent to (7.11) and (7.12) until $n' \nleqq n$ has been proved, but the remarks below apply to this version also.]

The recursive definition of $<$ is

(7.11) $$n \nless 1,$$

(7.12) $$m < n' \leftrightarrow m < n \vee m = n.$$

Then $m \leqq n$ is defined as $m<n \vee m=n$. We show

(7.13) $$n < n',$$

(7.14) $$1 < m \,\&\, m < n \rightarrow 1 < n,$$

(7.15) $$m < n \rightarrow m' \leqq n,$$

(7.16) $$n \neq 1 \rightarrow (Em)(n = m').$$

(7.13) is an immediate consequence of (7.12), and (7.14–16), provable by weak induction on n.

From this point on there are no examples where $\mathrm{WI}(P) \not\rightarrow \mathrm{SI}(P)$ with P involving only 1, $'$, $<$. The reason is that the argument used in the proof of (4.7) shows that for all such P.

$$\mathrm{WI}(P) \rightarrow \mathrm{SI}(P).$$

The main change needed in the proof, now that $<$ is not defined in terms of $'$,

is to show, using (7.11–5), that the ϕ defined there is an automorphism with respect to $<$ as well as 1, $'$. [In the definition of finitely based, the old 'descendant' definition of $\leqq$ is still used of course.]

When $+$ is introduced this argument breaks down and I can't think of any axioms whose addition would allow it to be patched up. There certainly are examples where $\mathrm{WI}(P) \not\rightarrow \mathrm{SI}(P)$, even first order ones, e.g.,

$$P(n) \leftrightarrow (Em)(n = 2m \vee n = 2m + 1 \vee n = 3m)$$

(i.e., n is even or odd or a multiple of three). But I don't think there are any natural first order ones involving 1, $'$, $+$, $<$ only. For all the textbook developments of the elementary properties of $+$ (commutativity, associativity, etc.) which I have seen, or any natural alternative I have tried, proceed by weak induction on these obvious properties to the point where one has proved all the axioms necessary for Presburger's decision method and first order completeness. This means that from then on all first order statements are provable without any further use of induction, so that strong induction is never called for.

Once $\cdot$ is introduced there are, as we have seen in Section 2, natural examples, even first order ones.

What we have done in this section is simply to give evidence of the fact that weak induction is usually sufficient, rather than an answer to the question in the section head, which calls for an explanation of why this is so. Is there one—or is it a meaningless question?

The connection between this and the ordering of the set $W = \{n \mid \neg P(n)\}$ is an odd one. $\mathrm{WI}(P) \rightarrow \mathrm{SI}^*(P)$ is equivalent to "If W is nonnull, it either has a least element or is closed under predecessor." [This equivalence is true as soon as we have existence of predecessor. The same is true of $\mathrm{SI}(P)$ as soon as we have also $m < n' \leftrightarrow m \leqq n$.]

Is there a good reason why this should usually be provable (without induction of course)?

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THE PICARD THEOREMS

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1. Introduction. The two theorems of Picard that we shall discuss are among the most famous results of complex function theory and have had a decisive effect on its development. They are classified according to size. The Small Picard Theorem states:

(1) *The values assumed by a nonconstant entire function, i.e., one which is holomorphic in the entire finite plane, cover the finite plane P with the exception of at most one point.*

The Great Picard Theorem is a local version of the above; it asserts:

(2) *The values assumed by a function holomorphic in the neighborhood of an isolated essential singularity cover P with the exception of at most one point.*

These theorems remind us of two elementary theorems, which they deepen in a remarkable way:

LIOUVILLE'S THEOREM. *The values assumed by a nonconstant entire function cover an unbounded portion of P.*

WEIERSTRASS-CASORATI THEOREM. *The values assumed by a function holomorphic in the neighborhood of an isolated singularity lie everywhere dense in P.*

Both of these theorems will be used in the proof of the Picard theorems.

Picard's own proof of his theorems made use of the elliptic modular function of Jacobi (referred to hereafter as the modular function), and appeared in 1879. In 1904 Landau and Schottky proved theorems, now known by their names, from which the Picard theorems can be derived and whose proofs avoid the modular function; Landau called this approach "elementary." In advanced complex analysis courses it is frequently the practice to prove the Small Picard Theorem by means of the modular function, which can be introduced in a variety of ways, but then to turn to the "elementary" approach for the derivation of the Great Picard Theorem.

It is the purpose of this lecture to present the view that there are some advantages in a unified approach that uses the modular function to prove both Picard theorems (in other words, Picard's original idea):

A. The idea of both proofs is the same; it is a very simple idea and easily remembered. The additional concepts required to prove the Great Theorem, once the properties of the modular function have been developed, are mostly known from the student's previous work or are useful additions to his knowledge.

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B. The modular function can be used to provide simple proofs of the main theorems of the "elementary" approach, and to prove the fundamental theorem in the theory of normal families (Montel's original proof). Cf. Carathéodory [3].

C. In any deeper study the modular function cannot be avoided, for it appears explicitly in the exact expression for the constant ("Landau radius") in Landau's theorem (see [3]). It is also used in deriving the best of the known bounds for the Bloch and Landau constants, which appear in further theorems of the "elementary" approach [2, 9].

D. The modular function is an essential tool in other branches of mathematics, such as analytic number theory, the theory of automorphic functions, and special conformal mappings.

In this lecture we shall observe the ground rule that material appearing in L. V. Ahlfors' textbook [1] may be quoted without detailed proof. Thus we shall list the essential properties of the modular function but refer for the proof of its existence to Ahlfors. We shall prove the Small Theorem—a very short proof indeed—and go on to show how the Great Theorem can be derived by the same basic method.

2. The Small Theorem [6]. In what follows we shall use certain abbreviations:

S = complex sphere (extended finite plane)
P = finite plane
H = upper half-plane
D = open disk, $0 \leqq |z-a| < r$
D' = deleted disk, $0 < |z-a| < r$.

Before introducing the modular function, let us recall some of the main properties of the exponential function $w = e^z$. This function is holomorphic in P, its range is $S - \{0, \infty\}$, and it has an inverse, $z = \log w$, which is locally holomorphic for $w \neq 0, \infty$.

The modular function, $w = \lambda(\tau)$, by analogy and contrast, has the following properties. These properties do not characterize λ uniquely, but any function possessing them will do for our present purposes. Later we shall need additional properties.

(i) λ *is defined and holomorphic in* H,

(3) (ii) *the range of* λ *is* $S - \{0, 1, \infty\}$,

(4) (iii) $w = \lambda(\tau)$ *has an inverse* $\tau = L(w)$ *that is locally holomorphic on the range of* λ, *i.e., for* $w \neq 0, 1, \infty$.

The inverse function $L(w)$ has infinitely many branches, which are connected by the transformations of a group M (to be defined in the next section). But whatever branch we are on, it is always true that $L(w)$ *lies in* H.

The existence of $\lambda(\tau)$ is proved in Ahlfors [1, pp. 269–274] as the culmination of his treatment of elliptic functions (beginning on p. 257).

Let us now assume that we have an entire function $g(z)$ that does not take on either of the distinct values α or β. Then

$$f(z) = \frac{g(z) - \alpha}{\beta - \alpha}$$

is an entire function that is never equal to 0 or 1, and f and g are both constant or both nonconstant. The inspiration of Picard was to consider the composite function

$$\phi(z) = L \circ f(z). \tag{5}$$

The function ϕ is not *a priori* single-valued, because of the many-valuedness of L. Starting with a definite value for $L(f(0))$ we can, however, continue ϕ analytically to any point z along any path in the z-plane, and ϕ will remain holomorphic at each point of the path. This is seen most easily by constructing the derivative $\phi'(z)$ by the chain rule; $L'(f(z))$ always exists by (4), because $f(z)$ never passes through 0, 1, or ∞, and $f'(z)$ always exists because f is entire. In this situation we can apply the Monodromy Theorem; since P is simply-connected, $\phi(z)$ is single-valued in the whole plane. So even though L is multiple-valued, the *composite* function $\phi = L \circ f$ is globally single-valued in P; *ϕ is an entire function of z.*

Now ϕ takes only values in H, hence

$$u(z) = e^{i\phi(z)}$$

is an entire function that is bounded ($|u| < 1$) and so, by Liouville's theorem, is a constant. This implies $\phi(z) \equiv$constant ($0 = u'(z) = e^{i\phi(z)}\phi'(z)$) and consequently also $f(z) \equiv$constant, for from (5),

$$f(z) = \lambda(\phi(z)).$$

This is the end of the proof.

Following J. E. Littlewood [5] we can write out this proof in one line:

$$f \text{ entire}, f \neq 0, 1 \Rightarrow e^{iL(f(z))} \text{ is entire and bounded}, \Rightarrow f = \text{constant}.$$

Littlewood also speculates as to what a referee of the day might have said about this result: "Exceedingly striking and a most original idea. But, brilliant as it undoubtedly is, it seems more odd than important; an isolated result, unrelated to anything else, and not likely to lead anywhere."!!

3. The Great Theorem. Beginning of the proof. In this exposition we follow mainly Osgood [6]. For other expositions see Picard [8] and Julia [4].

For the Great Theorem we need another property of $\lambda(\tau)$ analogous to one possessed by e^z. This function is periodic with period $2\pi i$, in other words, e^z is invariant with respect to the simply periodic group of linear-fractional transformations generated by $z \rightarrow z + 2\pi i$. Moreover, $e^{z_1} = e^{z_2}$ implies $z_2 = Tz_1$ for a transformation T in this group. The corresponding property of λ is as follows:

(6) $\lambda(T\tau) = \lambda(\tau)$ for each linear-fractional transformation

$$T\tau = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1,$$

where a, d are odd integers, b, c are even integers. The group of such transformations will be denoted by M. Furthermore,

$$\lambda(\tau_1) = \lambda(\tau_2) \quad \text{implies} \quad \tau_2 = T\tau_1 \tag{7}$$

for some T in M. Since L is the inverse function to λ we certainly have

$$\lambda(L(w)) = w, \ w \neq 0, 1, \infty. \tag{7a}$$

We now assume $f(z)$ to be holomorphic and $f(z) \neq 0$, 1 in D', an arbitrary deleted disk with center $z = a$. As before we form the function

$$\phi(z) = L \circ f(z).$$

The present situation differs from the previous one in two respects:

(i) We cannot apply the Monodromy Theorem to ϕ, for the domain D' is not simply-connected. In fact ϕ is in general not single-valued.

(ii) We cannot apply Liouville's Theorem to ϕ even if it were single-valued, for the domain of ϕ is a disk, not the whole plane.

The second difficulty can be overcome by using the Weierstrass-Casorati Theorem in place of Liouville's Theorem. The first difficulty can be treated only by studying the many-valuedness of ϕ.

For this purpose let us mention a version of the Monodromy Theorem that has the advantage of being usable in any region, simply connected or not.

LEMMA 1. *Let Ω be a region in P and let C_1, C_2 be continuous oriented curves that are homotopic in Ω and have the same initial point z_0 and the same endpoint z_1. If $(z_0, g(z_0))$ is a function element of the function $g(z)$, and if g can be continued analytically to $z \in \Omega$ along every path connecting z_0 and z and lying in Ω, then continuation along C_1 and along C_2 leads to the same function element $(z_1, g(z_1))$.*

A proof is in Ahlfors [1, pp. 285–287]. Two curves are called homotopic if one can be deformed continuously into the other. The precise definition makes use of a continuous function of two variables t, u mapping the "deformation square" $0 \leqq t, u \leqq 1$ onto a family of curves passing from $C_1(u=0)$ to $C_2(u=1)$.

Let C be a circle with center at $z=a$ that lies in D' and is oriented positively. Let us distinguish an initial point z_1 on C and a terminal point z_2; z_1 and z_2 are of course the same geometric point. Because $f \neq 0, 1, \infty$ in D', ϕ is locally holomorphic throughout D'. Let us fix the value $\tau_1 = \phi(z_1)$ at z_1; then by analytic continuation along C we obtain the value $\tau_2 = \phi(z_2)$. Now $\lambda(\tau_2) = \lambda(\phi(z_2)) = \lambda(L(f(z_2))) = f(z_2)$ and likewise $\lambda(\tau_1) = f(z_1)$. Since z_1 is the same point as z_2, we have $\lambda(\tau_1) = \lambda(\tau_2)$, and then (7) shows that

$$\tau_2 = T\tau_1 \tag{8}$$

for some $T \in M$. By Lemma 1, continuation along any path homotopic to C in D' would lead to the same value τ_2. Here we need

LEMMA 2. *Every closed path in D' starting at z_1 is homotopic to mC for some integer m. Here mC is the (oriented) circle C described m times if $m>0$; it is C^{-1} described $|m|$ times if $m<0$, where C^{-1} is the circle described with reverse orientation; and it is the path consisting of the point z_1 (the "null path") if $m=0$.*

Proof in Ahlfors [1, pp. 287–289]. In topological language the lemma says that the "fundamental group" of the deleted disk is the infinite cyclic group.

Starting from a fixed point z_0 in D', denote by $\phi_0(z)$ the value obtained by continuation along the straight line C_0 connecting z_0 and z. (If this line passes through the center of D', make a slight detour to the right of the center.) Let C_1 be any path in D' connecting z_0 and z, and let $\phi(z)$ be the value obtained by continuation from z_0 to z along C_1. By $C_0^{-1}C_1$ we mean the path obtained by traversing C_0^{-1} first, then C_1. Thus $C_0^{-1}C_1$ is a closed path that starts and ends at z, and so by the discussion following Lemma 1 we conclude that $\phi(z)=W\phi_0(z)$ for some W in M.

But, by Lemma 2, $C_0^{-1}C_1$ is homotopic to mC for some integer m, where C is the circle with center $z=a$ that passes through z_0. If continuation along C leads to the transformation T—see (8)—we have the following result: the values of $\phi(z)$ obtained by continuation from a fixed point z_0 to the point z along all possible paths in D' lie in the set

$$\{T^m\phi_0(z),\ m = \text{integer}\}, \tag{9}$$

where T is a definite transformation of the group M *that does not depend on* z, and $\phi_0(z)$ is the value obtained by continuation along the straight line connecting z_0 and z.

It is crucial here that T is a fixed transformation of M. But we do not know *which* transformation it is, and so we shall have to consider all possibilities.

In his first course in complex variables the student finds out a bit about linear-fractional transformations. He learns that there are 4 types, which can be distinguished by the value of the trace $a+d$:

$$a+d \text{ real}\begin{cases}\text{elliptic if } |a+d| < 2\\ \text{parabolic if } |a+d| = 2\\ \text{hyperbolic if } |a+d| > 2\end{cases}$$

$$a+d \text{ nonreal: loxodromic.}$$

Since, in M, $a+d$ is an integer, there are no loxodromic elements, and the elliptic elements, if any, have trace 0 or ± 1. But trace ± 1 is impossible because $a+d$ is even. And $a+d=0$ implies $ad-bc=-a^2-bc=1$; this is impossible with a odd and b, c even, for a square is never congruent to -1 modulo 4.

LEMMA 3. *Each $T\in M$ falls into exactly one of the following classes:*

$T_1\tau = \tau$

$T_2\tau = \tau + m, \qquad m = \textit{integer} \neq 0$

$$T_3\tau = \tau', \qquad \frac{1}{\tau' - r} = \frac{1}{\tau - r} + c, \qquad \begin{array}{l} r \text{ rational}, \\ c = \text{integer} \neq 0 \end{array}$$

$$T_4\tau = \tau', \qquad \frac{\tau' - \rho_1}{\tau' - \rho_2} = A\,\frac{\tau - \rho_1}{\tau - \rho_2}, \qquad \rho_1 > \rho_2, \qquad A > 0, \qquad A \neq 1.$$

The normalization in T_4 makes it possible to choose a branch for which

$$0 < \arg \frac{\tau - \rho_1}{\tau - \rho_2} < \pi. \tag{9a}$$

We need one additional property of $\lambda(\tau)$. It is proved in Ahlfors [1, p. 272] that λ tends to a definite limit (finite or infinite) as τ tends to -1, 0, or $i\infty$ (the so-called cusps of a "fundamental region" for the group M) within a "cusp sector" (the region bounded by two circular arcs orthogonal to the real axis and tangent at the point; in the case of $i\infty$, a vertical strip). We wish to show the same is true as τ approaches any rational point. For this purpose, it is only necessary to prove that each rational p/q is M-equivalent to either -1, 0 or $i\infty$. For if $V(\infty)=p/q$, for example, with $V\in M$, then $\lambda(V\tau)=\lambda(\tau)$ and we have only to let $\tau\rightarrow i\infty$.

There are 3 cases: p odd, q even; p even, q odd; p and q both odd; and in all cases p is prime to q. Let us, for example, show that in the first case p/q is M-equivalent to $i\infty$. Write $p=2a+1$, $q=2c$. We want to find b, d, such that

$$V = \begin{pmatrix} 2a+1 & 2b \\ 2c & 2d+1 \end{pmatrix} \in M, \quad \text{i.e.,} \quad (2a+1)(2d+1) - 4bc = 1.$$

This reduces to $(2a+1)d-2cb=-a$, and this can be solved in integers b, d since g.c.d.$(2a+1, -2c)=$g.c.d.$(p, -q)=1$. Now $V(i\infty)=(2a+1)/2c=p/q$, as required. The other cases are similar.

4. Proof of the Great Theorem. We recall that $f(z)$ is holomorphic in an arbitrary deleted disk D' about $z=a$, an essential singularity of f, and that f omits the values 0 and 1 in D'. Thus $f(z)\neq 0, 1, \infty$ for z in D'. The function

$$\phi(z) = L\circ f(z)$$

is holomorphic (but not single-valued) in D'. Starting with a definite value $\tau_0=\phi(z_0)$, we continue ϕ analytically to all points of D'. As we have seen, the various branches of ϕ are related by a unique transformation T of the group M; to be precise, if $\phi_0(z)$, $\phi(z)$ are two determinations of ϕ, then $\phi(z)=T^m\phi_0(z)$ for some integer m.

We can now treat the first difficulty mentioned at the beginning of Section 3. To restore single-valuedness, we shall compose ϕ with a further mapping ξ that is invariant under T. This will have the effect of wiping out the ambiguities of ϕ and so rendering the composed map $u=\xi\circ\phi$ singled-valued. Furthermore we can choose ξ so that the range of u is bounded, and so be in a position to apply Riemann's theorem and remove the singularity at $z=a$.

For each of the four classes T_i discussed in Lemma 3 we introduce a corresponding mapping ξ_i as follows:

$$
\begin{aligned}
\xi_1 &= e^{2\pi i\tau} \\
\xi_2 &= e^{2\pi i\tau/|m|} \\
\xi_3 &= e^{-2\pi i/|c|(\tau-r)} \qquad (10)\\
\xi_4 &= \exp\left[2\pi i \log\frac{\tau-\rho_1}{\tau-\rho_2}\Big/ |\log A|\right].
\end{aligned}
$$

In ξ_4 we can define a unique branch of the logarithm by virtue of (9):

$$\log\frac{\tau-\rho_1}{\tau-\rho_2} = \log\left|\frac{\tau-\rho_1}{\tau-\rho_2}\right| + i\arg\frac{\tau-\rho_1}{\tau-\rho_2}\cdot$$

The functions ξ_i have the all-important properties:

(11) ξ_i is holomorphic in H

(12) $\xi_i(T_i\tau) = \xi_i(\tau)$

(13) $|\xi_i(\tau)| < 1, \qquad \tau \in H.$

This is immediately checked from the definitions and (9).

Now set up the composite function

$$u_i(z) = \xi_i \circ \phi(z) = \xi_i \circ L \circ f(z), \qquad i = 1, 2, 3, 4. \tag{14}$$

Let U be the unit disk. Each u_i maps D' into U, by (12) and the fact that $L \circ f$ lies in H. Moreover, u_i is holomorphic at each point of D', for $f(z)$ is never 0, 1, ∞ (the singularities of L) and $L \circ f$ avoids the singularities of ξ_i, which are all real. Besides this, u_i is single-valued, for two determinations of $L \circ f$ differ by a power of T_i and are therefore mapped into the same point of U by ξ_i, because of (12). The function $u_i(z)$, then, is a single-valued holomorphic map of D' into U.

Since U is a bounded region we can extend u_i to the full disk D. We still call the extended function u_i and now have:

(15) *$u_i(z)$ is holomorphic in D for $i = 1, 2, 3, 4$.*

From this we wish to show that f is holomorphic at $z=a$, thus providing a contradiction.

For this purpose we solve for f from (14). Note first that $\xi_i(\tau_1) = \xi_i(\tau_2)$ implies $\tau_1 = T_i^m\tau_2$. This can be seen directly from the definitions (10). For example, $\xi_4(\tau_1) = \xi_4(\tau_2)$ yields

$$\log\frac{\tau_1-\rho_1}{\tau_1-\rho_2} = \log\frac{\tau_2-\rho_1}{\tau_2-\rho_2} + m|\log A|,$$

and exponentiation yields our statement. It follows that

$$\xi_i^{-1}(w) = \{T_i^m\tau \mid m = \text{integer}\},$$

where τ is any number such that $\xi_i(\tau) = w$. Now λ is invariant under T_i, since T_i belongs to the group M. Hence

$$\lambda(\xi_i^{-1}(w)) = \lambda(\tau), \qquad w = \xi_i(\tau).$$

Applying this to (14):

$$\lambda(\xi_i^{-1}(u_i(z))) = \lambda(L(f(z))), \tag{16}$$

since $\xi_i(L(f(z))) = u_i(z)$. But, as we noted in (7a), the right member of (16) equals $f(z)$, so we have

$$f(z) = \lambda(\xi_i^{-1}(u_i(z))). \tag{17}$$

Attention now centers on $u_i(a)$, which certainly exists because of (15). If $u_i(a) \neq 0$, it is an image of some $\tau(=L \circ f(z))$ lying in H. Since ξ_i' vanishes only for real τ, ξ_i^{-1} is continuous at $u_i(a)$ and λ is continuous in all of H, so $f(z)$ tends to a finite value as $z \to a$. This is always the situation for T_4, since by (9),

$$e^{-c} < |\xi_4(\tau)| < 1, \qquad c = 2\pi^2/|\log A|;$$

thus $u_i(a)$ cannot vanish.

If $u_i(a) = 0$, then $u_i(z) \to 0$ as $z \to a$ and $\xi^{-1}(u_i(z)) \to i\infty$, $i\infty$, or r, according as $i = 1, 2, 3$. Since λ unites the various branches of ξ^{-1}, we can assume that $\xi^{-1}(u_i(z))$ lies within a cusp sector at the point in question and can deduce by the remark at the close of Section 3 that $f(z) = \lambda \circ \xi^{-1}(u_i(z))$ tends to a definite (finite or infinite) limit as $z \to a$.

In every case, then, there is a deleted disk D' about $z = a$ such that the values $f(D')$ are not dense in the plane; in fact, $f(D')$ is confined to the neighborhood of a single point (which may be ∞). By the Weierstrass-Casorati theorem, f does not have an essential singularity at $z = a$. This contradicts the hypothesis, and the Great Picard Theorem is proved.

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FLYING IN A WIND FIELD, II

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1. Introduction. We shall be considering the round trip flight of an airplane flying between two cities. It will be assumed that air speed of the plane with respect to the wind and the wind velocity are constant, also that the wind velocity and the path of the plane lie in a horizontal plane. The corresponding 3-dimensional flight problems are more complicated and will be treated in a subsequent paper.

In the previous paper of the same title [1], it was shown geometrically that the shortest time flight is the same as the shortest distance flight (with respect to ground). Here we assume that the airplane is using ADF (automatic direction finder) [2], so at all times the plane is pointed to the city of destination. Consequently, we have the classical pursuit problem of the duck on the river [3], i.e., "A duck swims across a constant flowing river, always heading for a given point on the bank; find the path." We shall first show that the round trip flight time is independent of the wind direction. Although one of us had read of this result in a newspaper, we have not been able to trace a reference for it in the literature. Additionally, we shall relate this problem to another classic pursuit problem, the dog and his master [3], i.e., "A dog out in the field sees his master walking along the road and runs toward him, always heading for his master; find the path." Finally, we shall give several methods for calibrating the air speed (with respect to the wind) indicator when flying in a piecewise uniform wind field. One of the methods will use the results of the first problem.

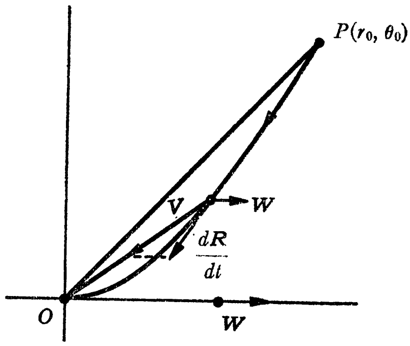

FIG. 1

2. The flight problem. We choose a polar coordinate reference system with the polar axis parallel to the wind velocity. We start from the point (r_0, θ_0), head to the origin and then back again. To show that the total time of flight is independent of the direction of the wind velocity is equivalent here to showing that the flight time is independent of θ_0.

Let R denote the position vector of the plane and W the wind velocity. Then the equation of motion for the trip from P to O is given by

$$\frac{dR}{dt} = -\frac{v}{r}R + W.$$

Here v is the speed of the plane with respect to wind and $r = |R|$. Set $R = re^{i\theta}$ to obtain

$$\frac{dr}{dt} = -v + w\cos\theta, \tag{1}$$

$$r\frac{d\theta}{dt} = -w\sin\theta \qquad (w = |W|). \tag{2}$$

Eliminate t:

$$\frac{dr}{r} = \left\{\frac{v}{w\sin\theta} - \frac{\cos\theta}{\sin\theta}\right\}d\theta.$$

Integrate:

$$r = \frac{r_0\sin\theta_0}{\sin\theta}\left\{\frac{\tan\theta/2}{\tan\theta_0/2}\right\}^{\lambda}, \tag{3}$$

where $\lambda = v/w$. Substitute (3) back in (2):

$$-\frac{d\theta}{dt} = \frac{w\sin^2\theta}{r_0\sin\theta_0}\left\{\frac{\tan\theta_0/2}{\tan\theta/2}\right\}^{\lambda}.$$

Thus the time of flight from P to O is

$$t_1 = \int_0^{\theta_0}\frac{r_0\sin\theta_0}{w\sin^2\theta}\left\{\frac{\tan\theta/2}{\tan\theta_0/2}\right\}^{\lambda}d\theta. \tag{4}$$

To integrate (4), we use the identity

$$\frac{d\theta}{\sin^2\theta} = \frac{(\tan^2\theta/2 + 1)}{2\tan^2\theta/2}d\tan\theta/2.$$

Whence,

$$2t_1 = \frac{r_0}{w}\sin\theta_0\left\{\frac{\tan\theta_0/2}{\lambda+1} + \frac{\tan^{-1}\theta_0/2}{\lambda-1}\right\}. \tag{5}$$

We have also assumed that $w<v$, otherwise the plane would never get to its destination.

The time of flight for the return trip can be obtained by also flying from (r_0, θ_0) to the origin, provided that we change W to $-W$. Thus the equations of motion are

$$\frac{dr}{dt} = -v - w\cos\theta, \tag{6}$$

$$r\frac{d\theta}{dt} = w\sin\theta. \tag{7}$$

Proceeding as before, we obtain

$$r = \frac{r_0 \sin\theta_0}{\sin\theta}\left\{\frac{\tan\theta_0/2}{\tan\theta/2}\right\}^{\lambda}, \tag{8}$$

$$t_2 = \int_{\theta_0}^{\pi} \frac{r_0 \sin\theta_0}{w \sin^2\theta}\left\{\frac{\tan\theta_0/2}{\tan\theta/2}\right\}^{\lambda} d\theta,$$

$$2t_2 = \frac{r_0}{w}\sin\theta_0\left\{\frac{\tan\theta_0/2}{\lambda - 1} + \frac{\tan^{-1}\theta_0/2}{\lambda + 1}\right\}. \tag{9}$$

Whence,

$$T = t_1 + t_2 = \frac{2r_0 v}{v^2 - w^2}, \tag{10}$$

which is independent of θ_0. This also shows that for the kind of round trip flight considered, the wind increases the time of flight. That this result is also true more generally, see [1].

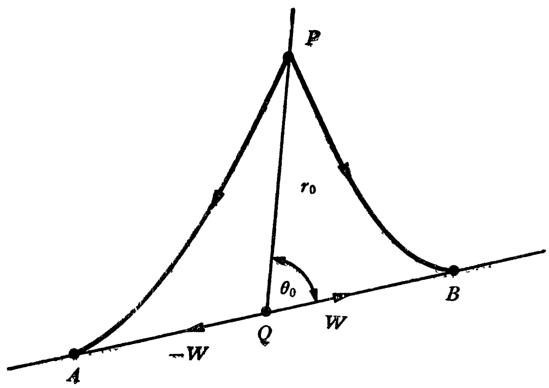

FIG. 2

3. Dog and master problem. We now relate the pursuit problem of the duck on the river with the one of the dog and his master. (This relationship appears in [4]. However, it is not very well known and, additionally, it has not been fully exploited.) To do this, we employ the same device of using a moving coordinate system as was done in [1]. The time of flight from P to O will be exactly the same if we superimposed a negative wind velocity $-W$ on the entire system.

This is equivalent to the airplane flying without being subjected to the wind. However, the point of destination is now moving with a velocity of $-\boldsymbol{W}$. Thus we have transformed the first type pursuit problem into the second type. Using the result in (10), we then have the equivalent result, as shown in Fig. 2.

If a dog starts running from P with a speed v and always heads directly for his master who started moving (at the same time as the dog) from Q along rays QB or QA with a speed of w, then the sum of the times for capture at A plus capture at B is independent of θ_0 and equals T as before. Equivalently, $AB=wT$, also independent of θ_0.

For $\theta_0=0$, the result is trivial since obviously

$$T = \frac{r_0}{v+w} + \frac{r_0}{v-w} = \frac{2r_0 v}{v^2 - w^2}\,.$$

However, for $\theta_0=\pi/2$, we obtain the nontrivial result that the time for capture at A or B is $T/2$ by symmetry.

The equation for the pursuit curve PB can be gotten immediately from the trajectory given by Eqs. (3) and (2) by superimposing a wind field of $-\boldsymbol{W}$. Referring back to the polar coordinate system in section 2, we find that the parametric equations for PB in terms of θ are

$$x = r\cos\theta - wt,$$
$$y = r\sin\theta,$$

where r is given by (3) and

$$t = \int_{\theta}^{\theta_0} \frac{r_0 \sin\theta_0}{w\sin^2\theta}\left\{\frac{\tan\theta/2}{\tan\theta_0/2}\right\}^{\lambda} d\theta$$

(the latter integral can be integrated as in (4)).

For the special case where $v=w$, it follows from (4) that $t_1=\infty$. Thus the airplane will not reach the origin but will approach the radial line $\theta=0$ asymptotically at $r=r_0\cos^2\theta_0/2$ (from (3)). Equivalently, the dog will approach the ray QB asymptotically such that his distance behind his master approaches $r_0\cos^2\theta_0/2$. This last result is fairly well known for $\theta_0=\pi/2$ but not apparently otherwise.

4. A calibration problem. As a related problem, we consider the calibration of an air speed indicator which should read the speed of the airplane with respect to the wind. As before, we first assume that the airplane is flying in a horizontal plane at a fixed throttle setting and that the wind is blowing uniformly in a direction parallel to the horizontal plane. One elegant solution to this problem was given by Von Mises [5] as follows:

The airplane flies a closed triangular course with respect to ground and the flight for each leg of the triangle is timed. The length of each leg is gotten by referring the visual flight plan to a scale map of the terrain. Alternatively one could fly a triangular trip whose vertices are airports using VOR (VHF omni-

range equipment) [2]. By keeping the left-right needle centered one could fly a straight line path (with respect to ground) to the respective airports. Now knowing the respective distances and times for each leg of the triangle, we can obtain the respective velocities (with respect to ground). Let $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$ denote these respective velocities, let $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ denote the corresponding velocities with respect to the wind, and let $\boldsymbol{W}$ denote the wind velocity. Then

$$\boldsymbol{U}_1 = \boldsymbol{V}_1 + \boldsymbol{W}, \qquad \boldsymbol{U}_2 = \boldsymbol{V}_2 + \boldsymbol{W}, \qquad \boldsymbol{U}_3 = \boldsymbol{V}_3 + \boldsymbol{W},$$

$$|\boldsymbol{V}_1| = |\boldsymbol{V}_2| = |\boldsymbol{V}_3| = v = \text{air speed with respect to wind.}$$

Since $|\boldsymbol{U}_1-\boldsymbol{W}| = |\boldsymbol{U}_2-\boldsymbol{W}| = |\boldsymbol{U}_3-\boldsymbol{W}|$, it follows that if we draw the vectors $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$, and $\boldsymbol{W}$ from the same origin, then the terminus of $\boldsymbol{W}$ must be equidistant from the termini of $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{U}_3$. Thus the terminus of $\boldsymbol{W}$ is the circumcenter of P_1, P_2, P_3, which is obtained by drawing the perpendicular bisectors of $\overline{P_1P_2}$ and $\overline{P_2P_3}$. Then $v = P_1Q$ and $\boldsymbol{W} = \boldsymbol{OQ}$.

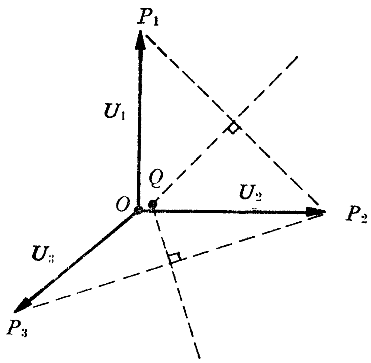

FIG. 3

Another way of determining v and w is to take two timed round trips POP and $P'OP'$ as in Figure 1 where, for simplicity, $OP' = r_0$ and $\angle P'OP = \pi/2$. (Note that even though θ_0 is unknown, one can choose a point P' such that $\angle P'OP = \pi/2$.)

We now use Eqs. (5) and (9). Eq. (5) can be rewritten as

$$t_1 = \frac{r_0}{w}\,\frac{\lambda + \cos\theta_0}{\lambda^2 - 1} \tag{5$'$}$$

Thus,

$$\frac{t_1}{t_2} = \frac{\lambda + \cos\theta_0}{\lambda - \cos\theta_0} \qquad \text{(for trip } POP\text{)},$$

$$\frac{t_1'}{t_2'} = \frac{\lambda - \sin\theta_0}{\lambda + \sin\theta_0} \qquad \text{(for trip } P'OP'\text{)}.$$

the wind along the successive four legs of the trip. W_1 and W_2 denote the two different constant values of the wind velocity.

Now consider a vector diagram where the U_i and the W_i are drawn from the same origin as in Figure 4. The termini of W_1 and W_2 must lie on the respective perpendicular bisectors AB and AC of P_1P_2 and P_3P_4. Since also, $U_2-U_3=W_1-W_2$, we continue the construction in Figure 5. Draw AD parallel and equal to P_3P_2; draw DE parallel to AC; draw EF parallel to AD. Then E and F are the respective termini of W_1 and W_2 and $v=P_1E$.

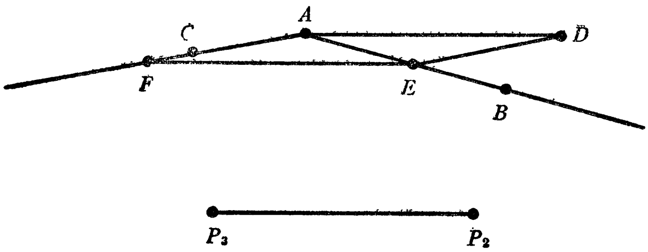

FIG. 5

It should be noted that the previous calibration procedures could be simplified if one also assumed an accurate compass was aboard the plane.

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ON THE ENUMERATION OF MATRICES OVER FINITE COMMUTATIVE RINGS

D. BOLLMAN AND H. RAMÍREZ, University of Puerto Rico

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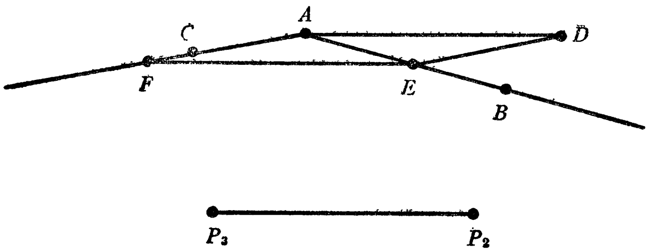

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2. Decomposition preserved matrix properties. For any commutative ring R and any pair (μ, ν) of positive integers we denote by $M_{\mu,\nu}(R)$ the module of $\mu\times\nu$ matrices over R. If R is a finite direct sum $\oplus_{i=1}^{k} R_i$, then each Π_i denotes the projection of R onto R_i. We extend each Π_i to $M_{\mu,\nu}(R)$ and to the power set of $M_{\mu,\nu}(R)$ in the obvious way.

By a *matrix property* we shall mean a set

$$P \subset \mathfrak{M} = \bigcup_{\substack{R\in\mathfrak{R} \\ (\mu,\nu)\in N\times N}} M_{\mu,\nu}(R),$$

where $\mathfrak{R}$ is the class of all commutative rings. We shall say that a matrix property P is *decomposition preserved* if and only if for each $\mu\times\nu$ matrix $A\in P$, each commutative ring R such that $A\in M_{\mu,\nu}(R)$, and each decomposition $\oplus_{i=1}^{k} R_i$ into a finite direct sum, $\Pi_i(A)\in P$ for each $i=1, 2, \cdots, k$. For example, one can verify that the properties of cyclicity, symmetry, persymmetry, skew-symmetry, and nilpotency are decomposition preserved.

For each commutative ring R, let

$$P_{\mu,\nu}(R) = P \cap M_{\mu,\nu}(R).$$

Now let P be any decomposition preserved property, and let $R=\oplus_{i=1}^{k} R_i$ be a finite commutative ring. Then $P_{\mu,\nu}(R)=\times_{i=1}^{k} P_{\mu,\nu}(R_i)$, so

$$| P_{\mu,\nu}(R) | = \prod_{i=1}^{k} | P_{\mu,\nu}(R_i) |, \tag{1}$$

where $|S|$ denotes the cardinality of the set S.

Various special cases of this formula have appeared in the literature. For example, in [12], Fine and Herstein have computed $|P_{n,n}(Z_m)|$, where P is the nilpotency property. In [31], Levine and Korfhage have computed $|P_{n,n}(Z_m)|$, where $P=\{X\in\mathfrak{M} \mid X^2=I\}$ and in [30], Korfhage has generalized this latter result to $|P_{n,n}(R)|$, where R is an arbitrary commutative ring with unity.

We shall consider a refinement of (1); namely, we shall derive a formula for $|P_{\mu,\nu,r}(R)|$, where P is any decomposition preserved property, r is any non-negative integer, R is a finite commutative ring, and $P_{\mu,\nu,r}(R)$ is the subset of elements of $P_{\mu,\nu}(R)$ having rank r.

3. Rank in a directly reducible matrix module. The *rank* of a matrix $A=[a_{ij}]$ over a commutative ring R is defined as follows: A matrix A has rank zero if and only if there is a nonzero $c\in R$ such that $ca_{ij}=0$ for each element a_{ij} of A. If A does not have rank zero then its rank is the greatest positive integer r such that $c \det A_{ij}=0$ for each $r\times r$ minor A_{ij} of A implies $c=0$.

Observe that for a commutative ring $R=\oplus_{i=1}^{k} R_i$ and a matrix A over R we have $\det A = (\det \Pi_1(A), \cdots, \det \Pi_k(A))$. We shall use this fact in the following

LEMMA. *Suppose R is a commutative ring and $R=\oplus_{i=1}^{k} R_i$. Then for each matrix $A\in M_{n,n}(R)$,*

$$rank\ A = \min\{rank\ \Pi_1(A), \cdots, rank\ \Pi_k(A)\}.$$

Proof. Let $r=\min\{\text{rank }\Pi_1(A), \cdots, \text{rank }\Pi_k(A)\}$. If $r=0$, it follows trivially from the definition of rank, that rank $A=0$ also. Suppose $r\neq 0$. Then there is an integer t, $1\leqq t\leqq k$, such that for each integer $s\geqq r+1$, there is a nonzero $c_t\in R_t$ such that c_t det $\Pi_t(A_{ij})=0$ for each $s\times s$ minor A_{ij} of A. Hence for any $s\geqq r+1$, there is a nonzero $c\in R$ such that c det $A_{ij}=0$ for each $s\times s$ minor A_{ij} of A. Thus rank $A\leqq r$. If rank $A<r$ then by a similar argument, it follows that rank $\Pi_t(A)<r$ for some integer t, $1\leqq t\leqq k$, which is a contradiction. Hence rank $A=r$.

COROLLARY. *Nonsingularity is decomposition preserved.*

4. A formula for the number of $\mu\times\nu$ nonsingular matrices over Z_m. As another application of Formula (1), we prove

THEOREM 1. *Let m be a positive integer and let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a prime factorization of m. Let N be the nonsingularity property. Then for each positive integer n,*

$$|N_{n,n}(Z_m)| = m^{n^2}\prod_{i=1}^{k}\prod_{j=1}^{n}(1-p_i^{-j}).$$

Proof. First observe that for any prime p, any positive integer α, and any $A\in M_{n,n}(Z_{p^\alpha})$, rank $A=$rank $h(A)$, where h is the natural homomorphism from $M_{n,n}(Z_{p^\alpha})$ onto $M_{n,n}(Z_p)$. Furthermore, every element of $M_{n,n}(Z_p)$ is the homomorphic image of $p^{n^2(\alpha-1)}$ distinct elements of $M_{n,n}(Z_{p^\alpha})$ and

$$|N_{n,n}(Z_{p^\alpha})| = p^{n^2(\alpha-1)}N_{n,n}(Z_p).$$

It is well known that

$$|N_{n,n}(GF(q))| = \prod_{i=0}^{n-1}(q^n-q^i),$$

for any finite field $GF(q)$. Hence

$$|N_{n,n}(Z_{p^\alpha})| = p^{n^2(\alpha-1)}\prod_{i=0}^{n-1}(p^n-p^i).$$

Now since $Z_m\cong\oplus_{i=1}^{k}Z_{p_i^{\alpha_i}}$, we have from Formula (1) and the corollary,

$$\begin{aligned}|N_{n,n}(Z_m)| &= \prod_{i=1}^{k}p_i^{n^2(\alpha_i-1)}\prod_{j=0}^{n-1}(p_i^n-p_i^j)\\ &= m^{n^2}\prod_{i=1}^{k}\prod_{j=1}^{n}(1-p_i^{-j}).\end{aligned}$$

5. Enumeration by rank of matrices having a decomposition preserved property.

THEOREM 2. *Let P be any decomposition preserved property and let R be a finite commutative ring with* $R=\oplus_{i=1}^{k} R_i$. *Then*

$$|P_{\mu,\nu,r}(R)| = \begin{cases} \displaystyle\sum_{i=1}^{k}\left[\prod_{j=1}^{i-1}\sum_{t=r+1}^{\min(\mu,\nu)} |P_{\mu,\nu,t}(R_j)|\right] |P_{\mu,\nu,r}(R_i)| \\ \displaystyle\qquad\times\left[\prod_{j=i+1}^{k}\sum_{t=r}^{\min(\mu,\nu)} P_{\mu,\nu,t}(R_j)\right] & \text{if } 0\leqq r<\min(\mu,\nu) \\ \displaystyle\prod_{i=1}^{k} P_{\mu,\nu,r}(R_i), & \text{if } 0<r=\min(\mu,\nu). \end{cases}$$

Proof. Using the lemma and the hypothesis that P is decomposition preserved, we readily see that

$$P_{\mu,\nu,r}(R) = \mathop{\times}_{i=1}^{k} P_{\mu,\nu,r}(R_i) \qquad \text{if } 0<r=\min(\mu,\nu)$$

and

$$P_{\mu,\nu,r}(R) = \bigcup_{i=1}^{k}\left\{\left[\mathop{\times}_{j=1}^{i-1}\bigcup_{t=r+1}^{\min(\mu,\nu)} P_{\mu,\nu,t}(R_j)\right]\times P_{\mu,\nu,r}(R_i)\times\left[\mathop{\times}_{j=i+1}^{k}\bigcup_{t=r}^{\min(\mu,\nu)} P_{\mu,\nu,t}(R_j)\right]\right\}$$

if $0\leqq r<\min(\mu,\nu)$, where each union which appears is the union of a disjoint family of sets. From this, the desired result follows immediately.

As applications of the above theorem, one can obtain a formula for $|P_{\mu,\nu,r}(\oplus_{i=1}^{k} GF(q_i))|$ (and in particular for $|P_{\mu,\nu,r}(B)|$, where B is any finite Boolean ring) for any decomposition preserved property for which $|P_{\mu,\nu,r}(GF(q))|$ is known. Similarly, one can obtain a formula for $|P_{\mu,\nu,r}(Z_m)|$, where m is an arbitrary positive integer, for any decomposition preserved property for which $|P_{\mu,\nu,r}(Z_{p^\alpha})|$, where p is a prime and α is a positive integer, is known.

For example, using the results of Bollman and Ramírez [2], Carlitz [4], Daykin [10], Landensburg [29], or other existing formulas found in the references cited here, one can derive formulas for $|P_{\mu,\nu,r}(R)|$, where P is either the nilpotent property, the symmetry property, the persymmetry property, or the property $P=\mathfrak{M}$, etc., and where R is either a residue class ring Z_m or a finite direct sum of finite fields.

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ON NONLINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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1. Introduction. Let H be a complex Hilbert space with inner product $(\ |\ )$ and norm $\|\ \|$, and let R^+ be the set of nonnegative real numbers. Suppose that f is a mapping from $R^+\times H$ into H, that is, for each (t, u) in $R^+\times H, f(t, u)$ is a vector in H. In this paper, we are interested in the problem of uniqueness of the solutions for the Cauchy problem:

$$du/dt = f(t, u), \qquad 0 \leqq t \leqq T, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where u_0 (the initial data) is a fixed vector in H. It is well known that Peano's methods can be applied to prove that the system (1), (2) has solutions when $H=R^n$, the n-dimensional Euclidean space, and f is a continuous mapping. This method cannot be generalized to the infinite dimensional case, as was shown by Dieudonné [2], even if we assume the continuity on f. Browder [1, Section 4, Theorem 7] has proved (with a convenient assumption on f) an infinite dimensional version of the theorem of Peano, whose ideas we shall summarize in the following.

Let H_w be the Hilbert space H endowed with the weak topology. We say that f, a mapping from $R^+\times H$ into H, is *weakly continuous* when f is continuous as a mapping from $R^+\times H_w$ into H_w. The theorem proved by Browder is the following:

THEOREM 1. *Let H be a Hilbert space and f a weakly continuous mapping from $R^+\times H$ into H. Then for each $r>0$, there exists $\alpha(r)>0$ such that for each u_0 in H with $\|u_0\|<r$, there exists a strongly C' solution u of system* (1), (2) *for* $0\leqq t\leqq\alpha(r)$.

The theorems of Nagumo [4] and Osgood [5] on uniqueness of the solutions for the Peano theorem in the finite dimensional case are well known. In this paper, we give an infinite dimensional version of those theorems in the case of a Hilbert space.

2. Uniqueness of Solutions.

THEOREM 2. *If in Theorem* 1, *besides the weak continuity of f, we suppose that*

$$\operatorname{Re}(f(t, u) - f(t, v) \mid u - v) \leqq \frac{1}{2t}\|u - v\|^2 \tag{3}$$

for all u, v in H, and $0<t\leqq\alpha(r)$, then the solution of (1) *and* (2) *is unique.*

Proof. In fact, let u, v be two solutions of (1) in $[0, \alpha(r)]$ with the same initial value u_0; then it follows

$$\frac{d(u-v)}{dt} = f(t, u) - f(t, v). \tag{4}$$

Taking the inner product of both sides of (4) with $u-v$, and taking real parts, we obtain:

$$\frac{d}{dt}\|u-v\|^2 = 2\ \mathrm{Re}(f(t, u) - f(t, v) \mid u-v).$$

It follows, by assumption (3), that

$$\frac{d}{dt}\|u-v\|^2 \leqq \frac{1}{t}\|u-v\|^2 \qquad 0 < t \leqq \alpha(r). \tag{5}$$

If we set $\Delta(t) = \|u(t) - v(t)\|^2$, the inequality (5) is equivalent to the ordinary differential inequality

$$\Delta'(t) \leqq \frac{1}{t}\Delta(t) \qquad 0 < t \leqq \alpha(r), \tag{6}$$

where the primes are ordinary derivatives. Integrating (6) from 0 to $t \leqq \alpha(r)$, observing that $\Delta(0) = 0$, we obtain

$$0 \leqq \Delta(t) \leqq \int_0^t \frac{\Delta(s)}{s}\, ds. \tag{7}$$

The proof of Theorem 2 follows from (7) and the following lemma.

LEMMA 1. (Hille [3]). *Suppose* $g \in C[0, t]$, $g(0) = 0$, $g'(0)$ *exists and* $g'(0) = 0$. *Then if*

$$0 \leqq g(t) \leqq \int_0^t \frac{g(s)}{s}\, ds \qquad 0 < t < T$$

it follows that $g(t) = 0$ *on* $[0, T]$.

Proof. In fact, let ϕ: $[0, T] \to R$ be defined by

$$\phi(s) = \begin{cases} g(s)/s & \text{if } s \neq 0 \\ 0 & \text{if } s = 0. \end{cases}$$

Since

$$\lim_{s \to 0} \phi(s) = \lim_{s \to 0} g(s)/s = g'(0) = 0,$$

it follows that ϕ is a continuous mapping from $[0, T]$ into R. If we define F on $[0, T]$ by

$$F(t) = \int_0^t \phi(s)\, ds,$$

it follows that F is continuously differentiable, $F'(t)=\phi(t)=g(t)/t$ for $t\neq 0$, and $F'(0)=0$. By hypothesis, $g(t)\leqq F(t)$ on $[0, T]$; it follows that

$$F'(t) = \frac{g(t)}{t} \leqq \frac{F(t)}{t},$$

that is,

$$\frac{d}{dt}\left[\frac{F(t)}{t}\right] \leqq 0.$$

Hence $F(t)/t$ is decreasing for $t>0$. Since $F(t)/t\to\phi(0)=0$, it follows $F(t)=0$ on $[0, T]$, because $F(t)\geqq 0$ for all t. This implies $g(t)=0$ on $[0, T]$ since $0\leqq g(t)\leqq F(t)$

In order to prove the infinite dimensional version of the theorem of Osgood, let us make the following definition.

DEFINITION 1. Let w be a positive real function defined on $[0, T]$. We say that w is a *permissible function*, if it is strictly increasing on $[0, T]$, if $w(0)=0$, and if

$$\int_\epsilon^a dz/w(z) \to \infty \quad \text{as} \quad \epsilon\to 0, \qquad \epsilon > 0, \qquad 0 < a < T.$$

THEOREM 3. *If in Theorem* 1, *besides the weak continuity of the function f, we suppose that*

$$2\ \mathrm{Re}(f(t, u) - f(t, v) \mid u - v) \leqq w(\|u - v\|^2), \qquad 0 \leqq t \leqq \alpha(r),$$

for some permissible function w, then the solution of (1), (2) *on* $[0, \alpha(r)]$ *is unique.*

Proof. Let u, v be two solutions of (1) on $[0, \alpha(r)]$ with the same initial data u_0. It follows that

$$\frac{d(u - v)}{dt} = f(t, u) - f(t, v). \tag{8}$$

By the same argument as used in the proof of Theorem 2, the hypotheses of Theorem 3, and (8), we obtain

$$\frac{d}{dt}\|u - v\|^2 \leqq w(\|u - v\|^2) \qquad \text{on } [0, \alpha(r)].$$

If we set $\Delta(t)=\|u(t)-v(t)\|^2$, the proof of Theorem 3 follows from the next lemma.

LEMMA 2. (Hille [3]). *Let w be a permissible function on* $[0, T]$. *If $g\in C[0, T]$ satisfies*

$$0 \leqq g(t) \leqq \int_0^t w[g(s)]ds, \qquad 0 < t \leqq T,$$

then $g(t)=0$ *on* $[0, T]$.

Proof. Suppose g is not identically zero on $[0, T]$. If h: $[0, T]\to \boldsymbol{R}$ is defined by $h(t)=\max_{0\leqq s\leqq t} g(s)$, we have $g(t)\leqq h(t)$ for each $0\leqq t\leqq T$. There exists $0\leqq t_1\leqq t$ such that $h(t)=g(t_1)$. We have

$$\begin{aligned} g(t) \leqq h(t) = g(t_1) &\leqq \int_0^{t_1} w[g(s)]ds \\ &\leqq \int_0^t w[g(s)]ds \leqq \int_0^t w[h(s)]ds. \end{aligned}$$

Since w is strictly increasing, we obtain

$$w[h(t)] \leqq w\left(\int_0^t w[h(s)]ds\right). \tag{9}$$

Taking $k(s)=w[h(s)]$, we have from (9)

$$k(t) \leqq w\left[\int_0^t k(s)ds\right]. \tag{10}$$

Thus for $0<\epsilon<a$, we obtain

$$\int_\epsilon^a \frac{k(t)dt}{w\left[\int_0^t k(s)ds\right]} < a. \tag{11}$$

If $v(t)=\int_0^t k(s)\, ds$, it follows that $v'(t)=k(t)$, and from (11) we have

$$\int_\epsilon^a \frac{dv}{w(v)} < a$$

which is a contradiction, because w is an admissible function.

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other. Hence, from Lemma 1, each member in the set $\{x_i^{x_i}, 1 \leq i \leq M\}$ is incongruent to every other member (applying Lemma 1 to all possible pairs in C'); there are M ($> [\sqrt{(p-1)/2}]$) members in this set. The theorem follows.

THEOREM 2. *For any p sufficiently large, the number of distinct residues of n^n (mod p) is at most $p-4$; in fact for $p=8x+3$, the number is at most $p-6$.*

Proof. For all odd primes, $(p-1)^{p-1} \equiv 1^1$ (mod p). If $p \equiv 1, 3$ (mod 8), then $[(p-1)/2]^{(p-1)/2}$ and 1 have (mod p) the reciprocals $(-2)^{(p-1)/2}$ and 1, which are congruent by Euler's criterion. If $p \equiv 1, 5$ (mod 8), then $[(p-1)/4]^{(p-1)/4}$ and 1 have (mod p) the reciprocals $(-4)^{(p-1)/4}$ and 1 which are congruent since

$$(-4)^{(p-1)/4} \equiv (-1)^{(p-1)/4} 2^{(p-1)/2} \equiv 1 \pmod{p}$$

by Euler's criterion. If $p \equiv 3, 5$ (mod 8), then $[(p+1)/2]^{(p+1)/2}$ and $(p-2)^{p-2}$ have (mod p) the reciprocals $2^{(p+1)/2}$ and -2, which are congruent by Euler's criterion. Again, if $p \equiv 7$ (mod 8), $[(3p-1)/4]^{(3p-1)/4}$ and $(p-2)^{p-2}$ have (mod p) the reciprocals $(-4)^{(3p-1)/4}$ and -2, which are congruent since

$$(-4)^{(3p-1)/4} \equiv -2^{(3p-1)/2} \equiv -2^{(p+1)/2} \equiv -2 \pmod{p}$$

by Euler's criterion. Finally, if $p \equiv 3$ (mod 8), $[(3p-1)/4]^{(3p-1)/4}$ and $[(p+1)/4]^{(p+1)/4}$ have (mod p) the reciprocals $(-4)^{(3p-1)/4}$ and $4^{(p+1)/4}$ which are congruent, since

$$(-4)^{(3p-1)/4} \equiv 2^{(3p-1)/2} \equiv 2^{(p+1)/2} \equiv 4^{(p+1)/4} \pmod{p}$$

by Fermat's Theorem.

Reference

1. R. Crocker, On a new problem in number theory, this MONTHLY, 73 (1966) 355–357.

SOME REMARKS ON ORDERED SEMIGROUPS

H. J. WEINERT, University of Florida, Gainesville

This note concerns some problematical parts of [3]. For a more detailed discussion and generalizations of the essential result of [3] (cf. 1. below), see [8].

1. Let G be a semigroup, and let S be a subsemigroup of G such that each element of S is central and cancellable in G. Let G_S be the unique semigroup which contains G and consists of all quotients a/s, $a \in G$, $s \in S$ (for example cf. [4], Section 47). From the proof of Theorem 1 in [3], it follows that a partial order (p.o.) $\leqq$ on G obeying

$$ax \leqq bx \text{ with } a, b \in G, x \in S \text{ implies } a \leqq b \tag{1}$$

(obviously $a=b$ in this formula in [3] is a misprint) can be extended to a p.o. on G_S.

other. Hence, from Lemma 1, each member in the set $\{x_i^{x_i}, 1 \leqq i \leqq M\}$ is incongruent to every other member (applying Lemma 1 to all possible pairs in C'); there are M $(>[\sqrt{(p-1)/2}])$ members in this set. The theorem follows.

THEOREM 2. *For any p sufficiently large, the number of distinct residues of n^n (mod p) is at most $p-4$; in fact for $p=8x+3$, the number is at most $p-6$.*

Proof. For all odd primes, $(p-1)^{p-1} \equiv 1^1 \pmod p$. If $p \equiv 1, 3 \pmod 8$, then $[(p-1)/2]^{(p-1)/2}$ and 1 have (mod p) the reciprocals $(-2)^{(p-1)/2}$ and 1, which are congruent by Euler's criterion. If $p \equiv 1, 5 \pmod 8$, then $[(p-1)/4]^{(p-1)/4}$ and 1 have (mod p) the reciprocals $(-4)^{(p-1)/4}$ and 1 which are congruent since

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$$(-4)^{(3p-1)/4} \equiv -2^{(3p-1)/2} \equiv -2^{(p+1)/2} \equiv -2 \pmod p$$

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6. H. J. Weinert, Über Halbringe und Halbkörper, II, Acta Math. Acad. Sci. Hung., 14 (1963) 209–227.

7. ———, Über Halbringe und Halbkörper, III, Acta Math. Acad. Sci. Hung., 15 (1964) 177–194.

8. ———, Extension of partial orders on semigroups to semigroups of quotients, to appear in Acta Math. Acad. Sci. Hung.

SOLUTIONS OF A FUNCTIONAL DIFFERENTIAL EQUATION

G. H. RYDER, Montana State University

1. The purpose of this paper is to study the existence of unique nontrivial solutions of the functional differential equation

$$\bar{f}'(x) = A\bar{f}(g(x)), \tag{1}$$

where A is a given constant $n\times n$ real matrix, $\bar{f}$ an n-vector function and $g(x)$ a given real function. W. R. Utz [1] has posed this problem for the scalar case of (1).

A great deal of attention has been devoted to the study of solutions of (1) in the case that $g(x)$ is a delay function of the form $x-\tau(x)$, for a given nonnegative $\tau(x)$ defined in some interval $[t_0, T]$. There a solution is sought which coincides with a given initial function $\phi(x)$ on an initial set $E_{t_0}=\{x \mid x-\tau(x)\leqq t_0, x\geqq t_0\}$ [2]. We shall instead examine (1) for solutions of the initial-value problem $\bar{f}(x_0)=\bar{f}_0$ when g is a more arbitrary function.

It is easy to see that such a problem can be translated to an initial-value problem at the origin by replacing x by $x+x_0$; i.e. if $\bar{w}'(x)=A\bar{w}(g^*(x))$ and $\bar{w}(x_0)=f_0$, then $\bar{f}(x)=\bar{w}(x+x_0)$ satisfies $\bar{f}'(x)=A\bar{f}(g(x))$ with $g(x)=g^*(x+x_0)-x_0$. Moreover, $\bar{f}(0)=\bar{w}(x_0)=\bar{f}_0$. We shall therefore assume we are dealing with the problem of finding solutions satisfying

$$\bar{f}(0) = \bar{f}_0, \tag{2}$$

$\bar{f}_0$ a given real constant n-vector.

2. To prove the existence and uniqueness of solutions we shall apply a fixed point theorem.

Let $g(x)$ be a real continuous function defined in some interval D_g including the origin, with corresponding range R_g, and satisfying $|g(x)|\leqq k$ for all x in D_g. Let S be the space of continuous functions $\bar{f}$ from D_g to R^n such that $\bar{f}(0)=\bar{f}_0$ and, for some constant L, $\|\bar{f}(x)-\bar{f}_0\|<L|x|$ for all $x\in D_g$. Then it is an elementary exercise to show the following:

LEMMA 1. *(S, ρ) is a complete metric space with the metric defined by*

$$\rho(\bar{f}_1, \bar{f}_2) = \inf.\{L: \|\bar{f}_1(x) - \bar{f}_2(x)\| \leqq L|x|, \forall x\in D_g\}. \tag{3}$$

If we now define the operator T on S by

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If we now define the operator T on S by

where $g(x)=1/a$, $\|Ak\|=|a(1/a)|=1$. Any solution valid in an interval including 0 and $1/a$ will therefore be of the form

$$f(x) = af(1/a)x + f_0.$$

Letting $x=1/a$, we see that f_0 must be zero, and the only solutions valid at $x=0$ are the straight lines through the origin $f(x)=cx$. Moreover, all of these are solutions.

(b) An example of a locally unique solution is furnished by the system

$$f'(x) = \tfrac{1}{2}f\left(\frac{-x}{x+1}\right), \qquad f(0) = 1.$$

If we let $D_g=[-\frac{1}{2}, 1]$, then

$$|g(x)| = \left|\frac{x}{x+1}\right| \leqq 1 \quad \text{in} \quad D_g, \qquad \|Ak\| = |\tfrac{1}{2}\cdot 1| = \tfrac{1}{2}, \quad \text{and} \quad R_g = D_g.$$

Therefore there exists a unique solution in D_g, namely $f(x)=\sqrt{(x+1)}$.

References

1. W. R. Utz, The equation $f'(x)=af(g(x))$, Bull. Amer. Math. Soc., 71 (1965) 138.
2. L. E. El'sgol'ts, Introduction to the Theory of Differential Equations with Deviating Arguments, Holden-Day, San Francisco, 1966.

ON NUMBERS RELATED TO PARTITIONS OF A NUMBER

J. M. GANDHI, University of Alberta, Edmonton, and York University, Toronto

1. Introduction. In this note we study the properties of the numbers $G(n)$ defined by

$$(1) \qquad -\frac{\Phi(x)}{(d/dx)[\Phi(x)]} = \sum_{n=0}^{\infty} (-1)^n G(n) x^n,$$

where

$$\Phi(x) = (1-x)(1-x^2)(1-x^3)\cdots.$$

We prove

THEOREM 1. *If* $n>1$ *and* $n\not\equiv 0 \pmod 5$, *then* $G(n)\equiv 0 \pmod 5$.

(In what follows all congruences are to be understood modulo 5 and n, m, k and t are rational integers.) From Theorem 1, the congruences

$$(2) \qquad p_{-1}(5m+4) \equiv 0 \pmod 5,$$

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follow immediately, where $p_{-1}(n)$ and $\tau(n)$ respectively denote the unrestricted partitions of a number and Ramanujan's τ function. Although a detailed investi-

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follow immediately, where $p_{-1}(n)$ and $\tau(n)$ respectively denote the unrestricted partitions of a number and Ramanujan's τ function. Although a detailed investi-

gation of the numbers $S(n)$ generated by

$$\frac{1}{-(d/dx)[\Phi(x)]} = \sum_{n=1}^{\infty} (-1)^n S(n) x^n \tag{4}$$

is the subject matter of a forthcoming paper, some formulae relating $S(n)$ and $G(n)$ are given.

2. Proof. Using the Euler's identity

$$\Phi(x) = \sum_{\beta \gtrless 0} (-1)^{\beta} x^{(\beta(3\beta-1))/2} \tag{5}$$

and multiplying both sides of (1) by $(d/dx)[\Phi(x)]$ and expanding and equating the coefficients of x^n, we can prove

$$\sum_{j} \frac{j(3j \pm 1)}{2} G\left[n + 1 - \frac{j(3j \pm 1)}{2}\right](-1)^{j+j(3j\pm1)/2} = (-1)^{\beta+n} \quad \text{or} \quad 0 \tag{6}$$

accordingly as

$$\frac{\beta(3\beta - 1)}{2} = n \quad \text{or} \quad \frac{\beta(3\beta - 1)}{2} \neq n.$$

From (6) we can calculate the values of G. The Table for first few $G(n)$ is given below

TABLE 1

$G(0) = 1$	$G(1) = 3$	$G(2) = 5$	$G(3) = 10$
$G(4) = 25$	$G(5) = 64$	$G(6) = 160$	$G(7) = 390$
$G(8) = 940$	$G(9) = 2270$	$G(10) = 5515$	$G(11) = 13440$
$G(12) = 32735$	$G(13) = 79610$	$G(14) = 193480$	$G(15) = 470306.$

As $G(0)$ and $G(1)$ are positive integers, it follows from (6) that all G's are positive integers. Then let us define $p_\gamma(n)$ by

$$[\Phi(x)]^\gamma = \sum_{n=0}^{\infty} p_\gamma(n) x^n. \tag{7}$$

Differentiating (7) with respect to x and using (1) we get

$$\gamma \sum_{n=0}^{\infty} p_\gamma(n) x^n = -\left(\sum_{n=1}^{\infty} n p_\gamma(n) x^{n-1}\right)\left(\sum_{n=0}^{\infty} (-1)^n G(n) x^n\right). \tag{8}$$

whence expanding and equating the coefficients of x^n we get

$$\gamma p_\gamma(n) = -\sum_{i=1}^{n} G(n + 1 - i) i (-1)^{n+1-i} p_\gamma(i). \tag{9}$$

From (9) with $\gamma=-2$ one has

$$(10)\qquad -2p_{-2}(n) = \sum_{i=1}^{n} (-1)^{n+1-i}G(n+1-i)ip_{-2}(i).$$

We note that the success of the proof of Theorem 1 depends critically on the numerical value $G(1)=3$.

We shall first prove that for $m>0$, $G(5m+1)\equiv 0$. From (10) we get

$$(11)\qquad -2p_{-2}(n) \equiv \sum_{t=0}^{[m/5]} (-1)^{n+1-5t-1}G(n+1-5t-1)p_{-2}(5t+1)$$

(all other terms in (10) drop out, those with $i\equiv 0$ because of the factor i, those with $i=2$, 3, 4 because of the congruences [1])

$$(12)\qquad p_{-2}(5m+i) \equiv 0 \qquad \text{for } i = 2, 3, \text{ and } 4.$$

Let $n=5m+1$ and use the induction assumption $G(5k+1)\equiv 0$ for $0<k<m$. Then (11) becomes

$$(13)\qquad -2p_{-2}(5m+1) \equiv (-1)^m p_{-2}(1)G(5m+1) + G(1)p_{-2}(5m+1).$$

Using $p_{-2}(1)=1$ and the fact that $G(1)=3$, (13) yields

$$G(5m+1) \equiv (-1)^{m+1}5p_{-2}(5m+1) \equiv 0.$$

The same method yields the proof of $G(5m+i)\equiv 0$ for $i=2$, 3 and 4 and the theorem is proved.

3. Discussion. It may be noted that if an independent proof of Theorem 1 can be given, then congruences (12) will follow from (10).

Now consider $\gamma=-1$ and $n=5m+4$; from (9) we get

$$(14)\qquad p_{-1}(5m+4) = \sum_{i=1}^{5m+4} G(5m+5-i)ip_{-1}(i)(-1)^{5m+5-i}.$$

From (14) and Theorem 1, congruences (2) follow by the method of induction.

Now Ramanujan's τ function is defined by

$$(15)\qquad x[\Phi(x)]^{24} = \sum_{n=1}^{\infty} \tau(n)x^n,$$

whence from (7) we have

$$(16)\qquad p_{24}(n) = \tau(n+1).$$

Therefore considering $\gamma=24$, and $n=5m-1$, in view of (16) and from (9) we get

$$(17)\qquad 24\tau(5m) = \sum_{i=1}^{n} G(5m-i)i(-1)^{5m-i}\tau(i+1).$$

From (17) and Theorem 1, congruence (3) immediately follows.

Now we give some formulae relating $G(n)$, $p_{-1}(n)$, and $\sigma(n)$ (the sum of the divisors of n).

It is well known that

$$\log[\Phi(x)] = -\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n. \tag{18}$$

Differentiating (18) with respect to x, we get

$$(d/dx)[\Phi(x)]/\Phi(x) = -\sum_{n=1}^{\infty} \sigma(n)x^{n-1}. \tag{19}$$

From (1) and (19), we have

$$1 \Big/ \sum_{n=1}^{\infty} \sigma(n)x^{n-1} = \sum_{n=0}^{\infty} (-1)^n G(n)x^n. \tag{20}$$

From (20) it is easy to prove

$$\sum_{\gamma=1}^{n+1} (-1)^{n+1-\gamma} G(n+1-\gamma)\sigma(\gamma) = 0 \qquad \text{for } \gamma > 0. \tag{21}$$

From (1), (4) and (7), it is evident that

$$G(n) = \sum_{\beta} (-1)^{\beta - \frac{1}{2}\beta(3\beta-1)} S[n - \tfrac{1}{2}\beta(3\beta-1)], \tag{22}$$

$$S(n) = \sum_{\gamma=0}^{n} p_{-1}(n-\gamma)G(\gamma)(-1)^{\gamma}. \tag{23}$$

In the end we give a small table for the numbers S, the properties of which will be discussed elsewhere.

TABLE 2

$S(0)=1$	$S(1)=2$	$S(2)=4$	$S(3)=8$
$S(4)=21$	$S(5)=52$	$S(6)=131$	$S(7)=316$
$S(8)=765$	$S(9)=1846$	$S(10)=4494.$	

My thanks are due to the referee for his valuable suggestions, which improved Theorem 1.

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ON CANCELLATION IN GROUPS

R. HIRSHON, Polytechnic Institute of Brooklyn

Let $A \times B$ represent the direct product of the groups A and B. We shall say that B may be cancelled in direct products if

$$A \times B \approx A_1 \times B_1, \qquad B \approx B_1$$

imply $A \approx A_1$ for any A.

It seems natural to inquire about those groups which may be cancelled in direct products. We will show in this paper that a finite group B may be cancelled in direct products. As far as we can determine, this result does not appear in any standard text in group theory or algebra, perhaps because it appears to have been discovered as recently as 1947 ([4], introduction), and apparently is still not well known. Good use of it might be made, for example, in proving that the decomposition of a finite group as a direct product of indecomposable groups is unique up to isomorphism.

We present a proof of the cancellation theorem which we feel is the simplest available and is suitable for undergraduates. We also present in this paper an outline of a proof that an infinite cyclic group may not, in general, be cancelled in direct products, thus giving an example of the "simplest" type of group which may not be cancelled.

CANCELLATION THEOREM. *If B is a finite group, B may be cancelled in direct products.*

Proof. We observe first that it suffices to show

$$G = D \times B = D_1 \times B_1, \qquad B \approx B_1, \quad \text{imply } D \approx D_1. \tag{1}$$

We prove (1) by induction on $|B|$, the order of B.

Clearly (1) is true if $|B| = 1$. Assume (1) is true for groups B, with $|B| < k$. We prove (1) is true if $|B| = k$. First observe that if $B \cap D_1 = 1$ then $G = B \times D_1$, so that $D \approx G/B \approx D_1$. Hence, without loss of generality, we may assume $B \cap D_1 \neq 1$. Also by symmetry we may assume

$$F = B \cap D_1 \neq 1, \qquad K = B_1 \cap D \neq 1.$$

Now from (1), we may see

$$G/(F \times K) = (B \times D)/(F \times K) = (B_1 \times D_1)/(K \times F). \tag{2}$$

By a standard isomorphism theorem, we see from (2)

$$(B/F) \times (D/K) \approx (B_1/K) \times (D_1/F).$$

Hence, since $B \approx B_1$, we may write

$$B \times (B/F) \times (D/K) \approx B_1 \times (B_1/K) \times (D_1/F). \tag{3}$$

However,

groups, may be cancelled in direct products. The proof is essentially the same as the one we have given for finite groups except that one uses induction on the length of a principal series. Some applications of this cancellation result appear in [2]. A sufficient condition for the cancellation of infinite groups which obey the maximal condition for normal subgroups, is given in [3].

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A MAXIMUM MODULUS PRINCIPLE FOR CLOSED ALGEBRAS OF LIPSCHITZ FUNCTIONS

R. M. Crownover, University of Missouri

Let us call a complex-valued function f on a metric space (X, d) an *LOC function* if f satisfies a uniform Lipschitz condition on each compact subset of X, i.e., if for each compact $E\subset X$, there is a constant $K_E(f)$ such that for $x, y\in E$,

$$|f(x)-f(y)| \leqq K_E(f)d(x, y).$$

For example, each analytic function on a plane domain is an LOC function.

In general, uniform limits of LOC functions are not LOC functions. However, if the functions are analytic functions on a plane domain, then, of course, the uniform limits are again LOC functions. In the direction of a converse of this result, we shall obtain a maximum modulus theorem for certain algebras of LOC functions which are closed under uniform limits, and indeed obtain analyticity in one special case.

Lemma. *Let A be a linear space of bounded functions on (X, d) which is closed in sup norm. If $E\subset X$, and each $f\in A$ satisfies a uniform Lipschitz condition on E, then there exists a constant K_E such that for any $f\in A$ with $\|f\|_\infty\leqq 1$,*

$$|f(x)-f(y)| \leqq K_E\, d(x, y) \qquad \textit{for } x, y \in E. \tag{1}$$

Proof. Let $S=\{f: f\in A$ and such that for any $x, y\in E$, $|f(x)-f(y)|\leqq d(x, y)\}$. Then $A=\bigcup_{n=1}^\infty (nS)$; since A is a complete metric space, the Baire category theorem applies, implying for some n, the set $\overline{nS}=nS$ has nonvoid interior. Consequently for some $f_0\in S$, and $r>0$, $S\supset f_0+N(0; r)$, where $N(0; r)=\{h: h\in A$ and $\|h\|_\infty<r\}$. Since S is symmetric, $-f_0+N(0; r)\subset S$, and since S is convex, for each $h\in N(0; r)$, $h=\frac{1}{2}(-f_0+h)+\frac{1}{2}(f_0+h)$ lies in S, i.e., $N(0; r)\subset S$. It follows that if $K_E=1/r$, then (1) holds for all $x, y\in E$.

We now prove the aforementioned maximum modulus theorem.

groups, may be cancelled in direct products. The proof is essentially the same as the one we have given for finite groups except that one uses induction on the length of a principal series. Some applications of this cancellation result appear in [2]. A sufficient condition for the cancellation of infinite groups which obey the maximal condition for normal subgroups, is given in [3].

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A MAXIMUM MODULUS PRINCIPLE FOR CLOSED ALGEBRAS OF LIPSCHITZ FUNCTIONS

R. M. CROWNOVER, University of Missouri

Let us call a complex-valued function f on a metric space (X, d) an *LOC function* if f satisfies a uniform Lipschitz condition on each compact subset of X, i.e., if for each compact $E\subset X$, there is a constant $K_E(f)$ such that for $x, y\in E$,

$$|f(x)-f(y)| \leqq K_E(f)d(x, y).$$

For example, each analytic function on a plane domain is an LOC function.

In general, uniform limits of LOC functions are not LOC functions. However, if the functions are analytic functions on a plane domain, then, of course, the uniform limits are again LOC functions. In the direction of a converse of this result, we shall obtain a maximum modulus theorem for certain algebras of LOC functions which are closed under uniform limits, and indeed obtain analyticity in one special case.

LEMMA. *Let A be a linear space of bounded functions on (X, d) which is closed in sup norm. If $E\subset X$, and each $f\in A$ satisfies a uniform Lipschitz condition on E, then there exists a constant K_E such that for any $f\in A$ with $\|f\|_\infty\leqq 1$,*

$$|f(x)-f(y)| \leqq K_E\, d(x, y) \qquad \textit{for } x, y \in E. \tag{1}$$

Proof. Let $S=\{f: f\in A$ and such that for any $x, y\in E$, $|f(x)-f(y)|\leqq d(x, y)\}$. Then $A=\bigcup_{n=1}^\infty (nS)$; since A is a complete metric space, the Baire category theorem applies, implying for some n, the set $\overline{nS}=nS$ has nonvoid interior. Consequently for some $f_0\in S$, and $r>0$, $S\supset f_0+N(0; r)$, where $N(0; r)=\{h: h\in A$ and $\|h\|_\infty<r\}$. Since S is symmetric, $-f_0+N(0; r)\subset S$, and since S is convex, for each $h\in N(0; r)$, $h=\frac{1}{2}(-f_0+h)+\frac{1}{2}(f_0+h)$ lies in S, i.e., $N(0; r)\subset S$. It follows that if $K_E=1/r$, then (1) holds for all $x, y\in E$.

We now prove the aforementioned maximum modulus theorem.

THEOREM. *If A is a closed sup norm algebra on a compact connected metric space (X, d), and if on some open set $\Omega \subset X$, each $f \in A$ is an LOC function, then for each $f \in A$*

$$\sup_{x \in X} |f(x)| = \sup_{x \in X - \Omega} |f(x)|.$$

Proof. Suppose the theorem false, i.e., that for some $f \in A$,

$$\sup_{x \in X} |f(x)| > \sup_{x \in X - \Omega} |f(x)|.$$

Let $x_0 \in \Omega$ be chosen so that $|f(x_0)| = \sup_{x \in X} |f(x)|$, and let $Q = f^{-1}(f(x_0))$. Then $Q \subset \Omega$. We may assume that $f(x_0) = 1$; if the constant function $1 \in A$, we may replace f by $(1+f)/2$ if necessary to obtain $|f(y)| < 1$ for $y \in X - Q$. The set

$$A_1 = \{f + c : f \in A \text{ and } c \text{ constant}\}$$

is closed, since A_1 is the direct sum in $C(X)$ of A and a one-dimensional space if $1 \notin A$ ([3], p. 22). Also, A_1 is an algebra of functions which are LOC on Ω. Thus it results in no loss of generality to assume $1 \in A$. Since the metric space (X, d) is normal, and since Q is a closed subset of the open set Ω, there is an open set Ω_1 such that $Q \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$. Let $E = \bar{\Omega}_1$ and apply the lemma, so that (1) holds. Since X is connected, ∂Q is not empty, and we choose an $x \in \partial Q$. Let $y \in \Omega_1$ be chosen so that $d(x, y) < 1/(2K_E)$ and $|f(y)| < 1$. By (1),

$$|f^k(x) - f^k(y)| < 1/2 \qquad (k = 1, 2, 3, \cdots),$$

but this is impossible for large k, since $|f^k(x)| = 1$ and $|f^k(y)| < 1$. This completes the proof of the theorem.

Now let U be the plane disk $\{z : |z| < 1\}$. Rudin ([1]; see also [2], Theorem 12.12) calls an algebra of complex-valued continuous functions A on $\overline{U}$ a *maximum modulus algebra* if for each $f \in A$, there is a point z_0 with $|z_0| = 1$ such that $|f(z)| \leqq |f(z_0)|$ for all $z \in \overline{U}$. He proves that if A contains z (i.e., the function ϕ defined by $\phi(z) = z$), then each $f \in A$ is analytic in U. As an immediate consequence of Rudin's theorem and the above theorem, we obtain the following corollary:

COROLLARY. *Suppose that A is a closed sup norm algebra of continuous functions on $\overline{U}$ and also that each $f \in A$ is an LOC function on U. If A contains z, then each $f \in A$ is analytic on U.*

An open question is whether the condition that A contain z can be weakened to the condition that A separate points in U, and contain a nonconstant analytic function.

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from $[0, 1]$ to R contains each in its graph.) Now according to a familiar and elementary variant of the Weierstrass approximation theorem, each element of C is approximable uniformly over $[0, 1]$—and hence surely over each of its finite subsets—by a polynomial with rational coefficients. The set of all such polynomials, which is a countable set, is then dense in P.

To deduce (*) from (**) we take any countable dense subset E of P and with each f in E we associate any function f' in Y for which $|f'(x)-f(x)| \leqq 1/2$ whenever $x \in A$. (For example, one may choose for $f'(x)$ the greatest integer not exceeding $f(x)+1/2$.) Since $f'(x)=n$ whenever n is an integer and $|f(x)-n| < 1/2$, the set $\{f':f\in E\}$ is dense in Y.

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SIGMA-FINITENESS AND HAAR MEASURE

R. E. HUFF, University of North Carolina

The purpose of this note is to point out the explicit form of the sesquiregular extension of Haar measure on a locally compact (Hausdorff) group.

The terminology follows [1] and [3]. Let G be a locally compact group, let μ be a left Haar measure on the class $\mathfrak{B}$ of Borel subsets of G, and let μ_s denote the unique sesquiregular extension of μ; i.e., μ_s is an extension of μ to a measure on the class $\mathfrak{B}_w$ of weakly Borel subsets of G such that every member of $\mathfrak{B}_w$ is outer regular and every open set is inner regular (see [1]).

THEOREM. *If A is in $\mathfrak{B}_w$, then A has σ-finite μ_s measure if and only if A is in $\mathfrak{B}$. Thus $\mu_s(A)=\mu(A)$ if A is in $\mathfrak{B}$, and $\mu_s(A)=\infty$ (non-σ-finite) if A is in $\mathfrak{B}_w\backslash\mathfrak{B}$.*

The theorem extends the well-known result that μ_s is totally σ-finite if and only if G is σ-compact (this follows from [3], p. 256, Exercise 9).

The proof is based on three facts: (1) a set B in $\mathfrak{B}_w$ is in $\mathfrak{B}$ if and only if B is σ-bounded [2]; (2) there exists a σ-compact, open subgroup H of G ([3] p. 251, Theorem 57.B); and (3) if W is any non-void open Borel set, then $\mu(W)>0$ ([3] p. 251, Section 58).

Suppose B is in $\mathcal{B}_w$ and that $\mu_s(B) < \infty$. Then there exists an open set U such that $B \subset U$ and $\mu_s(U) < \infty$. If gH is any left coset of H and if U intersects gH, then $U \cap gH$ is an open Borel set by fact (1), and $\mu_s(U \cap gH) = \mu(U \cap gH) > 0$ by fact (3). Since $\mu_s(U) < \infty$, U can intersect only countably many left cosets of H; thus B is σ-bounded. If A has σ-finite μ_s measure, then A is the countable union of σ-bounded sets, and hence A is σ-bounded. By fact (1), A must be in $\mathcal{B}$. The converse follows from (1) immediately.

Note that the proof depends only on μ_s extending μ and every weakly Borel set of finite measure being outer regular.

The result reported in this paper appeared in the author's masters thesis at the University of North Carolina at Chapel Hill (1967). This work was supported by a National Science Foundation Graduate Fellowship.

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

IS THERE AN ELEMENTARY PROOF OF PEANO'S EXISTENCE THEOREM FOR FIRST ORDER DIFFERENTIAL EQUATIONS?

HUBERT C. KENNEDY, Providence College

In 1886 Giuseppe Peano stated [8] that the initial value problem: $y' = f(x, y)$, $y(a) = b$, has a solution on the sole condition that f is continuous, and he gave an elementary proof of this. His theorem is correct, but a historical investigation into the work of Peano has led to the conclusion that his proof is invalid. This raises the question: Is there an elementary proof of Peano's Theorem?

It is clear that f must satisfy some condition if the existence of a solution of the differential equation $y' = f(x, y)$ satisfying the initial value $y(a) = b$ is to be guaranteed. Cauchy proved that there exists a solution of the initial value problem if f and f_y are continuous [6], or if f is synectic (continuous, monodrome, and monogenic). Charles Briot and Jean Bouquet improved the proof of the latter theorem [1]. Rudolf Lipschitz gave an existence theorem which imposed a less restrictive condition on f [3]. (The reference usually given for this is [4], which is essentially a French translation of the Italian article [3].)

Furthermore, Chartrand and Kapoor [2] have show that G^3-v is Hamiltonian for each point v in G^3.

We conclude with a related conjecture concerning the total graph of a nonseparable graph. The *total graph* $T(G)$ of a graph G is that graph whose points are in one-to-one correspondence with the elements (the set of points and lines) of G, such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are points or both are lines) or they are incident (if one element is a point and the other a line). An equivalent definition for $T(G)$ can be given in terms of the subdivision graph. The *subdivision graph* $S(G)$ is that graph obtained from G by replacing each line uv by a new point w and the two new lines uw and wv. It is then easy to show that the two graphs $T(G)$ and $(S(G))^2$ are isomorphic. (Thus, the graph T^2 shown in Figure 1 is the total graph of a star graph consisting of three lines incident at a common point.) Hence, if Conjecture 1 is true, then the following conjecture is also true:

CONJECTURE 2. *If G is a nonseparable graph, then $T(G)$ is Hamiltonian.*

We remark that Behzad and Chartrand [1] have shown that if G is any nontrivial connected graph, then $T(T(G))$ is Hamiltonian.

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

MEHLER'S INTEGRAL FOR $P_n(\cos \theta)$

RICHARD ASKEY, University of Wisconsin, Madison

There are many interesting results in mathematics which are not given in courses because the known proofs are either too complicated or too artificial for the particular class of students. One such example is the Dirichlet-Mehler

Furthermore, Chartrand and Kapoor [2] have show that $G^3 - v$ is Hamiltonian for each point v in G^3.

We conclude with a related conjecture concerning the total graph of a nonseparable graph. The *total graph* $T(G)$ of a graph G is that graph whose points are in one-to-one correspondence with the elements (the set of points and lines) of G, such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are points or both are lines) or they are incident (if one element is a point and the other a line). An equivalent definition for $T(G)$ can be given in terms of the subdivision graph. The *subdivision graph* $S(G)$ is that graph obtained from G by replacing each line uv by a new point w and the two new lines uw and wv. It is then easy to show that the two graphs $T(G)$ and $(S(G))^2$ are isomorphic. (Thus, the graph T^2 shown in Figure 1 is the total graph of a star graph consisting of three lines incident at a common point.) Hence, if Conjecture 1 is true, then the following conjecture is also true:

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We remark that Behzad and Chartrand [1] have shown that if G is any nontrivial connected graph, then $T(T(G))$ is Hamiltonian.

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RICHARD ASKEY, University of Wisconsin, Madison

There are many interesting results in mathematics which are not given in courses because the known proofs are either too complicated or too artificial for the particular class of students. One such example is the Dirichlet-Mehler

integral for the Legendre polynomial

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi - 2\cos\theta)^{1/2}}, \qquad 0 < \theta < \pi. \tag{1}$$

One of the most natural ways to introduce the Legendre polynomial, and the way it originated, is by the generating function

$$(1 - 2xr + r^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) r^n. \tag{2}$$

We will show that (1) is an immediate consequence of (2).

In (2), let $r = e^{i\phi}$ and $x = \cos\theta$. Then we see

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(\cos\theta)e^{in\phi} &= (1 - 2e^{i\phi}\cos\theta + e^{2i\phi})^{-1/2} \\ &= e^{-(i\phi/2)}(2\cos\phi - 2\cos\theta)^{-1/2},\end{aligned} \tag{3}$$

where we have used $2\cos\phi = e^{i\phi} + e^{-i\phi}$. Multiplying by $e^{i\phi/2}$, and taking the real part we see that

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(\cos\theta)\cos(n+\tfrac{1}{2})\phi &= (2\cos\phi - 2\cos\theta)^{-1/2} && 0 \leqq \phi < \theta \\ &= 0 && \theta < \phi \leqq \pi.\end{aligned} \tag{4}$$

But

$$\int_0^\pi \cos(n+\tfrac{1}{2})\phi\cos(m+\tfrac{1}{2})\phi\, d\phi = \frac{\pi}{2}\delta_{n,m} \tag{5}$$

so

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi - 2\cos\theta)^{1/2}}. \tag{6}$$

Taking the imaginary part of (3) we have

$$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi}{(2\cos\theta - 2\cos\phi)^{1/2}}\, d\phi. \tag{7}$$

For $0<\theta<\pi$ and $\phi \neq \theta$, the series in (3) converges and the function to which it converges is integrable. Thus we may integrate term-by-term as we did above.

There are many applications which follow from (6) and (7). One of the most interesting elementary applications concerns the zeros of $P_n(x)$. Since $P_n(x)$ is a polynomial of degree n it has n zeros. The fact that these are all real and lie between -1 and 1 is usually proven from Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{8}$$

This inequality is the basis for his theory of Cesaro summability of spherical harmonic series, this time $(C, 2)$ and not $(C, 1)$ as for ordinary Fourier series. See [5].

We should conclude with a few words about the usual proofs of Mehler's integral (1). As George Gasper pointed out to me, the above proof is very close to Dirichlet's original proof, the difference being his use of the orthogonality of $\cos n\phi$ instead of $\cos(n+\frac{1}{2})\phi$. Hermite has found a proof from a complicated real integral which is sometimes given [5]. Another method is to use a contour integral applied to the generating function [7]. But (2) is a result of real analysis and a real proof should be preferred. There is one natural real variable proof which is easy once hypergeometric series have been introduced and $P_n(\cos\theta)$ and $\cos(n+\frac{1}{2})\phi$ have been identified as the appropriate hypergeometric functions. But hypergeometric functions are not usually introduced in mathematics courses. Also the identification of $P_n(\cos\theta)$ and $\cos(n+\frac{1}{2})\phi$ as hypergeometric series is not the most obvious fact about them. For systematic treatments of fractional integrals of hypergeometric functions and the corresponding results for the classical orthogonal polynomials, see [1] and [4].

Supported in part by NSF grant GP 6764.

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APPROXIMATION TO A FUNCTION BY A POLYNOMIAL IN ANOTHER FUNCTION

J. L. WALSH, University of Maryland

Although the following theorem was first proved ([1], pp. 444–445) in the more general setting of complex variables, the theorem itself admits an immediate and direct proof, as is our purpose here to indicate. Of course this theorem is also a consequence of the well-known work [2, 3] of M. H. Stone on approximation, subsequent to [1], and may be considered as intermediate between the Weierstrass polynomial approximation theorem and that of Stone.

THEOREM. *Let the function $\phi(x)$ be continuous on the interval $0 \leq x \leq 1$, and let $f(x)$ be an arbitrary continuous function on $[0, 1]$. A necessary and sufficient*

This inequality is the basis for his theory of Cesaro summability of spherical harmonic series, this time $(C, 2)$ and not $(C, 1)$ as for ordinary Fourier series. See [5].

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THEOREM. *Let the function $\phi(x)$ be continuous on the interval $0 \leqq x \leqq 1$, and let $f(x)$ be an arbitrary continuous function on $[0, 1]$. A necessary and sufficient*

'RYE WHISKEY' IN CONTRAPOSITIVE

W. P. Cooke, West Texas State University
(Now at the University of Wyoming)

While discussing some elementary logic in a class in geometry, I found that the students enjoyed those problems which were based on "If-Then" couplets from popular songs. The most fun occurred when we attempted to write them using contraposition. The 'game' was not only to achieve the contrapositive, but also to preserve the rhyme and meter of the song. Following is an only slightly more ambitious undertaking which should illustrate the idea.

In a famous Western 'Classic,' elegantly sung, as I recall, by Tex Ritter, is found (perhaps imperfectly remembered) the verse:

Statements

If the ocean was whiskey and I was a duck,
I'd swim to the bottom and never come up.
But the ocean ain't whiskey and I ain't no duck,
So I'll play Jack-O-Diamonds and trust to my luck.
For it's whiskey, Rye whiskey, Rye whiskey I cry.
If I don't get Rye whiskey I surely will die.

These statements are 'naturals' for contraposition, as follows:

Contrapositives

If I never reach bottom or sometimes come up,
Then the ocean's not whiskey or I'm not a duck.
But my luck can't be trusted or the cards I'll not buck,
So the ocean is whiskey or I am a duck.
For it's whiskey, Rye whiskey, Rye whiskey I cry.
If my death is uncertain then I get whiskey (Rye).

LHOSPITAL'S RULE WITHOUT MEAN-VALUE THEOREMS

R. P. Boas, Jr., Northwestern University

The "rule" that goes by the name of the Marquis De Lhospital (to give him, for once, the spelling that he himself used [4]), but which was actually discovered by John Bernoulli (see [4] or [5]), is usually proved by using the generalized mean-value theorem. I shall show that, with slightly stronger hypotheses that suffice for all applications, it can be proved quite simply without any mean-value theorems at all; this proof seems to have some pedagogical advantages, as well as suggesting some results that are not covered by the usual formulation.

We are to prove that *if f and g are real functions with continuous derivatives, if $f(x)$ and $g(x)$ both approach 0 or both approach ∞ as $x \to a$, if $g'(x) \neq 0$, and if $f'(x)/g'(x) \to L$ as $x \to a$ then $f(x)/g(x) \to L$*; all limits are taken on one side of a. We shall take $a=\infty$ and L finite, but only formal modifications are required for other cases.

We shall need only (i) the definition of a limit; (ii) that a continuous function

'RYE WHISKEY' IN CONTRAPOSITIVE

W. P. COOKE, West Texas State University
(Now at the University of Wyoming)

While discussing some elementary logic in a class in geometry, I found that the students enjoyed those problems which were based on "If-Then" couplets from popular songs. The most fun occurred when we attempted to write them using contraposition. The 'game' was not only to achieve the contrapositive, but also to preserve the rhyme and meter of the song. Following is an only slightly more ambitious undertaking which should illustrate the idea.

In a famous Western 'Classic,' elegantly sung, as I recall, by Tex Ritter, is found (perhaps imperfectly remembered) the verse:

Statements

If the ocean was whiskey and I was a duck,
I'd swim to the bottom and never come up.
But the ocean ain't whiskey and I ain't no duck,
So I'll play Jack-O-Diamonds and trust to my luck.
For it's whiskey, Rye whiskey, Rye whiskey I cry.
If I don't get Rye whiskey I surely will die.

These statements are 'naturals' for contraposition, as follows:

Contrapositives

If I never reach bottom or sometimes come up,
Then the ocean's not whiskey or I'm not a duck.
But my luck can't be trusted or the cards I'll not buck,
So the ocean is whiskey or I am a duck.
For it's whiskey, Rye whiskey, Rye whiskey I cry.
If my death is uncertain then I get whiskey (Rye).

LHOSPITAL'S RULE WITHOUT MEAN-VALUE THEOREMS

R. P. BOAS, JR., Northwestern University

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We shall need only (i) the definition of a limit; (ii) that a continuous function

we really use in (ii) is that a derivative that is different from 0 on an interval has a fixed sign there.

Since our proof did not use any mean-value theorems, it opens up the possibility of extending the rule to cases where mean-value theorems are not available. As an illustration, I state the sequence analogue of Lhospital's rule (see [2], 1st ed., pp. 377 ff.; 2nd ed., pp. 413 ff.).

Let $\{a_n\}$ *and* $\{b_n\}$ *be two real sequences that both approach zero or both approach* ∞; *let* $\Delta g_n = g_n - g_{n+1}$ *have a fixed sign, and let* $\Delta a_n/\Delta b_n \to L$; *then* $a_n/b_n \to L$.

The proof is the same, substituting differences for derivatives; in going from (2) to (3) we use

$$\sum_{k=n}^{\infty} (c_k - c_{k+1}) = c_n,$$

which is true when $c_k \to 0$. For example, suppose that $\sum x_n$ and $\sum y_n$ are two convergent series of positive terms; put $a_n = \sum_n^\infty x_k$, $b_n = \sum_n^\infty y_k$, the remainders. Then $x_n/y_n \to L$ implies $a_n/b_n \to L$, a result that is sometimes useful in dealing with infinite series.

It is possible to formulate a theorem that includes both Lhospital's rule and this discrete analogue; the reader is invited to find such a theorem for himself.

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1. L. Bers, On avoiding the mean value theorem, this MONTHLY, 74 (1967) 583.

2. T. J. I'a. Bromwich, An introduction to the theory of infinite series, Macmillan, London, 1st ed., 1908; 2nd ed., 1926.

3. L. W. Cohen, On being mean to the mean value theorem, this MONTHLY, 74 (1967) 581–582.

4. O. Spiess (editor), Der Briefwechsel von Johann Bernoulli, Band I, Birkhäuser, Basel, 1955.

5. C. Truesdell, The new Bernoulli edition, Isis, 49 (1958) 54–62.

MATHEMATICAL EDUCATION

EDITED BY J. G. HARVEY AND M. W. POWNALL

Material for this Department should be sent to either of the editors: J. G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706; M. W. Pownall, Department of Mathematics, Colgate University, Hamilton, NY 13346.

MATHEMATICS INSTRUCTION OF THE CULTURALLY DISADVANTAGED YOUNG ADULT

MARGARET A. FARRELL, State University of New York at Albany

During the summer of 1968 I was the mathematics instructor for twenty young adults who had been accepted for fall admission to college, despite some deficiencies in academic background. Their learning problems and my instructional problems were not unique, and our solutions and failures were neither

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dramatic nor original. Nevertheless, I offer the following descriptive account of some of the summer's activities in the hope that others interested in mathematics education and the culturally disadvantaged may find it helpful.

The students. The twenty students with whom I had direct contact ranged in age from 17 years to 22 years. Twelve of them had taken no formal work in mathematics beyond a ninth grade course in "general," "basic," or "business" mathematics. Their knowledge of algebra was either nonexistent or negligible. The remaining eight students had been previously enrolled in an elementary algebra course although school records showed failing to borderline grades. In addition, two or three of these eight students had been enrolled in one or two semesters of geometry. Beyond these academic factors, consider that most of the twenty students had taken their last mathematics course anywhere from three to five years earlier. As for study skills, pre-tests indicated that most of these students were below the 50th percentile on national norms of the Kelly-Greene Reading Comprehension Test and the Brown-Carlson Listening Comprehension Test. Furthermore, since all these students were in need of economic assistance to pursue a college education and had, in general, no hope of attending college before the initiation of COP (College Opportunity Program), there was a large cluster of related psychological and cultural factors which affected them and their reaction to education in any form.

The objective. Our immediate goal was to diagnose and treat mathematical deficiencies in order to prepare these students for their entrance into college. (Mr. John Therrien, Associate Professor of Mathematics, instructed an additional thirteen students whose pre-college backgrounds included at least three years of high school mathematics. These students and their problems are not included in this paper.) In particular, I attempted to introduce algebraic concepts and skills to the group of twelve (T), and to upgrade skills and correct misunderstandings with the group of eight (E). It should be noted that my goal —not an especially ideal one—was to establish or improve knowledge and skills needed to pass the first college mathematics course.

Groups and subgroups. Since our time schedule had earlier been divided into five 50 minute and two 75 minute blocks for the purpose of mathematics instruction, I devised a schedule which allowed me to meet with T for two 50 minute blocks and one 75 minute block and with E the remaining blocks. Every other week the amounts of time were reversed. Within these group meetings various sized subgroups were utilized and considerable one-to-one instruction was possible. While one group worked on a problem sheet or a quiz, three, four, or six others might be sitting in an informal arrangement with the instructor in order to ask questions about an assignment or to discuss a new concept. At times a student chairman was assigned to a group with a specific charge. This last procedure was most successful after the entire group had taken a test. The subgroups in this case were heterogeneous and the individuals worked toward group understanding of the various concepts and relationships.

Labs. In addition to the above class groupings, the students were scheduled for individual tutoring with college student aides and for "labs" during the math block times when they were not in class. Admittedly an administrative device to alter class size, the "labs" required the students to meet as a group, be supervised by a college student aide, and work on individually assigned programmed texts, readings, or sets of problems. Although the aide was available to help the student, these sessions generally were held with no immediate feedback. The work was corrected at a later time and further instructions written, or a student-instructor conference requested.

Toward a text. We began the classes without a text and gradually hoped to make available Schaum's *Elementary Algebra* and *College Algebra* outlines, not as a text but as a resource. We knew we could not hope to accomplish a magnitude of work in five weeks and we anticipated student self-study in these outlines. The low reading and listening level of most of the students had been the major reason for this nontext approach. Eventually, the Schaum outlines were included in the lab folders, and serendipity in the form of student requests to borrow these for further work outside of lab time developed.

Other features. The administrative diversity described above was probably the most noticeable aspect of these classes but it was not the only aspect and certainly not the most important one. Listed below are some other features of the program and the accompanying difficulties:

1. The introductory work in algebra used the "frames" concept and concentrated on mathematical sentences. Inequality and order symbols, absolute value symbols, root and power notation were used in these sentences and seemed to create few difficulties. However, when the customary x, y, or z was substituted for the frame, there were obvious individual problems. This phenomenon is well known among younger children; it is apparently not affected by age or mental maturity. Or is it a function of the cultural background of these students?

2. Throughout the program, models and operational techniques were used on the assumption that effective teaching at any level is based on developmental psychology. Toward the end of the session I realized the existence of a limiting condition to this assumption. In the use of wax paper to construct a parabola, a device used with considerable success with eleventh graders, college seniors, and teachers, I became aware of an uncharacteristic silence from the participants—a silence which shouted their suspicion that they were being educationally pampered. This kind of suspicion is not dispelled by hinting at the sophistication of the technique. In this case, the completion and analysis of the problem relieved the situation.

3. As is customary in any population, there were preferred individual strategies. Some students were visually oriented; others seemed to be symbol oriented. It was still surprising when a student who had been barely holding his own in the simplifying of algebraic expressions suddenly went to the top of the class in interpretation and construction of graphs.

The results. To summarize, the students seemed to enjoy working in groups when the group task represented a challenge to them. In this respect, if the range of heterogeneity was not too great, the use of a high-achieving student as a chairman was accepted by the group and all seemed willing to work at raising the group level. The success of this kind of group work seems to depend on several complex factors: the age, the socio-economic level of the class, their recently acquired hopes for upward mobility. . . . The single failure, which occurred when the instructor was not present, was the result of the absence of a resource person, the absence of guidance at several critical times, and the presence of a degree of racial conflict. These experiences in subgroups seem to suggest that a mobile class structure has particular advantages in a diverse group of individuals. Unlike some other techniques discussed in this paper, a changing group structure apparently posed no threat to the students. On the contrary, communication improved in quality and quantity as mobility increased.

It is worth noting that the "labs" were most effective, from the point of view of sustained interest and calibre of work accomplished, with T. Among the possible causes must be listed a variety of individual factors (including motivation, enthusiasm for "new" tasks in contrast to review tasks), the presence of an accepted leader in T, and the highly supporting attitude of the aides assigned to T. In addition, those students who considered their mathematical deficiency to be merely quantitative resisted these problem-solving sessions and preferred lectures where they might assume a passive role. For these students (most of whom were in the E group) labs were a failure. Furthermore, since these students refused to recognize the limitations of receptive learning in mathematics (e.g., neglected assignments) their progress with respect to their background was considerably less than that of the T group. In spite of this negative result, my personal bias now favors the type of grouping described in these paragraphs although aides need to be guided in their duties and materials need to be structured more carefully. Under the best of circumstances, I would prefer that the lab supervisor be a competent teacher in his own right since many individual problems are exposed in these sessions.

In all cases, techniques and materials different from those the students had known were causes of suspicion and potential deterrents to learning. It is possible that the negative effects of such suspicion can be decreased or even eliminated if suitable preliminary approaches are incorporated.

Problems of the above kind and the others indicated in this article are not isolated and are more and more in need of solution. As the culturally disadvantaged of all ages become the concern of educators, it will be necessary to reassess our understanding of mathematics learning in terms of language, background, and emotion-charged factors. We need more centers studying mathematics learning and more scholarly attention to the problems of the mathematically disabled, whether the "injury" be the result of cultural, emotional, social, or educational factors.

IMPROVING ELEMENTARY INSTRUCTION IN MATHEMATICS

A Program to Train Secondary Personnel as Teachers for Elementary In-Service Programs.

S. S. BLAKNEY AND T. C. GIBNEY, University of Toledo

The joint talents of the Colleges of Arts and Sciences and Education were utilized at the University of Toledo in producing an experimental conference to train secondary mathematics teachers. The participants were high school mathematics personnel with a bachelor's degree and at least 18 semester hours in mathematics with a desire to direct (or teach) in-service programs for elementary mathematics teachers.

Eligibility for the Toledo Conference was restricted to secondary mathematics teachers who were already thoroughly grounded in mathematics and who could profit from an intensive program designed to prepare them for in-service work with elementary teachers. Each high school teacher who participated in the Toledo Conference had the endorsement of his local school system to return to his local district as a resource teacher and in-service leader of elementary school teachers. The four week conferences were supported by the National Science Foundation during the summers of 1967 and 1968.

The Toledo Conference was designed to familiarize secondary teachers with new materials, methods and approaches to teaching mathematics in the elementary school and to prepare them to work with the elementary teachers in their local school systems. Instruction during the conference was given jointly by professors in the Mathematics Department and the College of Education as they helped each participant prepare himself for a specific role the following academic year as an in-service mathematics leader in the elementary school of his district.

The objectives of a summer conference and the program for the conference will vary according to the needs of the geographical area served by the conference; therefore the following objectives and program that were utilized by the Toledo Conference are listed only to serve as an aid and a guide to others who have the responsibility to conduct a summer conference.

I. Objectives of Program

1. To build a background in the area of elementary mathematics content and use this knowledge in preparing in-service classes for elementary teachers.
2. To familiarize the participants with new materials, methods, and approaches to teaching mathematics in the elementary school and develop techniques of using these in accordance with varying needs and abilities of individual teachers.
3. To enable participants to diagnose and prescribe for the individual teacher's needs, concerns, interests, aspirations, and general mathematics development; and plan for effective classroom management in conducting in-service classes for elementary teachers to reign at their unique levels and according to their varying interests.

4. To emphasize unification and integration of mathematical ideas and practices.
5. To establish with each secondary participant a plan for implementing a new mathematics program in his respective elementary school with regard to the administration, faculty, lay public, etc.
6. To create an environment through demonstration classes by a master teacher in the elementary school during which the participants can observe student behavior and familiarize themselves with the climate of mathematics instruction in the elementary school.

II. Program. The content course of the program was a survey at best. The material was treated with this question in mind: What relevance did this material have for elementary school mathematics or new programs in elementary mathematics? The fulfillment of the objectives was sought through the following:

A. *Foundational Mathematics*
1. Basic notions of Sets and Logic.
2. Overview of the real number system.
3. The nature of mathematical systems and the system of real numbers as an ordered field.
4. Equations and inequalities.
5. Some aspects of number theory.
6. Geometry, stressing three levels:
   (a) Concrete
   (b) Semi-abstract (intuitive)
   (c) Abstract (deductive level).
7. The need and nature of proof.

B. *Analysis of New Mathematics Program.* This class was held part time in the Mathematics Laboratory and part time in the Curriculum Materials Center. An analysis of the School Mathematics Study Group materials appropriate for elementary teachers was performed along with examinations of contemporary major textbook offerings. Special projects such as the Madison Project were reviewed along with reports such as the Cambridge Report and CUPM's recommendations for required training of elementary teachers. Several sets of enrichment materials were examined and many different programmed units were reviewed. Materials specifically related to instruction of the slow learner in mathematics were housed in the Mathematics Laboratory, and the Laboratory was used as the center to demonstrate the use of this material. Time was spent in the Mathematics Laboratory discussing the various projects, materials, and recommendations with the instructor as a guide and research person.

C. *Seminar on how to organize an in-service class and put it into operation.* One of the basic features of the experimental conference was a seminar on how to organize an in-service class and put it into operation. The focus was on tying the

subject matter, methods of instruction, organization of materials, and curriculum planning to the needs of each individual school district. Each participant prepared a notebook and other teaching materials which served as aids for their in-service classes.

D. *Demonstration Class.* The major purpose of a demonstration class was to give the conference participant a picture of the environment within an elementary classroom. Another important purpose was to demonstrate developmental teaching by employing a master teacher to demonstrate promising techniques of instruction. Therefore, a two week demonstration class was scheduled during the second and third weeks of the conference and a master teacher was chosen to instruct a nongraded group of elementary students from ages seven through eleven and to answer questions daily from the participants.

E. *Problem Sessions.* Beginning the second week of the conference one hour per day was devoted to problem solving. During this period assigned problems were discussed and the participants received assistance from the graduate assistant and other staff members in applying the theory developed in the previous lectures. This part of the program was inaugurated as a result of the feedback from the 1967 conference and was highly praised in the written critique from the 1968 participants.

F. *Critique.* A weekly two-hour seminar was held each Friday afternoon during which participants, the seminar instructor, and the course instructors discussed specific problems as they arose during the course of the conference. On each Wednesday, the participants were given a critique form to complete which evaluated the activities and experiences from the standpoint of course work and the supervised seminar regarding the degree to which the conference objectives were being achieved. This weekly feedback was discussed by participants and staff members at the Friday meeting to determine ways to correct deficiencies and improve the experience of the participants.

G. *Evaluation and Follow-Up.* A major part of the evaluation of the experimental conference depended upon what use was made of the participants when they returned to their local districts. A follow-up letter was sent to the administrators of the participants in the Toledo Conference in which they were asked to list the participant's involvement with in-service instruction. This information plus subsequent visits by the conference staff revealed that most participants were engaged with in-service programs within their own districts while others were involved with in-service classes outside their districts.

The University of Toledo plans to continue the education of secondary mathematics teachers as in-service leaders of elementary teachers. Undoubtedly, more time and evidence is needed to properly evaluate the effectiveness of this group but the initial evaluations point strongly to a well prepared group of in-service leaders who are being actively utilized by the public and private schools.

Textbooks Required

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A REFUTATION OF THE ARTICLE, INSTITUTIONAL INFLUENCES IN THE GRADUATE TRAINING OF PRODUCTIVE MATHEMATICIANS

RALPH H. FOX, Princeton University

The *American Mathematical Monthly* a year and one-half ago published the article *Institutional influences in the graduate training of productive mathematicians* by B. R. Siebring, 74 (1967), 1126–1130, in which certain statistical data was used (or rather grossly misused) to support conclusions related to the ability of small mathematics departments to provide quality education for small numbers of Ph.D. candidates.

Mr. Siebring's article consists of four statistical tables, some discussion of the manner in which they were compiled, and one or two general conclusions to be drawn from them. These tables were claimed to be based on three figures compiled for each of a selected list of universities. These are as follows:

n = the number of students awarded the Ph.D. by the mathematics department of that university during the decade 1950–1959,

m = the number of these that published at least one paper that has been reviewed by *Mathematical Reviews* by the end of 1964.

p = the number of such papers.

Tables I and II of the "Siebring Report" purport to rank ten universities according to the values of m and p respectively, while Tables III and IV purport to present rankings (of about twenty universities) based on the values of m/n and p/n respectively.

However this is not exactly what happened. The value of n was taken from a U. S. Office of Education document and is accurate enough, but since that document does not list the names of the degree recipients involved, some other source for the values of m and p had to be found. What Siebring did was to take the names from *American Men of Science* and look them up in *Mathematical Reviews*. Thus the Tables I, II, III, IV are actually based respectively on the

E 2197. *Proposed by M. S. Klamkin, Ford Scientific Laboratory and D. J. Newman, Yeshiva University*

Solve the functional equation $F(x^m) = [F(x)]^n$.

E 2198. *Proposed by Michael Aissen, Fordham University*

If $r > 1$ is an integer and x is real, define

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{r-1} \left[\frac{x + jr^k}{r^{k+1}}\right],$$

where the brackets denote the greatest integer function.
Show that

$$f(x) = \begin{cases} [x] , & \text{if } x \geqq 0 \\ [x+1], & \text{if } x < 0. \end{cases}$$

E 2199.* *Proposed by F. J. Papp, University of Delaware*

For given n, determine all r and s such that

$$\frac{(r+1)(n+s)!}{(s+1)(n+r)!}$$

is integral.

E 2200.* *Proposed by J. M. Moser, Navy Electronics Laboratory, San Diego, Cal.*

Find the number of partitions of an integer n into m unequal parts such that each part must be not larger than $2m$, nor smaller than q, where $2 \leqq q \leqq m+1$.

E 2201. *Proposed by J. M. Quoniam, Saint-Etienne, France*

Given a triangle, find (with compass and straightedge) the points in which the inscribed Steiner ellipse of the triangle intersects the three Artzt parabolas of the triangle, and also construct the tangents to the curves at these points. (The *inscribed Steiner ellipse* of a triangle is the ellipse inscribed in the triangle and having the centroid of the triangle for its center; an *Artzt parabola* is the parabola tangent to two sides of the triangle at the endpoints of the third side.)

SOLUTIONS OF ELEMENTARY PROBLEMS

Minimal Decomposition of an Integer

E 2137 [1968, 1113]. *Proposed by R. A. Christiansen, University of Victoria, Canada*

Show that each integer q has a unique decomposition

$$q = \sum_{j=0}^{n} d_j 2^j, \quad \text{where} \quad d_j \in \{0, 1, -1\}$$

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$$q = \sum_{j=0}^{n} d_j 2^j, \quad \text{where} \quad d_j \in \{0, 1, -1\}$$

II. $a \in D$. Then 2^{a+1} appears on the left side of (5) and therefore also on the right side. As $a \in A \cap D$, we have $a+1 \in B' \cap C'$. This shows that 2^{a+1} on the right side of (5) must be the result of adding 2^a from $\sum_{b \in B} 2^b$ and 2^a from $\sum_{c \in C} 2^c$. Hence $a \in C$. We have now proved that $A \subset C$, and uniqueness is thus established.

Let us now consider an arbitrary decomposition

$$q = \sum_{s \in S} c_s 2^s - \sum_{t \in T} c_t 2^t, \tag{6}$$

where S and T are finite disjoint subsets of N, and c_u is a positive integer for all $u \in S \cup T$. Together with this decomposition we shall consider the number

$$\sum_{u \in S \cup T} c_u f(u), \tag{7}$$

where f is the nonincreasing function mentioned. In (6) and (7) we shall consider $c_u 2^u$ and $c_u f(u)$ as short for, respectively, $2^u + 2^u + \cdots + 2^u$ and $f(u) + f(u) + \cdots + f(u)$, with c_u terms in each. We can now reduce the decomposition (6) to the decomposition (2) by a succession of replacements of the following forms and the four forms with all signs reversed:

(i) $2^x + 2^x$ with 2^{x+1}, (ii) $2^x - 2^x$ with nothing,

(iii) $2^x + 2^{x+1}$ with $-2^x + 2^{x+2}$, (iv) $2^x - 2^{x+1}$ with -2^x.

With no one of these replacements will the number (7) increase, because, respectively:

(i) $f(x) + f(x) \geqq 0 + f(x+1)$, (ii) $f(x) + f(x) \geqq 0$,

(iii) $f(x+1) \geqq f(x+2)$, (iv) $f(x+1) \geqq 0$.

The stated inequality is now proved.

Also solved by W. D. Bouwsma, M. S. Klamkin, and the proposer.

An Even Integer

E 2138 [1968, 1114]. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha*

Show that for all positive integers n, the integer

$$\left[\frac{(n-1)!}{n(n+1)}\right]$$

is even.

Solution by E. P. Starke, Plainfield, N. J. Put $Q = (n-1)!/n(n+1)$ and note that $[Q] = 0$ for $n < 6$. In the sequel we consider only $n \geqq 6$.

Suppose first that both n and $n+1$ are composite. Then we can put $n = a \cdot b$, $n+1 = c \cdot d$, $(ab, cd) = 1$, with each of a, b, c, $d \geqq 2$ and $\leqq \frac{1}{2}(n+1)$. If $a \neq b$ and $c \neq d$ (so that $n > 13$), the four integers are distinct members of the set

$S=\{1, 2, \cdots, n-1\}$, whence $Q(=[Q])$ is an integer. Moreover, S has at least six even members, whence $[Q]=(n-1)!/abcd$ is even.

If $a=b$, we take the four integers a, $2a$, c, d; if $c=d$, take a, b, c, $2c$. In either case these are four distinct members of S, so that $(n-1)!/2n(n+1)=Q/2$ is an integer and $[Q]=Q$ is an even integer.

Now if $n=p$, a prime >5, consider

$$Q+\frac{1}{p}=\frac{(p-1)!+(p+1)}{p(p+1)}.$$

By Wilson's theorem, $(p-1)!+(p+1)$ is divisible by p. As above, $(p-1)!/(p+1)$ is an even integer, so that $(p-1)!+(p+1)$ is an odd multiple of $(p+1)$. Hence $Q+1/p$ is an odd integer, so that $[Q]$ is an even integer.

Similarly if $n+1=p$, a prime >5, $Q+1/p$ is an odd integer because $(p-2)!\equiv 1 \pmod p$. In this case again $[Q]$ is even. This completes the proof.

Also solved by Brother Alfred Brousseau, Arthur Gittleman, Emil Grosswald, M. G. Greening (Australia), M. S. Klamkin, D. H. Pilgrim, D. E. Searls, R. A. Jacobson, and the proposer.

An Equilateral Property of All Triangles

E 2139 [1968, 1114]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

Consider the following four points of the triangle: the circumcenter, the incenter, the orthocenter, and the ninepoint center. Show that no three of these points can be the vertices of a nondegenerate equilateral triangle.

Solution by John Leech, University of Stirling, Scotland. Let O, I, H, N, G denote the circumcenter, incenter, orthocenter, nine-point center and centroid, respectively, of the given triangle, and let R, r denote the radii of its circumcircle and incircle respectively. We have to show that no three of O, I, H, N form an equilateral triangle.

Since O, H, N, G are collinear, the only possibilities are for I to be one of the vertices. N is the midpoint of OH and G is a trisector of OH and of NO. We have

$$NI = \tfrac{1}{2}R - r \qquad \text{(Feuerbach)}$$

$$OI^2 = R^2 - 2Rr \qquad \text{(Euler)}$$

so that $OI^2=2R\cdot NI\geqq 4NI^2$ and $OI\geqq 2NI$, with equality only in the degenerate case $r=0$. The set of points P for which $OP=2NP$ is the circle on diameter GH, whence I lies within this circle. But it is immediately evident that an equilateral triangle with vertices at any two of O, H, N has its third vertex outside this circle, which gives the result.

Note that if G had been included, we could have an equilateral triangle with vertices G, N, I, but no other combination is possible. Triangles are possible if I is allowed to be the center of an escribed circle, provided that O is not chosen with either G or N. A simple case is that if OHI is equilateral, the original triangle has angles of 30°, 60°, 90°.

$S=\{1, 2, \cdots, n-1\}$, whence $Q(=[Q])$ is an integer. Moreover, S has at least six even members, whence $[Q]=(n-1)!/abcd$ is even.

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Also solved by D. A. Herrero, I. H. Singh (India), and by the proposer.

Herrero notes the "truncated tetrahedron" for $n=12$, the icosahedron with three pentagonal pyramids removed for $n=9$, and the cuboctahedron and the twisted cuboctahedron for $n=12$. The first two graphs appear in Figure 3, the third and fourth are formed by joining two triangular cupolas (for $n=9$) in Figure 1. The proposer notes that for even n, one may use an $n/2$ prism and an $n/2$ antiprism except for the 4-prism and the 3-antiprism (cube and octahedron). Goldberg and Herrero each exhibited the polyhedron of 45 vertices. Beyond this demonstration, no proof or disproof of part (2) was attempted.

A Number Theory Problem

E 2142 [1969, 82]. *Proposed by Erwin Just, Bronx (N. Y.) Community College*

Let k, b, and r be fixed integers. Call an integer n *special* if each member of $\{kb^n+i\}$, $i=1, 2, \cdots, r$, is composite. Prove that the number of special integers is infinite.

Solution by the proposer. Let p_i be a prime divisor of $kb+i$, $(i=1, 2, \cdots, r)$, such that $p_i \nmid b$. (This is possible for any i unless $b|i$. In this eventuality, kb^n+i is composite for all n, and therefore can be eliminated from consideration.) Then, for any positive integer m, $kb^{1+m(p_i-1)}+i$ will be divisible by p_i. This must be true since Fermat's theorem guarantees that $(b^m)^{p_i-1}-1$ is divisible by p_i, and we may write

$$kb^{1+m(p_i-1)} + i = kb[(b^m)^{p_i-1} - 1] + (kb + i).$$

It follows if $n=1+m\prod_{j=1}^{r} (p_j-1)$, then kb^n+i will be divisible by p_i, $(i=1, 2, \cdots, r)$. This yields the desired conclusion.

Also solved by M. G. Greening (Australia).

A Variation on an Old Problem

E 2144 [1969, 82]. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha*

Solve the equation

$$\frac{x}{y} = \frac{(x^2 - y^2)^{y/x} + 1}{(x^2 - y^2)^{y/x} - 1}$$

for positive integral values of x and y.

I. *Solution by Jinfu Feng, Dartmouth College.* Easy simplification gives

$$(x - y)^{x+y} = (x + y)^{x-y}.$$

$x-y$ has to be positive, since otherwise $(x+y)^{x-y}<1<(x-y)^{x+y}$. Considering prime factors of $x-y$ and $x+y$, we see that $x+y=k(x-y)$, where k is a positive integer greater than 1. Substituting this expression into the equation we

obtain $x-y=k^{1/(k-1)}$. If $k>2$, then $1<x-y<2$, so there is no solution. $k=2$ gives $x-y=2$ and $x+y=4$, whence $x=3$, $y=1$ is the unique solution.

II. *Solution by M. G. Greening, University of New South Wales, Australia*

The given equation easily reduces to

$$\frac{\ln(x+y)}{x+y}=\frac{\ln(x-y)}{x-y}.$$

Now $(\ln t)/t$ is continuous for $t>0$ and has a maximum at $t=e$, so that the only solution in positive integers of $a^b=b^a$, $a>b$, is $a=4$, $b=2$. So the only solution possible is $x=3$, $y=1$, ($x-y$ necessarily being positive).

Also solved by J. E. Adams, Jr., A. N. Aheart, Anders Bager (Denmark), W. J. Blundon, Clyde Campbell, Mannis Charosh, E. N. Christensen & Amos Nannini, M. S. Demos, F. J. Duarte (Venezuela), G. E. Engebretsen, W. F. Fox, Toyomasa Fujinawa (Japan), Michael Goldberg, Emil Grosswald, J. E. Hafstrom, R. J. Herbold, T. F. Hughes, Jr., J. A. H. Hunter, Geoffrey Kandall, Ivan King (Australia), M. S. Klamkin, L. Kuipers, Mary B. Lewin, S. H. Lipson, George Lowerre, D. C. B. Marsh, Armel Mercier, C. B. A. Peck, Stephen Pierce, M. Beth Ruskai, B. L. Schwartz, N. T. Sheth, J. S. Shipman, David Spear, J. S. Vigder, Charles Wexler, Gregory Wulczyn and the proposer.

Editorial Note. The equivalent problem $a^b=b^a$ is indeed well known. L. E. Dickson, *History of the Theory of Numbers*, II, 687, notes that Euler and at least eight others considered it prior to 1915. The problem was posed in the 1960 Putnam competition [1961, 635], has appeared in many articles and books, and may be found in this MONTHLY [1921, 141], [1931, 444], [1945, 278], [1961, 856], [1967, 298], and [1968, 1104].

Equiareal Quadrilaterals

E 2145 [1969, 83]. *Proposed by V. F. Ivanoff, San Carlos, California*

In an arbitrary quadrangle $ABCD$, let line CE parallel to DA cut AB in E, and line CF parallel to BA cut AD in F. Denote by K the point of intersection of BF and ED. Show that the quadrangles $AEKF$ and $KBCD$ have equal directed areas.

Solution by William Wernick, City College of New York. $AECF$ is a parallelogram, with area $2z$. Draw KA, KC, EF. It is known that for any point K the sum of the directed areas of AEK and KCF is z, as is the sum of ECK and KFA. Now, $ECK+KFA=z=ECF=ECD=ECK+KCD$; therefore $KFA=KCD$. Analogously, $AEK+KCF=z=FEC=FBC=BCK+KCF$, so that $AEK=BCK$ Therefore, $AEKF=AEK+KFA=BCK+KCD=BCDK$.

Also solved by Trygve Breiteig (Norway), H. Demir (Turkey), Jordi Dou (Spain), Michael Goldberg, Lew Kowarski, L. Kuipers, D. C. B. Marsh, Simeon Reich (Israel), A. W. Walker, and the proposer.

100th Power Residues

E 2146 [1969, 83]. *Proposed by A. M. Kirch, University of Missouri*

Find, for each positive integer m, the last three digits of m^{100}.

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* Since $(m+10k)^{100} \equiv m^{100}$ (mod 1000), we need consider only the set of least nonnegative residues of m modulo 10. Applying the Euler Theorem and the Chinese Remainder Theorem, we have

m(mod 10)	0	2, 4, 6, 8	5	1, 3, 7, 9
m^{100}(mod 8)	0	0	1	1
m^{100}(mod 125)	0	1	0	1
m^{100}(mod 1000)	0	376	625	1.

II. *Solution by J. H. Jordan, Washington State University.* According to Jordan, *The number of unrestricted k^{th} power residues* (this MONTHLY, 75 (1968) 521) the number of unrestricted 100^{th} power residues is 4. The four residues are 0, 1, 376, and 625 computed by a^{100}, for $a = 0, 1, 2, 5$.

Also solved by Marcia Asher, Anders Bager (Denmark), Merrill Barnebey, N. A. Borba, W. D. Bouwsma, R. J. Bridgman, Brother Alfred Brousseau, Mannis Charosh, Sarah L. M. Christiansen, J. E. Crick, D. M. Danvers, E. H. Davis, M. S. Demos, G. C. Dodds, G. E. Engebretsen, W. F. Fox, Robert Giese, Michael Goldberg, Michael Goodman, M. G. Greening (Australia), Emil Grosswald, Iris P. Hansen, P. M. Harma, C. V. Heuer, T. F. Hughes, Jr., Geoffrey Kandall, M. S. Klamkin, H. R. Leifer, H. S. Lieberman, T. M. Little, Lois M. R. Louden, G. Lowerre & G. Satlow, D. C. B. Marsh, R. L. McFarland, Armel Mercier, Margaret S. Oglesby, H. L. Nelson, Bob Prielipp, Simeon Reich (Israel), J. S. Shipman, Roy Smith, David Spear, G. C. Thompson, Gregory Wulczyn, D. E. Zitarelli, and the proposer.

A Generalization of the Triangle Inequality

E 2147 [1969, 83]. *Proposed by R. Shantaram, State University of New York at Stony Brook*

Let a and b be complex numbers and let $r \geqq 0$. Show that

$$|a+b|^r \leqq k_r(|a|^r + |b|^r),$$

where $k_r = 1$ if $r \leqq 1$, and $k_r = 2^{r-1}$ if $r \geqq 1$.

Solution by W. D. Bouwsma, Southern Illinois University. Suppose $r > 1$, and consider the function $f(x) = x^r$, $x \geqq 0$. Since $r > 1$, we have $f'' > 0$, so that f is convex. Hence if $x, y \geqq 0$, we have

$$\left(\frac{x+y}{2}\right)^r \leqq \frac{x^r + y^r}{2}.$$

Now, setting $|a| = x$, $|b| = y$, we have

$$|a+b|^r \leqq (|a| + |b|)^r \leqq 2^{r-1}(|a|^r + |b|^r).$$

Next suppose $r < 1$, and again let $f(x) = x^r$, $x \geqq 0$. Then f' is decreasing. Take $0 \leqq y \leqq x$. Then

$$f(x+y) - f(x) = yf'(\xi_2), \quad \text{and} \quad f(y) - 0 = yf'(\xi_1)$$

with $0<\xi_1<y\leqq x<\xi_2$. Since $f'(\xi_1)>f'(\xi_2)$, we have $f(x+y)-f(x)\leqq f(y)$, so that $(x+y)^r\leqq x^r+y^r$, and again

$$|a+b|^r \leqq (|a|+|b|)^r \leqq |a|^r+|b|^r.$$

Since $r=1$ gives the familiar triangle inequality, the proof is complete.

Also solved by Marcia Asher, Anders Bager (Denmark), W. O. Egerland, Graeme Fairweather (Scotland), Michael Goldberg, M. G. Greening (Australia), D. W. Hadwin, G. A. Heuer, M. S. Klamkin, L. Kuipers, Douglas Lind (England), Simeon Reich (Israel), Steve Rohde, G. S. Rogers, E. F. Schmeidel, David Wille, and the proposer.

Lind notes that the inequality could be further generalized as follows: Let (X, Σ, μ) be a totally finite measure space. Then if $r\geqq 1$ and f is any complex function on X,

$$\left|\int_X f d\mu\right|^r \leqq \mu(X)^{r-1}\int_X |f|^r d\mu.$$

Modification of a Conditionally Convergent Series

E 2148 [1969, 83]. *Proposed by G. R. MacLane, Purdue University*

Let

$$(1) \qquad \sum_{\nu=1}^{\infty} a_\nu$$

be a conditionally convergent series with real constant terms. The familiar examples (those that satisfy the Leibniz convergence criterion, for example) are such that

$$(2) \qquad \sum_{\nu=1}^{\infty} \operatorname{sgn}(a_\nu)\,|a_\nu|^\lambda$$

is convergent for each $\lambda>0$. Find a series (1) which converges, but such that (2) is divergent for each λ, $0<\lambda$, $\lambda\neq 1$.

Solution by R. J. Driscoll, Loyola University, O. P. Lossers, Technological University of Eindhoven, and the proposer (independently). Consider the series

$$\frac{1}{\log 2}-\frac{1}{2\log 2}-\frac{1}{2\log 2}+\cdots+\frac{1}{\log n}-\frac{1}{2\log n}-\frac{1}{2\log n}+\cdots$$

whose terms come in blocks of three, one block for each $n\geqq 2$. For $\lambda\neq 1$, the nth block contributes

$$\frac{1-2^{1-\lambda}}{(\log n)^\lambda}$$

to (2), so it is clear that (2) diverges to $+\infty$ $(-\infty)$ if $\lambda>1$ $(0<\lambda<1)$.

Also solved by Seymour Haber and by K. A. Post (Netherlands).

with $0<\xi_1<y\leqq x<\xi_2$. Since $f'(\xi_1)>f'(\xi_2)$, we have $f(x+y)-f(x)\leqq f(y)$, so that $(x+y)^r\leqq x^r+y^r$, and again

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Also solved by Seymour Haber and by K. A. Post (Netherlands).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers-The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before February 28, 1970. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

5695. *Proposed by Anon, Erewhon-upon-Wabash*

Let $R=Z[a_1, a_2, a_3, b_1, b_2, b_3]$ with the single relation $a_1b_1+a_2b_2+a_3b_3=1$. Let M be the R-module generated by x_1, x_2, x_3 with the single defining relation $b_1x_1+b_2x_2+b_3x_3=0$. Prove that M is not a free module, but is projective.

5696. *Proposed by P. J. Chase, Laurel, Maryland*

Show that a lattice L is Boolean if and only if it admits a unary operation $x\rightarrow x'$ such that $a\cap b\leqq c\leqq a\cup b$ implies $c\cap b'\leqq a\leqq c\cup b'$.

5697. *Proposed by Frederick Hammer, Paine College, Augusta, Ga.*

A set S is *transitive* if $x\in y\in S$ implies $x\in S$, *disjoint* if x, $y\in S$ implies $x\cap y=\emptyset$ or $x=y$. The *Axiom of Regularity* states that $x\neq\emptyset$ implies the existence of $y\in x$ with $y\cap x=\emptyset$. Show that in each finite cardinality there is exactly one transitive and disjoint set, and further, that there is only one transitive and disjoint infinite set, assuming the Axiom of Regularity.

5698. *Proposed by D. L. Lutzer, University of Washington*

It is well known that a subspace S of a separable Hausdorff space X need not be separable. Is there an example of this in which S is dense?

5699. *Proposed by G. J. Foschini, Bell Telephone Laboratories*

A derivation on a ring S is an additive mapping $s\rightarrow s'$ of S into itself satisfying $(pq)'=pq'+p'q$. Let $C[0, 1]$ be the ring of continuous real functions on $[0, 1]$ with the usual norm. Show that there exists a ring $R[0, 1]\subset C[0, 1]$ with a derivation $r\rightarrow r^{[1]}$ such that the following subsets of $R[0, 1]$ are dense in $C[0, 1]$:

(i) $\{r \mid r^{[1]} = 0\}$,

(ii) $\{r \mid r^{[1]} = r\}$,

(iii) $\{r \mid r^{[n]} = (r^{[n-1]})^{[1]} > 0, n = 1, 2, \cdots\}$

and such that $R[0, 1]\supset C^{\infty}[0, 1]$ whereon the derivation coincides with the usual derivative.

5700. *Proposed by R. E. Chandler and R. A. Struble, North Carolina State University at Raleigh*

Let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in (0, 1). For $x\in(0, 1)$ define $f(x)=\sum 1/2^n$, where the summation is over all n for which $x_n<x$. Evaluate $\int_0^1 f(x)dx$.

Free Groups with Two Generators I

5638 [1969, 1125]. *Proposed by Oswald Wyler, Carnegie-Mellon University*

Show that the free group G with two generators a, b and the three relations $a^4=b^4=abab=e$ is an extension of the free abelian group $Z\times Z$ (where Z is the additive group of integers) by the cyclic group Z_4.

Solution by D. Ž. Djoković and F. C. Y. Tang, University of Waterloo. Let H be the subgroup of G generated by a^3b and ba^3. H is abelian since

$$(a^3b)(ba^3) = (a^3b^3)(b^3a^3) = (ba)(ab) = (ba^3)(a^3b).$$

From

$$a(a^3b)a^{-1} = ba^3, \qquad a(ba^3)a^{-1} = aba^2 = b^3a = (a^3b)^{-1},$$
$$b(a^3b)b^{-1} = ba^3, \qquad b(ba^3)b^{-1} = b^2a^3b^3 = b^3a = (a^3b)^{-1},$$

we conclude that H is normal in G. Since $G=\langle H, a\rangle$ we infer that G/H is cyclic of order 1, 2 or 4.

Let $Z\times Z$ be generated by α and β and let $Z_4=\langle\gamma\rangle$. We can identify γ with the automorphism of $Z\times Z$ defined by

$$\alpha^\gamma = \beta^{-1}, \qquad \beta^\gamma = \alpha.$$

This automorphism has order 4. Let F be the semidirect product of $Z\times Z$ by Z_4. One can easily verify that

$$\gamma^4 = (\gamma\alpha)^4 = \gamma(\gamma\alpha)\gamma(\gamma\alpha) = 1.$$

Therefore there exists the unique homomorphism $f:G\to F$ such that $f(a)=\gamma$, $f(b)=\gamma\alpha$. Since γ and α generate F, f is an epimorphism. Since $f(H)=Z\times Z$ and $f(a)=\gamma$, we infer that $H\cong Z\times Z$ and $|G/H|=4$.

Also solved by M. G. Greening (Australia), R. C. Lyndon, W. O. J. Moser, S. S. Sampson, F. A. Sherk, J. H. van Lint, Kenneth Yanosko, and the proposer.

For additional two-dimensional space groups Sherk refers us to Coxeter and Moser, *Generators and Relations for Discrete Groups*, (Ergeb. der Math. NF 14, 1965, p. 40).

Free Groups with Two Generators II

5639, [1968, 1125]. *Proposed by Oswald Wyler, Carnegie-Mellon University*

Show that the free group G with two generators a, b and the three relations $a^4=b^4=aba^2b^2=e$ is an extension of the cyclic group Z_5 by the cyclic group Z_4.

I. *Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands.* The three relations imply

$$(1) \qquad b = ab^3a, \qquad (2) \quad ab = b^2a^2.$$

Define $x=ab$; we then get in turn

$$(3) \qquad x^2 = (b^2a^2)a(ab^3a) = ba,$$

$$x^5 = (ba)(ba)(ab) = b(aba^2b^2)b^3 = e, \tag{4}$$

$$axa^3 = a^2ba^3 = a^3b^3 = (ab)^{-2} = (ab)^3 = x^3, \tag{5}$$

$$bxb^3 = ba = x^2. \tag{6}$$

Thus x generates a cyclic subgroup Z_5 of G which is normal by (5) and (6). In any word of G we can replace b by a^3x and then (5) implies that each word of G is of the form a^mx^n with $0 \leqq m \leqq 3$, $0 \leqq n \leqq 4$, i.e., $G/Z_5 \cong Z_4$.

II. *Solution by Roberto Frucht, Santa Maria University, Valparaiso, Chile.* It is known (see, e.g., Coxeter and Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, (1.89)) that the relations

$$s^5 = t^4 = e, \qquad t^{-1}st = s^2 \tag{1}$$

define a group of order 20 which is an extension of the cylic group Z_5 by a cyclic group Z_4. It remains to be shown that the group defined by

$$a^4 = b^4 = aba^2b^2 = e$$

is isomorphic to the group defined by (1); this however is essentially an exercise in Carmichael, *Introduction to the Theory of Groups of Finite Order*, No. 29, p. 42.

Also solved by D. M. Bloom, D. Ž. Djoković, M. A. Ettrick, M. G. Greening (Australia), R. C. Lyndon, W. O. J. Moser, S. S. Sampson, Kenneth Yanosko, and the proposer.

Euler's Constant

5640 [1968, 1125]. *Proposed by R. E. Shafer, University of California at Livermore*

Show that

$$\gamma = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right) + \frac{1}{9}\sum_{k=n+1}^{\infty}(k-n)\int_0^1 \frac{dx}{(k+x)^2[(k+x)^4 - \frac{1}{3}(k+x)^2 + \frac{1}{9}]}.$$

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands. First we evaluate the integral and find that

$$\begin{aligned} I_k &= \frac{1}{9}\int_0^1 \frac{dx}{(k+x)^2[(k+x)^4 - \frac{1}{3}(k+x)^2 + \frac{1}{9}]} \\ &= \frac{1}{k} - \frac{1}{k+1} + \frac{1}{2}\log\frac{(k+1)^2 + (k+1) + \frac{1}{3}}{k^2 + k + \frac{1}{3}} \\ &\quad - \frac{1}{2}\log\frac{(k+1)^2 - (k+1) + \frac{1}{3}}{k^2 - k + \frac{1}{3}}. \end{aligned}$$

$$x^5 = (ba)(ba)(ab) = b(aba^2b^2)b^3 = e, \tag{4}$$

$$axa^3 = a^2ba^3 = a^3b^3 = (ab)^{-2} = (ab)^3 = x^3, \tag{5}$$

$$bxb^3 = ba = x^2. \tag{6}$$

Thus x generates a cyclic subgroup Z_5 of G which is normal by (5) and (6). In any word of G we can replace b by a^3x and then (5) implies that each word of G is of the form a^mx^n with $0 \leqq m \leqq 3$, $0 \leqq n \leqq 4$, i.e., $G/Z_5 \cong Z_4$.

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$$\gamma = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right)$$
$$+ \frac{1}{9}\sum_{k=n+1}^{\infty}(k-n)\int_0^1 \frac{dx}{(k+x)^2[(k+x)^4 - \frac{1}{3}(k+x)^2 + \frac{1}{9}]}.$$

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$$I_k = \frac{1}{9}\int_0^1 \frac{dx}{(k+x)^2[(k+x)^4 - \frac{1}{3}(k+x)^2 + \frac{1}{9}]}$$
$$= \frac{1}{k} - \frac{1}{k+1} + \frac{1}{2}\log\frac{(k+1)^2 + (k+1) + \frac{1}{3}}{k^2 + k + \frac{1}{3}}$$
$$- \frac{1}{2}\log\frac{(k+1)^2 - (k+1) + \frac{1}{3}}{k^2 - k + \frac{1}{3}}.$$

$$J_n = \frac{1}{n+1} - \frac{1}{2}\log\left(n^2 + 3n + \frac{7}{3}\right) + \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right)$$
$$+ \lim_{N\to\infty}\left\{\sum_{l=1}^{N}\frac{1}{n+l+1} + \tfrac{1}{2}\log\prod_{l=1}^{N}\left[\frac{(n+l+1)^2 - (n+l+1) + \frac{1}{3}}{(n+l+1)^2 + (n+l+1) + \frac{1}{3}}\right]\right\}.$$

Now

$$\frac{(n+l+1)^2 - (n+l+1) + \frac{1}{3}}{(n+l+1)^2 + (n+l+1) + \frac{1}{3}} = \frac{(l-\frac{1}{2})^2 + 2(n+1)(l-\frac{1}{2}) + (n+1)^2 + \frac{1}{12}}{(l+\frac{1}{2})^2 + 2(n+1)(l+\frac{1}{2}) + (n+1)^2 + \frac{1}{12}},$$

whence finally we get

$$J_n = \frac{1}{n+1} - \frac{1}{2}\log\left(n^2 + 3n + \frac{7}{3}\right) + \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right)$$
$$+ \lim_{N\to\infty}\left\{\sum_{l=1}^{N}\frac{1}{n+l+1} + \frac{1}{2}\log\left[(n+1)^2 + (n+1) + \frac{1}{3}\right]\right.$$
$$\left. - \frac{1}{2}\log\left[N^2 + (2n+3)N + n^2 + 3n + \frac{7}{3}\right]\right\}$$

and

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{2}\log\left(n^2 + n + \frac{1}{3}\right) + J_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$
$$+ \lim_{N\to\infty}\left\{\sum_{l=1}^{N}\frac{1}{n+l+1} - \frac{1}{2}\log\left[N^2 + (2n+3)N + n^2 + 3n + \frac{7}{3}\right]\right\}$$
$$= \lim_{N\to\infty}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{n+N+1}\right.$$
$$\left. - \frac{1}{2}\log\left[(N+n+1)^2 + (N+n+1) + \frac{1}{3}\right]\right\} = \gamma.$$

Also solved by D. J. Johnson, and by the proposer.

Using the notation for the logarithmic derivative of the gamma function the proposer offers the following generalization of his problem:

$$\psi(z) = \frac{1}{2}\log\left[(z+n)^2 + (z+n) + \frac{1}{3}\right] - \sum_{l=0}^{n}\frac{1}{z+l} - \frac{1}{9}\sum_{k=n+1}^{\infty}(k-n)A_k,$$
$$A_k = \int_0^1 (z+k+x)^{-2}\left[(z+k+x)^4 - \frac{1}{3}(z+k+x)^2 + \frac{1}{9}\right]^{-1} dx.$$

He also gives another, more rapidly converging series expansion for $\psi(z)$.

Indexing Continuous Functions

5644 [1969, 94]. *Proposed by Jerrold Siegel, Purdue University*

Prove that there does not exist a continuous function f of two real variables,

$f:R\times R\rightarrow R$, with the property that for any continuous $g:R\rightarrow R$, there exists a real t such that $g(x)=f(t, x)$ for all x.

Does such f exist if R is replaced by $[0, 1]$?

Solution by R. W. Chaney, University of California at Santa Barbara. (a) Suppose that such a function $f:R\times R\rightarrow R$ exists. Defining $g:R\rightarrow R$ by $g(x)=f(x,x)+1$, there is no t in R such that $g=f(t, \cdot)$, for otherwise we would get $f(t, t)=g(t)=f(t, t)+1$.

(b) No such f exists when R is replaced by $[0, 1]$. Suppose that such an f does exist and let K be the set of all continuous functions g: $[0, 1]\rightarrow[0. 1]$. If $(g_n)_{n=1}^{\infty}$ is a sequence in K, we can then write $g_n=f(t_n, \cdot)$, where $t_n\in[0, 1]$; a subsequence of (t_n) converges to some t and so a subsequence of (g_n) must converge to $f(t, \cdot)$. In short, we have shown that each sequence in K must admit a subsequence which converges pointwise to an element in K. But this is false; consider $g_n(x)=x^n$. The argument requires only that f be separately continuous in its variables.

Also solved by Einar Andresen (Norway), Robert Breusch, Michel Bousquet, David Boyd, G. E. Bredon, James Carrell & Ernst Ruh, D. E. Cooper, R. O. Davies (England), Crist Dixon, G. J. Foschini, Claus Gerhardt (Germany), D. A. Hejhal & R. K. Keinigs, C. V. Heuer & G. A. Heuer, Richard Johnsonbaugh, J. G. Jones & E. G. Grassman, Emmett Keeler, Douglas Lind, J. B. Linder & R. V. Fuller, O. P. Lossers (Netherlands), Ka Menehune, Steven Minsker, Hugh Noland, P. J. Owens (England), Charles Riley, Perry Smith, Alan Tschetter, P. van der Steen (Netherlands), Konrad Victor (Israel), D. A. Zave, and the proposer.

Johnsonbaugh raises the question of a possible indexing using relaxed conditions on $f(t, x)$, e.g., f measurable.

REVIEWS

EDITED BY KENNETH O. MAY

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

All unsigned material is written by the editors. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.*

Fundamentals of Abstract Analysis. By Andrew M. Gleason. Addison-Wesley, Reading, Massachusetts, 1966. xi+404 pp. $13.75 (Telegraphic Review, Oct. 1967)

The author's declared purpose in writing this book, intended for third or fourth year undergraduates, is to explain the relation of set theory to the rest of mathematics. The book devotes fewer than fifty pages to such explanation. The other three hundred and fifty odd pages are to be seen as artfully selected illustrative material. By themselves, they would constitute an excellent textbook in basic analysis. Interlaced as they are with the rich insights which moti-

$f: R \times R \to R$, with the property that for any continuous $g: R \to R$, there exists a real t such that $g(x) = f(t, x)$ for all x.

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defines a sequence as a function, its members as elements of its range, and order relations as antisymmetric, any nontrivial cycle such as $\cdots 0, 1, 0 \cdots$ causes trouble.) A more substantial difficulty exists: this is a book which directs attention to set-theoretic subtleties, which is self-consciously careful about the legitimacy of its set constructions. The author claims (p. 153) "All of the usual systems [for axiomatic set theory] will justify all of the reasoning used in this book up to and including section 11–4." Thus Gleason (p. 65) eschews calling equality (resp. isomorphism) a relation in his technical sense. Yet (p. 137) similarity and dominance of sets are admitted as relations. This apparent casualness enters into the discussion of cardinal numbers, and into the formulation of theorem 13–1.1. In each case, the set of all sets rears its dubious head. Some recasting of this material seems essential. It seems also that Theorem 11–1.8 and the paragraph which introduces it are not related as claimed without recourse to the choice axiom which follows ten pages later.

We should all read the book—student and teacher alike. Augmented by a suitable bibliography (why isn't it there?) it will make an ideal self-study text for summer reading between years 3 and 4 (or even 2 and 3). The first eleven chapters form an excellent source for students approaching the foundations of mathematics with a philosophic bent. The book should enjoy wide adoption as a text, in third or fourth year, but the teacher who uses it must expect to find himself upstaged by Gleason's performance at every turn.

L. T. GARDNER, University of Toronto

Introduction to the Methods of Real Analysis. By Maurice Sion. Holt, Rinehart and Winston, New York, 1968. x+134 pp. $8.95 (Telegraphic Review, Jan. 1969)

This book is designed as a text for "part of a year's course in analysis normally taken by senior undergraduates and first year graduate students." It draws on the student's presumably solid but unsophisticated understanding of limits, integration, and differentiation, and attempts to bring him along to a modern and deep appreciation of these concepts.

The book is divided into two parts. Part I, entitled Topological Concepts, has chapters on Elements of Set Theory, Spaces of Functions, Elements of Point Set Topology, and Continuous Functions; Part II, Measure Theory, has chapters on Measures in Abstract Spaces, Lebesgue-Stieltjes Measures, Integration, Differentiation, and Riesz Representation.

I have mixed feelings about this book. It is short, very short considering the amount of material it covers, and is written in a somewhat telegraphic style. Certainly the material is standard, and with not too much supplementing, could form the basis for a full year's course. But my main reservations concern the style, which students would find difficult to read on their own. For example Chapter 1 consists largely of extensive and rather complete lists of definitions which I find quite intimidating in their brevity and in their number. The lecturer would have to supply much of the motivation and somewhat fuller explanations.

In short, the book seems to supply a fairly complete skeleton, with the flesh left to be supplied by the lecturer and students. Those who like this approach would find this a teachable book. For other tastes, I think another choice of text would be preferable.

W. FULKS, University of Colorado

A Survey of Mathematics: Elementary Concepts and their Historical Development. By Vivian Shaw Groza (Sacramento City College). Holt, Rinehart and Winston, New York, 1968. xvi+327 pp. $8.50.

Recently it has become a not uncommon practice in college textbooks for terminal liberal arts courses in mathematics to include some history of the subject. In 1967 there was Morris Kline's substantial *Mathematics for Liberal Arts,* as well as Margaret Willerding's enrichment paperback on *Mathematical Concepts: A Historical Approach.* In 1968 we had William P. Berlinghoff: *Mathematics: The Art of Reason* and the book under review; but whereas in the Berlinghoff volume the historical element is found chiefly in an appendix, in Mrs. Groza's work the framework itself, made up of four "Parts," is historical. Part I, consisting of two chapters, is called "The Prehistorical Period," but this is in part misleading. Tallying, number words, finger counting, and simple grouping systems for numerals do indeed antedate history, but other topics in the two chapters do not. Development of a positional system probably was a product of the early historical period, and most of set theory has been the work of the past century. Again in Part II, "The Ancient Oriental Period," some of the material is properly so headed, including such things as Egyptian, Babylonian, and Chinese numerals, as well as Mesopotamian and Egyptian algebra and geometry; but the extensive chapter on "Arbitrary Bases" is thoroughly modern and Western. Chapters 8 and 9 of Part III, "The Greek Period," deal with Euclidean geometry and Greek arithmetic; but these are preceded by two chapters on logic and mathematical systems which again are historically misplaced. Part IV, comprehensively designated as "The Hindu-Arabic-European Period," is a kaleidoscopic miscellany of ancient and modern history and of elementary and advanced topics in mathematics. Here the chapter on algebra includes much history, especially of words and symbols, but less mathematics (mostly on the solution of linear equations). The last chapter, which is "optional," includes much mathematics in little space—probability, analytic geometry, calculus, non-Euclidean geometries, and topology, all in thirty pages.

Each of the four Parts of the book is introduced by an historical overview of from two to eighteen pages; and the volume closes with a three-page historical time chart and a two-page map of historically significant cities and sites. Historical allusions throughout the book are commendably accurate, although not impeccable. The date 4241 B.C., for example, is no longer generally accepted as that of the Egyptian calendar (p. 34); attention might well have been called (p. 88) to the Mesopotamian use of 3 1/8 for pi (as well as the value 3); and to refer (p. 204) to Arago (1786–1853) as a "contemporary" of Euler (1707–1783)

is stretching language a bit far. Occasionally an historical situation is oversimplified, as in the statement that "Mathematics was born with the appearance of the *Elements*, written by Euclid," (p. 99) and in the attribution to Descartes (p. 199) of the "invention of a coordinate system." One must bear in mind, however, that fidelity to history is not the primary object of this book. The purpose is, rather, "to acquaint the student with the various branches of mathematics and to develop an appreciation and understanding of the relationship of mathematics to the modern world." Intended for use in a course in the general education curriculum, the material, both mathematical and historical, is on an elementary level which presupposes no particular algebraic or geometric background on the part of the student. That the purpose is to teach mathematics rather than history is readily confirmed in that the exercises with which most chapters close call generally for simple computations or elementary mathematical analysis, rather than for historical perspective or recall. The author intended that history should serve "as a unifying thread weaving together the various topics of mathematics into a whole which is meaningful to the student." This is perhaps an unattainable goal for students at the level for which the book is intended; and in the last part of the book (mathematically the most ambitious part) the account seems to dissolve into a juxtaposition of historical and mathematical materials, rather than to form a seamless whole. If the author has not been entirely successful in the heroic attempt at integrating mathematics and history, she has nevertheless attractively sugar-coated what to the uninitiated and unprepared college freshman often appears to be the bitter pill of elementary mathematics.

C. B. BOYER, Brooklyn College

An Introduction to Algebraic Structures. By Azriel Rosenfeld. Holden-Day, San Francisco, 1968. xi+285 pp. $12.50. (Telegraphic Review, January 1969.)

The use of the word introduction in the title, and the first sentence of the cover jacket which reads, "A self-contained introduction to modern algebra, this book is suitable for a one- or two-semester course on the advanced undergraduate or graduate level," is misleading. Even the author's statement in the foreword that, "Most of the readers of this book will have some degree of prior exposure to algebraic structures," is too mild. The reader would probably experience great difficulty with this book unless he had completed a good course in algebra, possessed considerable mathematical maturity, and had a wealth of examples at his disposal.

The book covers many more topics than the usual introductory book and topics are covered more extensively than in introductory courses. It is written in a staccato, theorem-proof-theorem-proof style. Its strength lies in the areas of extensive coverage, conciseness, and the proving of results in their most general settings. Its weakness is in motivation, concrete examples, and applications. For example, many results about quotient groups are first proved for quotient sets and quotient groupoids. On the other hand the first examples of a finite

noncommutative group appear after 100 theorems or propositions and an equal number of corollaries about groups have been proved.

The book is fairly self-contained. It begins with sets, functions, and numbers, including the development of the natural numbers from Peano's axioms. After more material on sets it covers algebraic systems with one and two binary operations through Galois theory and the Wedderburn-Artin theorems on simple and semisimple rings. Chapter three is on lattices, and lattice theory is used throughout. The second to last chapter is on vector spaces, and here again it would be well for the student to have prior knowledge of linear algebra. The emphasis of the last half of the book is on finiteness conditions.

Exercises are interspersed within the body of the material and should be worked as they are encountered since references are made to these exercises in the ensuing material. The book also contains numerous, so called, "Examples," most of which are assertions requiring further proof or verification. Even these "Examples" are unevenly distributed and none appears between pages 120 and 205 (roughly 30% of the book).

The book seemed to be relatively free of errors, but those which occurred in the exercises and examples could be particularly vexing. For example, the student may wonder what response is expected of him when he is asked to prove: "Exercise 3. (page 66) A relation on $\mathcal{S}$ is an equivalence on a subset of $\mathcal{S}$ if and only if it is symmetric and transitive."

In the opinion of this reviewer this book is not suitable as a text for the overwhelming majority of pregraduate mathematics majors in American colleges today. However, it could serve as a good basis for an independent study or reading course for an able pregraduate student who has completed an algebra course similar to Herstein. Since many of the topics are not covered in the usual introductory course, the student can strengthen his background in algebra, accelerate his mathematical maturity, and gain an insight into the tone and pace of graduate work in mathematics. This book should be available in the library for use by faculty and students.

BERNARD JACOBSON, Franklin and Marshall College

Reguläres Parkettierungsproblem. By Heinrich Heesch. Westdeutscher Verlag, Köln and Opladen, 1968. 96 pp. $3.50 (paper).

In 1891, the Russian crystallographer E. S. Fedorov enumerated the seventeen two-dimensional space groups, that is, discrete groups of isometries including two independent translations. One way to illustrate such a group is to specify a fundamental region: a simply-connected region of such a shape and size that, when all the elements of the group are applied to it, the whole plane is exactly filled and covered. In other words, we take a kind of tile, and repeat it infinitely often to obtain a tessellation. The simplest case is the group $\boldsymbol{p1}$, generated by two translations. The obvious tile is a parallelogram, but infinitely many equally effective shapes can be derived by changing each side of the parallelogram into

a simple curve joining the same two vertices. The only restriction is that opposite sides must be changed in the same manner, so that neighboring tiles will fit together. Such variations are not geometrically significant, though the Dutch artist, M. C. Escher, makes use of them with a startling effect. (See, for instance, the reviewer's *Introduction to Geometry*, Wiley, New York, 1969, p. 57.) However, a significantly different tessellation is obtained when the same group $p1$ is generated by *three* translations, so that the "obvious" fundamental region is a centrally symmetrical hexagon instead of a parallelogram. Analogous variations occur in other groups, such as pg and $p3$, which Escher illustrated with the same skill.

Any triangle or quadrangle can be used as a tile if its neighbors are derived by half-turns about the mid-points of its sides (*op. cit.*, pp. 55, 56). A variant, having artistic possibilities, is obtained by changing each side into a curved arc such as the integral sign $\int$, symmetrical by the half-turn that interchanges its end-points. In other words, the group $p2$ can be generated by either three or four half-turns. It can also be generated by two translations along with two or three or four half-turns; therefore this group yields five significantly different tessellations. In this manner, those of Fedorov's groups that contain no reflections (namely $p1$, $p2$, pg, pgg, $p4$, $p3$, $p6$) yield $2+5+4+8+3+2+4=28$ basic types of tessellation. The enumeration was first carried out by Heinrich Heesch and Otto Kienzle in their book *Flächenschluss* (Springer, Berlin, 1963).

The present book compares three approaches to the classification of such tessellations. One approach yields 93 types which are later reduced to the 28 basic types.

Page 15 describes a solution for the two-dimensional version of Hilbert's eighteenth problem: to find a tile that can be repeated to fill and cover the Euclidean plane in an essentially irregular manner, that is, without being a fundamental region for any one of the 17 groups. The simplest instance is a heptagon having angles 315°, 225°, 45°, 90°, 45°, 45°, 135°.

The book is beautifully printed, well illustrated, and includes tables, a bibliography, summaries in English and French, and a discussion in which questions by Günter Ewald, E. F. Peschl, Hans Hermes, Hans Töpfer, and Günter Bergmann are answered by the author.

H. S. M. COXETER, University of Toronto

A History of Vector Analysis: The Evolution of the Idea of System. By M. J. Crowe. University of Notre Dame Press, 1967. xvii+270 pp. $12.95. (Telegraphic Review, May 1969.)

We are living in an exciting age for a vast number of reasons. One of these, and perhaps one of the less obvious, is that the History of Science as a scholarly field is coming of age. As a scholarly friend of mine recently commented, "The History of Science entered this century with concentration on the history of scientific prefaces. In the ensuing decades, we moved into a period where the

a simple curve joining the same two vertices. The only restriction is that opposite sides must be changed in the same manner, so that neighboring tiles will fit together. Such variations are not geometrically significant, though the Dutch artist, M. C. Escher, makes use of them with a startling effect. (See, for instance, the reviewer's *Introduction to Geometry*, Wiley, New York, 1969, p. 57.) However, a significantly different tessellation is obtained when the same group $p1$ is generated by *three* translations, so that the "obvious" fundamental region is a centrally symmetrical hexagon instead of a parallelogram. Analogous variations occur in other groups, such as pg and $p3$, which Escher illustrated with the same skill.

Any triangle or quadrangle can be used as a tile if its neighbors are derived by half-turns about the mid-points of its sides (*op. cit.*, pp. 55, 56). A variant, having artistic possibilities, is obtained by changing each side into a curved arc such as the integral sign $\int$, symmetrical by the half-turn that interchanges its end-points. In other words, the group $p2$ can be generated by either three or four half-turns. It can also be generated by two translations along with two or three or four half-turns; therefore this group yields five significantly different tessellations. In this manner, those of Fedorov's groups that contain no reflections (namely $p1$, $p2$, pg, pgg, $p4$, $p3$, $p6$) yield $2+5+4+8+3+2+4=28$ basic types of tessellation. The enumeration was first carried out by Heinrich Heesch and Otto Kienzle in their book *Flächenschluss* (Springer, Berlin, 1963).

The present book compares three approaches to the classification of such tessellations. One approach yields 93 types which are later reduced to the 28 basic types.

Page 15 describes a solution for the two-dimensional version of Hilbert's eighteenth problem: to find a tile that can be repeated to fill and cover the Euclidean plane in an essentially irregular manner, that is, without being a fundamental region for any one of the 17 groups. The simplest instance is a heptagon having angles 315°, 225°, 45°, 90°, 45°, 45°, 135°.

The book is beautifully printed, well illustrated, and includes tables, a bibliography, summaries in English and French, and a discussion in which questions by Günter Ewald, E. F. Peschl, Hans Hermes, Hans Töpfer, and Günter Bergmann are answered by the author.

H. S. M. COXETER, University of Toronto

A History of Vector Analysis: The Evolution of the Idea of System. By M. J. Crowe. University of Notre Dame Press, 1967. xvii+270 pp. $12.95. (Telegraphic Review, May 1969.)

We are living in an exciting age for a vast number of reasons. One of these, and perhaps one of the less obvious, is that the History of Science as a scholarly field is coming of age. As a scholarly friend of mine recently commented, "The History of Science entered this century with concentration on the history of scientific prefaces. In the ensuing decades, we moved into a period where the

nions were a peculiarly British invention and it is astonishing to read to what a degree chauvinistic feelings played a role in shaping the polemics over the worth of various vector-like systems. Professor Crowe does not make much of national differences and *a priori* there is no reason why he should; but they are so strikingly present that I found myself a little uncomfortable not finding more explicit reference to that phenomenon.

But all in all Professor Crowe's study is a worthy addition to the small number of histories of the development of mathematical ideas. Hopefully it augurs well for further studies of its kind. It should be made available not just to students of the History of Science and Mathematics but to anyone interested in the struggle surrounding the development of new ideas in science and mathematics.

STANLEY GOLDBERG, Antioch College

Lectures on Rings and Modules. By Joachim Lambek. Blaisdell, Waltham, Mass., 1966. vii+183 pp. $8.50. (Telegraphic Review, April 1967.)

Lambek's book is an introduction to associative rings and modules which requires of the reader only the mathematical maturity which one would attain in a first-year graduate algebra covering such topics as the homomorphism theorems for groups, the Jordan-Hölder theorem, and some applications of the axiom of choice. In order to make the contents of the book as accessible as possible, the author develops all the fundamentals he will need.

In addition to covering the basic topics of rings with DCC (including the Wedderburn structure theorems, the radical, lifting idempotents, and some generalizations to semiperfect rings), completely reducible modules, projective and injective modules, "tor" and "ext," the author covers some topics not so readily available to the nonspecialist: There is a detailed discussion of generalized (complete) quotient rings, first in the commutative case, then in general; a proof, for the case of a finite number of summands, of Azumaya's generalization of the Krull-Schmidt theorem (p. 78) together with an interesting connection between it and the Jordan-Holder theorem; a very simple proof of Osima's (?) theorem that in a ring with DCC any two full sets of primitive, orthogonal idempotents are conjugate (Prop. 3, p. 377, generalized to semiperfect rings); and a simple proof that a ring is regular if and only all of its right modules are flat.

The chapters are written to be as independent as possible. As a result the specialist may experience some annoyance at the overly computational nature of some of the proofs, an annoyance not shared by students making their first acquaintance with the subject.

One of the most successful features of the book is that it can be read by graduate students with little or no help from a specialist.

L. S. LEVY, University of Wisconsin

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L. S. LEVY, University of Wisconsin

TELEGRAPHIC REVIEWS

Telegraphic reviews are intended to give prompt notice of books with sufficient information to assist in deciding whether to order an examination copy or suggest library purchase. Possible uses are coded as follows: T = textbook, S = supplementary student reading, P = professional reading, TT = teacher training, L = library purchase, 13 to 18 = freshman to second graduate year level, 1 to 4 = one to four semesters. An asterisk is used for emphasis. Books covering standard high school material are called "remedial." All textbooks are examined carefully, and mention is made of noteworthy features that are not evident from the title and coding. Publishers are indicated by the standard abbreviations used in *Books in Print* (which gives full names and addresses). P = paperback.

ALGEBRA, *P, *L. *Proceedings of the Conference on Transformation Groups. New Orleans, 1967.* Ed. by Paul S. Mostert. Springer-Verlag, 1968. 568 p. $15. Emphasis on review, exposition, possible new directions, unsolved problems. Also many new results.

ALGEBRA, P, L. *Boolean Algebra.* Roman Sikorski. 3rd ed. Springer-Verlag, 1969. 247 p. $9.50. Ergebnisse 25. Minor corrections of 1967 edition.

ANALYSIS, P, L. *The Confluent Hypergeometric Function, with Special Emphasis on its Applications.* Herbert Bucholz. Springer-Verlag, 1969. 256 p. $16. Comprehensive treatise. Appendix summarizes all functions that can be considered as special cases. *Bibliography (17p).*

ANALYSIS, P, L. *Four Papers on Functions of Real Variables.* K. K. Golovkin, V.P. Il'in, V. A. Solonnikov. AMS Transl. Ser. 2, Vol. 81. Am Math, 1969. 283 p. $14.20. Topics: Inequalities in function spaces, Convergence of variation processes, Differentiable functions of several variables, equivalent norms for fractional spaces.

ANALYSIS, P, L. *Capacity Functions.* L. Sario, K. Oikawa. Springer-Verlag, 1969. 373 p. $24. The first monograph in a field less than two decades old. Includes analytic tools (presupposing advanced graduate level training in complex analysis), exposition, applications.

ANALYSIS, P. *La Totale-D de Denjoy et la Totale-S Symetrique.* W. J. Trijitzinsky. *Memorial des Sciences Math. 166.* Gauthier-Villars, Paris, 1968. 104 p. 36 F. The purpose is a deeper development of abstract totalization growing out of the notions first introduced by Denjoy. The terminology "Totale-D" is unfortunate, though in keeping with the current fashion of reckless symbolization. In English "Denjoy total" and "symmetric total" would be better. The use of letters doesn't always improve communication!

ART, REPRINT, S, P, L. *Pattern and Design with Dynamic Symmetry.* Edward B. Edwards. Dover, 1967. 142 p. $2 (P). A reprint of the original work, entitled *Dynamarhythmic Design,* published in 1932. Based on ideas of J. Hambidge.

ART, REPRINT, S, P, L. *The Elements of Dynamic Symmetry.* Jay Hambidge Dover, 1967. 150 p. $2 (P). A reprint of a famous little book first published in 1926 from papers originally appearing in 1919, in which the author applied very elementary (and somewhat naive) geometry to artistic composition.

CALCULUS, T(13-14; 3-4). *Calculus with Analytic Geometry.* Edward M. J. Pease, George P. Wadsworth. Ronald, 1969. 1084 p. $13.50. A welcome innovation is the early introduction and frequent use of probability. The style is often turgid. For example (page 230): "If u and v are variables, each dependent on the same independent variable x, and if, for some constant c, the quantities x - c, u, and v are related infinitesimals, and if A and B are constants, or if A and B are variables, such that when x is sufficiently close to c, each of A and B is numerically less than some fixed positive number N, then Au + Bv (if not zero) is an infinitesimal related to u and v."

COMPUTERS, SCIENCE FICTION. *Forecast 1968 - 2000 of Computer Developments and Applications.* Coordinated by Chresten A. Bjerrum. Parsons & Williams, Nyropsgade 43, Copenhagen, Denmark. 64 p. $12.50. The opinions of 88 delegates to an International Seminar on file organization held in Denmark in Nov. 1968.

COMPUTER, *P, *L. *Symbol Manipulation Languages and Techniques.* Ed. Daniel G. Bobrow. North-Holland, 1968. 497 p. $19.50. Proceedings of a conference organized by the International Federation of Information Processing in 1966. Symbol manipulation as opposed to numerical calculation.

*COMPUTERS, S, *P, *L. *Computers in Mathematical Research.* Ed. R. F. Churchhouse, J.-C. Herz. Intro. J. Dieudonne. North-Holland, 1968. 196 p. $9. Fifteen papers beginning with D. H. Lehmer, ending with S. M. Ulam and covering a wide variety of uses. *Bibliography* (13p).

COMPUTER SCIENCE, T(17; 1-2), S, P. *Formal Languages and their Relation to Automata.* John E. Hopcroft, Jeffrey D. Ullman. A-W, 1969. 249 p. $11.95. Broad treatment of a rapidly growing field that "sprang to life around 1966 when Noam Chomsky gave a mathematical model of a grammar in connection with his study of natural languages..."

*COMPUTERS, S, *P, L. *Perceptrons. An Introduction to Computational Geometry.* By Marvin Minsky, Seymour Papert. M.I.T. Pr, 1969. 264 p $12, $4.95 (P). In a long review in *Science*, 165 (22 Aug. 1969), 780-782, Allen Newell calls this a "great book" that makes important contributions not only to the problem of pattern recognition ("a perceptron is a predicate that can be represented as whether a weighted sum of other predicates exceeds a specified threshold") but also to theoretical issues associated with the achievement of intelligent machines and the broader question of the nature of computer science as a separate discipline.

*CRYPTANALYSIS, *S, *TT, *P, *L. *Elementary Cryptanalysis. A Mathematical Approach.* Abraham Sinkov. *New Math. Lib. 22.* Published for the monography project of the SMSG. Random and Singer, 1968. 198 p. $1.95 (P). Another addition to this distinguished series for which every library should have a standing order, and which are all useful as supplementary reading for undergraduates as well as for the "large audience of high school students and laymen" for which they were written. There are probably very few people interested in mathematics who would not enjoy this volume.

DIFFERENTIAL EQUATIONS, P. *Integral Operators in the Theory of Linear Partial Differential Equations.* 2nd rev. prntg. Stefan Bergman. *Ergebnisse 23.* Springer-Verlag, 1969. 150 p. $9. (1st ed. 1961).

DIFFERENTIAL EQUATIONS, S(13), L. *Equations Differentielles Au Secondaire.* A. Ronveaux. Lidec, Montreal, 1969. 83 p. $1.75. A sequel to the author's *Introduction Aux Equations Aux Differences Finies*,

Lidec, 1966. Intended for high school students, this booklet presents derivatives and differential equations without introducing the definite integral. Physical, biological and social applications. Worked and unworked exercises. (TR by D. E. Richmond, Williams College).

EDUCATION, S, P, L. *Die Neugestaltung des Mathematikunterrichtes an den Hoeheren Schulen.* Selected reports from the Vienna Seminar of the International Commission on Mathematical Education in the summer of 1966. Ed. H. Bhenke, et al. Klett, Stuttgart, 1969. 142 p. 12.50 DM (P). General problems, probability and statistics, algebra and number theory, topological notions and analysis. Elegant four color illustrations.

EDUCATION, TT, P, L. *Elementary School Mathematics. A Guide to Current Research.* Vincent J. Glennon, Leroy G. Callahan. Association for Supervision and Curriculum Development, NEA, Washington D.C. 1968. 135 p. $2.75 (P). *Bibliography* (296 titles).

EDUCATION, TT, P, L. *Teaching Secondary School Mathematics.* Kenneth B. Henderson. *(What Research Says to the Teacher 9).* Association of Classroom Teachers, NEA, Washington, D.C., 1969. 32 p. 25¢ (P). A brief summary of recent research. *Bibliography.*

EDUCATION, HISTORY, *P, TT, S, *L. *An Investment in Knowledge. The First Dozen Years of the National Science Foundation's Summer Institutes Programs to Improve Secondary School Science and Mathematics Teaching 1954-1965.* Hillier Krieghbaum, Hugh Rawson. NYU Pr, 1969. 341 p.

EDUCATION, *P, TT. *Supplementary Chapters in Mathematical Analysis. Materials for Students in the Physical-Mathematical Faculties of Pedagogical Institutes.* (Russian). I. T. Makarov. Moscow, 1968. 310 p. 67 Kopeks. Functions of a real variable, from set theory to Lebesgue integration, elements of functional analysis, elementary complex analysis. For those concerned with teacher training courses of high quality.

EDUCATION, *TT(13), S, P, L. *More Topics in Mathematics for Elementary School Teachers. Thirtieth Yearbook.* NCTM, 1969. 598 p. $7.50. The 29th Yearbook (1964) brought together 8 previously published booklets: Sets, The Whole Numbers, Numeration Systems for the Whole Numbers, Algorithms for Operations with Whole Numbers, Numbers and their Factors, The Rational Numbers, Numeration Systems for the Rational Numbers, Number Sentences. The present volume presents 10 more booklets (available separately at 65¢ each with discounts for quantity orders) with the following titles: The System of Integers; The System of Rational Numbers; The System of Real Numbers; Logic; Graphs, Relations, and Functions; Informal Geometry; Measurement; Collecting, Organizing, and Interpreting Data; Hints for Problem Solving; Symmetry, Congruence and Similarity. Exercises, answers, and suggestions for further reading.

FINITE DIFFERENCES, S(13). *Introduction aux Equations aux Differences Finies.* A. Ronveaux. Lidec, Montreal, 1966. 103 p. $1.75 (P). Written for high school students and used in a mathematical summer camp in the province of Quebec. Exercises, solved and unsolved.

FOUNDATIONS, T(17), S, *P, *L. *Models and Ultraproducts: An Introduction.* J. L. Bell, A. B. Slomson. North-Holland, 1969. 331 p. $15, $10 (P). Model Theory = "the study of the relationship between formal languages and abstract structures." Assumes set theory, in-

cludes needed logic, Boolean Algebra. Subject is closely related to non-standard analysis. Historical remarks, *bibliography*.

FOUNDATIONS, S, *P, *L. *Foundations of Mathematics. Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel.* Ed. Jack J. Bulloff, Thomas C. Holyoke, S. W. Hahn. Springer-Verlag, 1969. 207 p. $9.75. Portrait, biographical data, description of the symposium, greetings from J. R. Oppenheimer, tribute from J. von Neumann, bibliography of Gödel, nine papers.

FOUNDATIONS, P, L. *Demonstration, Verification, Justification.* Proceedings of the International Institute of Philosophy, Liège, Sept. 1967. Editions Nauwelaerts, Louvain-Paris, 1968. 368 p.

FOUNDATIONS, P, *L. *Dictionary of Symbols of Mathematical Logic.* Ed. Robert Feys, Frederic B. Fitch. North-Holland, 1969. 179 p. $9.50. Arrangement systematic, style discursive. Covers all symbols used in 9 standard treatises, the *Journal of Sumbolic Logic* since 1950, and other sources.

FOUNDATIONS, S, P, *L. *Einfuhrung in die operative Logik und Mathematik.* Paul Lorenzen. 2nd ed. Springer-Verlag, 1969. 298 p. $13.50. Classic (first ed. 1965). Minor changes.

FOUNDATIONS, S, *P, *L. *Constructible Sets with Applications.* A. Mostowski. North-Holland, 1969. 278 p. $7.70. "Gödel's theory of constructible sets and Cohen's construction of models by means of generic sets..."

GENERAL, T(13-15; 1; 2), *S, P, L. *Sets, Lattices and Boolean Algebras.* James C. Abbott. Allyn, 1969. 282 p. $11.50. An exposition of the subjects and of their role in mathematics. Topics include distributive and modular lattices, semi-boolean algebras and implication algebras (new elementary theory developed in the author's undergraduate research seminar), Von Neuman-Gödel-Bernays axioms (the author uses Zermelo-Fraenkel-Skolem). *Bibliography*, table of symbols. An interesting looking book, possibly useful in many ways: rugged introductory course for general students, part of a course on some of its topics, supplementary reading for all mathematics students, readable introduction for teachers.

GEOMETRY. *Introduction to Geometrical Transformations.* Edward H. Barry. Prindle, 1966. 105 p. A supplement to start the student "along the road that leads to Klein's famous Erlanger Program and thence to many fields of current interest and research..." Unfortunately there are numerous errors and obscurities.

**GENERAL, P, L. *The Mathematical Sciences: A Report.* Committee on Support of Research in the Mathematical Sciences (COSRIMS) of the National Research Council, for the Committee on Science and Public Policy, National Academy of Sciences. Publication 1681, NAS, Wash., D.C. 1968. 270 p. $6. Under the chairmanship of Lipman Bers, COSRIMS has presented a report that deals with major issues effecting mathematics in the immediate future. Five sections (Summary, The state of the mathematical sciences, The mathematical sciences in education, Level and forms of support, Conclusions), references, five appendices. Related publications are: *The Mathematical Sciences: Undergraduate Education* reviewed below and *The Mathematical Sciences: A Collection of Essays* (TR Aug. 1969).

**EDUCATION, P, L. *The Mathematical Sciences: Undergraduate Education.* COSRIMS of the NRC for the CSPP of the NAS. Publication 1682, NAS,

Washington, D.C., 1968. 122 p. $4.25. An important document by the panel on undergraduate education chaired by John G. Kemeny. Chapters: Recommendations, A quarter century of change: Eight case histories, Clients of the mathematician, Problems of staffing, Special areas of concern, Support for the college mathematics teacher. (See TR above).

GENERAL, S(13-14). *Exploring University Mathematics 3. Lectures given at Bedford College, London.* Ed. N. J. Hardiman. Pergamon, 1969. 129 p. $4.75, $3 (P). Lectures given at the 1967 Easter Conference for students about to begin a university course in mathematics (previous volumes contain the 1965 and 1966 lectures). Contents: Symmetry of Pyramids and Prisms, Mathematics and the Physicist, Cushion Craft, Logic, Space and Spaces, Some Applications of the Taylor Series in Numerical Analysis, Some Irrational Numbers.

*GENERAL, S(15-17), P, L. *Ueberblicke Mathematik. Band 1.* 1968. Ed. Delef Laugwitz. Bibliographisches Institut, Mannheim/Zurich, 1968. 213 p. First of a series planned to present high quality expositions for non-specialists in the mathematical community. In this issue: *History of Mathematics* by Christoph J. Scriba, *The Time Number Theorem* by Wolfgang Schwarz, *Foundations of Geometry* by Hanfried Lenz, *Numerical Integration of Ordinary Differential Equations* by Hans Knapp and Gerhard Wanner (single step procedures), *Information Theory* by Peter Weiss, *Lie Series* by Gerhard Wanner, *Local Rings* by Henrich Reitberger, *Categories* by Dieter Pumpluen, *Kinematics* by Joseph Hoschek. *Bibliographies.* A good candidate for translation!

GENERAL, P, L. *National Science Foundation. Eighteenth Annual Report for the Fiscal Year Ended June 30, 1968.* Superintendant of Documents, GPO, Washington, D.C. 20402. 295 p. $1.25 (P). Brief, but interesting, comments on mathematics on pages 76-80.

GENERAL, *S, TT, P, *L. *Reprint Series.* Ed. William L. Schaaf. SMSG, Stanford Univ., 1969. Dist. by Vromans, 2085 E. Foothill Blvd, Pasadena, Calif. 91109. 40¢ each. (RS-1 through RS-10 TR Aug. 1968). RS-11: Memorable Personalities in Mathematics: Nineteenth Century (Laplace, Gauss, the Bolyais, Galois, Gibbs). RS-12: Memorable Personalities in Mathematics: Twentieth Century (Ramanujan, Minkowski, Banach, Whitehead, Sierpinski, von Neumann). RS-13: Finite Geometry. RS-14: Infinity. RS-15: Geometry, Measurement and Experience.

GENERAL. *Dictionnaire raisonné de mathématiques.* Andre Warusfel. 3rd ed. Editions du Seuil, Paris, 1966. Also from Hachette Université, Montreal. 523 p. $13.45 (Can). In spite of a good plan (about 2,000 concepts described in 64 articles with a complete index), an ordinary handbook of mathematics thru calculus with some intermingling of traditional and Bourbaki terminology.

GEOMETRY, T(14; 1-2), TT. *Linear Algebra and Geometry.* James A. Murtha, Earl R. Willard. HR & W, 1969. 245 p. $7.95. Developed from a course in geometry at an academic year institute for high school teachers. Chapters are finite dimensional spaces, affine geometry, multilinear algebra, and projective geometry.

GRAPH THEORY, T(15-17; 1-2), S, *L. *Graph Theory.* Frank Harary. A-W, 1969. 283 p. $12.50. An introductory, but fairly comprehensive treatment by Mr. Graph Theory, who displays his usual lively style. *Bibliography* (25 p).

GROUP THEORY, HISTORY, *P, L. *Die Genesis des abstrakten Gruppenbegriffes. Ein Beitrag zur Entstehungsgeschichte der abstrakten Gruppentheorie.* Hans Wussing. VEB Deutscher Verlag, 1969. 258 p.

The history of the group concept to about 1920. *Bibliography* of 747 titles. A careful scholarly piece of work that belongs in every serious mathematical library.

HEURISTIC, P, *L, *Psychological Investigations in Creativity. A Bibliography (1954-1965).* Lynda J. Martin. Richardson Foundation, Greensboro, N.C., 1965. 121 p. Free. Covers titles in *Psychological Abstracts* and many others. It follows a comprehensive bibliography of over 1900 titles up to 1954 that appeared under the title *Bibliography on Creativity.* (Industrial Research Institute, 1955).

*HISTORY, *S, *P, *L, *Éléments d'histoire des Mathématiques.* Nicolas Bourbaki. 2nd ed. Revised, corrected, augmented by notes and an index. Hermann, Paris, 1969. 323 p.

HISTORY, *S, P, L, *Science: Men, Methods, Goals. A Reader: Methods of Physical Science.* Ed. Boruch A. Brody, Nicholas Capaldi. W. A. Benjamin, 1968. 351 p. $10, $3.95 (P). Anthology of great writers, including Poincare, Reichenbach, Popper and C. S. Peirce.

HISTORY, P,*L, *Mechanics in Sixteenth-Century Italy. Selections from Tartaglia, Benedetti, Guido Ubaldo, and Galileo.* Trans., annotated by Stillman Drake, I. E. Drabkin. U of Wisc Pr, 1969. 440 p. $12.50. Superbly done.

HISTORY, P, *L, *Oeuvres de Jacques Hadamard.* Editions du Centre National de la Recherche Scientifique, Paris, 1968. Four volumes. 2296 p. $32.66. Facsimile reprints of all Hadamard's publications exclusive of books, popularizations, papers relating to elementary pedagogy, some notes that were later developed in more complete form and that do not have exceptional historical interest, and translations. The first volume has a portrait and a brief chronology of his 98 years, the last a complete list of his publications, from 1884-1964!

HISTORY, S, TT, *Zaehlen und Rechnen Einst und Jetzt.* Wlodzimierz Krysicki. Teubner, Leipzig, 1968. 106 p. $1.15 (P). Translated from the Polish of 1958. Counting and calculation are carried from ancient times to the electronic computer in simple style. There should be more little books of this kind in English.

HISTORY, *P, *L, *Modern Mathematics. The Genesis of a School in Poland.* Sister Mary Grace Kuzawa. College and Univ, Pr, 1968. 143 p. $4.50. Development of Polish Mathematics between the two World Wars. *Bibliography.*

HISTORY, P, L, *Benjamin Peirce and the U.S. Coast Survey.* V. F. Lenzen. San Francisco Pr, 1968. 61 p. $2.75. Included are a general biographical sketch and some discussion of the contributions to algebra of this first American mathematician to play a significant role in the world history of mathematics.

LINEAR ALGEBRA, T(14-15, 1), *Linear Algebra.* George D. Mostow, J. H. Sampson. McGraw, 1969. 306 p. $8.95. To follow a first calculus course. Last three chapters are on Hermitian forms, spectral decompositions, triangulation of matrices, Jordan normal form, multilinear algebra, tensors.

LINEAR ALGEBRA T(13-14), *Linear Algebra.* John De Pillis. HR & W, 1969. 528 p. $8.95. Begins concretely, includes both classical and axiomatic definition of determinants, ends with structure of operators. Chart of theorems.

NUMBER THEORY, *S(13), *L. *An Adventurer's Guide to Number Theory* Richard Friedberg. McGraw, 1968. 228 p. $5.95. A pleasing, elementary, historical exposition through the early 19th century, including a last chapter on quadratic reciprocity. First in the McGraw-Hill History of Science Series edited by Daniel A. Greenberg (like the author of this book, a physicist), "designed to involve young readers in some of the excitement of scientific thought and development through the centuries." *Bibliography.*

PHYSICS, P, L. *Transport Theory.* Ed. Richard Bellman, Garrett Birkhoff, Ibrahim Abu-Shumays. Proceedings of a Symposium in Applied Mathematics of the AMS and the SIAM held in New York City, April, 5-8 1967. (SIAM-AMS Proceedings 1). Am Math, 1969. 334 p. $11. Neutron transport theory related to problems of analysis and probability.

PHYSICS, P, L. *Plane Elastic Systems.* L. M. Milne-Thomson. 2nd corrected ed. *Ergebnisse Angew. Math. 6.* Springer-Verlag, 1968. 219 p. $12. (First edition 1960). Good survey of "plane elastic systems in equilibrium or steady motion, within the framework of the linear theory." Complex analysis the main tool.

PHYSICS, P, L. *Topics in Nonlinear Physics. Proceedings of the Physics Session International School of Nonlinear Mathematics and Physics. A NATO Advanced Study Institute. Max-Planck-Institute for Physics and Astrophysics (Munich, 1966).* Ed. Norman J. Zabusky. Springer-Verlag, 1968. 50 Figs. 754 p. $13.50. Papers by W. Heisenberg, C. Truesdell, I. Prigogine, M. Baus, N. Bloembergen, P. G. Saffman, J. A. Wheeler. Mathematics sessions will be reported in a separate volume.

PRECALCULUS, T(13; 2). *Precalculus: Elementary Functions and Relations.* Donald R. Horner. HR & W, 1969. 416 p. $9.95. Full coverage, including sets, some logic, real number system.

PRECALCULUS, T(13; 1). *Elementary Functions and Coordinate Geometry.* Marvin Marcus, Henryk Minc. HM, 1969. 416 p. $8.95. Algebra, analytic geometry, and trigonometry "from a functional standpoint". Some overlap with the authors' *New College Algebra* (TR Dec. 1968).

PROBABILITY, S, P. *Saetze und Aufgaben ueber Markoffsche Prozesse.* E. B. Dynkin, A. A. Juschkewitsch. Transl. from Russian (1967), by K. Schurger. Springer-Verlag, 1969. 232 p. $3.70 (P). Assumes calculus and a little probability. Probabilistic solutions, partial differential equations, connections with potential theory, boundary values, optimal problems of Markov chains.

PROBABILITY, OPTIMIZATION, P, L. *Progress in Mathematics. Vol. 3.* Ed. R. V. Gamkrelidze. Plenum Pub, 1969. 120 p. $15. Transl. of *Itogi Nauki--Seriya Matematika* 1967, containing *Markov Processes and Differential Equations* by M. I. Friedlin and *Discrete Problems in Mathematical Programming* by A. A. Korbut and Yu. Yu. Finkelshtein. *Bibliographies.* (Vol. 2, TR Aug, 1968).

PROBABILITY, PHYSICS, S(16, 17), *P. *Correlations and Entropy in Classical Statistical Mechanics.* J. Yvon. Transl. H. S. H. Massey. Ed. D. Ter Haar. Pergamon, 1969. 201 p. $8. Good motivation and commentary.

PROBLEMS, S(13). *Sequences, Combinations, Limits.* S. I. Gelfand, M. L. Gerver, A. A. Kirillov, N. N. Konstantinov, A. G. Kushnirenko. Lib. School Math. 3. Transl. and adapted from the Russian by Leslie Cohn, Joan Teller. Survey of Recent East European Mathematical Liter-

ature, a project conducted by Izaak Wirszup. M.I.T. Pr, 1969. 151 p. $6, $1.50 (P). Brief exposition and 148 problems with sections giving solutions and hints.

RECREATIONS, S, P, L. *The Master Book of Mathematical Puzzles and Recreations.* Fred Schuh. Transl. by F. Goebel. Transl. ed. T. H. O'Beirne. Dover, 1968. In Canada, General Publishing Co., Toronto. 416 p. $3 (P). *Wonderlijke Problemen; Leerzaam Tijdverdrijf Door Puzzle en Spel* (Zutphen 1943). On a lower level than Ball's standby, but there is a tremendous number of amusing and interesting problems that will appeal to a very wide range of readers. On page 2, the author defines a "pure puzzle" as one that is invariant under translation from one language to another.

REFERENCE, *L. *The Random House Dictionary of the English Language.* Unabridged. Random, 1966. 2091 p. $25. Many up to date definitions. Best general dictionary for a mathematics library.

REMEDIAL ARITHMETIC, *S. *Arithmetic: A Review.* J. Louis Nanney, Richard D. Shaffer. Wiley, 1969. 315 p. $3.95. A "write-in" text with exposition, exercises with spaces for work and answers to odd number problems. Begins with reading and writing counting numbers and ends with an introduction to the language of algebra. Highly adaptable for students with arithmetical deficiencies.

REMEDIAL, T(13), TT. *Introduction to Mathematics. 2nd ed.* Bruce E. Meserve, Max A. Sobel. P-H, 1969. 431 p. $7.95. Under the slogan "mathematics can be fun!" the authors touch many precalculus topics.

REPRINT, NUMBER THEORY, S, P, *L. *Niedere Zahlentheorie.* By Paul Bachmann. Corrected reprint in one volume of two originals (1902, 1910). Chelsea Pub, 1968. 902 p. $15. Classic. An index would help!

REPRINT, P, *L. *Richard Dedekind. Gesammelte mathematische Werke.* Ed. Robert Fricke, Emmy Noether, Oystein Ore. Originally published in three volumes 1930-1932. Reprinted in two volumes. Chelsea Pub, 1969. $25. Unaltered reprint of the original, except that the first 222 pages of the original volume three (the eleventh supplement to the Dirichlet-Dedekind *Vorlesungen ueber Zahlentheorie*) has been omitted, and the remainder of the original volume three has been bound with the original volume two. The omitted supplement is included in the reprint of Dirichlet-Dedekind published simultaneously.

STATISTICS, T(13), S. *Mathematics for Statistics.* W. L. Bashaw. Wiley, 1969. 342 p. $8.50, $4.95 (P). Arithmetic, algebra, matrix algebra, sets, probability, graphing, for pre-calculus.

STATISTICS, PROBABILITY, T(13). *Introduction to Probability and Statistics. 2nd ed.* William Mendenhall. Wadsworth Pub, 1967. 406 p. $9.50. Formerly titled *Introduction to Statistics.* For arts, sciences, and business. New chapters on analysis of variance and nonparametric statistics.

TOPOLOGY, S, P. *Piecewise Linear Topology.* By J. F. P. Hudson. Univ. of Chicago Lecture Notes prepared with assistance of J. L. Shaneson, J. Lees. W. A. Benjamin, 1969. 291 p. $15, $4.95 (P). Presupposes only "a very little algebraic topology."

HISTORY, L. *Pierre Sergescu, 1893-1954.* E. J. Brill, Leiden, 1968. 73 p. Twelve papers on his work and life, originally published in Janus, Vol. 55, no. 1. 1968. *Bibliography.*

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor W. R. Ballard, University of Montana, represented the Association at the inauguration of President E. M. Grossell of the College of Great Falls on April 12, 1969.

Professor W. F. Cassidy, St. John's University, Jamaica, New York, represented the Association at the inauguration of President H. Schueler of Richmond College on May 9, 1969.

Professor Bernard Greenspan, Drew University, represented the Association at the inauguration of President J. K. Olsen of Paterson State College on May 6, 1969.

Professor D. W. Hall, SUNY at Binghamton, represented the Association at the inauguration of President L. Park of Mansfield State College on April 26, 1969.

Professor H. M. MacNeille, Case Western Reserve University, represented the Association at the inauguration of President W. G. Caples of Kenyon College on April 15, 1969.

Professor Dwight Paine, Wells College, represented the Association at the inauguration of President L. R. Schoenhals of Roberts Wesleyan College on April 19, 1969.

Professor E. B. Shanks, Vanderbilt University, represented the Association at the inauguration of President M. G. Scarlett of Middle Tennessee State University on May 1, 1969.

University of Maryland: Dr. L. J. Goldstein, Yale University, has been appointed Associate Professor; Dr. M. H. Powell, University of California at Santa Barbara, has been appointed Assistant Professor; Associate Professor Gertrude Ehrlich has been promoted to Professor.

Assistant Professor Ethel M. Cain, Simpson College, died on February 7, 1969. She was a member of the Association for three years.

Mr. C. W. Carter, Brookfield Center, died on September 22, 1968. He was a member of the Association for six years.

Assistant Professor Edwin Goldfarb, Stevens Institute of Technology, died on May 5, 1968. He was a member of the Association for thirteen years.

Assistant Professor Emeritus Helen B. Owens, Pennsylvania State University, died on June 6, 1968. She was a member of the Association for forty-nine years.

Dr. J. L. Scott, Denver, died on November 19, 1968. He was a member of the Association for nine years.

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN THE MATHEMATICAL SCIENCES

In its annual brochure on Fellowship and Research Opportunities in the Mathematical Sciences, the Division of Mathematical Sciences of the National Research Council calls attention to a number of fellowships and other kinds of support for research in the mathematical sciences at both the predoctoral and postdoctoral levels to be awarded during the year 1969–70. Copies of this brochure are available from: Division of Mathematical Sciences, National Research Council, 2101 Constitution Avenue, N. W., Washington, D. C. 20418.

The following program was presented:

1. *On semigroups of functions on topological spaces*, by A. G. Haddock and T. L. Hicks, University of Missouri, Rolla (presented by T. L. Hicks).

2. *The Cartan-Brauer-Hua theorem*, by Franklin Haimo, Washington University.

3. *Semirings and their homomorphisms*, by Elbert Pirtle, University of Missouri, Kansas City.

4. *What computers are doing to college mathematics*, by R. V. Andree, University of Oklahoma (invited address).

5. *Functional analysis and linear operator theory in linear spaces with quaternion and Cayley-number scalars*, by A. J. Penico, University of Missouri, Rolla.

6. *The limits of functions in terms of sequences*, by Henry Polowy, Lincoln University.

7. *Perturbations of a matrix by additive rank-one matrices*, by J. R. Foote, University of Missouri, Rolla.

8. *Uniform differentiation*, by Sam Lachterman, Saint Louis University.

9. *A recursion formula for finite partition lattices*, by T. J. Brown, University of Missouri, Kansas City.

VIRGINIA M. KERN, *Secretary-Treasurer*

MAY MEETING OF THE NEW JERSEY SECTION

The fourth joint meeting of the New Jersey Section of the MAA and the Association of Mathematics Teachers of New Jersey was held at Drew University on May 3, 1969. The meeting was chaired by Bernard Greenspan, Drew University.

The morning program was as follows:

1. *A classification system for assembly of mathematics tests from a computerized data bank*, by Marian Epstein, Educational Testing Service.

2. *Computers in teaching*, by Kenneth Iverson, T. J. Watson Research Center, IBM.

The afternoon program consisted of a panel discussion moderated by William Brower, Newark College of Engineering. The following members served on the panel: Linda Alvord, Scotch Plains—Fanwood High School; Samuel Greitzer, Rutgers University (Newark); Martin Moskowitz, Valesburg High School; Sheldon Myers, Educational Testing Service; Dorothy Roberts, Director of Mathematics, Scotch Plains—Fanwood; Malcolm Simpson, West Essex High School.

JOHN RECKZEH, *Secretary-Treasurer*

PROPOSED AMENDMENT TO THE BY-LAWS

At the meeting of the Board of Governors held on August 24, 1969, in Eugene, Oregon, the Secretary was instructed to submit to a vote of the membership an amendment to Article III of the By-Laws which will give the membership the right of referendum by mail ballot.

In accordance with these instructions of the Board, a motion will be made at the business meeting of the Association to be held in San Antonio on Sunday, January 25, 1970, to add Section 9 to Article III of the By-Laws to read as follows:

"The Board may refer a matter to a referendum mail vote of the entire membership of the Association and shall make such reference if a referendum is requested, prior to final action by the Board, by three hundred or more members. The taking of a referendum shall act as stay upon Board action until the votes have been canvassed, and thereafter no action may be taken by the Board except in accordance with a plurality of the votes cast in the referendum."

HENRY L. ALDER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fifty-third Annual Meeting, San Antonio, Texas, January 24–26, 1970. **This is a change from the location previously announced.**

Fifty-first Summer Meeting, University of Wyoming, Laramie, August 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, May 2, 1970.

FLORIDA, Rollins College, Winter Park, March 20–21, 1970.

ILLINOIS, Loyola University, Chicago, May 8–9, 1970.

INDIANA

IOWA, Grinnell College, Grinnell, April 17, 1970.

KANSAS, Kansas State Teachers College, Emporia, March 1970.

KENTUCKY, University of Kentucky, Lexington, Spring 1970.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Wagner College, Staten Island, Spring 1970.

MICHIGAN, Wayne State University, Detroit, April 4, 1970.

MISSOURI, Central Missouri State College, Warrensburg, May 2, 1970.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 24–25, 1970.

NEW JERSEY

NORTH CENTRAL

NORTHEASTERN

NORTHERN CALIFORNIA, Diablo Valley College, Concord, February 7, 1970.

OHIO, Bowling Green State University, Bowling Green, Spring 1970.

OKLAHOMA-ARKANSAS, Southwestern State College, Weatherford, Oklahoma, March 1970.

PACIFIC NORTHWEST, Pacific Lutheran University, Tacoma, Washington, June 19–20, 1970.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, May 8–9, 1970.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, March 20–21, 1970.

SOUTHERN CALIFORNIA, University of California, Irvine, March 21, 1970.

SOUTHWESTERN, University of Texas at El Paso, March 27–28, 1970.

TEXAS, Sam Houston State College, Huntsville, April 10–11, 1970.

UPPER NEW YORK STATE

WISCONSIN, University of Wisconsin, Waukesha, May 1970.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, Mass., December 26–31, 1969.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1970.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Ohio State University, June 22–25, 1970.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 26–28, 1970.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hilton Hotel, Washington, D. C., April 20–22, 1970.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of Denver, Colorado, June 1970.

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PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of Denver, Colorado, June 1970.

A brief commentary on the present condition of calculus courses and texts

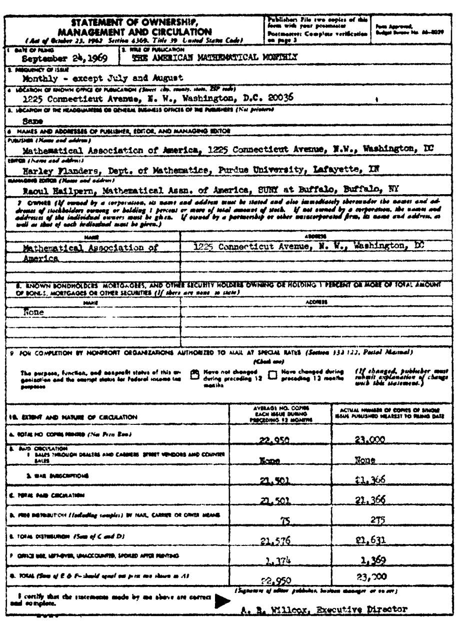
by Kenneth Hoffman
Massachusetts Institute of Technology

"Do we need more calculus books? In spite of the recent deluge, it seems to me that we do need more—in the sense that we need still better calculus texts.

"Since World War II the applications of mathematics have increased dramatically. Scientific and technical fields have come to encompass much more mathematics in their theoretical fabrics. The social sciences have developed stronger mathematical components. These factors have created a significant increase in the number of college students taking serious mathematics courses. In particular, it has become necessary to teach calculus to a very large group of students, at a faster pace than we used for the elite student group not so many years ago.

"So, we need better textbooks—not more complete scholarly treatises on calculus, but texts which are better written to be used in conjunction with classroom teaching to help a broad spectrum of students understand something about calculus and its applications. No calculus book can be all things to all people, but it seems to me that there are some general guidelines.

"It goes without saying that a calculus text should be mathematically correct and thus should be written by someone with a deep understanding of the subject; however, the text should not be written to impress the potential instructor. It should be written for use by the students, in the way students work outside of class. There is a significant gap between the way most books are written and the way they are read (or not read) by students. Often, too much effort has gone into producing books which repeat a series of lectures. The theoretical part of calculus is difficult, as anyone who has tried to teach a rigorous calculus course will testify. The role of the lecturer in discussing the theoretical aspects of calculus cannot be overemphasized. Even the most gifted lecturer needs help in the form of a text which carries students step-by-step



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MANAGING EDITOR *(Name and address)*: Raoul Hailpern, Mathematical Assn. of America, SUNY at Buffalo, Buffalo, NY

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A. B. Willcox, Executive Director

THE AMERICAN MATHEMATICAL MONTHLY, Vol. 76, No. 9, November 1969, 989–1100

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN

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Names of authors are in ordinary type, those of reviewers in capitals and small capitals.

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NECROLOGY

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A PROBLEM IN CARTOGRAPHY

JOHN MILNOR, Massachusetts Institute of Technology

1. Introduction. The central problem of mathematical cartography is the problem of representing a portion of the curved surface of the earth on a flat piece of paper without introducing any more distortion than is absolutely necessary. This note will propose a quantitative definition for the term "distortion," and then study the mathematical problem of choosing a method of mapping which minimizes distortion.

To simplify the problem we first replace the rather irregular surface of the earth by a perfect sphere.

DEFINITIONS. Let S be the sphere of radius r consisting of all points x in the 3-dimensional euclidean space with distance r from the origin, and let U be any nondegenerate subset of S. (By "nondegenerate" we mean that U must contain at least two distinct points.)

A map projection f on the domain U will mean a function which assigns to each point x of U some point $f(x)$ of the euclidean plane E.

Let $d_S(x, y)$ denote the geodesic distance between two points x and y of the sphere S. By definition, this is equal to the length of the shorter great circle arc joining x to y. The euclidean distance between two points a and b of the plane E will be denoted analogously by $d_E(a, b)$.

The scale of a map projection f with respect to a pair of distinct points x and y in the domain U is defined to be the ratio

$$d_E(f(x), f(y))/d_S(x, y).$$

Ideally we would like this scale to be the same for all pairs of points x and y in U, but this is not usually possible. So we must introduce the *minimum scale* σ_1, defined to be the infimum of the ratio $d_E(f(x), f(y))/d_S(x, y)$ as x and y vary over all pairs of distinct points in U, and the *maximum scale* σ_2, defined to be the supremum of the ratio $d_E(f(x), f(y))/d_S(x, y)$. In other words σ_1 and σ_2 are the

---

Professor Milnor was graduated from Princeton in 1951 and received his Ph.D. in 1954 under Ralph Fox. After a year as Higgins Lecturer, he was appointed Assistant Professor and also was made an Alfred P. Sloan Research Fellow. He was promoted to Associate Professor in 1957, to Professor in 1960, and to Putnam Professor in 1962. He was a visiting professor at the University of California, Berkeley for 1959–60, at U.C.L.A. for 1967–68, and presently is Sloan Professor at MIT. He received honorary Sc.D. degrees from Syracuse University in 1965 and the University of Chicago in 1967. He is a member of the American Academy of Arts and Sciences and the Philosophical Society.

In 1962 Professor Milnor received the Fields Medal, the highest honor for a mathematician, at the International Congress of Mathematicians. In 1963 he was elected to the National Academy of Sciences, one of the youngest ever thus honored. In 1966 he received the President's National Medal of Science, with the citation: "For clever and ingenious approaches in topology which have solved long outstanding problems and opened new exciting areas in this active branch of mathematics." *Editor.*

"best" possible constants such that the inequality

$$\sigma_1 d_S(x, y) \leqq d_E(f(x), f(y)) \leqq \sigma_2 d_S(x, y)$$

holds for all points x and y in U.

To measure the extent to which scale fails to be constant we propose the following:

DEFINITION. *The distortion of the map projection f is the natural logarithm*

$$\delta = \log(\sigma_2/\sigma_1)$$

of the ratio of maximum scale to minimum scale.

Thus $0 \leqq \delta \leqq \infty$, where δ is finite if and only if both σ_1 and σ_2 are positive and finite numbers. If δ is finite, notice that the function f is continuous and one-to-one.

We would like to find a map projection f with no distortion at all ($\delta = 0$). Since this is not possible except in a few special and uninteresting cases (e.g., the case of a domain U consisting of only three points), the best we can actually do is to try to find a map projection for which δ is as small as possible.

DEFINITION. *A minimum distortion map projection f_0 on U will mean a map projection whose distortion δ_0 is less than or equal to the distortion of every other map projection on U.*

PRELIMINARY THEOREM. *For every nondegenerate set of points U on the sphere there exists a minimum distortion map projection f_0 with domain U.*

The proof of this theorem, which is quite elementary, will be deferred until Appendix A.

Unfortunately the proof will fail to suggest answers to a number of relevant questions: Is this minimum distortion map f_0 unique in some sense? Is f_0 differentiable (assuming that U is a nice enough set so that differentiability makes sense)? How can one actually construct f_0, or even a reasonable approximation to f_0? How can one estimate the minimum possible distortion δ_0 associated with a given set U?

This note will succeed in answering these questions only in one very special case, namely, the case of the region bounded by a circle on S.

Given a fixed point x_0 of S, let D_α denote the closed *disk* of geodesic radius $r\alpha$, consisting of all points x in S for which $d_S(x, x_0) \leqq r\alpha$. Here α can be any number in the interval $0 < \alpha < \pi$.

MAIN THEOREM. *There is one and, up to similarity transformations of the plane, only one minimum distortion map projection f_0 on the domain D_α. This map projection is infinitely differentiable, and has distortion δ_0 equal to* $\log(\alpha/\sin \alpha)$.

This minimum distortion projection f_0, known to cartographers as the "azimuthal equidistant projection," can be characterized by the fact that it pre-

serves both distances and directions from the central point x_0. The explicit formula $\delta_0 = \log(\alpha/\sin\alpha)$ shows that the distortion δ_0 is small for small values of α, being asymptotically equal to

$$\alpha^2/6 \sim \tfrac{2}{3} \text{ area } D_\alpha/\text{area } S$$

as $\alpha \to 0$. However δ_0 tends to infinity as $\alpha \to \pi$.

This theorem will be proved in Section 2. The problem of estimating the δ_0 associated with a more general domain U is discussed in Section 3. There are two appendices, one proving that minimum distortion map projections exist, and a second discussing a corresponding problem for conformal map projections, following Chebyshef.

2. The azimuthal equidistant projection. Again let D_α denote a spherical disk of geodesic radius $r\alpha$ centered at x_0.

LEMMA 1. *The distortion δ for any map projection f with domain D_α satisfies $\delta \geqq \log(\alpha/\sin\alpha)$.*

Proof. We may assume that f has finite distortion. Hence the "Lipschitz inequality"

$$d_E(f(x), f(y)) \leqq \sigma_2 d_S(x, y) \tag{1}$$

is satisfied, where σ_2 is a finite constant, and it follows that f is continuous. Furthermore f is one-to-one.

Let C_α denote the boundary of the disk D_α. Clearly the image $f(C_\alpha)$ is a simple closed curve in the plane. We shall first prove:

ASSERTION A. *Every half-line emanating from the point $f(x_0)$ in the plane must intersect the simple closed curve $f(C_\alpha)$ at least once.*

Proof. The Jordan Curve Theorem asserts that the simple closed curve $f(C_\alpha)$ cuts the plane into two components

$$E - f(C_\alpha) = E_1 \cup E_2,$$

one of these components, say E_1, being bounded, and the second unbounded. But the bounded component E_1 is just the image, under the continuous one-to-one function f, of the interior of the disk D_α. This is proved, for example, in Newman [**10**, Theorem 12.2, p. 121]. In particular it follows that the point $f(x_0)$ must belong to the bounded component E_1. Hence every half-line emanating from x_0 must cross $f(C_\alpha)$, since otherwise it would lie completely within the bounded set E_1 which is impossible. This proves Assertion A.

Since the curve C_α on S has finite length $2\pi r \sin\alpha$, it follows easily from the Lipschitz inequality (1) that $f(C_\alpha)$ also has finite length L, where

$$L \leqq 2\pi\sigma_2 r \sin\alpha. \tag{2}$$

(The *length* of a not necessarily smooth curve is defined for example in [6, p. 36].)

Now let us make use of the inequality

$$d_E(f(x), f(y)) \geqq \sigma_1 d_S(x, y). \tag{3}$$

Since every point of C_α has geodesic distance exactly $r\alpha$ from x_0 it follows that every point of $f(C_\alpha)$ has euclidean distance $\geqq \sigma_1 r\alpha$ from $f(x_0)$.

Thus $f(C_\alpha)$ is a simple closed curve of finite length L which lies outside an open disk D^* of radius $\sigma_1 r\alpha$ in the plane, and cuts every half-line through the center of this disk.

ASSERTION B. *This implies that $L \geqq 2\pi\sigma_1 r\alpha$, where equality holds if and only if $f(C_\alpha)$ is precisely equal to the boundary of D^*.*

Proof. Cut $f(C_\alpha)$ by a straight line through the center of D^* and choose intersection points, say a and b, which lie on opposite sides of D^*. Let A be either one of the two arcs of $f(C_\alpha)$ from a to b. Introducing polar coordinates ρ and θ about the center of D^*, first assume that the arc A can be described, in terms of a parameter t, by piecewise smooth functions

$$\rho = \rho(t), \qquad \theta = \theta(t).$$

Then

$$\text{length } A = \int (\dot\rho^2 + \rho^2\dot\theta^2)^{1/2} dt \geqq \int \rho \, | \dot\theta | \, dt,$$

where the dot denotes differentiation. Since

$$\rho \geqq \sigma_1 r\alpha \quad \text{and} \quad \int | \dot\theta | \, dt \geqq \left| \int \dot\theta dt \right| \geqq \pi,$$

this proves that length $A \geqq \pi\sigma_1 r\alpha$, and therefore $L \geqq 2\pi\sigma_1 r\alpha$, as required.

If A is not piecewise smooth, then an extra step is needed. For each $\epsilon > 0$ it is possible to approximate A by a polygonal path A'_ϵ from a to b which lies outside the disk of radius $\sigma_1 r\alpha - \epsilon$ and satisfies

$$\text{length } A \geqq \text{length } A'_\epsilon \geqq \pi(\sigma_1 r\alpha - \epsilon).$$

Letting $\epsilon \to 0$, we obtain length $A \geqq \pi\sigma_1 r\alpha$, as before.

Now suppose that the length of A is precisely equal to $\pi\sigma_1 r\alpha$. Then any portion of A which has distance greater than $\sigma_1 r\alpha$ from the center of D^* must be a straight line segment. Otherwise, replacing some small portion of A by a straight line segment we could decrease its length, which is impossible.

Any maximal line segment A_0 which forms a part of A must lead from one of the end points a or b of A to a point of the circle bounding D^*. The only other possibility would be that both end points of A_0 lie on the circle, which is impossible. Thus A consists of a line segment (possibly degenerate) from a to the

circle, followed by a circle arc, followed by a line segment to b. Elementary geometry now shows that the minimal length $\pi\sigma_1 r\alpha$ is achieved only if A is the semicircle. Hence L can equal $2\pi\sigma_1 r\alpha$ only if $f(C_\alpha)$ is the full circle. This completes the proof of Assertion B.

Combining Assertion B with the inequality (2) we obtain

$$2\pi\sigma_1 r\alpha \leqq 2\pi\sigma_2 r \sin \alpha$$

or

$$\alpha/\sin \alpha \leqq \sigma_2/\sigma_1$$

and hence $\log(\alpha/\sin \alpha) \leqq \delta$, which completes the proof of Lemma 1.

LEMMA 2. *If the distortion of f is precisely equal to* $\log(\alpha/\sin \alpha)$, *then f is an azimuthal equidistant projection.*

By definition this means that f carries each great circle passing through x_0 into a straight line in the plane, the angle between two great circles being equal to the angle between the corresponding straight lines, and that f carries each circle C centered at x_0 to a circle $f(C)$ centered at $f(x_0)$, the radius of $f(C)$ being proportional to the geodesic radius of C.

To differential geometers, this means that f is the inverse of the so called exponential map. It follows that f is infinitely differentiable, even at x_0. See for example [**9**, p. 147].

Proof of Lemma 2. If $\delta = \log(\alpha/\sin \alpha)$, then according to Assertion B the image $f(C_\alpha)$ must be precisely equal to the circle of radius

$$\sigma_1 r\alpha = \sigma_2 r \sin \alpha$$

centered at $f(x_0)$. Hence the image $f(D_\alpha)$ must be precisely the closed disk bounded by this circle. (Compare the proof of Assertion A.)

Now consider an arbitrary point x of D_α. Construct a great circle segment from x_0 through x to a point $\bar{x}$ on the boundary C_α of D_α. If c denotes the geodesic distance $d_S(x_0, x)$, note that x has geodesic distance precisely $r\alpha - c$ from $\bar{x}$, and geodesic distance strictly greater than $r\alpha - c$ from every other point of C_α. Hence, using inequality (3), the image $f(x)$ must

(a) have distance at least $\sigma_1 c$ from $f(x_0)$,

(b) have distance at least $\sigma_1(r\alpha - c)$ from $f(\bar{x})$, and

(c) have distance greater than $\sigma_1(r\alpha - c)$ from every other point of $f(C_\alpha)$.

Clearly there is one and only one point in the disk $f(D_\alpha)$ which satisfies these three conditions: namely, the point which lies at distance $\sigma_1 c$ along the line segment from $f(x_0)$ to $f(\bar{x})$. Thus the map projection f on D_α is completely determined by what is does to boundary points of D_α.

To complete the proof of Lemma 2 we need only verify that f carries the circle C_α to the circle $f(C_\alpha)$ by a similarity transformation which multiplies all

lengths by the constant factor σ_2. Suppose that we cut C_α into two arcs A and A', so that

$$\text{length } A + \text{length } A' = \text{length } C_\alpha = 2\pi r \sin \alpha.$$

The Lipschitz inequality (1) implies that

$$\text{length} f(A) \leqq \sigma_2 \text{ length } A, \qquad \text{length} f(A') \leqq \sigma_2 \text{ length } A'. \tag{4}$$

But

$$\text{length} f(A) + \text{length} f(A') = \text{length} f(C_\alpha)$$

is precisely equal to σ_2 times the length $2\pi r \sin \alpha$ of C_α. So both of the inequalities (4) must actually be equalities. This proves Lemma 2.

Now we must prove the converse.

LEMMA 3. *The azimuthal equidistant projection on the disk D_α has distortion δ precisely equal to* $\log(\alpha/\sin \alpha)$.

Proof. Centering D_α at the north pole, we will use the longitude $0 \leqq \theta \leqq 2\pi$ and the colatitude $0 \leqq \gamma \leqq \alpha$ as coordinates. Suppose that f maps the point with colatitude γ and longitude θ to the point with cartesian coordinates $(r\gamma \cos \theta, r\gamma \sin \theta)$ in the plane. The length of any smooth curve $\gamma = \gamma(t)$, $\theta = \theta(t)$ in D_α is given by the integral

$$L = r \int (\dot{\gamma}^2 + \dot{\theta}^2 \sin^2 \gamma)^{1/2} dt,$$

and the length of the corresponding curve in $f(D_\alpha)$ is

$$L' = r \int (\dot{\gamma}^2 + \dot{\theta}^2 \gamma^2)^{1/2} dt.$$

But, since $\gamma/\sin \gamma$ is a monotone increasing function of γ, we have

$$\sin \gamma \leqq \gamma \leqq (\alpha/\sin \alpha) \sin \gamma,$$

from which it follows easily that

$$L \leqq L' \leqq (\alpha/\sin \alpha) L. \tag{5}$$

Starting from this inequality (5) we will prove that

$$d_S(x, y) \leqq d_E(f(x), f(y)) \leqq (\alpha/\sin \alpha) d_S(x, y)$$

for every x and y in D_α. Clearly this will imply that $\delta \leqq \log(\alpha/\sin \alpha)$ and hence, by Lemma 1, that $\delta = \log(\alpha/\sin \alpha)$.

Proof that $d_S(x, y) \leqq d_E(f(x), f(y))$. Join $f(x)$ to $f(y)$ within the convex set $f(D_\alpha)$ by a line segment of length L' precisely equal to $d_E(f(x), f(y))$. The corresponding curve in D_α will have length $L \geqq d_S(x, y)$. Since $L \leqq L'$, we obtain $d_S(x, y) \leqq d_E(f(x), f(y))$, as required.

Proof that $d_E(f(x), f(y)) \leqq (\alpha/\sin\alpha) d_S(x, y)$. First suppose that $\alpha \leqq \pi/2$, so that the disk D_α is "geodesically convex." Then the proof is quite analogous. Join x to y, *within* D_α, by a great circle segment A of length $L = d_S(x, y)$. Then $f(A)$ has length $L' \geqq d_E(f(x), f(y))$, so the inequality $L' \leqq (\alpha/\sin\alpha)L$ implies that $d_E(f(x), f(y)) \leqq (\alpha/\sin\alpha) d_S(x, y)$, as required.

If $\alpha > \pi/2$, so that the disk D_α is not geodesically convex, then a more complicated argument is necessary. Suppose that the shortest great circle arc from x to y does not lie completely within D_α, but rather crosses out of D_α at a boundary point $\bar{x}$, and then crosses back in at another boundary point $\bar{y}$. We shall show that

$$d_E(f(x), f(\bar{x})) \leqq (\alpha/\sin\alpha) d_S(x, \bar{x}), \tag{6}$$

$$d_E(f(\bar{x}), f(\bar{y})) \leqq (\alpha/\sin\alpha) d_S(\bar{x}, \bar{y}), \tag{7}$$

$$d_E(f(\bar{y}), f(y)) \leqq (\alpha/\sin\alpha) d_S(\bar{y}, y). \tag{8}$$

Adding these three inequalities, we shall clearly obtain the required inequality.

But (6) and (8) can be proved by the argument above. To prove (7) we introduce an auxiliary azimuthal equidistant projection g whose domain is the complementary disk $D'_{\pi-\alpha}$ centered at the south pole. Since $\pi - \alpha \leqq \pi/2$ we have

$$d_E(g(\bar{x}), g(\bar{y})) \leqq ((\pi - \alpha)/\sin(\pi - \alpha)) d_S(\bar{x}, \bar{y}).$$

Multiplying this by $\alpha/(\pi - \alpha)$ we obtain the required inequality (7). This completes the proof of Lemma 3.

Clearly Lemmas 1, 2, and 3 imply the "Main Theorem" of Section 1.

3. Discussion. How can one estimate the minimum possible distortion δ_0 for map projections on a given set U? Here is a crude estimate. Define the *angular width* w of a set U as follows. Choose a smallest possible "lune" (figure bounded by two great semicircles) containing U, and let w be the angle at the vertex of this lune.

ASSERTION. *Any set with angular width* $w < \pi$ *possesses a map projection with distortion* $\delta \leqq \log \sec(w/2)$.

This is proved by rotating so that the lune is centered on the equator, and then using the latitude and longitude of x as the cartesian coordinates of $f(x)$. The computations are similar to those in the proof of Lemma 3.

It is conjectured that this estimate gives the right order of magnitude in the case of a small geodesically convex region, in the sense that δ_0 is greater than say one sixth of $\log \sec(w/2)$. But $\log \sec(w/2)$ is not a really good estimate for δ_0, except perhaps in the case of a long narrow region.

It would be more interesting to find a relation between δ_0 and area.

PROBLEM. *Among all geodesically convex regions of given area, does the disk* D_α *require the largest distortion?*

In other words, if $\text{area}(U)=\text{area}(D_\alpha)$ does it follow that U has a map projection with distortion $\delta \leqq \log(\alpha/\sin\alpha)$? If true this would imply the existence of map projection with smaller distortion than any which are actually known for many regions on the sphere. A test case which would be particularly interesting would be that of a small "rectangular" region on the sphere.

Slightly cruder is the following possible estimate.

PROBLEM. *Does every geodesically convex region U possess a map projection with distortion less than the normalized area,*

$$\delta < \text{area } U/\text{area } S?$$

As an example, for the continental United States with about 1.5 percent of the earth's area, does there exist a map projection with scale errors of no more than 1.5 percent (or perhaps $1.5+\epsilon$ to allow for the lack of geodesic convexity)? All standard map projections for the continental United States seem to have scale errors of at least 2.2 percent.

Appendix A. Minimum distortion projections always exist. We shall first prove the following. Let U be a subset of the sphere S and let $\overline{U}$ denote the topological closure of U.

LEMMA 4. *Any map projection f on U with distortion $\delta < \infty$ extends uniquely to a map projection $\bar{f}$ on $\overline{U}$ having the same distortion δ.*

Proof. The inequalities

$$\sigma_1 d_S(x, y) \leqq d_E(f(x), f(y)) \leqq \sigma_2 d_S(x, y)$$

show that f is uniformly continuous, and hence extends uniquely to a continuous function $\bar{f}$ on $\overline{U}$. (See [3, p. 55].) Clearly $\bar{f}$ will also satisfy these inequalities.

Now, given some fixed set U, consider all possible map projections f with domain U, and let δ_0 denote the infimum of the corresponding distortions $\delta(f)$. We must construct a map projection f_0 whose distortion is precisely equal to δ_0. We may assume that $\delta_0 < \infty$, since otherwise there is nothing to prove.

REMARK. Note that there exists a map projection with finite distortion on U if and only if the closure $\overline{U}$ is not the entire sphere. For if U is not everywhere dense on S then U is contained in some disk $D_{\pi-\epsilon}$ and hence possesses a map projection with distortion $\delta \leqq \log((\pi-\epsilon)/\sin(\pi-\epsilon)) < \infty$. But if $\overline{U}=S$ then a map projection with finite distortion on U would extend to a map projection with finite distortion on S, which is impossible since $S \supset D_\alpha$ for all α, or since S is not homeomorphic to any subset of E. (See for example [10, p. 122].)

Choose a sequence of map projections $\{f_1, f_2, f_3, \cdots\}$ on U so that the corresponding sequence $\{\delta_1, \delta_2, \delta_3, \cdots\}$ of distortions tends to the limit δ_0. We may assume that each f_i has been chosen so as to have maximum scale equal to 1, and so that the image $f_i(U)$ contains the origin.

Choose a countable dense subset

$$U' = \{x_1, x_2, x_3, \cdots\}$$

of U. Since the points $f_1(x_1), f_2(x_1), \cdots$ all have distance $\leqq \pi r$ from the origin, we can choose a convergent subsequence. That is there exists an infinite set I_1 of positive integers so that the sequence of points $f_i(x_1)$, where i tends to infinity through the set I_1, converges to some limit a_1 in E. Similarly we can find an infinite set $I_2 \subset I_1$ so that the limit

$$\lim\{f_i(x_2) \mid i \to \infty, i \in I_2\}$$

exists. Call this limit a_2. Continuing inductively we can define a function f from U' to the plane by $f(x_j) = a_j = \lim\{f_i(x_j) \mid i \to \infty,\ i \in I_j\}$. Since the inequalities

$$e^{-\delta_i} d_S(x, y) \leqq d_E(f_i(x), f_i(y)) \leqq d_S(x, y)$$

hold for all i, it follows, taking the limit as i tends to infinity through an appropriate I_j, that

$$e^{-\delta_0} d_S(x, y) \leqq d_E(f(x), f(y)) \leqq d_S(x, y)$$

for all x and y in U'. Thus f is a map projection on U' with distortion δ_0.

Now applying Lemma 4 we obtain the required map projection on U with distortion δ_0.

Appendix B. Conformal map projections. Recall that a map projection f, defined on an open set U, is called *conformal* (cartographers prefer the term "orthomorphic") if it is differentiable and preserves angles. (That is, f transforms any pair of curves in U, whose tangent vectors at a point of intersection span the angle α into a pair of curves in E, whose tangent vectors at the corresponding intersection point span the same angle α.)

It follows that f has a well defined *infinitesimal-scale* $\sigma(x)$ at each point x of U. By definition $\sigma(x)$ is the limit of the ratio $d_E(f(x), f(y))/d_S(x, y)$ as y tends to the limit x. (Compare [**1**, p. 74].)

We shall make use of the *Laplace-Beltrami operator* Δ, a second order partial differential operator which assigns to each twice differentiable real valued function g on a Riemannian manifold a new real valued function Δg. In euclidean space this is the familiar Laplace operator. We shall use Δ only on the sphere S of radius r. Using latitude λ and longitude θ as coordinates, the operator Δ on the sphere takes the form

$$r^2 \Delta g = g_{\lambda\lambda} - g_\lambda \tan \lambda + g_{\theta\theta} \sec^2 \lambda.$$

(Compare [**14**, p. 160]. The subscripts denote partial derivatives.)

Suppose now that U is a simply connected open subset of the sphere S.

LEMMA 5. *The infinitesimal-scale function $\sigma(x)$ associated with a conformal map projection f on U determines f up to an* (*orientation preserving or reversing*)

rigid motion of the plane. A given positive real valued function σ on U is the infinitesimal-scale function associated with some conformal f if and only if σ is twice differentiable and satisfies the differential equation $r^2\Delta \log \sigma = 1$.

As an example, the function $\sigma(x) = \sec(\text{latitude } x)$ provides a solution to this equation $r^2\Delta \log \sigma = 1$, except at the north and south poles. The corresponding f turns out to be the familiar Mercator projection.

(Note that our differential equation cannot have any solution which is defined and smooth throughout the entire sphere, since the condition $\Delta \log \sigma > 0$ implies easily that σ cannot have any local maximum.)

Proof of Lemma 5. More generally, consider a smooth surface M provided with a Riemannian metric, expressed in terms of local coordinates u and v as $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Let Δ denote the associated Laplace-Beltrami operator, and let K denote the Gaussian curvature of M. Consider a second Riemannian metric of the form $\sigma^2 ds^2$ on M, where σ is a positive twice differentiable function. Computation (using for example [**14**, pp. 113, 160]) shows that the Gaussian curvature K' associated with this new Riemannian metric is given by the formula $K' = (K - \Delta \log \sigma)/\sigma^2$.

If σ is the infinitesimal-scale function associated with a conformal mapping f from M to M', then clearly $K'(x)$ is just the Gaussian curvature of M' at $f(x)$. Thus if M' is the euclidean plane, with $K' \equiv 0$, we see that the differential equation

$$\Delta \log \sigma = K$$

must be satisfied. In particular, taking M to be the open subset U of S, with $K \equiv 1/r^2$, we obtain the required equation

$$r^2 \Delta \log \sigma = 1.$$

Conversely, given any solution σ to the differential equation $\Delta \log \sigma = K$, the Riemannian metric $\sigma^2 ds^2$ has curvature K' identically zero. Hence any sufficiently small connected open subset of M, with the metric $\sigma^2 ds^2$, can be mapped isometrically onto an open subset of the plane ([**14**, p. 145]). This isometry is unique up to rigid motions of the plane, since any isometry ϕ from one connected open subset of the plane to another extends to an isometry of the entire plane. (Assuming that ϕ preserves orientation, we can think of ϕ as a complex analytic function [**1**, p. 74] with $|d\phi/dz| \equiv 1$. Hence $d\phi/dz$ is constant and $\phi(z) = cz + c'$ with $|c| = 1$.)

Now if M is simply connected then a monodromy argument shows that these local isometries can be chosen so as to fit together to yield a smooth mapping f from all of M to E.

(Compare [**8**, p. 1297]. The "Monodromy Theorem" says that if we are given connected open sets U_α covering a simply connected manifold M, and for each U_α a collection F_α of functions from U_α to Y satisfying the following condition, then there exists a function from M to Y whose restriction to each U_α belongs

to F_α. The condition is that for each f_α in F_α and each x in $U_\alpha \cap U_\beta$ there should exist one and only one f_β in F_β which coincides with f_α throughout some neighborhood of x. Compare [**1**, p. 285], [**12**, p. 67].)

In the large, this mapping f from M to E may not be one-to-one, but locally it carries M, with the metric $\sigma^2 ds^2$, isometrically to E. Hence it carries M with the original metric ds^2 conformally to E, the infinitesimal-scale function of f being precisely equal to σ. This completes the proof of Lemma 5.

Chebyshef [**2**] studied conformal map projections, using the ratio $\sup \sigma(x)/\inf \sigma(x)$ of maximum infinitesimal-scale to minimum infinitesimal-scale as a measure of distortion.

REMARK. If the domain U is geodesically convex, note that the maximum infinitesimal-scale $\sup \sigma(x)$ is equal to the maximum scale σ_2 of Section 1. (Compare the proof of Lemma 3.) Similarly, if f is one-to-one and $f(U)$ is convex, then $\inf \sigma(x) = \sigma_1$.

CHEBYSHEF THEOREM. *If U is a simply connected region bounded by a twice differentiable curve, then there exists one and, up to a similarity transformation of E, only one conformal map projection which minimizes this ratio* $\sup \sigma/\inf \sigma$. *This "best possible" conformal map projection is characterized by the property that its infinitesimal-scale function $\sigma(x)$ is constant along the boundary of U.*

This result has been available for more than a hundred years, but to my knowledge it has never been used by actual map makers.

Proof. Setting $g(x) = \log \sigma(x)$, first note that the differential equation $r^2 \Delta g = 1$ has a unique solution satisfying the boundary condition $g(x) = 0$ for $x \in bd(U)$. See for example [**5**, p. 288]. If h is any other function which is twice differentiable and satisfies the equation $r^2 \Delta h = 1$ throughout the interior of U, then we shall show that

$$\sup h - \inf h \geqq \sup g - \inf g, \tag{9}$$

where equality holds if and only if

$$h = g + \text{constant}.$$

(Note that $\sup g - \inf g$ is just the logarithm of the ratio $\sup \sigma/\inf \sigma$ which we want to minimize.) Clearly this will complete the proof.

Since $\Delta g > 0$, an easy argument shows that the function g cannot attain its maximum at any interior point of U. Since g must achieve a maximum at some point of the compact set $\overline{U}$, it follows that the maximum must be attained on $bd(U)$. Thus $\sup g(x) = 0$.

The difference $h - g$ satisfies the homogeneous equation $\Delta(h-g) = 0$, and so cannot achieve its maximum at an interior point of U unless $h - g = \text{constant}$. (See [**5**, p. 232].) Hence any sequence of points $x_1, x_2, \cdots$ for which

$$\lim_{i \to \infty} (h(x_i) - g(x_i)) = \sup(h - g)$$

must be a sequence tending to the boundary of U, unless $h-g=$constant.

Setting $c=\sup h$ (we may assume that c is finite since otherwise (9) would trivially be satisfied), we have

$$g(x_i) \to 0, \qquad h(x_i) \leqq c,$$

hence

$$\sup(h - g) = \lim(h(x_i) - g(x_i)) \leqq c,$$

or in other words

$$h(x) \leqq g(x) + c$$

for all x. Therefore $\inf h \leqq \inf g+c$, which proves (9).

If equality holds, then at the interior point x_0 of U where g achieves its minimum we have

$$h(x_0) = g(x_0) + c.$$

Thus $h-g$ achieves its maximum c at an interior point, and hence is constant. This completes the proof.

REMARK. The "best possible" conformal map projection f, although locally well behaved, may not be one-to-one in the large. However, if U is geodesically convex, then it can be shown that f is one-to-one and that $f(U)$ is also convex.

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SOUSLIN'S CONJECTURE

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In the first volume of *Fundamenta Mathematicae*, which appeared in 1920, in the problem section at the end, Problem 3 presents a conjecture of a young Russian mathematician named Souslin [**1**]. The conjecture is very natural and elementary; and it seems reasonable that the conjecture was made in pure innocence of its difficulty, not to say significance.

However, although Souslin died at the age of 25 and published only one paper, he made other basic set theoretic contributions to mathematics [**3**]. Finding an error in a "proof" to the contrary by Lebesgue, Souslin gave an example of a Borel set in the plane whose projection on the line is not Borel. He went on to define *analytic sets* (occasionally called *Souslin sets*) and gave a unique scheme known as the *Souslin schema* for constructing these sets. And he proved several other fundamental theorems about analytic sets, extended and improved by his teacher Lusin [**2**] and many others. Recently they have been in the news again because several of the still unsolved problems about analytic sets have yielded to attack by logicians (such as R. Solovay, D. Martin, and Y. Moschovakis, to mention a few) using P. Cohen's methods.

But if Souslin's only contribution to mathematics had been this conjecture, his name would still survive mathematically, for *Souslin trees* have become the standard term for a certain kind of partially ordered set.

Souslin's conjecture sounds simple. Anyone who understands the meaning of countable and uncountable can "work" on it. It is in fact very tricky. There are standard patterns one builds. There are standard errors in judgement one makes. And there are standard not-quite-counter-examples which almost everyone who looks at the problem happens upon. S. Tennenbaum and others have shown that it is consistent with the axioms of Zermelo-Fraenkel set theory that Souslin's conjecture be either true or false. In fact, there is a fast growing wealth of consistency results about Souslin trees, and I shall state and give references for some of these results at the end of this paper. But I shall not prove any of these theorems. Instead I shall describe the standard pattern and prove the one elementary theorem, and I shall give one of the not-quite-counter-examples. My aim is to make a Souslin tree recognizable in case one finds the pattern in another problem. I have found it useful in the past [**8**].

So what is Souslin's conjecture? Suppose that:

(1) L is a totally ordered set without a first or last element,

(2) L has the interval topology induced by the total ordering and L is connected,

(3) each collection of disjoint open intervals in L is countable.

Souslin conjectures that L is topologically the real line.

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It is well known that a space satisfying (1) and (2) is topologically a line if and only if it is separable. So we call L a *Souslin line* if L satisfies (1), (2), (3), and

(4) L is not separable.

We have our first equivalence: the existence of a Souslin line is equivalent to Souslin's conjecture being false.

Let us assume the existence of a Souslin line L and see what patterns we can build. Observe that if a subset C of L is countable, the closure of C is not L, because of (4). So $L-\overline{C}$ is the union of a (nonempty) collection $T(C)$ of nonempty disjoint open intervals. Thus, by (3), $T(C)$ is countable. Using transfinite induction, we perform this operation for each countable ordinal. That is, let c_0 be a countable subset of L. Now suppose that for some countable ordinal α we have selected a countable subset c_β of L for each $\beta<\alpha$. The union C_α of all c_β for $\beta<\alpha$ is countable and $T(C_\alpha)$ is countable. Define c_α to consist of one point from each member of $T(C_\alpha)$.

Observe that the union C of all c_α has cardinality $\aleph_1$. And the union T of all $T(C_\alpha)$ has cardinality $\aleph_1$.

Now let us prove that $L\subset\bigcup\overline{C}_\alpha$. Otherwise there is a point p of L not in $\overline{C}_\alpha$ for any α; hence for each α there is a term I_α of $T(C_\alpha)$ containing p. For each α, $I_\alpha\supset I_{\alpha+1}$ but $I_\alpha\neq I_{\alpha+1}$. Hence there is an open interval J_α of L in $I_\alpha-I_{\alpha+1}$. But $\{J_\alpha\}_{\alpha<\omega_1}$ is a collection of disjoint open intervals which by (3) cannot be uncountable; this is a contradiction.

So while L has no countable dense subset, the next best thing is true, C is dense in L and of cardinality $\aleph_1$.

But it is not C that is important really. It is T. Let me just use T_α for $T(C_\alpha)$. The T_α's form successive layers of countable sets of disjoint intervals, each layer containing the following ones (like nested tin cans to use an old simile of Burton Jones).

We can partially order T by inclusion and show the following facts about this partially ordered set:

A. T has cardinality $\aleph_1$.

B. Each chain (totally ordered subset) of T is countable.

C. Each antichain (pairwise unordered subset) of T is countable.

D. T is a tree.

A partially ordered set P is called a *tree* if for each t in P, the set of all elements of P which precede t is well ordered. Under our inclusion order, x precedes y means x contains y. If $t\in T$, then $\{x\in T \mid x\supset t\}$ is totally ordered by inclusion exactly as the ordinals $\{\alpha \mid x\in T_\alpha\}$ are ordered by size. Hence T is a tree and D is satisfied.

Each antichain in T is a collection of disjoint open intervals in L and hence is countable; so we have C.

It remains to show that B is satisfied.

We can index the elements of a chain in T by the subscripts of the T_α to which they belong. So a chain $\{I_\alpha\in T_\alpha\}$ for uncountably many α implies again

the existence of uncountably many disjoint open intervals $\{J_\alpha \subset (I_\alpha - \bigcup_{\beta>\alpha} I_\beta)\}$, and hence is impossible. So chains in T are countable.

A partially ordered set satisfying A, B, C, and D is called a *Souslin tree of cardinality* $\aleph_1$. Given a cardinal K it is not hard to guess that one might call a tree of cardinality K without any chains or antichains of cardinality K a *Souslin tree of cardinality* K. If K is a singular cardinal (one such as $\aleph_\omega$ which is cofinal with a smaller cardinal) there are trivial fan shaped Souslin trees of cardinality K. Assuming the generalized continuum hypothesis, there is [**7**] a model A of set theory and an extension B of A such that there are Souslin trees of cardinality K in B for every cardinal K in A. The cardinals for which there are no Souslin trees are called "weakly compact", and there are many open questions [**14**] in this area.

Now forget L and the T we constructed from L and start all over. Just assume that we have a tree T (with partial order $\leqq$ instead of $\supset$). To be sure that T looks like a tree instead of a forest, we can add a trunk, i.e., add one element to T preceding all of the rest. (This corresponds to adding L to the members of T in the previous construction.) For $t\in T$, the set $\{x\in T \mid x\leqq t\}$ is a well-ordered sequence. So there is an ordinal α, called the *level* of t such that $\{x\leqq t\}$ and $\{\beta\leqq\alpha\}$ can be put into one-to-one order preserving correspondence. We define the αth *level of the tree* to be $T_\alpha = \{t\in T \mid \text{the level of } t \text{ is } \alpha\}$. Thus T_0 is the trunk, T_1 is the set of branches from the trunk, each member of T_2 is a branch from one and only one member of T_1, etc. Trees look like trees. Notice that the members of T_α form an antichain. Assume that T is a Souslin tree. Then A says that T is of uncountable height, since C tells us that T is not of uncountable width. But B says we cannot start at the trunk and proceed up the tree through uncountably many branches without turning around or jumping sideways.

The uninitiated mathematician usually feels strongly either that there can be no Souslin tree or, just as strongly, that he should be able to describe an algorithm for the construction of a Souslin tree. It is a simple matter to visualize a tree with countably many branches at each level, but one must decide which branches of the tree should continue. One common fallacy is to assume that having no uncountable antichain is equivalent to having each level of the tree countable. There are [**9**] uncountable trees without uncountable chains in which T_α is countable for each α. I call such trees *fake Souslin trees*; this is probably bad notation for such trees are *not* Souslin trees.

I shall describe one fake Souslin tree. M. Aronszajn [**10**] and B. Jones discovered the example independently. Set theorists and topologists, unlucky enough to fall under the spell of Souslin's conjecture, discover the same examples again and again.

Let R be the set of all rational numbers between 0 and 1. Let X be the set of all countable well ordered sequences of terms of R. If x and t belong to X define $t\leqq x$ to mean t is an initial segment of x. Then X is a tree. Define Y to be the subtree of X consisting of those t in X such that the *sum* $s(t)$ of the terms of t belongs to R. The $s(t)$ for the terms t of a chain in Y form a strictly increasing

well-ordered sequence of numbers less than 1; so every chain in Y is countable. But if T is any uncountable subtree of Y, there is some r in R such that uncountably many terms t of T have $s(t)=r$. Since $\{t \mid s(t)=r\}$ is an antichain in Y, there is an uncountable antichain in T. So when we define an uncountable subtree T of Y in which every T_α is countable, T is not a Souslin tree; T is fake.

Define $T_0=R=X_0=Y_0$. For each countable ordinal α we will define T_α inductively. Our induction hypotheses are:

(1) T_α is a countable subset of $X_\alpha \cap Y$,

(2) if $\gamma<\alpha$ and $t\in T_\gamma$ and $r\in R$ and $r>s(t)$, then there is a $y\in T_\alpha$ such that $t\leqq y$ and $s(y)=r$.

Assume that α is a countable ordinal and that T_β satisfying the hypotheses has been defined for all $\beta<\alpha$. Suppose α is not a limit ordinal. If $t\in T_{\alpha-1}$ and $r\in R$ and $r>s(t)$, then there is precisely one $y\in X_\alpha$ such that $s(y)=r$ and $t\leqq y$. Let T_α be the set of all such y for all t and r. Suppose α is a limit ordinal. If $t\in T_\beta$ and $\beta<\alpha$ and $r\in R$ and $r>s(t)$, then select $\beta<\alpha_1<\alpha_2< \cdots$ having α as a limit and $s(t)<r_1<r_2< \cdots$ having r as a limit and $r_n\in R$. Using (2) select $t\leqq y_1\leqq y_2\leqq \cdots$ such that $y_n\in T_{\alpha_n}$ and $s(y_n)=r_n$. Then there is precisely one $y\in X_\alpha$ such that $t\leqq y_1\leqq y_2\leqq \cdots \leqq y$; observe that $s(y)=r$. Let T_α be a countable set of such y, precisely one for each pair t and r.

I call the fake Souslin tree T defined above the *bush*. If α is not a limit ordinal we can think of t as a straight branch of length the last term of t (and $s(t)$ as its distance back to the ground). If $t\in T_\alpha$ for a limit ordinal α, then t has no length, but is a nodule from which new branches of the tree will spring. And the branches in T_0 spring from a point on the ground. The resulting picture is a fat little tree of height less than 1.

Just as one can ask about fake Souslin trees of cardinality $\aleph_1$, one can ask for larger cardinals K, if there exists a tree of cardinality K in which there are no chains of cardinality K and each level has cardinality less than K (but some antichain has cardinality K)? Except when K is the successor of a singular cardinal, using the generalized continuum hypothesis, one ([**11**], [**12**], [**13**]) can construct an example of a fake Souslin tree of cardinality K by generalizing an example of a fake Souslin tree of cardinality $\aleph_1$. And K. Prikry [**14**] has shown that the existence of a fake Souslin tree of cardinality $\aleph_{\omega+1}$ is consistent with the axioms of Zermelo-Fraenkel set theory.

The following theorem was first proved by E. W. Miller [**3**], another promising set theorist who died very young:

The existence of a Souslin line is equivalent to the existence of a Souslin tree.

In order to prove the second half of the theorem, assume a Souslin tree T. Observe that any subset of T is a tree and any uncountable subset of T is a Souslin tree. I first discuss two modifications of T.

Let us *prune* T, i.e., discard its short limbs (the terminology is strictly mine). For t in T define

$$f(t) = \{y \in T \mid t \leqq y\} \quad \text{and} \quad T' = \{t \in T \mid f(t) \text{ is countable}\}.$$

Observe that

(1) the first level T_0' of T' is an antichain in T,

(2) the union of all $f(t)$ for $t \in T_0'$ is countable, and

(3) if $y \in T'$, then $y \in f(t)$ for some t in T_0'.

Hence T' is countable. The pruned tree $T-T'$ is a Souslin tree, and each term of $T-T'$ is followed by uncountably many terms of $T-T'$.

It is not necessary to prune a Souslin tree before constructing a Souslin line from it. The unpruned tree just gives rise to nontrivial separable intervals in the line (and to special cases in the proof that the line is Souslin). If one prunes, however, there are no countable branches or loose ends, and the situation is more homogeneous. The line could be pruned just as well as the tree. Let M be the set of all maximal separable open intervals in some Souslin line L. Since the terms of M are disjoint, M is countable and the union M^* of the terms of M is separable. The Dedekind completion of $L-\overline{M}^*$ is a Souslin line without any separable open intervals. An old question among Souslin conjecture addicts has been: is a pruned Souslin line homogeneous? R. Jensen has proved that more or less anything goes in some models of set theory as the last paragraph of this paper indicates.

Now we *normalize* T; this bad notation is standard. Let us say that an element t of T *branches* if there is another member of T having exactly the same predecessors in T as t. The normalization $\mathfrak{T}$ of T consists of the branching members of T. To see that $\mathfrak{T}$ is uncountable and hence a Souslin tree, suppose there were a countable ordinal α such that $\mathfrak{T} \cap T_\beta = \varnothing$ for all $\beta \geqq \alpha$. Select $t \in T_\alpha$ such that $f(t)$ is uncountable. There is no branching after T_α. Therefore $f(t)$ is an uncountable chain, which is a contradiction.

Without loss of generality, we assume that T is pruned and normalized. That is

(1) for $t \in T$, the set $f(t)$ is uncountable and

(2) for $t \in T$, the set $g(t)$ of all members of T having exactly the same predecessors as t has at least two members.

The picture T gives us is very much like the one we had when constructing T from L. We were careful then to select a point of c_α in each term of T_α, and this normalized our tree. What we want to do now is project T back on the line, which we do by defining a total order $\preceq$ on T, the Dedekind completion of which is a Souslin line.

When we draw a tree on paper we flatten T and automatically assign some total ordering to $g(x)$ for each $x \in T$. We use $g(x) = g_1(x), g_2(x), \cdots$; since $g(x)$ is countable there is such a simple well ordering. But T itself is unprejudiced, and there is no reason to feel that the Souslin lines constructed from the same T using different total orderings of $g(x)$ should be homeomorphic.

Suppose that $\alpha \leqq \beta$, $x \in T_\alpha$ and $y \in T_\beta$.

Case 1: $x = y$. Define $x \preceq y$.

Case 2: $\alpha < \beta$ and $x \leqq y$. There is a unique $t \in T_{\alpha+1}$ such that $t \leqq y$. Define $x \preceq y$ if and only if $t \neq g_1(t)$.

A_3 = *any two open intervals in* $\mathcal{S}$ *are isomorphic.*
A_4 = *no two distinct open intervals in* $\mathcal{S}$ *are isomorphic.*

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HERMITE-BIRKHOFF INTERPOLATION PROBLEMS WITH COMPLEX NODES

MURRAY SCHECHTER, Lehigh University

Introduction. Let k complex numbers $z_1, z_2, \cdots, z_k$ be given and also let a $k\times n$ matrix E be given with each element of E zero or one. Suppose exactly n elements of E are ones and no row of E is made up entirely of zeros. Let ϵ_{ij} denote the element in the ith row and jth column of E. Let π_n denote the set of all polynomials of degree $\leqq n$. The points $z_1, \cdots, z_k$ and the matrix E describe what I. J. Schoenberg [1] has called a *Hermite-Birkhoff* (HB) *interpolation* problem: determine $p\in\pi_{n-1}$ such that $p^{(j-1)}(z_i)$ have prescribed values for each i, j such that $\epsilon_{ij}=1$. The points $z_1, \cdots, z_k$ and the matrix E are called the *nodes* and the *incidence matrix* respectively of the HB problem. For example, the incidence matrix $(1, 1, 1, \cdots, 1)$ corresponds to Taylor interpolation, and the incidence matrix with first column all ones and all other entries zero corresponds to Lagrange interpolation.

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Introduction. Let k complex numbers $z_1, z_2, \cdots, z_k$ be given and also let a $k\times n$ matrix E be given with each element of E zero or one. Suppose exactly n elements of E are ones and no row of E is made up entirely of zeros. Let ϵ_{ij} denote the element in the ith row and jth column of E. Let π_n denote the set of all polynomials of degree $\leqq n$. The points $z_1, \cdots, z_k$ and the matrix E describe what I. J. Schoenberg [1] has called a *Hermite-Birkhoff* (HB) *interpolation* problem: determine $p\in\pi_{n-1}$ such that $p^{(j-1)}(z_i)$ have prescribed values for each i, j such that $\epsilon_{ij}=1$. The points $z_1, \cdots, z_k$ and the matrix E are called the *nodes* and the *incidence matrix* respectively of the HB problem. For example, the incidence matrix $(1, 1, 1, \cdots, 1)$ corresponds to Taylor interpolation, and the incidence matrix with first column all ones and all other entries zero corresponds to Lagrange interpolation.

An HB problem is said to be *poised* if an interpolating polynomial exists no matter what values are assigned to the prescribed quantities. A polynomial of any degree is said to *satisfy the homogeneous* HB *problem* if it satisfies the n conditions when the prescribed values are all zero. By reducing the problem of finding the interpolating polynomial to a linear system of n equations in n unknowns, it is easily verified that an HB problem is poised if and only if the only polynomial in π_{n-1} which satisfies the homogeneous HB problem is the zero polynomial.

The incidence matrix of an HB problem does not alone determine whether or not the problem is poised. For a fixed incidence matrix the problem may be poised for some choice of nodes and not poised for another. (See [1] for an example.) Some incidence matrices, on the other hand, do determine poised problems for any choice of nodes; for instance, Lagrange interpolation. In general the determination of whether or not a given incidence matrix always gives rise to a poised problem may be reduced to determining whether some polynomial in $z_1, \cdots, z_k$ has any zeros with all k components distinct. One would like to avoid this cumbersome or impossible task and be able to find out by some simple criterion whether a given incidence matrix always determines a poised problem. Pólya [2] has shown how to do this if just two nodes are involved, and Schoenberg [1] has shown how to do this for a special class of incidence matrices if the nodes are, in addition, required to be real. In this note we give a method for constructing a class of incidence matrices which together with any set of distinct complex nodes constitute a poised HB problem.

The Construction.

LEMMA. *Let a poised* HB *problem with $k\times n$ incidence matrix be given. Then there exists a unique monic polynomial in π_n which satisfies the homogeneous* HB *problem. This polynomial is of degree n.*

Proof: Let E be the $k\times n$ incidence matrix and $z_1, z_2, \cdots, z_k$ the nodes. Since the problem is poised there is a polynomial $p\in\pi_{n-1}$ such that

$$p^{(j-1)}(z_i) = \frac{d^{j-1}}{dx^{j-1}}(x^n)\Big|_{x=z_i} \qquad \text{if } \epsilon_{ij} = 1.$$

The polynomial $q(x)=x^n-p(x)$ is the desired monic polynomial. Uniqueness follows from the fact that if two polynomials satisfy the homogeneous HB problem, so does their difference.

Now let q be the polynomial whose existence we have just proved, so $q^{(j-1)}$ has $n-j+1$ zeros in the complex plane. The number of zeros of $q^{(j-1)}$ among the nodes $z_1, \cdots, z_k$ is not easily determined, but a lower bound for this number is just the total number of ones appearing in horizontal strings of consecutive ones with the leftmost one in the jth column of the incidence matrix. This number is $\leqq n-j+1$. If it is just $n-j+1$ we say that the jth column of E is *effective*. For example, let

$$E = \begin{bmatrix} 1 & \underline{1} & \underline{1} & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 0 \end{bmatrix}.$$

The second column of E is effective. The underlined ones are those that are to be counted in determining this.

Now suppose we are given a $k\times n$ incidence matrix E which gives a poised problem for any choice of k nodes. We describe two ways in which we can get a new incidence matrix having this same property.

1. Adjoin to E an $(n+1)$-th column consisting entirely of zeros. In this new matrix change some zero to one so that some column of the new matrix is effective.

2. Adjoin to E an $(n+1)$-th column consisting entirely of zeros and insert anywhere a row of $n+1$ zeros. Change some zero in the new row to one so that some column becomes effective.

We remark that these constructions are always possible because the last column of an incidence matrix is effective if it contains a one, hence we can always make the last column effective by inserting a one in it. If E has an effective column we can choose another way to make our new matrix have an effective column.

Now we show that these two procedures yield an incidence matrix with the stated property. Let k or $k+1$ nodes, corresponding to carrying out steps 1 or 2 above, be chosen arbitrarily and suppose that HB problem with these nodes and the newly constructed $n+1$ column incidence matrix is not poised. Then there exists a polynomial $q\in\pi_n$, $q\neq 0$, which satisfies the new homogeneous HB problem. Then q also satisfies the homogeneous HB problem with incidence matrix E and some set of nodes; by the lemma therefore, q is of degree n, so $q^{(j-1)}$ is of degree $n-j+1$ for $j=1, 2, \cdots, n$. For some j the jth column of the new incidence matrix is effective; therefore for this column $q^{(j-1)}$ has $(n+1)-j+1 > n-j+1$ zeros, which is impossible.

Examples. Starting with the 1×1 incidence matrix (1) we may by repetition of step 1 arrive at the incidence matrix $(1, 1, 1, \cdots, 1)$; i.e., Taylor interpolation. Starting again with (1) and applying repeatedly step 2, always inserting ones in the first column, we get the incidence matrix corresponding to Lagrange interpolation. Starting with the incidence matrix for Lagrange interpolation and performing step 1 repeatedly, we can arrive at any incidence matrix for a Hermite interpolation problem. Similarly we can get Abel-Gontscharoff interpolation and Lidstone interpolation. (See [3], page 28, for definition of these terms.) Finally, we give an example of a "nonstandard" interpolation problem. Starting with the incidence matrix (1), we get from step 2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

from step 2 again

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and finally from step 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

These last two represent interpolation problems which are not obviously poised.

In conclusion, we make some observations on the set of poised problems which may be obtained by this construction, starting say with the 1×1 incidence matrix (1). First, any incidence matrix we obtain has an effective column; however, there exist problems which are poised for every choice of nodes but whose incidence matrices have no effective column. An example of this is the incidence matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

An HB problem with this incidence matrix is poised by the result of Pólya [2] or its generalization by Schoenberg [1]. (These results are stated for real nodes, but in the case of two nodes a linear change of variable shows that the results also hold for complex nodes.) Yet this incidence matrix has no effective column. Finally we note that there exist incidence matrices with effective columns which give rise to poised problems for any choice of nodes and which cannot be obtained from the above construction. As an example take the incidence matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By the result cited above, this represents a poised problem and the last column is effective. Since it is the only effective column, it follows that if this matrix were obtained from our construction, the incidence matrix from which it is obtained would be just this matrix with the last column deleted; but the last described matrix has no effective column, hence can not be obtained by our construction.

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MATHEMATICAL NOTES

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COMPLEMENTS AND COMMENTS

DAVID DRASIN, Purdue University

The Editor receives much correspondence about articles which have appeared in the Notes sections, and wishes to initiate an annual review based on them.

H. R. Krall, in the Nov. 1960 issue, pp. 876–878, derived the general self-adjoint differential expression of order $2n$, in terms of Bernoulli numbers. Harald K. Wimmer writes that an alternative form is

$$L(y) = \sum_{k=0}^{2m} \sum_{i=0}^{m} \binom{m-i}{k-2i} P_i^{(k-2i)} y^{(2m-k)},$$

where $P_0, \cdots, P_m$ have sufficient differentiability properties to make the above expression meaningful. He states that this representation is an immediate consequence of a theorem in M. A. Neumark's *Lineare Differentialoperatoren* (1960), p. 8, which proves that every self-adjoint differential expression with real coefficients is of the form

$$L(y) = (P_0 y^{(m)})^m + (P_1 y^{(m-1)})^{m-1} + \cdots P_m y.$$

Two readers have commented on A. Waksman's article "On the distribution of primes," which appeared on pp. 764–765 of the August–September 1968 issue. The problem considered is seen to be:

Find all integers K such that all integers less than K and relatively prime to K are primes.

Stated thus, the problem is solved in Rademacher and Toeplitz, *The Enjoyment of Mathematics* (1957), p. 187, or in Uspensky and Heaslet's *Elementary Number Theory* (1939), p. 89, as noted by Glenn Engebretsen and Paul Catlin, respectively.

Hugh N. Edgar comments that a method suggested in Problem 1, p. 169 of Borevich and Shatarevich's *Number Theory* (1966) allows further examples of non-Euclidean subdomains of Euclidean domains. More precisely, if K is an algebraic number field, $[K]$ the integral domain of algebraic integers of K, and θ an order of K which is properly contained in $[K]$ then θ not only fails to be a Euclidean subdomain of $[K]$, but even fails to be a unique factorization domain of $[K]$. Of course, $[K]$ can be chosen to be a Euclidean domain. These comments relate to the problem posed by R. J. Arpaia in the October 1968 issue, pp. 864–865.

SOME IRREDUCIBLE POLYNOMIALS WHICH ARE REDUCIBLE MOD p FOR ALL p

M. A. LEE, University of Texas

While settling a conjecture of Chowla, Ankeny, and Rogers [1] produce a class of polynomials $f(x)$ with rational integer coefficients with the property that $f(x)$ has a zero mod p for all primes p but $f(x)$ has no integral zero. The object of this note is to prove the following simple theorem which provides a class of polynomials which are reducible mod p for all p but which are irreducible over the integers.

THEOREM. *Let a be a squarefree rational integer* $\neq 1$ *or* -1. *Then the polynomial* $x^4+2(1-a)x^2+(1+a)^2$ *is irreducible over the rational integers, but reducible* mod p *for every prime* p.

Proof. Let $f(x)=x^4+2(1-a)x^2+(1+a)^2$. The element $\sqrt{-1}+\sqrt{a}$ is a root of $f(x)$ and it is easy to see that $\sqrt{-1}+\sqrt{a}$ is a primitive element for the fourth degree extension $Q(\sqrt{-1}, \sqrt{a})$ where $Q=$ the rationals. Hence $f(x)$ is irreducible over Q.

Modulo 2, the polynomial $f(x)$ is equivalent to either x^4+1 or x^4, which are reducible. If p is a prime divisor of a, then $f(x)\equiv x^4+2x^2+1\equiv(x^2+1)^2 \pmod p$.

Now let p be odd and suppose $(a/p)=1$. Choose b such that $b^2\equiv 4a \pmod p$. Then $f(x)\equiv(x^2-bx+(1+a))(x^2+bx+(1+a)) \pmod p$.

Now suppose $(a/p)=-1$. If $(-1/p)=1$, choose b such that $b^2\equiv -4 \pmod p$. Then $f(x)\equiv(x^2-bx-(1+a))(x^2+bx-(1+a)) \pmod p$. If $(-1/p)=-1$ then $(-a/p)=1$. In this case $f(x)\equiv(x^2+(1-a)+2b)(x^2+(1-a)-2b) \pmod p$, where b satisfies $b^2\equiv -a \pmod p$.

Reference

1. N. C. Ankeny and C. A. Rogers, A conjecture of Chowla, Ann. Math., 53 (1951) 541–550.

CONTINUOUS FUNCTIONS AND SPACES IN WHICH COMPACT SETS ARE CLOSED

J. E. JOSEPH, Howard University

It is the purpose of this note to characterize those compact spaces in which compact subsets are closed and to improve the following important theorem for Hausdorff spaces: *Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism.* The result follows:

THEOREM. *In a compact space X every compact subset is closed if and only if every continuous bijection from a compact space to X is a homeomorphism.*

Proof. Suppose every compact subset of X is closed. Let g be a continuous bijection from a compact space Y to X and $A\subset Y$ be closed. Then A is compact,

so $g(A)$ is compact and thus closed in X. So g is a homeomorphism. Now, suppose every continuous bijection from a compact space to X is a homeomorphism. Let A be a compact subset of $\{X, T\}$. Then, if Q is the supremum of T and $\{X, X-A, \phi\}$, it is easy to show that $Q=\{V \cup W \cap (X-A): V, W \in T\}$ and (X, Q) is compact. Since $T \subset Q$, the identity function from (X, Q) to (X, T) is a continuous bijection and thus a homeomorphism. Consequently, since A is Q-closed in X, A is T-closed in X.

COROLLARY. *Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof. Compact subsets of a Hausdorff space are closed.

Finally, we remark that a compact space may fail to be Hausdorff even though all compact subsets are closed. One example is the one point compactification of the rationals. Thus compact Hausdorff is not characterized by the condition that all continuous bijections to the space from a compact space are homeomorphisms.

Reference

1. J. L. Kelly, General Topology, Van Nostrand, Princeton, 1955.

A CONJECTURE ON CONSECUTIVE COMPOSITE NUMBERS

C. A. GRIMM, South Dakota School of Mines and Technology

If we examine a sequence of consecutive composite numbers, for example

$$24, 25, 26, 27, 28, \tag{1}$$

we notice that a different prime can be factored from each number in the sequence: 2, 5, 13, 3, and 7. For the sequence

$$32, 33, 34, 35, 36, \tag{2}$$

we factor out 2, 11, 17, 7, 3. The factoring in the sequence (2) was accomplished by factoring out the largest prime divisor of each number which is not possible in (1) due to the numbers 24 and 27. Consider one other example, the sequence

$$1802,\ 1803,\ 1804,\ 1805,\ 1806,\ 1807,\ 1808,\ 1809,\ 1810. \tag{3}$$

From these composites we may factor the primes 53, 601, 41, 19, 43, 139, 113, 67, and 181. The factoring in (3) was also accomplished by factoring out the largest prime divisor of each number. In this case notice that the largest prime divisor is larger than the length of the sequence so that there is no question of it being a divisor of any other number in the sequence. On the strength of these three examples, together with the factoring of any other sequence of consecutive composites which the author has examined, we make the following

CONJECTURE. *Given n consecutive composite numbers $C+1, C+2, \cdots, C+n$*

so $g(A)$ is compact and thus closed in X. So g is a homeomorphism. Now, suppose every continuous bijection from a compact space to X is a homeomorphism. Let A be a compact subset of $\{X, T\}$. Then, if Q is the supremum of T and $\{X, X-A, \phi\}$, it is easy to show that $Q=\{V\cup W\cap(X-A): V, W\in T\}$ and (X, Q) is compact. Since $T\subset Q$, the identity function from (X, Q) to (X, T) is a continuous bijection and thus a homeomorphism. Consequently, since A is Q-closed in X, A is T-closed in X.

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CONJECTURE. *Given n consecutive composite numbers* $C+1, C+2, \cdots, C+n$

$$P_{1k}^{\alpha_{1k}} P_{2k}^{\alpha_{2k}} \cdots P_{jk}^{\alpha_{jk}} = C + k > C > n^{n-1}.$$

Since there are at most $n-1$ factors, $P_{ik}^{\alpha_{ik}}$, at least one of them must be $>n$. Thus for each $C+k$ with fewer than n prime factors we can choose at least one factor $P_{ik}^{\alpha_{ik}}>n$. If it were possible to choose the same prime power in two cases, say for $C+k$ and $C+j$ with $k>j$, then the smaller of the two powers, $P^{\lambda}>n$, would divide the difference $(C+k)-(C+j)=k-j<n$, an impossibility. Thus a different prime can be associated with each number, $C+k$, in all cases and the proof is complete.

The conjecture, if true, implies that it always requires at least n distinct primes to form n consecutive composites.

In conclusion it might be remarked that there is some heuristic evidence in favor of the conjecture. This is obtained by taking into account one of the estimates for the distance between consecutive primes along with the fact that most numbers have few distinct prime power divisors. From this we attempt to argue that in general the largest prime factor exceeds the length of the sequence for $n>N$, but the author has not been able to work out a complete proof.

Reference

1. P. Erdös and P. Turán, On a problem in the elementary theory of numbers, this MONTHLY, 41 (1934) 608–611.

RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

CAN THE COMPLETE GRAPH WITH $2n+1$ VERTICES BE PACKED WITH COPIES OF AN ARBITRARY TREE HAVING n EDGES?

R. A. DUKE, University of Washington

By the term *graph* we shall mean a finite set of points called vertices together with a collection of *edges* each of which joins two distinct vertices, its endpoints, and no two of which join the same pair of vertices. A *path P* in a graph G is an alternating sequence of vertices and edges of G such that each edge of P is directly preceded in P by one of its endpoints and followed by the other. A *tree* is a graph T such that for each two distinct vertices U and V of T there is a path beginning with U and ending at V, while no path in T having distinct edges begins and ends at the same vertex. The *complete graph* G_p has p vertices, each two of which are joined by an edge.

$$P_{1k}^{\alpha_{1k}} P_{2k}^{\alpha_{2k}} \cdots P_{jk}^{\alpha_{jk}} = C + k > C > n^{n-1}.$$

Since there are at most $n-1$ factors, $P_{ik}^{\alpha_{ik}}$, at least one of them must be $>n$. Thus for each $C+k$ with fewer than n prime factors we can choose at least one factor $P_{ik}^{\alpha_{ik}}>n$. If it were possible to choose the same prime power in two cases, say for $C+k$ and $C+j$ with $k>j$, then the smaller of the two powers, $P^{\lambda}>n$, would divide the difference $(C+k)-(C+j)=k-j<n$, an impossibility. Thus a different prime can be associated with each number, $C+k$, in all cases and the proof is complete.

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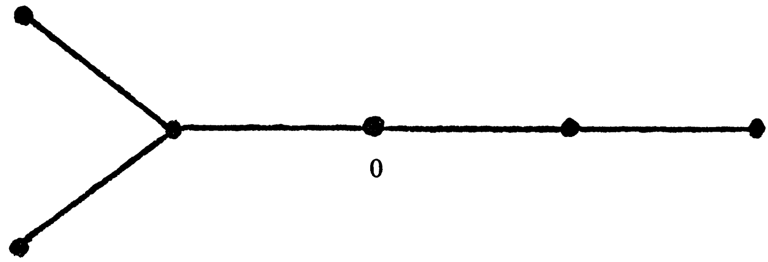

which trees such a preassignment is possible is not known. Again, several special cases have been investigated, some by mathematicians at the RAND Corporation with the aid of a new programming language. There are no examples known in which a vertex adjacent to an end vertex cannot be assigned zero.

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CLASSROOM NOTES

EDITED BY DAVID DRASIN

Manuscripts for this Department should be sent to David Drasin, Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

ON SELECTING SEPARATED OBJECTS FROM A ROW

H. D. ABRAMSON, University of St. Andrews, Scotland

The proof normally given that

$$\binom{n-k+1}{k}$$

is the number of ways of selecting k objects, no two of them consecutive, from n objects arranged in a row is the recurrence argument originally given by Kaplansky and reproduced in both Riordan's and Ryser's books on combinatorial mathematics. This simple combinatorial fact may also be verified as a special case (for $d=2$) of the following fact.

The number of ways of selecting k objects, no two less than d apart, $2 \leqq d$, from n objects arranged in a row is

$$\binom{n-(d-1)(k-1)}{k}.$$

This may be proved quite easily and directly. Let the n objects be the integers $1, 2, \cdots, n$. A selection of the desired type $\{a_1, a_2, \cdots, a_k\}$ then satisfies $1 \leqq a_1$ and $a_{j-1}+d \leqq a_j$ for $j=2, \cdots, k$. Such a selection is in one-one correspondence with the following k-combination of the integers $1, \cdots, n-(d-1)(k-1)$:

$$\{a_1, a_2-(d-1), a_3-(d-1)2, \cdots, a_k-(d-1)(k-1)\}.$$

The number of the latter combinations is of course

$$\binom{n-(d-1)(k-1)}{k},$$

completing the proof.

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1. I. Kaplansky, Solution of the problème des ménages, Bull. Amer. Math. Soc., 49 (1943).
2. John Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958.
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SOME APPLICATIONS OF A MORPHISM

Richard Singer, Webster College, St. Louis

The beginning student in abstract algebra is introduced to a wide variety of concepts and terminology which he may have difficulty in relating to his previous mathematical knowledge. This paper uses some of these concepts to prove two theorems about polynomials with integer coefficients. While these theorems were originally proved without these concepts, more concise proofs can be given by using them. Both proofs use the natural morphism $Z[X]$ onto $Z_p[X]$ to eliminate extraneous information. The proofs also use the fact that when p is prime, $Z_p[X]$ is an integral domain and in particular a unique factorization domain.

The system of integers will be denoted by Z, the system of integers modulo p by Z_p.

Theorem (Gauss). *Let $f, g \in Z[X]$. If f and g are primitive then fg is primitive.*

Proof. If fg is not primitive, then there is a prime p in Z such that $p|fg$. Letting α be the morphism from $Z[X]$ onto $Z_p[X]$ with kernel (p) it follows that

$$(\alpha f)(\alpha g) = \alpha(fg) = 0,$$

and since $Z_p[X]$ is an integral domain either $\alpha f=0$ or $\alpha g=0$. Thus either $p|f$ or $p|g$, contradicting the hypothesis that f and g are primitive.

One application of this theorem, which should relate to the student's previous mathematical experience, is to show that if a polynomial with integral coefficients cannot be factored into polynomials of lower degree with integer coefficients then a factorization into polynomials of lower degree with rational coefficients is also impossible.

THEOREM (Eisenstein). *Let* $f=a_0+\cdots+a_nX^n\in Z[X]$. *If there exists a prime* $p\in Z$ *such that* p *is not a factor of* a_n, $p|a_i$ *for all* $i<n$, *and* p^2 *is not a factor of* a_0, *then* f *is irreducible in* $Z[X]$.

Proof. Suppose $f=gh$, where $g=b_0+\cdots b_jX^j$, and $h=c_0+\cdots+c_kX^k$, with j and k greater than 1. Let α be the morphism from $Z[X]$ to $Z_p[X]$ with kernel (p) then αa_n, αb_j and αc_k are not zero and,

$$(\alpha g)(\alpha h) = \alpha f = (\alpha a_n)X^n.$$

Since $Z_p[X]$ is a *UFD*, $X|\alpha(g)$ and $X|\alpha(h)$. Thus $\alpha(b_0)=0=\alpha(c_0)$, and we have $p^2|b_0\cdot c_0=a_0$.

MATHEMATICAL EDUCATION

EDITED BY J. G. HARVEY AND M. W. POWNALL

Material for this Department should be sent to either of the editors: J. G. Harvey, Department of Mathematics, University of Wisconsin, Madison, WI 53706; M. W. Pownall, Department of Mathematics, Colgate University, Hamilton, NY 13346.

THE SECONDARY SCHOOL MATHEMATICS CURRICULUM IMPROVEMENT STUDY

H. F. FEHR, Teachers College, Columbia University and
JAMES FEY, University of Maryland

The Secondary School Mathematics Curriculum Improvement Study (SSMCIS) is in the fourth year of a proposed six year program of development and experimentation designed to produce a new unified secondary school mathematics curriculum for use by college-capable students. This is a brief report of the objectives, activities, and plans of the project.

Background. During the past decade, the United States has been engaged in revising the elementary and secondary school mathematics curriculum—primarily by up-dating the existing traditional curriculum. Modest recommendations by the CEEB Commission on Mathematics have been largely accepted by syllabus bodies and by writers of commercially produced textbooks. Implementation of this program by the SMSG has had wide acceptance and experimental use.

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Throughout our reform movements, the traditional division of school math-

ematics instruction into separate years of arithmetic, algebra, and geometry has been maintained. Beyond token introduction of new concepts and some rearrangement of the sequence of courses, little has been gained in bringing more advanced study into the high school through more efficient methods of organizing the subject matter. Recently, bolder and more radical recommendations for the improvement of secondary school education in mathematics have been made—both in this country and in Europe, notably in Belgium, Switzerland, and Denmark.

What has been called for is reconstruction of the entire curriculum from a global point of view eliminating the barriers separating the traditional branches of mathematics and unifying the subject through study of its fundamental concepts (sets, relations, operations, mappings) and structures (groups, rings, fields, and vector spaces). Such a curriculum would reflect the spirit of contemporary mathematics as well as permit introduction into the school program of much that was previously considered undergraduate mathematics.

In September, 1965, the Office of Education approved support of the SSMCIS, an experimental study whose objective would be the construction of a unified school mathematics curriculum for grades seven through twelve. In July, 1969, the National Science Foundation approved support for continuation of the study into the senior high school. The project is located at Teachers College, Columbia University.

Design of the curriculum. Long range planning of the proposed six year study was begun at a meeting of chief consultants in November 1965. The conferees outlined procedures for subsequent syllabus conferences, writing of experimental textbooks, teacher preparation, and pilot testing. In June 1966, a group of eighteen leading United States and European mathematicians and educators met for twenty days to outline the scope and sequence of a six year unified school mathematics program. Then specific recommendations for the mathematical content of the first course (seventh grade) were given to a team of eight mathematical educators (all with secondary school teaching experience) who wrote this first experimental course during July and August.

In subsequent years, shorter June syllabus conferences have considered revisions of previously written material and made specific recommendations for the content and outline of new courses—one each year. Eight week writing sessions followed these planning conferences.

Structure of the emerging curriculum. That formal mathematics can be organized in terms of the fundamental concepts of sets, relations, functions, and operations and structures such as groups, rings, fields, vector spaces, lattices, etc., was well known at the outset of the study—having been established by work in foundations at the turn of the century and by the Bourbaki analysis begun in the 1930's. What was not known, was how this organization could be presented in teachable form to secondary school students. Guidelines for such a development were available in the form of recent experimentation and reports

of syllabus conferences in Europe (for example, *Synopses for Modern Secondary School Mathematics*), but nowhere had a total 7–12 unified mathematics program been designed, produced, and tested.

The program that has emerged from successive syllabus conferences and writing sessions in SSMCIS has a kind of helical organization in which the abstract concepts and structures develop in coordination with the most important realizations of these structures—the number systems, geometry, probability, and analysis. The basic concepts and structures are introduced in an informal intuitive way to seventh graders and then developed with increasing depth and formality as need arises in later study.

This spiral development in which learning of concepts and structures arises from, and contributes to, learning of examples of these ideas, is evident in the organization of the first three completed courses. For instance, Course I begins with an investigation of finite number systems—comparing and contrasting their properties with those of the familiar whole numbers. Experience in these simple concrete situations is preparation for the next unit which examines the general concept of a binary operation and its properties.

From operational systems (groupoids), in what might be called the abstract strand, the study returns to an important example of a group—the integers under addition. This algebraic strand of course proceeds later to units on multiplication of integers, the rational number system, groups, fields, the real number system, matrices, and an introduction to the idea of vector space—the last two topics entering early in the ninth grade (Course III).

By making use of coordinates, vectors, and transformations (and synthetic methods), the study of geometry is successfully integrated into the total program. A unit on mappings appears early in Course I, and this concept is made central to later study of transformations in the plane—where the group concept, applied earlier in algebra, is also used. Coordinates are introduced in an early unit on lattice points in the plane and then used extensively in later study of affine coordinate geometry in the plane and in space. By the ninth grade (Course III), the notions of vector and vector space are sufficiently developed to begin study of euclidean space as a three dimensional vector space with inner product.

These illustrations convey the spirit of global organization that is at the heart of the SSMCIS curriculum—important mathematical systems unified by a core of fundamental concepts and structures common to all. A more complete picture of the scope and sequence of Courses I–IV emerges from the list of chapter titles appended to this paper. However, two points of special importance merit further explanation—the roles of formal logic and applications in this new school mathematics program.

Although abstract concepts and structures form the core of the total program, teaching of these concepts is begun at an intuitive, manipulative level in Course I, then broadened and formalized gradually as needed in succeeding courses. The first discussion of logical issues occurs at the beginning of Course II. It consists of an informal introduction to linguistic usage (connectives,

quantifiers, etc.) and proof strategy sufficient to *begin* formal work in group theory and affine geometry. Deeper and more formal examination of logic will be undertaken only when the need arises out of basic problems in mathematics.

Applications are not the major motivation or focus of the mathematical development in Courses I–IV. Teaching techniques for solving specific traditional problems is waived in favor of preparation for three important aspects of future applied mathematics:

1. Thorough understanding of (a) the real number system and its role in measurement situations, (b) the geometry of space, and (c) probabilistic thinking and its role in modelling chance situations.
2. Appreciation of axiomatic mathematical systems as models of physical systems.
3. Facility in using a time sharing computer and numerical methods as tools in solving mathematical problems.

With these conceptual and technical tools available, Courses V–VI will focus more heavily on using mathematical methods to solve "real problems" in areas as diverse as game theory, linear programming, statistical experimentation, biology, and physics.

Teacher preparation and pilot testing. The objective of the project is a teachable school mathematics program. An important phase in the development activities has been classroom testing of the written text materials.

In 1966, nine junior high schools in the Metropolitan New York area and one in Carbondale, Illinois, were selected to participate in experimental teaching. In each school, two teachers who had received special summer instruction at Teachers College were assigned to teach a single pilot class of college capable seventh graders (upper 15–20% in mathematical ability). Then, in succeeding summers, the teachers returned for preparation to teach the forthcoming experimental course, at the same time using their classroom experience to assist in preparation of detailed teacher commentaries for previously tested courses. At the end of the third summer program a report was written outlining the mathematical preparation found essential as background for any teacher who would implement the junior high school phase of the new curriculum.

The pilot teaching has been evaluated in three ways: (1) Proximity of most classes to Teachers College allows frequent personal visits by project staff members to observe classes in action. (2) The correctness, teachability, and appeal of text materials are criticized at periodic conferences involving the teachers and project consultants. (3) Student achievement is measured by special tests designed to probe understanding of important new topics as well as mastery of indispensable traditional topics. This evaluation has so far shown that capable students can learn and enjoy studying many of the new topics without appreciable decline in achievement on standardized measures of mathematical progress.

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1. Goals for School Mathematics, Report of the Cambridge Conference on School Mathematics, Houghton-Mifflin, Boston, 1963.
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5. Unified Modern Mathematics, Courses I, II, III, IV, SSMCIS, Teachers College Press, New York, 1969.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N.J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before March 31, 1970. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2202. *Proposed by Necdet Üçoluk, Clarion (Pa.) State College*

Since a binary relation on a set A is a subset of $A \times A$, every subset of the Euclidean plane E_2 is a binary relation on the reals. Define the *transitive span* of a relation ρ to be the smallest transitive relation containing ρ. (a) Determine the transitive span of the relation ρ, consisting of all points interior to or on the triangle with vertices (3, 5), (2, 7), and (6, 0). (b) What if the region is the set bounded by the ellipse $(x-1)^2+4y^2=16$?

References

1. Goals for School Mathematics, Report of the Cambridge Conference on School Mathematics, Houghton-Mifflin, Boston, 1963.

2. Erik Kristensen and Ole Rindung, *Matematik I, II, III*, G. E. C. Gads Forlag, Copenhagen, 1964.

3. G. Papy, Mathématique Moderne, Didier, Paris, 1965.

4. Synopses for Modern Secondary School Mathematics, Organization for Economic Cooperation and Development, Paris, 1961.

5. Unified Modern Mathematics, Courses I, II, III, IV, SSMCIS, Teachers College Press, New York, 1969.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, N.J. 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before March 31, 1970. Contributors (in the United States) who desire acknowledgement of receipt of their solutions are asked to enclose self-addressed stamped postcards.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2202. *Proposed by Necdet Üçoluk, Clarion (Pa.) State College*

Since a binary relation on a set A is a subset of $A \times A$, every subset of the Euclidean plane E_2 is a binary relation on the reals. Define the *transitive span* of a relation ρ to be the smallest transitive relation containing ρ. (a) Determine the transitive span of the relation ρ, consisting of all points interior to or on the triangle with vertices (3, 5), (2, 7), and (6, 0). (b) What if the region is the set bounded by the ellipse $(x-1)^2+4y^2=16$?

E 2203.* *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

It is known that if $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n$, $(n \geqq 3)$, then

$$x_1^{x_2} x_2^{x_3} \cdots x_n^{x_1} \geqq x_2^{x_1} x_3^{x_2} \cdots x_1^{x_n}.$$

Are there any other nontrivial permutations $\{a_i\}$ and $\{b_i\}$ of the $\{x_i\}$ such that

$$a_1^{a_2} a_2^{a_3} \cdots a_n^{a_1} \geqq b_2^{b_1} b_3^{b_2} \cdots b_1^{b_n}?$$

E 2204. *Proposed by A. C. Segal, University of Alabama, Birmingham, and Basil Lepp, Rust Engineering Co.*

Let $f(n) = \sum_{k=1}^{\infty} 1/k$, where k has no zeros in its n-ary expansion. Prove or disprove: $\lim_{n\to\infty}[f(n) - n \log n] = 0$.

E 2205. *Proposed by S. M. Farber, D. W. Walkup, and R. J. B. Wets, Boeing Scientific Research Laboratories*

Suppose nonnegative integers m and n are given in their representations to a prime base p, i.e.,

$$m = (r_k \cdots r_1 r_0)_p = \sum_{i=0}^{k} r_i p^i, \qquad 0 \leqq r_i < p,$$

$$n = (s_k \cdots s_1 s_0)_p = \sum_{i=0}^{k} s_i p^i, \qquad 0 \leqq s_i < p.$$

Find a simple expression for the binomial coefficient $C(m, n) = \binom{m}{n}$ mod p. In particular find necessary and sufficient conditions for

(a) $\binom{m}{n} \equiv 0 \mod p$, (b) $\binom{2m-1}{m-1}$ odd.

E 2206. *Proposed by R. C. Lyness, Blackpool, England*

Each of three hyperbolas has for its foci a different two of three noncollinear points. Each pair of hyperbolas has a set of six common chords. Show that three pairs of chords, a pair from each set, form the six lines of a quadrangle. Show, further, that the minor axes of the hyperbolas bisect the sides of a triangle whose vertices are three of the four points of the quadrangle.

E 2207. *Proposed by Anon, Erewhon-upon-Wabash*

Suppose $f(x, y)$ vanishes on the boundary of the square S: $0 \leqq x, y \leqq 1$, and that

$$\left| \partial^4 f / \partial x^2 \partial y^2 \right| \leqq B.$$

Prove that

$$\left| \iint_S f(x, y) dx\, dy \right| \leqq \frac{1}{144} B.$$

A Quadratic Diophantine Equation

E 2151 [1969, 187]. *Proposed by E. P. Starke, Plainfield, N. J.*

If a and b are consecutive integers, then $a^2+b^2+(ab)^2$ is always a perfect square. Find other integer pairs having this property. Indeed, show that corresponding to an arbitrary choice of a there are infinitely many values of b such that $a^2+b^2+(ab)^2$ is a square.

Solution by Charles Wexler, Arizona State University. Set up the "Pell" equation $x^2-(a^2+1)y^2=-1$. It has the (smallest) solution $x=a$, $y=1$. Hence

$$(x+y\sqrt{a^2+1})^{2n}(x-y\sqrt{a^2+1})^{2n}=(-1)^{2n}=1,$$

$n=1, 2, 3, \cdots$, gives infinitely many solutions of the companion Pell equation $X^2-(a^2+1)Y^2=+1$, where we take for X_{2n} the rational part of the expansion of $(a+\sqrt{a^2+1})^{2n}$, and for Y_{2n} the coefficient of $\sqrt{a^2+1}$ in this expansion. Hence

$$X_{2n}^2a^2=a^2+(a^2+1)Y_{2n}^2a^2=a^2+(Y_{2n}a)^2+(a\cdot Y_{2n}a)^2.$$

The infinitely many values of b are the values of $Y_{2n}a$, $n=1, 2, \cdots$. These are by no means all the solutions of the problem since $b=a+1$ does not belong to this set.

Also solved by Marcia Ascher, Günter Bach (Germany), Anders Bager (Denmark), W. J. Blundon, Robert Breusch, Ezra Brown, K. H. Byron, L. Carlitz, Mannis Charosh, G. E. Engebretsen, W. F. Fox, Arthur Gittleman, Michael Goldberg, M. G. Greening (Australia), Emil Grosswald, Charles Heuer & Gerald Heuer, J. A. H. Hunter, Bernard Jacobson, Free Jamison, Eleanor G. Jones, R. L. Jow, Edgar Karst, J. F. Leetch, Mary B. Lewin, Alice P. Meyer, D. C. B. Marsh, Norman Miller, A. J. Roques, Bro. Raymond Schnepp, Michael Stolnicki, E. W. Trost (Switzerland), W. G. Wild, Gregory Wulczyn, and the proposer.

A Polygonal Property not Shared by Polyhedra

E 2152 [1969, 188]. *Proposed by H. T. Croft, Peterhouse, Cambridge, England*

(1) Does a given closed planar polygon necessarily contain an edge E and a vertex V such that the foot of the perpendicular from V to E falls within (closed) E?

(2) Does a given closed polyhedron necessarily contain a face F and a vertex V such that the foot of the perpendicular from V to F falls within (closed) F?

Solution by Wayne G. Wild, Wisconsin State University, Stevens Point. (1) The answer is "yes." Consider a pair of perpendiculars drawn from opposite ends of a maximal edge. The remaining boundary of the polygon must cross these perpendiculars at least once and if no vertex is in this (closed) section the maximal edge condition is contradicted.

(2) The answer is "no." The convex polyhedron having vertices $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$ is an example which provides an exception.

Also solved by Michael Goldberg, Charles Heuer & Gerald Heuer, J. R. Kuttler, R. C. Lyndon, and the proposer.

An Efficient Construction

E 2153 [1969, 188]. *Proposed by Michael Warren, Constantine College, Middlesbrough, England*

Given n points in the Euclidean plane. Find a ruler and compass construction which will locate the point such that the maximum distance to any of the n points is minimized.

Solution by Michael Goldberg, Washington, D. C. The problem is the same as finding the center of the smallest circle which encloses all the given points. See Problem E 1866 [1967, 726]. The published solution to Problem E 1866 stated that "the trick is to do it efficiently."

Construct the polygon which is the convex hull of the n given points. In the following procedure, ignore those points which are not vertices of this polygon. Construct the circle which has the longest diagonal of the polygon as its diameter. If this circle contains all the vertices of the polygon, then this is the sought circle.

If it does not contain all the vertices, then construct the circumscribing circles of the triangles made by taking the vertices three at a time. One of these circles contains all the vertices of the polygon. This follows from the fact that if only two vertices lie on the circle, and they are not the ends of a diameter, then the circle can be reduced until it touches a third vertex. The final circle must be a circumscribing circle of one of the triangles. In some cases, other vertices of the polygon may also lie on the circumference.

Also solved by P. M. Berry, Jordi Dou (Spain), Dan Marcus, Simeon Reich (Israel), and the proposer.

Jester Moves on a Double Chessboard

E 2154 [1969, 188]. *Proposed by Marlow Sholander, Case Western Reserve University*

Consider a double chessboard with $2n^2$ unit squares (x, y, z), $1 \leqq x \leqq n$, $1 \leqq y \leqq n$, $z=0$ or 1 (in which each $(x, y, 1)$ is superimposed upon $(x, y, 0)$). A piece called a jester can move (only) as follows:

1) From $(a, b, 0)$ to $(x, y, 1)$ where x is a or $a+1$, y is b or $b+1$ and $x+y > a+b$.
2) From $(a, b, 1)$ to $(x, y, 0)$ where x is a or $a-1$, y is b or $b-1$ and $x+y < a+b$.

Let P_1 be the set of squares on a path of jester moves from the edge $(1, y, 1)$ to the edge $(n, y, 0)$. Let P_2 be the set on a path from edge $(x, 1, 1)$ to edge $(x, n, 0)$. Prove that $P_1 \cap P_2$ is not empty.

Solution by William Fox and Howard Hulen, Moberly (Mo.) Junior College. Suppose the jester starts at $(1, a, 1)$ and ends at $(n, b, 0)$ in traversing the set of squares P_1. Similarly for P_2 suppose the jester starts at $(c, 1, 1)$ and ends at $(d, n, 0)$.

First, we observe that since $(n, b, 0)$ can be reached only from $(n, b+1, 1)$ and that $(d, n, 0)$ can be reached only from $(d+1, n, 1)$, then we have $(n, b+1, 1) \in P_1$ and $(d+1, n, 1) \in P_2$.

Let a_i $(i=1, \cdots, k)$ denote the center of the ith square on the upper board occupied by the jester as it traverses P_1 from $(1, a, 1)$ to $(n, b+1, 1)$. Let b_j $(j=1, \cdots, m)$ denote the center of the jth square occupied by the jester in traversing P_2 from $(c, 1, 1)$ to $(d+1, n, 1)$.

Note (by inspection) that a_i and a_{i+1} are centers of adjacent squares, as are b_j and b_{j+1}. Construct the polynomial paths $A = a_1a_2, a_2a_3, \cdots, a_{k-1}a_k$ and $B = b_1b_2, b_2b_3, \cdots, b_{m-1}b_m$. As a consequence of the Jordan curve theorem, we see that A and B must intersect. Thus for some i, j the segment a_ia_{i+1} intersects the segment b_jb_{j+1}. If this intersection does not occur at an endpoint, then both segments are diagonal. This requires the jester to reach $(x+1, y+1, 1)$ from $(x, y, 1)$, or vice versa, in two moves which is seen to be impossible upon examination of all possible jester moves. Hence, the intersection occurs at an end point. This end point is then the center of a square contained in both P_1 and P_2. Hence $P_1 \cap P_2 \neq \varnothing$. Actually, since the same argument applies to the lower board (flip the board over and run the jester in reverse) $P_1 \cap P_2$ must contain at least two squares.

Also solved by Charles Heuer & Gerald Heuer, James Paggione, and the proposer. Paggione notes neither x nor y can be 1 or n.

A Bound for an Integral

E 2155 [1969, 188]. *Proposed by Anon, Erewhon-upon-Wabash*

Suppose $f(x)$ has a continuous $(2n)$-th derivative on $a \leqq x \leqq b$, that $|f^{(2n)}(x)| \leqq M$, and that $f^{(r)}(a) = f^{(r)}(b) = 0$ for $r = 0, 1, \cdots, n-1$. Show that

$$\left| \int_a^b f(x)dx \right| \leqq \frac{(n!)^2 M}{(2n)!(2n+1)!} \cdot (b-a)^{2n+1}.$$

Solution by Alberto Torchinsky, University of Chicago. Let $f(x)$ be as in the hypothesis and let $g(x) = (x-a)^n(b-x)^n$. Then successive integration by parts yields

$$\int_a^b f^{(2n)}(x)g(x)dx = \int_a^b f(x)g^{(2n)}(x)dx$$
$$= \int_a^b f^{(2n)}(x)(x-a)^n(b-x)^n dx = \int_a^b f(x)(2n)!\, dx.$$

It follows that

$$\left| \int_a^b f(x)dx \right| \leqq \frac{M}{(2n)!} \cdot (b-a)^{2n+1} \cdot \int_0^1 x^n(1-x)^n dx$$
$$= \frac{M}{(2n)!} \cdot (b-a)^{2n+1} \cdot \frac{(n!)(n!)}{(2n+1)!}\,.$$

Also solved by Robert Breusch, Graeme Fairweather (Scotland), D. S. Greenstein, D. K. Kahaner, Simeon Reich (Israel), Steve Rohde, Samuel Schechter, J. S. Shipman, and the proposer.

An Equation Involving Euler's Totient

E 2156 [1969, 188]. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha*

Find necessary and sufficient conditions on m, n in each of the following cases involving $\phi(n)$, Euler's totient function:

(1) $m\phi(n) = n\phi(m)$, (2) $n\phi(n) = m\phi(m)$.

Solution by Simeon Reich, The Technion, Israel Institute of Technology. (1) $m\phi(n) = n\phi(m)$ if and only if the distinct prime divisors of m and those of n are identical. For sufficiency note

$$m\phi(n) = mn \prod_{p|n} (1 - p^{-1}) = nm \prod_{p|m} (1 - p^{-1}) = n\phi(m).$$

To prove necessity, assume that our claim is false, let $\{p_i\}$ be the set of distinct prime divisors of n satisfying $(p_i, m) = 1$, and let $\{q_j\}$ be the set of distinct prime divisors of m satisfying $(q_j, n) = 1$. Now, $m\phi(n) = n\phi(m)$ implies that

$$\prod (1 - p_i^{-1}) = \prod (1 - q_j^{-1}),$$

or $\prod(p_i - 1)\prod q_j = \prod(q_j - 1)\prod p_i$. Because $q_j \nmid \prod p_i$ for every j, and $p_i \nmid \prod q_j$ for every i, we must have

$$\prod q_j \mid \prod (q_j - 1) \quad \text{and} \quad \prod p_i \mid \prod (p_i - 1),$$

an impossible situation.

(2) $n\phi(n) = m\phi(m)$ if and only if $m = n$. The sufficiency is obvious. To prove necessity, let $(m, n) = d$, and let $m = m'd$, $n = n'd$. Let r_i be the distinct prime divisors of d. Then we have

$$n'dn'd\left(\prod (r_i - 1)/\prod r_i\right) \prod_{\substack{p|n \\ p \neq r_i}} ((p_i - 1)/p_i)$$
$$= m'dm'd\left(\prod (r_i - 1)/\prod r_i\right) \prod_{\substack{q|m \\ q \neq r_i}} ((q_i - 1)/q_i),$$

or

$$n'(n'/\prod p_i)\prod (p_i - 1) = m'(m'/\prod q_i)\prod (q_i - 1).$$

Clearly, $n'/\prod p_i$ and $m'/\prod q_i$ are natural numbers. Now $(n', m') = 1$; therefore $n' \mid \prod(q_i - 1)$ and $m' \mid \prod(p_i - 1)$. Thus $m'n' \mid \prod(q_i - 1)\prod(p_i - 1)$. But $m'n' \geqq \prod p_i \prod q_i$. It follows that the p_i and q_i do not exist. Hence $m' = n'$ and $m = n$.

Also solved by Glenn Aston-Reese, Anders Bager (Denmark), D. M. Bloom, A. R. Bolder, Robert Breusch, L. Carlitz, Mannis Charosh, Josef Daneš (Czechoslovakia), G. C. Dodds, H. M. Edgar, G. E. Engebretsen, Charles Vanden Eynden, Neal Felsinger, Ray Glenn, Michael Gold-

berg, Charles Heuer, R. L. Jow, Lew Kowarski, H. S. Lieberman, Douglas Lind (England), P. A. Lindstrom, Graham Lord, D. C. B. Marsh, Arthur Marshall, C. B. A. Peck, Bob Prielipp, E. J. F. Primrose (England), R. Sivaramakrishnan (India), Al Somayajulu, D. P. Sumner, E. W. Trost (Switzerland), and Charles Wexler.

Both parts of the problem follow quite easily from results given (without proof) in Niven and Zuckerman, *An Introduction to the Theory of Numbers*, p. 37, Problems 7, 17, 19.

Coprime Integers in a Set of Consecutive Integers

E 2157 [1969, 300]. *Proposed by D. M. Bloom, Brooklyn College*

Prove that S_n is false for all n such that $17 \leqq n \leqq 1000$, where S_n is the statement: Every set of n consecutive integers contains an integer which is relatively prime to the others in the set.

Solution by Dan Marcus, Harvard University. We construct counterexamples C_n for $17 \leqq n \leqq (5008193)^2$.

Suppose that p and $2p+1$ are primes and $3p+2 \leqq n \leqq p^2$. Let q be the product of all primes less than n, excluding p and $2p+1$. The Chinese Remainder Theorem guarantees the existence of an integer x satisfying

$$x \equiv 0 \pmod{q}, \qquad x \equiv -3p-1 \pmod{p(2p+1)}.$$

Let $C_n = \{x, x+1, \cdots, x+n-1\}$. C_n contains $x+1$ and $x+2p+1$, which have a common factor p. Moreover $x+p$ has the factor $2p+1$ in common with $x+3p+1$, which is also in C_n. Any other member of C_n has some divisor of q in common with x. Thus C_n is a counterexample to S_n.

Taking $p=5$, we obtain C_n for $17 \leqq n \leqq 25$. $p=11$, 29, 251, 1013, 49919, and 5008193 generate C_n for $35 \leqq n \leqq (5008193)^2$.

Finally we handle the cases $26 \leqq n \leqq 34$ separately. Let C_n begin with y satisfying

$$y \equiv 0 \pmod{2 \cdot 5 \cdot 11 \cdot 17}, \qquad y \equiv -1 \pmod{3},$$
$$y \equiv -2 \pmod{7 \cdot 19}, \qquad y \equiv -3 \pmod{13}, \qquad y \equiv -4 \pmod{23}.$$

Note that Brauer (Bull. Amer. Math. Soc. 47 (1941) 328–331) showed that S_n is false for all $n \geqq 17$.

Also solved by G. A. Heuer & C. V. Heuer, Joel Spencer, and the proposer.

Several other readers communicated the reference given above and also S. S. Pillai, *On m consecutive integers*, Proc. Indian Acad. Sciences, Vol XI (1940) 6–12 and 73–80; Vol. XIII (1941) 530–533. The case $n=10$ was the subject of a problem on the 1966 Putnam Prize Competition.

A Diophantine Equation

E 2158 [1969, 300]. *Proposed by Gregory Wulczyn, Bucknell University*

For what integral values of $n>1$ will there be a finite or infinite number of solutions a, b, to the Diophantine equation

$$(1) \qquad 1 \cdot a^2 + 2(a+1)^2 + 3(a+2)^2 + \cdots + n(a+n-1)^2 = b^2.$$

berg, Charles Heuer, R. L. Jow, Lew Kowarski, H. S. Lieberman, Douglas Lind (England), P. A. Lindstrom, Graham Lord, D. C. B. Marsh, Arthur Marshall, C. B. A. Peck, Bob Prielipp, E. J. F. Primrose (England), R. Sivaramakrishnan (India), Al Somayajulu, D. P. Sumner, E. W. Trost (Switzerland), and Charles Wexler.

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Let $C_n = \{x, x+1, \cdots, x+n-1\}$. C_n contains $x+1$ and $x+2p+1$, which have a common factor p. Moreover $x+p$ has the factor $2p+1$ in common with $x+3p+1$, which is also in C_n. Any other member of C_n has some divisor of q in common with x. Thus C_n is a counterexample to S_n.

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$$(1) \qquad 1\cdot a^2 + 2(a+1)^2 + 3(a+2)^2 + \cdots + n(a+n-1)^2 = b^2.$$

Analogous study of the case $n\equiv 0 \pmod 3$ gives a similar result. Hence, in all cases, at least alternate members of the infinite sequence of solutions of (3) give integral values of a, b which satisfy (1). It is noted that for almost all values of $n(\notin N)$ there are yet other solutions of (3) which are not members of the indicated sequence.

One may determine explicitly the set N of those exceptional values of n which satisfy $n(n+1)=2m^2$. In fact with $n_0=1$, $m_0=1$, all n_i and m_i are given by repetitions of

$$n_{i+1} = 3n_i + 4m_i + 1, \qquad m_{i+1} = 2n_i + 3m_i + 1.$$

Also solved by Anders Bager (Denmark), Leon Bankoff, Robert Breusch, Bernard Jacobson, F. W. Saunders, and the proposer.

Several of the submitted solutions were incomplete in that it was not shown that solutions of (3) do indeed lead to infinitely many solutions of (1). J. H. Conway and M. G. Greening stated the answer, all $n>1$, without explanation.

Some New Triangle Inequalities

E 2160 [1969, 300]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let p_i, x_i be the distances of an interior or a boundary point P of a triangle $A_1A_2A_3$ from the vertex A_i and from the side opposite to A_i, $i=1, 2, 3$, with r the inradius. Prove the inequalities

$$\text{(a)} \qquad \sum_{i=1}^{3} p_i(\tfrac{1}{2}\sin A_i) \leqq \sum_{i=1}^{3} x_i \leqq \sum_{i=1}^{3} p_i \sin(\tfrac{1}{2}A_i).$$

$$\text{(b)} \qquad p_2p_3 + p_3p_1 + p_1p_2 \geqq 8x_1x_2x_3/r.$$

Solution by M. G. Greening, University of New South Wales, Australia. Let a_i be the side opposite A_i, let P_i be the angle $A_{i-1}PA_{i+1}$, $B_{i,j}$ be the angle between p_i and a_j at A_i, so that $B_{i,i+1}+B_{i,i-1}=A_i$. (All additions of subscripts are modulo 3.) Then $x_i=p_{i+1}\sin B_{i+1,i}=p_{i-1}\sin B_{i-1,i}$ and

$$\sum_i x_i = \tfrac{1}{2}\sum_j p_j(\sin B_{j,j+1} + \sin B_{j,j-1})$$

$$= \tfrac{1}{2}\sum_j p_j\cdot 2\sin(\tfrac{1}{2}A_j)\cos\tfrac{1}{2}(B_{j,j+1} - B_{j,j-1}).$$

The inequality $0\leqq |B_{j,j+1}-B_{j,j-1}| \leqq A_j$ then yields (a).

As $p_1p_2 \sin P_3 = x_3a_3$, we obtain

$$\sum_i p_ip_{i+1} = \sum_i \frac{x_ia_i}{\sin P_i} \geqq \frac{3\left(\prod_i a_i\cdot \prod_i x_i\right)^{1/3}}{\left(\prod_i \sin P_i\right)^{1/3}}$$

$$\geqq 3\left(\prod_i a_i \prod_i x_i\right)^{1/3}\left(\prod_i \sin P_i\right)^{-1/3} \geqq 2\sqrt{3}\left(\prod_i a_i\cdot \prod_i x_i\right)^{1/3}.$$

The last statement follows from the fact that $\prod_i \sin P_i$ with $\sum_i P_i = 2\pi$ has a maximum when $P_1 = P_2 = P_3$.

For (b) we now show

$$\text{(i)} \qquad 2\sqrt{3}r\left(\prod_i a_i\right)^{1/3} \geqq 8\left(\prod_i x_i^2\right)^{1/3}.$$

As $\sum_i x_i = 2\Delta$, $\prod_i x_i$ has a maximum when $a_1x_1 = a_2x_2 = a_3x_3 = 2\Delta/3$, so that max $8(\prod_i x_i^2)^{1/3} = 2^5\Delta^2 3^{-2}(\prod_i a_i)^{-2/3}$. (i) will follow if $3^{5/2}r\cdot \prod_i a^i \geqq 2^4\Delta^2$, or

$$\text{(ii)} \qquad 3^{5/2}R \geqq 4s,$$

as $\prod_i a_i = 4R\Delta$, where R is the circumradius. But the triangle of largest perimeter which can be inscribed in a given circle is equilateral and the inequality (ii) is certainly true then, so that (b) is established. In fact, 8 could be replaced by 12 in (b).

Also solved by Simeon Reich (Israel), T. Tamura (Japan), C. S. Venkataraman (India), A. W. Walker and the proposer.

The improved inequality for part (b) was conjectured by Walker and proved by Reich. It is interesting to note that aside from a solution to part (a) by L. Carlitz, all solvers and the proposer reside outside the United States of America.

Determinant of Binomial Coefficients

E 2161 [1969, 301]. *Proposed by T. Kaucký, Slovak Academy of Sciences, Bratislava, Czechoslovakia*

Let α be an arbitrary number. Evaluate the determinant

$$D(\alpha, s) = \begin{vmatrix} 0 & 0 & \cdots 0 & 1 & \binom{\alpha+1}{1} \\ 0 & 0 & \cdots 1 & \binom{\alpha+2}{1} & \binom{\alpha+2}{2} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \binom{\alpha+s-1}{1} & \cdots & \binom{\alpha+s-1}{s-2} & \binom{\alpha+s-1}{s-1} \\ \binom{\alpha+s}{1} & \binom{\alpha+s}{2} & \cdots & \binom{\alpha+s}{s-1} & \binom{\alpha+s}{s} \end{vmatrix}.$$

Solution by L. Carlitz, Duke University. Subtract the $(s-1)$th row of $D(\alpha, s)$ from the sth row, then subtract the $(s-2)$th from the $(s-1)$th, and so on, finally the first row from the second. We get

$$D(\alpha, s) = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & \alpha+1 \\ 0 & 0 & \cdots & 1 & \alpha+1 & \binom{\alpha+1}{2} \\ 0 & 0 & \cdots & \alpha+2 & \binom{\alpha+2}{2} & \binom{\alpha+2}{3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha+s-2 & \cdots & & \binom{\alpha+s-2}{s-2} & \binom{\alpha+s-2}{s-1} \\ \alpha+s-1 & \binom{\alpha+s-1}{2} & \cdots & & \binom{\alpha+s-1}{s-1} & \binom{\alpha+s-1}{s} \end{vmatrix}.$$

Expanding by the first row, we find

$$D(\alpha, s) = (-1)^{s-1}(\alpha + 1)D(\alpha, s - 1) + (-1)^{s-2}D(A),$$

where A is the minor of 1 in the first row. Now if we construct $D(\alpha-1, s)$ and expand it by the first row, we find

$$D(\alpha - 1, s) = (-1)^{s-1}(\alpha)D(\alpha, s - 1) + (-1)^{s-2}D(A).$$

It follows that

$$(*) \qquad D(\alpha, s) = D(\alpha - 1, s) + (-1)^{s-1}D(\alpha, s - 1).$$

Now $D(\alpha, 1) = \alpha+1$, $D(\alpha, 2) = -\binom{\alpha+2}{2}$. We shall prove by induction that

$$(**) \qquad D(\alpha, s) = (-1)^{\binom{s}{2}}\binom{\alpha + s}{s}.$$

Clearly (**) holds for $s=1, 2$. Assuming it holds up to the value $s-1$, we have, by (*),

$$D(\alpha, s) = (-1)^{\binom{s}{2}}\binom{\alpha + s - 1}{s} + (-1)^{\binom{s-1}{2}+s-1}\binom{\alpha + s - 1}{s - 1}$$

$$= (-1)^{\binom{s}{2}}\left[\binom{\alpha + s - 1}{s} + \binom{\alpha + s - 1}{s - 1}\right] = (-1)^{\binom{s}{2}}\binom{\alpha + s}{s}.$$

Also solved by M. T. Bird, Robert Breusch, Arnold Hammel, T. L. Markham, M. Stieglitz (Germany), E. Szekeres (Australia), and the proposer.

A Necessary Condition for Perfect Numbers

E 2162 [1969, 301]. *Proposed by D. Rameswar Rao, Osmania University, India*

Let p_i be distinct prime numbers. Show that

$$D(\alpha, s) = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & \alpha+1 \\ 0 & 0 & \cdots & 1 & \alpha+1 & \binom{\alpha+1}{2} \\ 0 & 0 & \cdots & \alpha+2 & \binom{\alpha+2}{2} & \binom{\alpha+2}{3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha+s-2 & \cdots & & \binom{\alpha+s-2}{s-2} & \binom{\alpha+s-2}{s-1} \\ \alpha+s-1 & \binom{\alpha+s-1}{2} & \cdots & & \binom{\alpha+s-1}{s-1} & \binom{\alpha+s-1}{s} \end{vmatrix}.$$

Expanding by the first row, we find

$$D(\alpha, s) = (-1)^{s-1}(\alpha+1)D(\alpha, s-1) + (-1)^{s-2}D(A),$$

where A is the minor of 1 in the first row. Now if we construct $D(\alpha-1, s)$ and expand it by the first row, we find

$$D(\alpha-1, s) = (-1)^{s-1}(\alpha)D(\alpha, s-1) + (-1)^{s-2}D(A).$$

It follows that

$$(*) \qquad D(\alpha, s) = D(\alpha-1, s) + (-1)^{s-1}D(\alpha, s-1).$$

Now $D(\alpha, 1) = \alpha+1$, $D(\alpha, 2) = -\binom{\alpha+2}{2}$. We shall prove by induction that

$$(**) \qquad D(\alpha, s) = (-1)^{\binom{s}{2}}\binom{\alpha+s}{s}.$$

Clearly (**) holds for $s=1, 2$. Assuming it holds up to the value $s-1$, we have, by (*),

$$\begin{aligned} D(\alpha, s) &= (-1)^{\binom{s}{2}}\binom{\alpha+s-1}{s} + (-1)^{\binom{s-1}{2}+s-1}\binom{\alpha+s-1}{s-1} \\ &= (-1)^{\binom{s}{2}}\left[\binom{\alpha+s-1}{s} + \binom{\alpha+s-1}{s-1}\right] = (-1)^{\binom{s}{2}}\binom{\alpha+s}{s}. \end{aligned}$$

Also solved by M. T. Bird, Robert Breusch, Arnold Hammel, T. L. Markham, M. Stieglitz (Germany), E. Szekeres (Australia), and the proposer.

A Necessary Condition for Perfect Numbers

E 2162 [1969, 301]. *Proposed by D. Rameswar Rao, Osmania University, India*

Let p_i be distinct prime numbers. Show that

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} > \left\{ \prod_{i=1}^{n} \frac{p_i}{p_i - 1} \right\}^{2/3} > 2^{2/3} > 3/2.$$

Thus we obtain the following result: If $A = \prod_{i=1}^{n} p_i^{k_i}$ is perfect, then

$$\frac{3}{2} < \prod_{i=1}^{n} \frac{p_i + 1}{p_i} \leqq 2 < \prod_{i=1}^{n} \frac{p_i}{p_i - 1} \leqq 3.$$

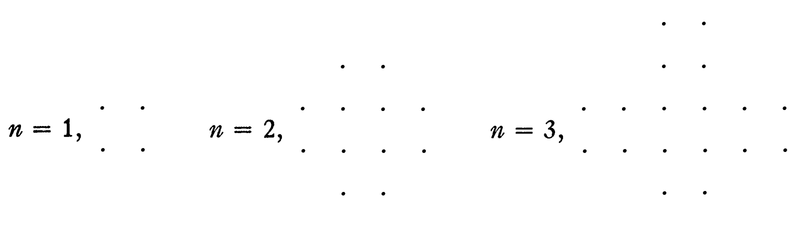
Also solved by Anders Bager (Denmark), Robert Baillie, A. R. Bolder, Robert Breusch, Orin Chein, G. C. Dodds, W. F. Fox, Ray Glenn, M. G. Greening (Australia), Emil Grosswald, Robert Heller, M. Hirschhorn (Australia), H. S. Lieberman, Douglas Lind (England), P. A. Lindstrom D. C. B. Marsh, Bob Prielipp, J. F. Reiser, Henry Ricardo, E. F. Schmeichel, A. G. Shannon (Papua), Stephen Spindler, Philip Trauber, E. W. Trost (Switzerland), R. L. Vogt, C. R. Wall, R. E. Whitney, and the proposer.

Editorial Note. The condition given above, even as strengthened by Reich, is not sufficient as is shown by taking $A = 15$. Also several solvers pointed out our error of using weak inequalities on both sides of the given condition.

Number of Squares on a Cross

E 2163 [1969, 301]. *Proposed by E. F. Bell, Washington and Jefferson College*

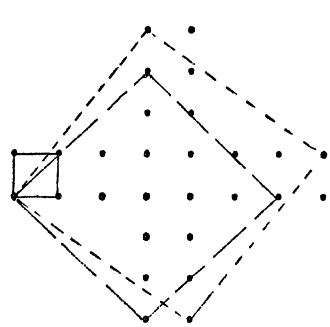
Given $8n-4$ points arranged in the form of a cross, e.g.



What is the largest number of squares which can be superimposed on the nth cross figure with each vertex of each square on one of the $8n-4$ points?

Solution by J. D. Baum, Oberlin College

In passing from $n=k$ to $n=k+1$ eight points are added to the star. There are



four small squares which include these new points in pairs. There are four intermediate squares which also include these new points in pairs, and there are two squares which include these new points in sets of four, as indicated in the figure. Thus the addition of the eight new points implies the addition of ten new squares. The total number of squares is thus the number present when $n=1$, namely 1, plus $10(n-1)$, which is finally $10n-9$.

Also solved by Anders Bager (Denmark), Orin Chein, Sarah Christiansen, M. G. Greening (Australia), G. A. Heuer, M. Hirschhorn (Australia), Thomas Hughes, S. Wu-Wei Liu, C. B. A. Peck, E. J. F. Primrose (England), E. F. Schmeichel, Zalman Usiskin, and the proposer.

A Property of the Series of Primes

E 2164 [1969, 301]. *Proposed by R. S. Luthar, University of Wisconsin at Waukesha*

Let $x_n=p_1+p_2+\cdots+p_n$, where $p_1, p_2, \cdots, p_n$ are the first n primes. Prove that between x_n and x_{n+1} there always lies a square number.

Solution by E. W. Trost, Technikum Winterthur, Switzerland. Let $2<3<5<7<q_5<q_6<\cdots$ be a sequence of positive odd integers satisfying the condition

$$q_{n+1} > 2n + 1 \qquad \text{for } n \geqq 4. \tag{1}$$

Putting $y_n=q_1+q_2+\cdots+q_n$ we have $y_i>i^2$ for $i=1, 2, 3, 4$, and from (1) we infer by mathematical induction that this is true for all positive integers i. Now we suppose

$$(n + k + 1)^2 > y_n \geqq (n + k)^2, \tag{2}$$

where k is a nonnegative integer. From (1) and (2), it follows that, for $k=0$ and $n\geqq 4$,

$$y_{n+1} > (n + 1)^2 > y_n. \tag{3}$$

This is true also for $n=1, 2, 3$. If $k\neq 0$, (2) implies

$$q_{n+1} > 2(n + k) + 1; \tag{4}$$

otherwise we would have $q_{n-j}\leqq 2(n+k)-(2j+1)$ for $j=0, 1, 2, \cdots, n-1$. Therefore, in opposition to (2),

$$y_n \leqq n^2 + 2nk < (n + k)^2.$$

Now we get from (2) and (4)

$$y_{n+1} > (n + k + 1)^2 > y_n. \tag{5}$$

By (3) and (5) we see that the assertion of the problem is valid for our generalized sequence.

Also solved by Anders Bager (Denmark), Merrill Barnebey, Robert Breusch, Orin Chein, S. C. Currier, Jr., Neal Felsinger, W. F. Fox, Arthur Gittleman, Ray Glenn, Emil Grosswald,

Heiko Harborth (Germany), Robert Heller, G. A. Heuer, Erwin Just, H. Kestelman, Lew Kowarski, A. M. Kriegsman, J. R. Kuttler, Douglas Lind (England), C. F. Marion, R. B. McNeill, Kanitta Meesook, D. S. Newman, K. K. Norton, Jan Pachl (Czechoslovakia), Bob Prielipp, Simeon Reich (Israel), Ira Rosenholtz, E. F. Schmeichel, Philip Trauber, C. S. Venkataraman (India), R. L. Vogt, C. R. Wall, Evelyn Woolley, and Alexander Zujus.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before March 31, 1970. Contributors (in the United States) who desire acknowledgment of receipt of their solutions are asked to enclose self-addressed stamped postcards.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5701. *Proposed by G. J. Foschini, Bell Telephone Laboratories*

Show that if P is a subset of the real line then P is homeomorphic to and has the same order type as a set of the form $G\cup Z$ where G is open and Z has Lebesgue measure zero.

5702*. *Proposed by A. A. Mullin, Warren, Michigan*

Let $g(m, n)$ be the class of all groups defined by not more than m generators and not more than n defining relations. Consider whether the word problem for all groups G in $g(m, n)$ is recursively solvable. If $m=0$, G is trivial; if $m=1$, G is cyclic and therefore abelian; if $n=0$, G is free; these cases can be shown to be solvable. Indeed, even if $n=1$, G is solvable by a result of Magnus. However, $g(7, 32)$ is not solvable. Is $g(2, 2)$ solvable? If so, what is the least n for which $g(2, n)$ is unsolvable?

5703. *Proposed by Erwin Just, Bronx (N. Y.) Community College*

If $n>1$ and k is any integer, can there exist solutions to the Diophantine equation:

$$\sum_{i=1}^{2n+1-1} (x_i^{2^n} - y_i^{2^n}) = (2k+1)2^{n+1}?$$

5704*. *Proposed by S. B. Maurer, Princeton University*

Does every uncountable subset of the real line contain a closed uncountable set?

5705. *Proposed by M. S. Klamkin, Ford Scientific Laboratory*

Find the general solution of the differential equation

$$[xD^{n+1} + 2nD^n - xD - n]y = 0.$$

5706. *Proposed by J. H. B. Kemperman, University of Rochester*

Let H be a Hamel basis of a field R over a subfield Q. Show that for each $a\in R$, $a\neq 1$, there exists an element $x\in H$ with $ax\notin H$.

SOLUTIONS OF ADVANCED PROBLEMS

Increasing and Decreasing Subsequences

5641 [1968, 1125]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

From the set $\{1, 2, 3, \cdots, n^2\}$ how many arrangements of the n^2 elements are there such that there is no subsequence of $n+1$ elements either monotone increasing or monotone decreasing?

Solution by Richard Stanley, Harvard University. Let A be an arrangement of $\{1, 2, \cdots, n^2\}$ with the desired property. Then the longest increasing and longest decreasing subsequences of A have length n (which is proved in the same way that one proves Erdös problem: every sequence of length n^2+1 has an increasing or a decreasing sequence of length $n+1$). It follows from a theorem of Schensted (*Increasing and decreasing subsequences*, Canadian J. of Math., 13 (1961) 179–191. Thm. 3), using the fact that the only partition of n^2 into n parts with largest part n is $n^2=n+n+\cdots+n$, that the number of such A is given by

$$\left[\frac{n^2!}{1^1\cdot 2^2\cdot 3^3\cdots n^n\cdot(n+1)^{n-1}(n+2)^{n-2}\cdots(2n-1)^1}\right]^2.$$

Schensted in fact solves the more general problem of finding the number of arrangements of $\{1, 2, \cdots, n\}$ whose longest increasing subsequence has a given length a and whose longest decreasing subsequence has a given length b.

Also solved by Joel Spencer.

A Delightful Inequality

5642 [1968, 1125, 1969, 422]. *Proposed by Raymond Redheffer, University of California at Los Angeles*

If x is real, show that

$$(\sin \pi x)/\pi x \geqq (1-x^2)/(1+x^2).$$

Solution by J. P. Williams, Indiana University. We need consider only $x\geqq 0$. Suppose first we assume that $x\geqq 1$. Starting from the well-known inequality $(\sin \pi y)/\pi y\leqq 1$ we have

$$\frac{1-x^2}{1+x^2}-\frac{\sin \pi x}{\pi x}=\frac{1-x^2}{1+x^2}+\frac{\sin \pi(x-1)}{\pi(x-1)}\left(\frac{x-1}{x}\right)\leqq\frac{1-x^2}{1+x^2}+\frac{x-1}{x}$$
$$=-\frac{(1-x)^2}{x(1+x^2)}\leqq 0.$$

Consider now $0<x<1$. Since

$$\frac{\sin \pi x}{\pi x}=\prod_{n=1}^{\infty}(1-x^2/n^2),$$

it is enough to prove that $(1+x^2)\cdot P_n \geqq 1$ for $n \geqq 2$, where

$$P_n = \prod_{k=2}^{n} (1 - x^2/k^2).$$

Actually we get $(1+x^2)P_n > 1+x^2/n$, $0<x<1$, by a simple induction argument based on the relation $P_{n+1} = (1-x^2/(n+1)^2)P_n$.

Also solved by Joel Anderson, Anders Bager (Denmark), D. Borwein, David Boyd, F. A. Butter, Jr., J. R. Campbell, L. Carlitz, L. E. Clark (England), Leon Gerber, Emil Grosswald, J. R. Hatcher, D. A. Hejhal & R. K. Keinigs, M. S. Klamkin, Margaret LaSalle, Beatriz Margolis (Argentina), Jernej Polajner (Yugoslavia), Al Somayajulu, T. Tamura (Japan), and J. H. van Lint (Netherlands).

Editorial Note. Solutions contributed by Gerber, Hejhal & Keinigs, Klamkin, and Margolis are more "elementary" than the one given above in that only differential calculus and a few calculations were used. It would still be nice to be able to offer an elegant elementary solution for the inequality, but none was made available to the editor.

Periodic Solutions for a Differential Equation

5645 [1969, 94]. *Proposed by Henry Guggenheimer, Polytechnic Institute of Brooklyn, N. Y.*

Given $y' = f(y) + p(x)$, with f Lipschitzian, p continuous and periodic of period T, assume sgn $f(y)$ sgn $y < 0$ for $|y| > a$ and $|f(y)| > \max|p(x)|$ for $|y| > a$.

Prove there exists a periodic solution $y(x)$ of period T.

Solution by Roy O. Davies, the University, Leicester, England. (1) By standard theory, there is a unique solution with $y(0) = y_0$—call it $y(y_0, x)$—and $y(y_0, T)$ is continuous in y_0.

(2) If $y(x)$ is a solution and $y(x_1) > a$, then the hypothesis implies $y'(x_1) < 0$, and so $y(x) > y(x_1) > a$ for all neighboring $x < x_1$; hence $y(x) > a$ for all $x < x_1$. Similarly, $y(x_1) < -a$ implies $y(x) < -a$ for all $x < x_1$. Therefore if $|y_0| \leqq a$ then $|y(x_0, x)| \leqq a$ for all $x \geqq 0$.

(3) In particular, $y(a, T) \leqq a$, and $y(-a, T) \geqq -a$. Hence there exists y_0 with $|y_0| \leqq a$ such that $y(y_0, T) = y_0$. The solution $y(y_0, x)$ has period T.

Also solved by Alan Berger, Robert Breusch, O. P. Lossers (Netherlands), Steve Rodhe, G. C. Schmidt, Klaus Schmitt, and the proposer.

Schmitt generalizes the result by proving the following theorem: Consider the ordinary differential equation

$$y' = f(x, y), \tag{1}$$

where f is continuous on $[0, T] \times R$. Assume there exist two functions $\alpha(x), \beta(x) \in C^1[0, T]$, $\alpha(x) \leqq \beta(x)$ such that $\alpha'(x) - f(x, \alpha(x)) \leqq 0 \leqq \beta'(x) - f(x, \beta(x))$, $\alpha(0) - \alpha(T) \leqq 0 \leqq \beta(0) - \beta(T)$. Furthermore let there exist a positive constant L such that $|f(x, y) - f(x, z)| \leqq L|y - z|$, whenever $\alpha(x) \leqq y, z \leqq \beta(x)$, $x \in [0, T]$. Then there exists a solution $y(x)$ of (1) such that $y(0) = y(T)$ and $\alpha(x) \leqq y(x) \leqq \beta(x)$.

An Invalid Proposition

5646 [1969, 94]. *Proposed by Henry Guggenheimer, Polytechnic Institute of Brooklyn, N. Y.*

If

$$\int_{-\pi/4}^{\pi/4} f(\theta)d\theta = \int_{-\pi/4}^{\pi/4} f(\theta)\left[1+\frac{1}{\cos 2\theta}\right]^{1/2} d\theta = \int_{-\pi/4}^{\pi/4} f(\theta)\left[\frac{1}{\cos 2\theta}-1\right]^{1/2} d\theta = 0,$$

then $f(\theta)$ changes sign at least three times in $(-\pi/4, \pi/4)$, and five times if, in addition,

$$\int_{-\pi/4}^{\pi/4} f(\theta)\begin{Bmatrix}\sin 2\theta\\ \cos 2\theta\end{Bmatrix} d\theta = 0.$$

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands. Both assertions are false. As to the first, the function $f(v)=v$ (or any odd function) gives a counterexample. As to the second, when we construct a function $f(v)$ with $f(v)=0$ if $v=0$, which has one positive zero such that $\int_0^{\pi/4} f(v)\sin 2v\, dv=0$, and to which we give an odd continuation for $v<0$, then this function $f(v)$ is a counterexample.

Also disproved by Robert Breusch.

On Ideals of a Polynomial Ring

5647 [1969, 94]. *Proposed by Pascual Llorente, Universidad Nacional de Ingenieria, Lima, Peru*

Let A be a Euclidean domain, and let $A[x]$ be the polynomial ring in one variable over A. Prove that for every ideal $\mathfrak{a}\subset A[x]$ there is an ideal $\mathfrak{b}\subset A[x]$ such that $\mathfrak{b}$ admits a system of generators of at most 2 elements and $\sqrt{\mathfrak{b}}=\sqrt{\mathfrak{a}}$. (Here $\sqrt{\mathfrak{b}}$ denotes the radical of $\mathfrak{b}$.)

Solution by Robert Gilmer, Florida State University. The following more general result is true: If D is a principal ideal domain with identity, and if A is any ideal of $D[X]$, then $\sqrt{A}$ has a basis of two elements. The solution depends largely on the prime ideal structure of $D[X]$. Since $D[X]$ is a unique factorization domain, each minimal prime of $D[X]$ is principal. Each proper nonminimal prime M of $D[X]$ is maximal and is of the form $(p, f(X))$, where p is a prime element of D and where $f(X)$ is irreducible modulo p; this result is known, but we sketch an elementary proof.

If $M\cap D=(0)$, then M extends to a proper prime ideal of $K[X]=D_S[X]=(D[X])_S$, where S is the multiplicative system of nonzero elements of D, and K is the quotient field of D. Hence the extension of M to $K[X]$ is minimal in $K[X]$ so that M is minimal in D. Consequently $M\cap D$ is a proper prime ideal of D; say $M\cap D=pD$, where p is a prime element of D. Then $M/pD[X]$ is a proper prime ideal of $D[X]/pD[X]\simeq(D/pD)[X]$, where D/pD is a field. It follows

that $M/pD[X]$ is principal and is generated by an element $f(X)+pD[X]$, where $f(X)$ is irreducible modulo p.

Since $D[X]$ is Noetherian, the radical of any proper ideal of $D[X]$ is a finite intersection $\bigcap_1^n P_i$ of proper prime ideals of $D[X]$, where $P_i \not\subseteq P_j$ for $i \neq j$. If $\{P_i\}_1^k$ is the subset of $\{P_i\}_1^n$ consisting of principal ideals, then $\bigcap_1^k P_i = P_1P_2 \cdots P_k$. Further, any P_j, for $k+1 \leqq j \leqq n$, is maximal in $D[X]$. Thus $P_1P_2 \cdots P_k$ and P_j are comaximal for each j so that $P_1P_2 \cdots P_k$ and $(\bigcap_{k+1}^n P_j)$ are also comaximal. Therefore $\bigcap_1^n P_i = (P_1P_2 \cdots P_k)(\bigcap_{k+1}^n P_j)$, and in order to show that $\bigcap_1^n P_i$ has a basis of two elements, we need only show that $\bigcap_{k+1}^n P_j$ has a basis of two elements.

Observe that if $M_1, \cdots, M_t$ are nonminimal proper primes of $D[X]$, each lying over the same prime ideal of pD of D, then $(\bigcap_{i=1}^t M_i)/pD[X]$ is a proper ideal of $D[X]/pD[X]$, and hence $\bigcap_{i=1}^t M_i$ is principal modulo p so that $\bigcap_{i=1}^t M_i = (p, b(X))$ for some $b(X)$ in $D[X]$. Thus, any finite intersection of nonminimal proper primes of $D[X]$ can be written in the form $\bigcap_{i=1}^s (p_i, a_i(X))$, where $p_1, p_2, \cdots, p_s$ are distinct prime elements of D and $a_i(X)$ is a nonzero element of $D[X]$ for each i. The ideals $p_1D[X], \cdots, p_sD[X]$ of $D[X]$ are pairwise comaximal. Therefore, the system of congruences $f(X) \equiv a_i(X)$ (modulo $p_iD[X]$), $1 \leqq i \leqq s$, has a solution $a(X)$ in $D[X]$. It then follows that $(p_i, a_i(X)) = (p_i, a(X))$ for each i. Then in $D[X]/(a(X))$,

$$\{(p_i, a(X))/(a(X))\}_{i=1}^s$$

is a set of principal, pairwise comaximal ideals. Consequently

$$\left(\bigcap_{i=1}^s (p_i, a(X))/a(X)\right) = \bigcap_1^s [(p_i, a(X))/(a(X))]$$
$$= \prod_{i=1}^s [(p_i, a(X))/(a(X))] = (p_1p_2 \cdots p_s, a(X))/(a(X)),$$

so that $\bigcap_1^s (p_i, a(X)) = (p_1p_2 \cdots p_s, a(X))$.

Also solved by the proposer.

The Set of Squares in a Group

5648 [1969, 95]. *Proposed by John Shafer, University of California at Davis*

When is the set of all squares of elements in a finite group a subgroup? When is it a proper subgroup?

Partial Solution by Simeon Reich, The Technion, Haifa, Israel. The subgroup of all squares coincides with the entire group if and only if the order of the group is odd. If the order is odd, then the order of any element g is odd. Therefore $g^{2n-1} = e$, or $(g^n)^2 = g$. If the order of the group is even, then by Cauchy's theorem there is an element of order 2. Since this means that $a^2 = e^2$ for $a \neq e$, the mapping $g \to g^2$ is not one to one. Therefore the number of squares is strictly less than the order of the group.

Also solved (first part) by Al Somayajulu, and by the proposer.

Editorial note. A sufficient condition in the second part is that the group be Abelian. Professor Barbara Ossofsky has pointed out that a necessary condition is that every product of commutators of G be a square, i.e., $D(G)$ contain the commutator subgroup of G. But it is not clear that this condition should also be sufficient.

Order of Normal Subgroups

5649 [1969, 95]. *Proposed by Erwin Just, Bronx Community College, New York*

Let G be a group with order $p^n m$ in which p is a prime, $m<2p$ and $n>1$. Prove that G contains a normal subgroup whose order is either p^n or p^{n-1}.

Solution by E. F. Schmeichel, College of Wooster, Ohio. By Sylow's first theorem, G has at least one subgroup of order p^n. If there is only one such subgroup, it is necessarily normal by Sylow's second theorem. Otherwise, let H and K be distinct subgroups of G with order p^n. Then $H\cap K$ is a proper subgroup of both H and K, and so $|H\cap K|$ divides p^n. If $|H\cap K| \leqq p^{n-2}$, it would mean

$$|HK| = \frac{|H|\,|K|}{|H \cap K|} \geqq \frac{p^n\cdot p^n}{p^{n-2}} = p^{n+2} \geqq p^n\cdot 2p > p^n m = |G|,$$

a contradiction. So $|H\cap K| = p^{n-1}$, and thus $H\cap K$ is normal in both H and K.

So if we let N denote the normalizer of $H\cap K$ in G, it follows immediately that H and K are both subgroups of N. Hence $|N| \geqq |HK| = p^{n+1} > p^n m/2 = |G|/2$. But this implies $N=G$, and thus $H\cap K$ is normal in G. This completes the proof.

Also solved by S. R. Alpert, D. M. Bloom, Theodore Chang, Stuart DeSousa, Ronald Evans, Jerry Fischer, Ralph Freise, Robert Gilmer & Joe Mott, M. G. Greening (Australia), Geoffrey Kandall, Yan Lee, H. S. Lieberman, Donna Martellotto & Kathy Shiple, Claire Parkinson, Al Somayajulu, and the proposer.

In his solution, Evans notes that the proposition is still true if $m=2p$.

Non-measurable Sets on the Unit Circle

5651 [1969, 199]. *Proposed by Maurice Machover, St. John's University*

Consider one of the standard nonmeasurable sets, say the set E formed by taking one representative from each one of the equivalent classes formed from the points of the circumference of a circle of diameter 1 by calling two points equivalent if the distance between them, measured along the arc, is an integer. It is easily seen that the inner measure of E is zero. What is its outer measure?

Solution by Merrill Barnebey, Wisconsin State University, LaCrosse. The answer is: arbitrarily small. In any arc $\theta_1 \leqq \theta \leqq \theta_2$ there is an element of each equivalence class. So the set E may be chosen in (θ_1, θ_2).

Also solved by E. P. Del Norte, D. A. Hejhal, G. A. Heuer, and Philip Trauber.

Meromorphic Functions Preserving the Unit Circle

5652 [1969, 200]. *Proposed by Joseph Lehner, University of Maryland*

What is the most general function that is meromorphic in $|z| \leqq 1$ and that maps the unit circle ($|z| = 1$) into itself?

Solution by Douglas Campbell, University of North Carolina. Since $f(z) \not\equiv \infty$, $f(z) \not\equiv 0$, $f(z)$ can have only a finite number of poles $\beta_1, \beta_2, \cdots, \beta_n$ and zeros $\alpha_1, \alpha_2, \cdots, \alpha_m$ in $|z| \leqq 1$ (counting multiplicities). Thus

$$g(z) = f(z) \cdot \prod_{j=1}^{m} \left(\frac{z - \beta_j}{1 - \bar{\beta}_j z} \right) \cdot \prod_{k=1}^{n} \left(\frac{1 - \bar{\alpha}_n z}{z - \alpha_k} \right)$$

is a meromorphic function without poles or zeros in $|z| \leqq 1$. Clearly $g(z)$ maps $|z| = 1$ into itself. From the minimum and maximum principles we get $|g(z)| = 1$ for all $|z| \leqq 1$. Consequently $g(z)$ must be a constant of modulus one. Hence

$$f(z) = e^{i\theta} \prod_{k=1}^{n} \left(\frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right) \Big/ \prod_{j=1}^{m} \left(\frac{z - \beta_j}{1 - \bar{\beta}_j z} \right).$$

REMARK. As is suggested by a problem in W. H. J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, Problem 2, p. 18, the conclusion is still true if $f(z)$ is meromorphic in $|z| < 1$ and if $|f(z)| \to 1$ as $|z| \to 1$.

Also solved by Sidney Birnbaum, W. M. Causey, M. R. Cullen, W. O. Egerland, D. S. Greenstein, Robert Goldstein (England), D. A. Hejhal, D. A. Herrero, O. P. Lossers (Netherlands), D. E. Myers, E. B. Saff (England), and the proposer.

Fixed Relations in a Collection of Orderings

5655 [1969, 200]. *Proposed by Paul Erdös and E. C. Milner, University of Calgary*

If $<_1, <_2, <_3, \cdots$ are countably many well orderings of the set of reals, must there be a pair x, y such that $x <_n y$ for all n?

Solution by G. A. Heuer, Concordia College. No. Put the rational numbers in a sequence $(q_1, q_2, \cdots)$. Let A_i be the set of reals less than q_i, B_i the complement of A_i. Let $<_{2i-1}$ be a well ordering of the reals such that $A_i <_{2i-1} B_i$, and let $<_{2i}$ be a well ordering such that $B_i <_{2i} A_i$. If $x < y$ in the natural order, choose q_i such that $x < q_i < y$. Then $x <_{2i-1} y$ but $y <_{2i} x$.

Also solved by M. A. Ettrick, Fred Galvin, L. Haddad & G. Sabbagh (France), J. R. Isbell, M. G. Laplaza (Puerto Rico), J. B. Linder & R. V. Fuller, Dan Marcus, E. C. Milner, J. C. Morgan II, Charles Riley, D. L. Silverman, and Charles Vanden Eynden.

REVIEWS

EDITED BY KENNETH O. MAY

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Printed materials for review should be sent to Kenneth O. May, American Mathematical Monthly, Department of Mathematics, University of Toronto, Toronto 5, Canada. Films and correspondence relating to films should be sent to Seymour Schuster, Carleton College, Northfield, Minnesota 55057.

*All unsigned material is written by the editors. A boldface capital **C** in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should first inform the editor in order to avoid duplication.*

Some Vistas of Modern Mathematics. Dynamic Programming, Invariant Imbedding, and the Mathematical Biosciences. By Richard Bellman. Univ. of Kentucky Press, Lexington, Kentucky, 1968. viii+141 pp. $7.00 (cloth) $3.95 (paper).

This is a remarkable book, written in the tradition of Poincaré *Science and Method*, Courant and Robbins *What is Mathematics*, and Khinchin *The Teaching of Mathematics* (Telegraphic Review June 1969). Dr. Bellman lays before us three personal and penetrating essays devoted to modern control theory, invariant imbedding, and mathematical biosciences. Each of these areas has, of course, undergone pronounced developments at his hands.

The aim throughout is to lead the reader to the frontiers of current mathematical research. This is done by stressing ideas and the role of the modern computing machine, rather than trivial details. The first two essays emphasize the transformation of two-point boundary-value problems into initial-value problems. Semi-group notions come to the fore.

The last essay deals with cancer chemotherapy, pattern recognition, and electrocardiography. In a book brimming with quotable sentences, Dr. Bellman offers this prescription for curing discontent in mathematics departments: "This involvement has been seriously lacking over the last twenty years and has been responsible for much discontent on college campuses. The rise of the mathematical biosciences will change much of this situation by providing the intellectual with the proper combination of challenge and service."

R. KALABA, University of Southern California

Où vont les mathématiques? Réflexions sur l'enseignement et la recherche. By Jean Kuntzmann. Illustrations by Avoine. Hermann, Paris, 1967. 168 pp. 15 F.

Dans cet ouvrage passionnant l'auteur nous livre ses réflexions sur la situation actuelle des mathématiques, leur avenir probable, l'enseignement de celles-ci à tous les niveaux et l'organisation de la recherche théorique et technique en mathématiques. Il insiste sur l'importance actuelle et future des mathématiques appliquées et sur le rôle que peut jouer l'ordinateur dans la recherche scientifique. Bien sûr, tous ne seront pas d'accord avec M. Kuntzmann sur ce point ou sur l'évolution probable des mathématiques d'ici 1980, mais il n'en reste pas

moins que chacun trouvera ce livre intéressant et stimulant. L'auteur s'adresse à tous ceux qui s'intéressent de près ou de loin aux mathématiques et il explique en des notes abondantes le sens de tous les termes qui ne font pas partie du bagage d'un bachelier.

Ce livre, dont les illustrations du dessinateur Avoine amuseront le lecteur, est le premier de la nouvelle collection "Science Publique" et il faut conclure que c'est un très bon départ.

R. CLÉROUX, Université de Montréal

Introduction to Modern Calculus. By Herman Meyer. McGraw-Hill, New York, 1969. 521 pp. $10.50. (Telegraphic Review June 1969.)

In its organization, in the way it is intended to be used, and in its development of the calculus itself, this book provides a new approach to a first course in the calculus. It clearly separates concepts from techniques and applications. Concepts are developed in the first 30 sections, while techniques and applications are presented in the last 21 sections. These latter sections are in programmed form, with brief descriptions followed by exercises in frames, through which the student is led with the help of detailed solutions provided later in the book. They are independent of and are intended to be studied concurrently with the earlier sections. In this two-track approach the Instructor can concentrate in the classroom on the theorems of the calculus while the student at the same time is learning on his own to differentiate, integrate, and apply the calculus in the usual ways.

With the assumption only of the properties of the real number system, all the calculus theorems are proved formally in all detail. They are developed through the Moore-Smith general theory of limits. For a function with domain a set of real numbers, an associated function is defined with domain a directed set, the set directed by a relation that is endless and transitive. Thus a single theory of limits is applied successively to the concepts of continuity, to the derivative, to the definite integral, to improper integrals, and to infinite series.

This is exemplary mathematics and should appeal to the best students and to instructors. In addition, the second track development of differentiation from slope curves and of integration from area curves is appealing and effective. The frames and exercises are well constructed for the student who can learn independently of classroom instruction. The proofs of the first track, however, may be a large amount of theory for many students who are just starting the calculus.

The style is clear and direct. A short "Instructor's Manual" provides an outline for scheduling the two tracks simultaneously. Through its several features the text makes a significant contribution to the teaching of the calculus.

D. H. BALLOU, Middlebury College

An Introduction to Mathematical Logic. By Gerson B. Robison (SUNY at New Paltz). Prentice-Hall, Englewood Cliffs, N. J. 1969. xi+212 pp. $5.95.

This is an introduction to the elementary part of mathematical logic, compa-

rable in level and scope, though not in quality, to Kalish and Montague *Logic Techniques of Formal Reasoning* (1964). Many readers will like the author's style ("Where do little axioms come from?"). The competent reader, however, will soon begin to worry about the author's disregard of distinctions commonly made in logic. It may be all right not to fuss about use and mention. After all, the distinction becomes vital only at stages far beyond the horizon of the text, e.g., in Gödel's incompleteness theorems. Less harmless is the neglect to distinguish between statements and statement forms, a distinction which is nowadays even taught in high school. Thus the reader will not be surprised that accidents happen. To mention only one example of several: By the definition of a demonstration (p. 66) and using the rule of choice (p. 71), while obeying all precautions, we obtain a one-line demonstration: $\exists x 3<x \Rightarrow 3<y$. By the definition on p. 75 we are allowed to call this a true statement of (almost) any mathematical system Γ. Even worse, by the rule of generalized forms on p. 112 we can state as a second line of an abbreviated demonstration: $\forall y(\exists x 3<x \Rightarrow 3<y)$; the latter we are even allowed to call a theorem of Γ. A teacher may try to repair those two slips and many others. The reviewer recommends that instead he should take a respectable logic book, e.g., *Elementary Logic* by Benson Mates (Oxford University Press, 1965. See the review in the Monthly, 73 (1966) p. 432).

G. FUHRKEN, University of Minnesota

Dialogues on Mathematics. By Alfred Renyi (Hungarian Acad. of Science) Holden-Day, San Francisco, 1967. 100 pp., \$4.95 (cloth) \$2.50 (paper). (Telegraphic Review, April 1969.)

Here we have an expert's attempt to explain the nature of mathematics to a nontechnical audience by historically plausible mock dialogues led by Socrates, Archimedes, and Galileo. The explanation is effective because of its wit, literary skill, and mathematical taste. A reader can almost hear the voice of some venerable colleague who fancies himself to be a reincarnation of one of these great men. (Libel laws prevent the reviewer from nominating his choices.)

This work is outstanding even in this age of popularization because it should be taken seriously by every mathematician including such marginal cases as a high school freshman wondering what distinguishes a mathematician, and a chairman preparing a confrontation with a dean of engineering.

The first and most convincing dialogue is between Socrates and Hippocrates, a neophyte mathematician, on the nature of mathematics and its relation to the outside world. Socrates, of course, only "assists as a midwife in the birth" of the student's ideas. Thus it is the student who jubilantly discovers: "One can have a much more certain knowledge about nonexistent things—for instance, about the objects of mathematics—than about the real objects of nature. . . . The main aim of the mathematician is to explore the secrets and riddles of the sea of human thought. They exist independently of the mathematician, though not of humanity as a whole. . . . The world of mathematics is nothing else but a reflection in our mind of the real world." (To quote any further would be to detract from the reader's pleasure!)

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Computing Methods for Scientists and Engineers. By L. Fox and D. F. Mayers. Oxford, New York, 1968. xii+225 pp. $7.50. (Telegraphic Review, March 1969.)

For the indicated audience this book offers a clear and gentle exposition of a variety of numerical methods. Throughout there are persistent reminders that numerical problems usually have several different approaches which—equally attractive from a mathematical point of view—fare quite differently in the computational process. In effect, the (floating point) computational process is presented almost as a laboratory experiment, successful only within the limitations of the precision of the instrumentation and the propriety of the techniques employed.

The topics covered include recurrence relations, polynomial zeros, matrix equations, Chebyshev approximation, interpolation, and integration. The exposition is laced with examples—brief, transparent, well-chosen—which is fortunate since an exposition essentially devoid of proofs runs a tangible risk: the *proof* that a procedure works as claimed often encloses its own clarifying aspect. It is to the authors' credit that their presentation ends up more or less unscathed.

R. M. BAER, University of California—Berkeley

C *A History of Mathematics.* By Carl B. Boyer (Brooklyn College). Wiley, New York, 1968. xv+717 pp. $11.95. (Telegraphic Review, Jan. 1969.)

This book is intended to be a textbook for a course in the history of mathematics for upper division college students. It can, then, be evaluated as a history of mathematics whose content and emphases are conditioned by the textbook function, and as a textbook which has some of the elements of a historical treatise.

In a historical treatise one looks for scholarship in marshalling the facts, for insight and a broad view in portraying the changing trends, themes, and interrelationships both within mathematics and between mathematics and other aspects of the culture. One may also be concerned for the coverage and for the quality of the evaluation of forces directing progress. This book scores high in all of these respects.

Recent historical analyses, such as those of Gillis with reference to familiar Egyptian materials, and new data, such as Aydin Sayili's publication of the algebraic work of Abd al Hamid ibn Turk, are integrated into standard historical materials. Both these newer materials and classical data are analyzed for their significance. Thoughtful conjectures as to implications for the history of mathematics are included. For example, the author not only discusses the facts of Viète's contributions to algebra, trigonometry, and symbolism but also points out the significance of his association of higher algebraic topics with higher geometric problems as typifying a trend leading toward analytic geometry. At the same time Boyer questions why Viète did not perceive the periodicity of the

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able in this book. Further the audience for which the book is designed is not prepared to read such a story. Over four chapters, however, are devoted to the nineteenth century, covering Gauss, Cauchy, "The Heroic Age in Geometry," "The Arithmetization of Analysis," and "The Rise of Abstract Algebra." The author does an excellent job of bringing out the changing views held by mathematicians of their subject and the recent growth of axiomatics and of abstraction in all fields, as well as actually giving some discussion of the growth of modern algebra, set theory, logic, and functional analysis.

The book is remarkably free of errors. Most of them are fairly obvious and easily corrected in a reprinting. Euler's constant is labeled transcendental on page 484 but on page 687 it is listed, properly, with other numbers whose algebraic or non-algebraic character has not yet been determined.

In conclusion, the book rates an excellent score as both a scholarly product which sets a good example for young scholars, and as a textbook in an undergraduate course. Although it lacks the range and the occasional flamboyance of E. T. Bell's *Development of Mathematics*, mathematicians too will find it good reading and the best book in English from the viewpoint of historical scholarship.

P. S. JONES, University of Michigan

Euclidean Geometry and Convexity. By Russell V. Benson. McGraw-Hill, New York, 1966. xiv+265 pp. $9.95. (Telegraphic Review, Feb. 1967.)

The study of convex sets at the undergraduate level has been impeded by a shortage of textbooks which are neither too elementary nor too advanced. This book should meet the needs of many teachers. A wide range of topics allows considerable flexibility in the design of a course: basic properties of convex sets, the theorems of Radon, Helly, and Carathéodory, sets of constant width, decompositions of polygonal and polyhedral regions, the isoperimetric problem and questions of convergence, mixed volumes and symmetrization, and convex functions.

The author's approach is usually to develop a concept in the plane and then immediately to extend it to three dimensions or, wherever possible, to treat the two- and three-dimensional cases together. Results are stated so as to be readily generalizable to higher dimensions, but the formal extension to n-space is deferred to a separate chapter. Much of the treatment is synthetic, but coordinates are also used freely.

Some of the discussion of Euclidean transformations seems irrelevant, and the terminology is sometimes unorthodox. It is not customary, for example, to regard "motions" and "rotations" as including opposite isometries. The analogue of the group generated in E^2 by translations and point reflections is given as the group generated in E^3 by translations and line reflections; the latter is just the group of all direct isometries.

As the author observes in a final chapter devoted to Minkowski geometry, in which an arbitrary centrally symmetric convex body plays the role of the

unit disk, "a great deal of the theory of convex bodies . . . does not depend on the fact that we work in Euclidean space." It is to be regretted that he does not go further to point out how much of the theory does not depend on any kind of metric and would therefore be equally valid in affine space.

Two excellent features of the book are the many clear and helpful diagrams and the copious number of exercises, most of which call either for proofs or for the construction of examples.

N. W. JOHNSON, Wheaton College

Differential and Difference Equations. By Louis Brand. Wiley, New York, 1966. xvi+698 pp. $11.95. (Telegraphic Review, June/July 1967.)

The dust cover of and the advertisements for this book are, in a way, deceptive for they suggest that the text might be studied primarily because it explores the parallelism that exists between differential equations and difference equations. Valid though this aspect may be, the advertisement does this work an injustice. The author presents to the reader an unusual, fine monograph on differential equations. That it also interrelates with difference equations (as it should in the reviewer's opinion) is all the better.

The author wisely begins by explaining a few standard methods (exact equations, separation, integrating factors) to give the student a manipulative exposure to the subject. Some local existence-uniqueness results are stated early; their proofs are deferred to Chapter 12.

In addition to the usual systematic study of solutions, the author includes a delightful section on Hurwitz type stability plus stability for linear autonomous systems, some Liapunov theory, limit cycles, and periodicity. The entire text contains an unusually large number of applications and examples worked in meticulous detail. In particular there are a substantial number of fascinating graphs that illuminate the stability theory.

Difference equations are studied midway in the text beginning with difference calculus and emphasizing series summation by the telescope effect of antidifference sums. The development exposes quite fully and beautifully the duality that exists between difference equations and differential equations. However, there is no mention of their coupling: differential-difference equations. This omission seems to be most unfortunate.

Chapter 10 (titled *Solutions in Series*) contains a bit over 100 pages that are chock full of classical analysis goodies: singularities, indicial equations, two and three term recurrence relations, hypergeometrics, orthogonality, mini-max Chebyshev polynomials, Legendre, Laguerre, Hermite, Bessel, Weber, etc., plus a plethora of examples.

The final 65 pages of text are devoted to elementary numerical methods. The lecturer should be cautioned that Newton-Cotes quadrature is introduced without mention of the convergence dilemma. Several classical methods for the numerical solution of ODE are presented, but without a discussion of convergence or error propagation. There is one example ($y' = -y$ with the midpoint

rule) that shows the growth of a spurious numerical solution. The author does not apply his earlier development of difference equations so as to analyze the propagated error and discuss matters like growth parameters and the Dahlquist stability theorems for linear multistep methods, although the students would be well prepared for this. Some lecturers may wish to so supplement the course.

There are a large number of exercises (perhaps 1000) of varying difficulty and importance. The Bendixson Theorem, for example, is an exercise (page 226).

This text contains very good balance. It has an ample supply of existence-uniqueness-continuity theory, enough for the non-specialist, plus a large quantity of applications from which to choose. Elements of Liapunov stability, of operational calculus, and of numerical methods (all of which currently are fashionable) are presented. There is substantial special function theory. The material is certainly accessible to anyone with an advanced calculus background. (The author, by the way, refers at will to his own advanced calculus text for theorems and proofs. This might be disconcerting to some.) This text is appropriate for advanced undergraduates. It could also be used in a graduate course for students in the sciences. It is well-suited for either a one or a two semester course. Those in charge of suitable classes should certainly review this textbook for potential adoption particularly if they seek one with an applied flavor. The price of the book (1.7¢ per page) is quite reasonable.

ALAN FELDSTEIN, University of Virginia

Mathematics for Technology. A New Approach. By M. Bruckheimer, N. W. Gowar, and R. E. Scraton. American Elsevier, New York, 1968. xiv+558 pp. $8.75. (Telegraphic Review, Aug./Sept. 1969.)

This is a delightfully refreshing book because of its unique organization and content. Each topic is treated four times under theory, technique, and notes on theory and technique. The topics covered provide a development of analysis through integration of functions of one variable and differentiation of functions from E_n to E_m, laced together with a liberal application of terminology from algebra that emphasizes the common structure among many topics. There are chapters on infinite series, differential and difference equations, and discrete and continuous probability.

The text is meant for first year undergraduate courses in England aimed at "users" of mathematics. The notation seems to be unduly heavy, e.g., different symbols for corresponding operations on real numbers, vectors, and matrices. Its scope is surprisingly broad. Complex functions are carried to the point of the Cauchy-Riemann equations. These are identified on page 439 but are not noted in the index. In general the index is woefully inadequate, considering the multiple treatment of topics.

The level of sophistication expected from the student varies widely. On page 220 we have ". . . , whereas exp with domain R is an isomorphism of $(R, +)$ onto $(R^+, \times)$, the extension to C is a homomorphism of $(C, +)$ onto $(C$

without zero, $\times$).” In contrast, on page 250 the algorithm for the division of polynomials is given in considerable detail.

Some terminology differs markedly from usage in American texts; mappings may be one-many, many-one, one-one, or many-many. If $\mathbf{x} = \lambda \boldsymbol{a} + \mu \boldsymbol{b} + \nu \mathbf{c}$ is a representation of vector $\mathbf{x}$ in terms of a basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\mathbf{c}$, then $\lambda \boldsymbol{a}$, $\mu \boldsymbol{b}$, $\nu \mathbf{c}$ are called *resolutes* of $\mathbf{x}$; λ, μ, ν being named *components*. An heuristic argument for successive approximation to $\sqrt{b}$ is called “jiggery-pokery.”

The discussion of maximum and minimum could have been clarified by the use of “relative” or “local” terminology. No particular attention is given to the domain of the function in this connection or to end-point considerations. On page 189 the word “gradient” is used incorrectly. In the coverage of discrete probability a sample space is defined as a set of possible results of an action which is subject to chance, then 7 is included in a sample space for throwing an unbiased die.

Some inclusions of particular interest are: experience with matrices in many different contexts, good informal discussions of limits, attention to accuracy and approximations of computations, the use of spiral similarity in complex numbers, sound functional notation using the arrow form, a generally informal and readable style.

How this book would fit any present course offering in the U. S. is not clear but it provides a challenging outline of topics for consideration in a unified approach to introductory college mathematics. As a text it would need supplementary problem sets and considerable pedagogical footwork to cope with the sophisticated notation. The reviewer recommends it heartily to all teachers concerned with the proper content and treatment of introductory college mathematics, particularly for a terminal course.

D. W. WESTERN, Franklin and Marshall College

Modern Projective Geometry. By Robert J. Bumcrot. Holt, Rinehart and Winston, New York, 1969. viii+149 pp. $8.95. (Telegraphic Review, Aug./Sept. 1969.)

This is an excellent book for an advanced undergraduate or first year graduate course in modern projective geometry. It presupposes somewhat sophisticated introductory courses in linear algebra, vector spaces, and arbitrary fields, though Chapter I contains a summary of the more important terms and concepts needed from these areas. It seems clear that an introductory course in the intuitive (perhaps naive) concepts of projective geometry should be presupposed also. Otherwise, the student may be bewildered, if not bored, by the really beautiful exposition of the structure of projective geometry, the mapping of projective spaces, and the relation of both to fields and division rings.

Chapter II introduces the abstract structure of projective spaces by means of the real projective plane and its coordinatization. Mappings of projective spaces, especially isomorphisms, are given. Affine spaces are defined. In Chapter III the coordinatization of an arbitrary projective plane is exhibited. Many

configurations are studied, as are the conditions that a plane be Desarguian or Pappian. Chapter IV defines the coordinate set of a point P in an r-dimensional subspace P_r of P_nF (an analytic projective space whose points are $(n+1)$-tuples chosen from field F) with respect to a frame of reference $(Q_0Q_1 \cdots Q_r | U)$ where $(Q_0Q_1 \cdots Q_r)$ is a coordinate simplex of P_r and U is the unit point. The affine coordinate of point P relative to the frame is defined and for collinear points P_1, P_2, P_3, P_4 of P_nF the cross ratio $(P_1P_2P_3P_4)$ becomes the affine coordinate of P_4 in the frame $(P_1P_2 | P_3)$. It is proved that an automorphism of F determines a unique isomorphism α_μ from P_r to P_r' of P_nF such that for four collinear points P_1, P_2, P_3, P_4 in P_r and their images $P_{1\mu}, P_{2\mu}, P_{3\mu}, P_{4\mu}$ in P_r' the crossratio $(P_1P_2P_3P_4)$ is transformed into the crossratio $(P_{1\mu}P_{2\mu}P_{3\mu}P_{4\mu})$. Projectivities are seen as those isomorphisms for which the associated automorphism is the identity function. Harmonic tetrads, correlations, order, and continuity are studied. Chapter V presents briefly plane curves (especially conics and cubics), their singularities and intersections. Chapter VI considers orders of projective planes, polarities, involutions, and perspectivities.

The book should be valuable as a textbook in the presentation of modern projective geometry, though not as the first introduction to such a study. One must beware of misprints, as is so often the case with first printings.

JOSEPHINE H. CHANLER, University of Illinois

Lattice Theory. By Thomas Donnellan. Pergamon, New York, 1968. xii+283 pp. \$6.00. (Telegraphic Review, January 1969.)

This pleasant little book would merit its existence by the sheer force of the author's enthusiasm for the subject area; it is refreshing to read a tract written by someone so obviously delighted with the material at hand.

The author gives, in six chapters, a strictly elementary, self-contained account of the foundations of lattice theory. The chapters, listed in order, are: Set and Relations, Definition of a Lattice, Lattices in General, Modular Lattices, Semi-Modular Lattices, and Distributive Lattices. The style is semi-telegraphic; in the space of 280 pages, the author manages to insert 70 definitions, 191 theorems, 108 examples, 146 exercises, and 37 figures. Proofs are supplied for almost all theorems, usually in explicit detail.

The matter covered by the first three chapters is quite basic; the motivating examples given are, for the most part, finite. In the latter chapters, the standard non-modular and non-distributive five element lattices are presented and the importance of their positions is amply justified. Semi-modular lattices receive a surprising amount of attention, and are treated in considerable detail. The Kurosh-Ore theorem is proved in Chapter 4, apparently to illustrate that a deep theorem can have proof using only elementary properties.

The printing is remarkably free of typographical errors. In view of current usage, the reviewer disagrees with the author's decision to eschew $\wedge$ and $\vee$ in favor of $\cdot$ and $+$, but this is certainly not a crippling alteration.

In short, this book is an appetite-whetter. I believe it would be a worthwhile

text for an undergraduate elective course, or for an undergraduate independent reading assignment. Its value in teacher-training courses would be greatly enhanced by presentation of additional material stressing applications of the elementary theory.

D. R. BROWN, University of Houston

The Elements of Probability. By Simeon M. Berman. Addison-Wesley, Reading, Mass., 1969. xiii+224 pp. $6.50. (Telegraphic Review, October 1969.)

All too often, elementary books on probability give students the impression that probability is a collection of formulas and tricks and, therefore, not worthy of further study. This excellent little book should, on the other hand, instill in the readers the desire to pursue further studies in probability. The author has successfully accomplished his purpose which, according to the preface, is " . . . the presentation of the more profound, interesting, and useful results of classical probability in elementary mathematical forms." For example: Bernoulli's Law of Large Numbers, the Glivenko-Cantelli theorem on convergence of the empirical distribution function, the gambler's ruin, random walks, branching processes, Markov chains. All of this, and more, is presented in a clear and rigorous fashion. I recommend this book as a textbook for college freshmen and high school seniors.

"Two features of the exposition are the presentation of proofs in Euclidean style and the minimization of symbolic notation." (From the jacket of the book.) Unfortunately, these features make some of the proofs unnecessarily long and arduous. However, this is not a serious drawback since the instructor can (if he so desires) easily introduce the traditional symbolism and rewrite the proofs. The book contains very few misprints and no serious errors. It also contains a sufficient number of examples and exercises, although many of the latter are repetitive.

E. M. BOLGER, Miami University

NOTABLE PAPERS

In the *IEEE Transactions on Engineering Writing and Speech*, Vol. EWS-11 (1968), pp. 17–19, E. M. Scheuer in "The Secretary's Friend, or A Form for the Preparation of Mathematical Expressions for Typing," describes a special printed form (available from CODEX Book Company, Inc., 74 Broadway, Norwood, Massachusetts 02062) that should prove helpful to authors writing copy for typists. The form makes it easy to indicate vertical and horizontal spacing, sizes of expression, and location on the page.

Acknowledgements. The following have generously helped in evaluating books: HUBERT E. CHRESTENSON, JOHN H. HODGES, WENDELL G. JOHNSON, R. KOZELKA, ARTHUR P. MATTUCK, BARRY M. MITCHELL, DAVID C. MURDOCH, LEE A. RUBEL, PETER SCHERK, THOMAS L. SHERMAN, MARVIN SHINBROT, MALCOLM F. SMILEY, RICHARD L. VAN DE WETERING, RICHARD E. WILLIAMSON, PAUL B. YALE, PAUL H. YEAROUT.

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TELEGRAPHIC REVIEWS

Telegraphic reviews are intended to give prompt notice of books with sufficient information to assist in deciding whether to order an examination copy or suggest library purchase. Possible uses are coded as follows: T = textbook, S = supplementary student reading, P = professional reading, TT = teacher training, L = library purchase, 13 to 18 = freshman to second graduate year level, 1 to 4 = one to four semesters. An asterisk is used for emphasis. Books covering standard high school material are called "remedial." All textbooks are examined carefully, and mention is made of noteworthy features that are not evident from the title and coding. Publishers are indicated by the standard abbreviations used in *Books in Print* (which gives full names and addresses). P = paperback.

ANALYSIS, T(17-18), S *P, *L. *Critical Point Theory in Global Analysis and Differential Topology. An Introduction.* Marston Morse, Stewart S. Cairns. Academic, 1969. 297 p. $18. "Mathematicians are finding that the study of global analysis for differential topology requires a knowledge not only of the separate techniques of analysis, differential geometry, topology, and algebra, but also a deeper understanding of how these fields can join forces."

COMPLEX ANALYSIS, *T(17-18) *S, *P, *L. *Function Theory in Polydiscs.* Walter Rudin. W. A. Benjamin, 1969. 188 p. $12.50, $3.95 P. Lecture notes on "the new phenomena that occur when the theory of holomorphic functions in the unit disk (in one variable) is carried over to several variables." Assumes one variable theory, Lebesgue integration, some functional analysis. Exercises, problems, open questions.

COMPUTER, P, L. *Simulation Programming Languages.* Ed: J. N. Buxton. North-Holland, 1968. 471 p. $18. Proceedings of an International Federation for Information Processing Conference in 1967. Broad coverage including related topics.

DIFFERENTIAL EQUATIONS, T(17), S, P, L. *Partial Differential Equations.* Avner Friedman. HR & W, 1969. 262 p. $10.95. Assumes Lebesgue integration, Banach spaces but no PDE. Elliptic equations, evolution equations, selected topics. Recent developments. *Bibliography.*

EDUCATION, S, *P, L. *Mathematical Concepts of Elementary Measurement.* A. L. Blakers. SMSG Studies in Math. 17 (1967). 420 p. $4 P, from Vromans, 2085 E. Foothill Blvd, Pasadena, Cal. Written with high school and elementary mathematics teachers in mind, but of considerably broader interest.

EDUCATION, TT, *P, *L. *Soviet Studies in the Psychology of Learning and Teaching Mathematics.* Ed: Jeremy Kilpatrick, Izaak Wirszup. Vol. 1. *The Learning of Mathematical Concepts. Vol. 2. The Structure of Mathematical Abilities.* School Mathematics Study Group, Stanford Univ. and Survey of Recent East European Mathematical Literature, Univ. of Chicago, 1969. 225 p. 133 p. $2 per vol, from Vromans. Selected translations "from the extensive Soviet literature in the past 25 years on research in the psychology of mathematical instruction."

FOUNDATIONS, *S, *P, L. *Simplified Independence Proofs. Boolean Valued Models of Set Theory.* J. Barkley Rosser. Academic, 1969. 232 p.

$10. An exposition of what appears to be an extraordinarily interesting simplification of methods of proving independence. Among interesting results: "even if one adjoins the axiom of choice to the axioms of set theory, one still cannot write down an explicit formula by means of which the real numbers can be well ordered."

FOUNDATIONS, S, *P, *L. *Logic, Methodology and Philosophy of Science III. Proceedings of the Third International Congress for Logic, Methodology and Philosophy of Science, Amsterdam, 1967.* Ed: B. Van Rootselaar and J. F. Staal. North-Holland, 1968. 565 p. $22.50. Mostly on foundations and philosophy of mathematics.

GEOMETRY, T(14-15; 1-2), TT. *College Geometry.* David C. Kay. HR & W, 1969. 383 p. $9.75. Not traditional "college geometry." Topics include famous theorems (25 p.), foundations, absolute geometry, non-Euclidean geometries, development of geometry from models (154 p.)

HISTORY, GEOMETRY, S, P, *L. *Le Axiome de Paralleles de Euclides a Hilbert. Un probleme Cardinal in le Evolution del Geometrie. Excerptes in facsimile ex le principal ovres original e traduction in le lingue international auxiliari Interlingue.* Introduction e commentarie de C. E. Sjostedt. Interlingue-Fundation, Uppsala, 1968. 981 p. You have just read some Interlingue. The facsimile reproductions and translations of twenty authors from Euclid to Hilbert and Einstein are beautifully printed.

LINEAR ALGEBRA, T(13-14; 1-2), S. *An Introduction to Linear Algebra.* A. Mary Tropper. Am Elsevier, 1969. 151 p. $6.50. An elementary, informal treatment beginning with groups and fields, ending with orthogonal and unitary transformations, quadratic and hermitian forms, simultaneous reduction of two forms. A possibility for the linear algebra portion of calculus.

MECHANICS, P, L. *Creep Problems in Structural Members.* Yu. N. Rabotnov. Trans. ed: F. A. Leckie. North-Holland & Wiley, 1969. 836 p. $33.60. Creep theory is concerned with time dependent relationships between stress and strain. The author is on the Univ. of Moscow faculty of mathematics and mechanics.

NUMBER THEORY, P, *L. *Number Theory and Analysis. A Collection of Papers in Honor of Edmund Landau (1877-1938).* Ed: Paul Turan. Plenum Pub, 1969. 355 p. $19.50. Twenty two papers by distinguished practitioners, portrait, bibliography of Landau's publications, but regretably no biographical article.

NUMERICAL ANALYSIS, S(17), P, L. *Constructive Aspects of the Fundamental Theorem of Algebra.* Ed: Bruno Dejon, Peter Henrici. Proc. of a Symposium at IBM Research Lab. Zurich-Ruschlikon, Switzerland, June 5-7 1967. Wiley, 1969. 344 p. $9.95. Articles by 21 specialists make up "the first book devoted exclusively to the problem of determining all zeros of a given polynomial to arbitrary accuracy."

OPTIMIZATION, T(15-17; 1), S, P, L. *Nonlinear Programming.* Olvi L. Mangasarian. McGraw, 1969. 232 p. $12.50. Assumes only advanced calculus. Finite number of variables and constraints.

PHYSICS, P, *L. *Problems of Hydrodynamics and Continuum Mechanics.* Contributions in honor of the sixtieth birthday of I. Sedov. SIAM, 1969. 826 p. $24.50. Eighty eight authors from twelve countries. Biography and *bibliography.*

COMPUTER, T(ELEM), S. *A Course on Programming in FORTRAN IV.* V. J. Calderbank. Chapman Hall (B&N in US), 1969. 94p. $4, $2.50 P.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Bradley University: Messrs. J. S. Haverhals and G. R. Tippett have been promoted to Assistant Professors.

University of Georgia: Dr. T. J. Cooney, University of Illinois, has been appointed Assistant Professor; Professor John Wagner, Michigan State University, has been appointed Visiting Professor for the winter quarter, 1970.

Hofstra University: Associate Professor Robert Bumcrot has been appointed Chairman of the Mathematics Department; Associate Professor Harry Siller has been promoted to Professor; Assistant Professors Stanley Kertzner and Alexander Weiner have been promoted to Associate Professors.

Marquette University: Associate Professor W. E. Lawrence has been promoted to Professor; Assistant Professor K. W. Weston, University of Notre Dame, has been appointed Associate Professor; Drs. M. Deshpande, University of Wisconsin, Milwaukee, and J. D. Harris, University of Kansas, have been appointed Assistant Professors.

St. John's University, Jamaica, New York: Dr. M. F. Capobianco has been appointed Chairman of the Mathematics Department; Dr. R. C. Morgan has been promoted to Professor; Drs. Edward Miranda and J. B. Frechen have been promoted to Associate Professors.

Wells College: Professor J. M. Perry has been appointed Dean of the College; Assistant Professor Ray Shiflett has been appointed Chairman of the Mathematics Department.

Professor M. A. A. Al-Bassam, Head of the Department of Mathematics, University of Baghdad, has been appointed Professor at the University of Kuwait.

Mr. D. A. Breault has resigned as Staff Analyst at the Harvard Computing Center to accept a position as Senior Analyst with Cyber, Inc. of Cambridge, Massachusetts.

Associate Professor L. M. Collister, William Rainey Harper College, has been appointed Chairman, Division of Mathematics and Physical Sciences.

Dr. A. F. Gilman III, College of the Virgin Islands, has been appointed Professor and Assistant Vice President for Academic Affairs at Western Carolina University.

Associate Professor H. W. Gould, West Virginia University, has been promoted to Professor.

Professor Bertram Mond, La Trobe University, Melbourne, has been appointed Editor of the Journal of the Australian Mathematical Society.

Professor Emeritus P. L. Armstrong, Clemson University, died on November 23, 1968. He was a member of the Association for seventeen years.

A REPORT ON THE DUBINSKY AFFAIR

N. D. KAZARINOFF, University of Michigan

Edward L. Dubinsky received his Ph.D. in 1962 from the University of Michigan. His thesis received the Sumner B. Myers Award. He was Lecturer at the University College of Sierra Leone in 1962–63 and at the University of Ghana in 1963–64. He joined the Tulane Mathematics Department as Assistant

Professor in 1964, and was promoted to Associate Professor in 1968, and received tenure July 1, 1969. On August 25, 1969, Prof. Dubinsky received a letter from the Tulane Board of Administrators informing him of his dismissal as of September 4, 1969 with one year's severance pay. (*Added in proof:* The Tulane administration regards the case as closed. Professor Dubinsky has requested an AAUP investigation.)

At the Summer Meetings of the AMS and MAA in Eugene, Oregon, Prof. Dubinsky's case received considerable attention. What happened at Tulane and at Eugene I shall describe briefly in this report. The events are of importance, I believe, and the description conveys that, but it cannot convey the drama of what happened in New Orleans and Eugene. The meetings buzzed with talk of the case, resolutions, and possible censure actions. This past spring, Tulane was shaken by demonstrations that scandalized a portion of the Tulane community including the president's wife and ROTC officers.

THE DEMONSTRATIONS: There were three, on April 24, April 29, and May 6. Students and a few professors marched about the ROTC drill ground situated at the center of the drill field, and were charged with entering the actual drill area and physically interfering with the drill. The last protest was at an ROTC awards assembly at which protestors booed and shouted.

THE CHARGES: The University Senate's Committee on Academic Freedom, Tenure, and Responsibility served as the Hearing Committee. It held hearings on May 29, 30, and 31 on charges preferred by the ROTC, Dubinsky's Dean, and the President, Herbert E. Longenecker.

CHARGE 1: "That on April 24, 1969 you interfered with a scheduled AFROTC drill period on the University Center Quadrangle by physically entering into the ROTC formations, which action coupled with similar action by others, caused a cancellation of the drill."

CHARGE 2: The same on April 29 "despite prior warnings to remain clear of the area, and you continued such interference in the face of orders to move, forcing your physical removal by members of the Tulane Security Police."

CHARGE 3: "That on May 6, 1969 you interfered with the regularly scheduled drill period and Awards Ceremony of the Combined Army, A.F., and Naval ROTC at McAllister Auditorium by shouting, booing, and otherwise vocally disturbing the proceedings."

THE FINDINGS: There were a majority of eight and a minority of one reporting on June 4th and 3rd, respectively. The majority found that: as to Charge 1, Prof. Dubinsky did interfere with the drill, but his asserted physical entry into the ROTC formations, etc., was not proved; as to Charge 2, the same as found under Charge 1; as to Charge 3, the charge was unproved. The minority report sustained all the charges.

THE RECOMMENDATIONS: "In light of the above findings, the majority of the Committee concluded that Associate Professor Dubinsky's actions were a serious violation of University policy but that these actions were not sufficient to warrant dismissal at this time. Therefore the majority recommends the following:

cott of Tulane by mathematicians and the Society. The business meeting voted to table the placing of a version of the M.A.G. motion on the agenda for the January 1970 San Antonio meeting by a count of 112–88.

Prof. Dubinsky is spending the academic year at McMaster University in Hamilton, Ontario.

August 29, 1969,
Eugene, Oregon

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FIFTIETH SUMMER MEETING OF THE ASSOCIATION

The Fiftieth Summer Meeting of the Mathematical Association of America was held at the University of Oregon, Eugene, Oregon, from Monday, August 25, to Wednesday, August 27, 1969, in conjunction with meetings of the American Mathematical Society, the Society for Industrial and Applied Mathematics, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta. There were registered 1398 persons including 588 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning, and on Wednesday afternoon. All sessions were held in the ballroom of the Erb Memorial Union of the University of Oregon. Presiding officers at the three Earle Raymond Hedrick Lectures were President G. S. Young, First Vice-President Victor Klee, and Second Vice-President S. A. Jennings; at the lecture by Professor Halmos, Professor C. B. Allendoerfer; at the lecture by Professor Forsythe, Professor A. T. Lonseth; at the lecture by Professor Klee, Professor G. D. Chakerian; at the lecture by Professor Feynman, Professor R. S. Pierce. The Panel Discussion on Mathematics in Two-Year Colleges was introduced by Professor J. W. Jewett. The eighteenth series of Earle Raymond Hedrick Lectures was delivered by Professor E. A. Bishop of the University of California, San Diego. The Program Committee consisted of Ivan Niven, Chairman; C. B. Allendoerfer, Ronald Harrop, A. T. Lonseth, and R. S. Pierce.

FIRST SESSION OF THE ASSOCIATION

Welcome on behalf of the University by Dean of the Faculties C. T. Duncan of the University of Oregon.

The Earle Raymond Hedrick Lectures: *The Constructive Point of View*, Lecture I, by Professor E. A. Bishop, University of California, San Diego.

Constructivism involves the development of mathematics as a tool for the discovery and communication of meaningful numerical information. Brouwer showed that classical mathematics and its logic are defective for this purpose. That it has been possible to redo much of abstract analysis constructively, without sacrificing the basic structures and results, indicates the constructivist program has good chances of success. Constructivism has historically been impeded by the imposition of restrictive dogmas: for example, Brouwer's intuitionism contains such dogmas, and the restric-

tion to recursive functions is such a dogma. The realist's dogma, that all such dogmas be eschewed, seems most fruitful for the constructivist program.

Finite-Dimensional Hilbert Spaces, by Professor P. R. Halmos, Indiana University.

Operator theory keeps shedding light on more and more new aspects of linear algebra and succeeds in keeping that classical subject alive and exciting. The purpose of this report was to illustrate the point by describing three nontrivial parts of finite-dimensional linear algebra, the original impetus for which came from infinite-dimensional Hilbert space. The subjects are (1) an algebraic characterization of pairs of subspaces of a finite-dimensional Hilbert space, (2) a geometric characterization of linear transformations in terms of rotations (dilation theory), and (3) a statement of some fragmentary results and challenging open problems about lattices of invariant subspaces.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Bishop.

Pitfalls in Automatic Computation, or Why a Math Book Isn't Enough, by Professor G. E. Forsythe, Stanford University.

Except for a minority interested in becoming mathematicians, students take mathematics to learn to apply it in their own fields of interest. Such application of mathematics is most likely to occur on a digital computer. The speaker examined some algorithms explicitly or implicitly given in pure mathematics courses (e.g., quadratic formula, solution of linear systems, integration, solving differential equations), and showed the pitfalls of carrying them out on a digital computer. The purpose of numerical analysis courses is to discuss explicit algorithms without these pitfalls. Sometimes (not always) these are only small modifications of methods of pure mathematics, and could easily have been included in pure mathematics books. In the long run is it wise or healthy for mathematics education to be split into the conceptual and the applicable?

Panel Discussion on Assistance to Developing Colleges

A panel discussion with Professor L. L. Clarkson, Texas Southern University, Professor A. J. Scavella, Tuskegee Institute, and Professor W. E. Marsh, Talladega College, moderated by Professor F. M. Stewart, Brown University and Tougaloo College.

Professor Clarkson gave an overview of some of the major problems in the undergraduate mathematics programs in developing institutions, with particular emphasis on the so-called Black institutions. He then made a set of recommendations to the Association which he had discussed with some of his colleagues, namely that:

1. it call a series of two-day regional conferences for the specific purpose of cataloguing the problems and programs of Black institutions, including special programs such as those of ISE and The Southern Education Foundation,

2. that it appoint an *ad hoc* curriculum committee to recommend to developing institutions meaningful alternatives to the current CUPM undergraduate curriculum recommendations,

3. that it adopt as policy a motion that states that adequate representation from Black institutions, and indeed from developing institutions, be included in initial planning sessions and subsequent sessions when important issues concerning the obligations of the Association are to be considered,

4. that it use its power to influence Congressmen, funding agencies, etc., to include representation from Black institutions in the initial sessions where legislation and guidelines for various programs concerning mathematics education are being considered,

5. that it survey present efforts for training the next generation of Black mathematics professors and then appoint an *ad hoc* committee with the specific charge of recommending to government, industry, funding agencies, etc. new ways to get more young Blacks involved,

6. that it research the needs in developing institutions concerning a variety of instruction related problems, such as providing adequate graduate faculty when necessary, graduate fellowships when necessary, staff and student recruitment, travel expenses for professional activities, upgrading libraries, student assistants, visiting professors and lecturers, development of applied

mathematics offerings, computers in education, precollege centers, curriculum development and other professional activities, research in learning and teaching mathematics, proper physical facilities, research instructorships, etc.

7. that it do what it can to stop the negative publicity the developing institutions, particularly the Black institutions, are getting; instead announce that these institutions are valuable assets to the national education network, that they do serve the communities, that they do conduct some basic research, that they do serve to provide educational opportunities to a significant segment of the population who would otherwise be denied such an opportunity and that they do turn out educated citizens. What more can you ask?

Professor Scavella reported on the mathematics curriculum in Black colleges. Most Black colleges have the awesome task of offering a mathematics curriculum for students generally poorly prepared, and with teaching faculties lacking in quantity and quality. Recognizing the key role that these colleges must continue to assume in the production of Black mathematicians, the Ford Foundation sponsored a Conference on Mathematics Curriculum last April at Morgan State College, with the view toward improving the individual mathematics programs at these institutions.

The speaker then recommended that (1) the MAA plan a followup to the Ford Conference, (2) membership on MAA committees include professors from Black colleges, (3) universities continue intensive recruitment of mathematics graduates from Black colleges.

Professor Marsh commented on two problems of a mathematician whose first teaching position is at a developing college. The first is that, unless he has studied at a state university, he is unlikely to have had experience with remedial or compensatory mathematics courses; nor is he likely to know which of the many texts proposed for such courses are considered good. A booklet like the CUPM *Basic Library List* or *Basic Curriculum* which gives a reasonably detailed list of necessary and recommended topics and texts would be very valuable. More extensively, comments on teaching methods appropriate to such courses would be valuable to a mathematician whose only recent course experience was in advanced courses, where motivation, basic competence, and self-confidence were not problems.

The second problem is that the isolation of many developing colleges, along with heavy teaching loads and the extra work on exposition needed by a new teacher, leaves little time for study and research. In his opinion, the developing colleges will employ a fair number of new teachers on a short-time basis, at least for the next few years; also it has been his experience that teachers who have finished their programs and are willing to spend two or three years at a developing college are much more effective than graduate students who take a year off to teach there. Thus, to help attract strong people to these colleges for an extended stay, larger colleges and universities might consider making one-year deferred appointments to persons going to these colleges. With such an appointment, a young mathematician would be assured of a chance to catch up in mathematics and to look for another job from an established school with the usual contacts.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Bishop.

Business Meeting of the Association; presentation of Lester R. Ford Awards.

What is Convexity? by Professor Victor Klee, University of Washington.

By a treatment that is somewhat impressionistic in nature, the speaker attempted to convey a picture of the present state of convexity theory, of its relevance to other parts of mathematics and to various areas of science and technology, and of the present limitations of the theory. The qualitative and combinatorial parts of the theory were emphasized.

FOURTH SESSION OF THE ASSOCIATION

The Application of Mathematics to Mathematics, by Professor R. P. Feynman, California Institute of Technology.

Before we prove or disprove a mathematical conjecture can we assign to it a certain probability of its being true? Problems arising in definition and calculation of such probabilities were discussed

informally. One important question which arises is this: Can we find a measure of minimum "distance" between a theorem and the axioms from which it is deduced, so that, for example, we can conclude that a given complicated proof of the theorem does not contain many more steps than are necessary, and that no appreciably shorter proof can ever be found.

Panel Discussion on Mathematics in Two-Year Colleges

A Nonmathematician Looks at Junior College Mathematics, by Professor J. F. Ellis, Simon Fraser University.

The mathematics taught in junior colleges is the result of confused and conflicting objectives externally imposed. Seldom, if ever, is mathematics taught for its own sake: as an area of pleasurable and legitimate inquiry which can yield insight into life, its meaning, its quality and its beauty.

Current approaches imply that mathematics should be studied only by the potential Ph.D.-mathematician or by the technologist who has mechanical problems to solve. This leaves the vast majority of the population either mathematically illiterate or frustrated by failure in studying mathematics or both.

The question is: Is it possible to teach mathematics in such a way as to yield both competence in process and awareness of relationships in an increasingly quantitative world?

A Comprehensive Community College Mathematics Program, by Professor J. C. Knutson, Portland Community College.

The "open door" policy of a metropolitan community college invites many students whose needs are as varied as their mathematical abilities. The mathematics program in such an institution must try to meet the needs of all these students. Mathematics courses are needed for basic education, numerous vocational courses, various engineering technology programs, and transfer mathematics to four-year institutions. The greatest demand by students is for courses in basic arithmetic and elementary algebra with applications in numerous fields. The mathematics department of a metropolitan "open door" community college must be continually revising its philosophy and curriculum to meet all of the needs of all the students if we wish to keep the "open door" policy from becoming a "revolving door" policy. The two- and three-week module approach to teaching mathematics is one technique that does have possibilities of keeping the "open door" philosophy "open".

What Is a Successful College Program?, by Professor G. O. Roberts, Shoreline Community College, Seattle.

A program's success is often judged by its output in terms of application, or production of mathematics, criteria not readily applicable to the community college. More appropriate is the achievement of each student relative to his ability and the assessment of self relative to mathematics.

The essential ingredients in a successful program are the engagement of staff with attention given to preparation, philosophy, and personality in that order, but in inverse order of importance, and the provision of a climate within which that staff can achieve its maximum potential.

This was followed by a general discussion by the panel and the audience, moderated by Professor J. F. Ellis.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the ballroom of the Erb Memorial Union on Sunday, Monday, and Tuesday, beginning at 7:00 P.M. The following films not previously shown at an Association meeting were shown on Sunday:

Allendoerfer Films on Arithmetic and Set Theory (animated and in color)

7:00–7:10 P.M. BINARY OPERATIONS AND THE COMMUTATIVE PROPERTY
7:11–7:19 P.M. ASSOCIATIVE PROPERTY
7:20–7:30 P.M. DISTRIBUTIVE PROPERTY

Films of the University of Illinois Arithmetic Project

7:45–8:29 P.M. SOME ARTIFICIAL OPERATIONS, a fourth grade taught by Miss Phyllis Klein
8:40–9:07 P.M. GRAPHING WITH SQUARE BRACKETS, a fifth grade with Professor David A. Page

Monday evening was devoted to films of the MAA MATHEMATICS TODAY Series and Tuesday evening to films made by Bruce and Katharine Cornwell, pioneers in the making of animated mathematical films.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Sunday at 9:00 A.M. in Room 101 of the Erb Memorial Union with thirty-five members present. Among the items of business transacted were the following:

The Board reelected Professor H. L. Alder of the University of California, Davis, as Secretary of the Association for the five-year term 1970–74.

The Board voted to invite Professor Harry Kesten of Cornell University to deliver the nineteenth series of Earle Raymond Hedrick Lectures at the 1970 summer meeting.

The Board approved an increased annual budget of $3,000 to be available to the Sections which, although they already collect fees at Section meetings, are still in need of additional funds to improve Section activities. Such Sections can apply for these funds in an amount not to exceed $300 per Section per year by writing to the Executive Director, justifying the need for these additional funds and describing in some detail the purposes for which they are to be used. Requests of Sections for these funds will be granted on approval by the Treasurer, the Executive Director, and the Chairman of the Committee on Sections, up to the limit of $3,000 budgeted for this purpose annually.

The Board approved that three types of contributing memberships be established in the Association and also that any member of the Association may become a Life Member after he has passed the age of 60 by paying a lump sum of $150. For details, see the October issue of this MONTHLY (page 981).

The Board approved the recommendation of its Committee on the Role of the Two-Year College Mathematics Teachers in the Association that a major effort be made to involve the two-year college teachers in the activities of the Association. Specifically, the Board voted that the Association give continued encouragement to the proposed junior college mathematics journal to be published by Prindle, Weber and Schmidt. This journal, which will begin publication in the spring of 1970 will be curriculum and pedagogically oriented. The MAA will assist this journal by appointing two members of the six-man Editorial Board and by cooperating with the publishers in finding other members of the Board.

The Board approved submission to NSF of a proposal by CUPM for continuation of its work for the two-year period beginning July 1, 1970. This 129-page document envisions intensive efforts by CUPM in several new areas, as well as continuation of some, although not all, of the existing activities.

The Board also approved a proposal prepared by the Committee on Secondary School Lecturers for support of a Secondary School Lecturer Program to be conducted on a modest scale by the Association. The proposal suggests that such a national program might provide funds during its first year of operation for lecturers in each of eight Sections of the Association. Hopefully, the scope of the program might be enlarged in subsequent years. The proposal suggests further that contributions from some of the schools visited would serve to augment any subvention from a foundation or other sources.

The Board approved the following schedule of future meetings of the Association: Miami, Florida, January 24–26, 1970 (by subsequent vote of the Board of Governors by mail ballot, this meeting will now be held in San Antonio, Texas, January 24–26, 1970); University of Wyoming, August 24–26, 1970, Atlantic City, New Jersey, January 23–25,

1971; Las Vegas, Nevada, January 19–21, 1972; Dallas, Texas, January 27–29, 1973; San Francisco, California, January 17–19, 1974; Washington, D.C., January 1975.

The Board approved holding joint meetings with NCTM at the time of their Annual Meetings in April in even-numbered years, beginning in 1970. The joint MAA-NCTM meetings at the time of the Annual Meetings of the MAA in January in odd-numbered years will be continued.

The Executive Director reported the membership of the Association as 17,865 individual members, an increase of 1.1 per cent over the comparable figure for the preceding year, 3 corporate members, and 250 academic members.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Young presiding.

The fifth set of Lester R. Ford Awards was presented by President Young to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE in 1968. The Awards, in the amount of $100 each, were presented for six articles (for further details on these Awards, see the September issue of this MONTHLY, pages 864–65).

The Secretary then reported on some of the actions taken by the Board of Governors on Sunday. He reported on some of the new activities which CUPM has initiated, specifically that it has set up an *ad hoc* Committee on Special Problems of Minority Groups. This group has decided that, for the moment, its main thrust should be in the direction of increasing the number of Blacks in the mathematical sciences. The immediate goal is not that of increasing the number of Black Ph.D.'s in the mathematical sciences, although this is a longer range goal. Instead, it is to attract more young Black students to the profession, making them aware of the careers and possibilities at all levels—programmers, mathematical technicians, teachers, etc. As has been pointed out, a major obstacle has been a cultural tradition in the Black community against going into the sciences and mathematics. The first goal is to attenuate this tradition by making Black youth aware of the variety and nature of opportunities open to them in the mathematical sciences.

CUPM has also set up an *ad hoc* Group on Problems of College Teaching. This was set up as a result of opinions expressed by several members of CUPM that the quality of college teaching in mathematics has deteriorated in recent years. It is hoped that the *ad hoc* Group on Problems of College Teaching will arrive at a consensus on what might be done toward steering the teaching of college mathematics in more fruitful directions. The first task of the committee will be to identify particular aspects of the problem. For example, does too much emphasis on an axiomatic approach create a false impression of what real mathematics is? Does the omission in much of present day teaching of the connections of a mathematical subject with other subjects either in science or mathematics mitigate against balanced learning? Is there in fact too much formal teaching and slavish adherence to syllabi and textbooks, so that students do not get a sense of purposeful involvement in mathematics?

The Secretary announced receipt of a grant from NSF for continued support of the Visiting Lecturers Program for Colleges for 1969–70. He also announced receipt of a bequest from Charles C. Morris in accordance with his will.

On behalf of the Association, the Secretary paid tribute to the excellent facilities and arrangements which had been provided for this meeting by the University of Oregon and its helpful Department of Mathematics. He introduced Professor F. W. Anderson, Chairman of the Committee on Arrangements, as the one deserving the major share of the credit for the success of the meeting. The audience expressed its appreciation by a round of applause.

A motion made by the Secretary to amend the By-Laws of the Association as printed on pages 581–586 of the May issue of this MONTHLY was approved.

Professor R. L. Wilder called attention to the comfortable foam cushions in the ballroom which had been provided at his own expense by Professor A. F. Moursund, Chairman of the Department of Mathematics at the University of Oregon. He expressed the gratitude of the participants in the meeting for Professor Moursund's thoughtfulness.

Professor K. O. May moved adoption of the following resolution:

"Whereas Edward L. Dubinsky, a tenured Associate Professor of Mathematics at Tulane University, has been dismissed after a faculty committee investigated the charges at the request of the President and recommended Professor Dubinsky's retention.

Therefore be it resolved that the business meeting of the Mathematical Association of America in Eugene, Oregon, on 26 August 1969, deplores the decision of the Board of Administrators of Tulane University to go against the advice of its faculty committee and urges reconsideration of this action.

And further be it resolved that the Secretary send by telegram a copy of this resolution to the President and Board of Administrators of Tulane University."

To inform the reader on the background concerning this motion, a report on the facts of the case, as gathered by Professor N. D. Kazarinoff, is given on pages 1173–1176 of this MONTHLY.

President Young then requested First Vice-President Klee to take the chair due to the fact that Professor Dubinsky is a member of the Department of Mathematics at Tulane University and also a close friend.

With Professor Klee in the chair, Professor May explained that the only issue in the resolution just presented was the one of appropriate procedures. He noted that it simply deplores the actions of administrators going against the judgment of faculty committees.

In answer to a question raised by Professor Moursund whether the case had been brought to the attention of the AAUP, First Vice-President Klee replied that the Tulane chapter of the AAUP is presently considering the case.

In view of this, several speakers favored deferring action until the AAUP had completed its investigation. Professor Chandler Davis then described the AAUP procedures as he saw them and raised the question whether, in the view of the average member of the AAUP, they are satisfactory. He submitted that they are not, and urged action by the MAA at this business meeting.

Professor C. M. Larsen, who had worked in the Washington office of the AAUP, did not feel that the previous speaker had accurately described some of the procedures of the AAUP and made some corrections. He also offered an amendment to the motion of Professor May, which the latter, however, did not accept.

Professor E. F. Beckenbach then stated his support of the statements made by previous speakers regarding proper procedures for accumulating evidence and doing this through proper channels, which he thought surely in this case would be the AAUP. He, therefore, moved to table the resolution.

The tabling resolution carried by a standing vote of 135 in favor of tabling and 92 against.

MEETING OF SECTION OFFICERS

The meeting of representatives of the Sections was held on Monday evening in Room 101 of the Erb Memorial Union. Professor L. E. Mehlenbacher, Chairman of the Committee on Sections, presided. Fifty-nine persons were present, representing twenty-six of the twenty-eight Sections of the Association.

President Young welcomed the Section Officers. He reported that he had visited many Sections in the past year and hoped very much to work out better means of aiding

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the Sections. He felt that many of the problems facing the MAA are best handled at the Section level. The strengthening of the work of the Sections, he felt, will become more and more urgent in the next few years.

Dr. Willcox, Executive Director of the Association, extended an invitation to members of the MAA to visit the Washington office when they are in the city or to write or telephone whenever his office can be of assistance in providing information on the Washington scene. He emphasized that, as the person responsible for administrative liaison with the Sections, he has a vital interest in strengthening the ties between the national organization and the Sections. In order to become better acquainted with Section Officers and members, he plans to visit as many Section meetings as he can each year, hoping to visit all Sections over a period of several years. He hopes that this will enable the Association office to be more responsive to the special needs of the Sections.

He reminded the Section Officers of the "Plan of Visits by National Officers to Section Meetings," under which national officers are available as speakers at Section meetings with travel expenses paid by the national office of the Association. Invitations should be sent directly to the officer (President, President-Elect, First and Second Vice-Presidents, Secretary, Treasurer, Editor of the MONTHLY, Executive Director) with copy to Secretary Alder. If the Section has no particular choice of an officer as a speaker, Secretary Alder will be happy to make the arrangements.

Professor H. L. Alder, Secretary of the Association, led a discussion on "How One Can Organize a Good Section Meeting." He began by presenting sixteen suggestions of his own. This was followed by numerous recommendations by the attending Section Officers. The Secretary announced that all useful suggestions received would be assembled in a small brochure of two or three pages and sent to all current Section Officers and future ones as they are elected.

Professor Harley Flanders, Editor of the MONTHLY, reported on the current status of his efforts to make the MONTHLY more attractive to its readers. He asked for comments and criticisms from those present. Dr. E. C. Posner of the Southern California Section stated his belief that the MONTHLY had vastly improved and would enable the officers of his Section to recruit more members. He added that everyone he had spoken to in his Section was very pleased with the MONTHLY.

Professor S. A. Jennings, Editor of the MATHEMATICS MAGAZINE, reported that most of the articles published in the MAGAZINE so far this year have been carryovers from the previous Editor. He expressed his hope to avoid articles which presume knowledge at the graduate level. His objective was to direct the MAGAZINE to people with a major in mathematics and who have a continuing interest in mathematics.

Professor V. O. McBrien, Chairman of the Committee on Institutes, reported on the work of that committee. In particular, he announced the tentative plans for a 1971 Summer Seminar patterned after the previous Seminars held from 1964 to 1966.

Professor H. M. Bacon, Chairman of the Committee on Secondary School Lecturers, reported on the work of that committee and, in particular, the proposal prepared for support of a Secondary School Lecturer Program to be conducted on a modest scale, nationally, by the Association.

Professor Arnold Wendt, Chairman of the Illinois Section, reported on "Articulation between Junior Colleges and Senior Colleges." The Illinois Section at its 1968 meeting established a Junior College Committee. During 1968–69, this Committee, consisting of junior college teachers, initiated two state-wide meetings dealing with the problem of coordinating mathematics programs in Illinois colleges and universities. As a consequence of these meetings, the Section established the Articulation Committee, which has as its primary responsibility the writing of a handbook incorporating the consensus voiced on various problems at the aforementioned articulation conferences. The Articulation Committee will also consider the problem of testing and placement, especially tests for placement in precollege algebra courses.

on the work of that Committee. The 20th Annual High School Mathematics Contest was held on March 11, 1969, in more than 7000 high schools with about 335,000 participating students. Coordinators abroad administered the test in twelve countries in Europe, Africa, and the Middle East; 20,000 copies of the test were used in England, and translations into several foreign languages were made.

Professor J. L. Tilley, Chairman of the Louisiana-Mississippi Section, reported on the work of that Section to encourage membership in the Association and participation in its Section meetings. During the fall semester, the Louisiana Vice-Chairman plans to visit the new branches of Louisiana State University, as well as several other small colleges in his area, to carry out this objective. Mississippi has been divided in such a way that Dr. Noel Childress, as well as the Mississippi Vice-Chairman and the Section Chairman, will visit almost every junior college in the State and several of the four-year colleges. Each visit will include an hour's lecture to faculty and/or students as well as a personal appeal to each staff member to join the Association and participate in the Section activities. Due to the size of the State, the Section Chairman has reserved for himself six overnight trips to include visits to 3 to 5 colleges on each trip.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. Two sets of Colloquium Lectures, each consisting of four lectures, were presented. Professor Raoul Bott of Harvard University delivered one set entitled "On the Periodicity Theorem of the Classical Groups and its Applications" on Tuesday at 1:30 p.m. and on Wednesday, Thursday, and Friday at 8:30 a.m. Professor Harish-Chandra of the Institute for Advanced Study delivered the other set entitled "Harmonic Analysis on Semisimple Lie Groups" on Tuesday at 2:45 p.m. and on Wednesday, Thursday, and Friday at 9:40 a.m. Invited addresses were given by Professor R. C. Kirby of the University of California, Los Angeles, on Thursday at 1:30 p.m., entitled "On the Existence and Uniqueness of Triangulations of Manifolds," by Professor Murray Gerstenhaber of the University of Pennsylvania on Thursday at 2:45 p.m., entitled "Algebraic Deformation Theory," and by Professor P. F. Baum of Brown University on Friday at 1:30 p.m. entitled "Vector Fields and Gauss-Bonnet."

The Society for Industrial and Applied Mathematics presented the John von Neumann Lecture on Wednesday at 8:00 p.m. It was delivered by Professor George Carrier of Harvard University on "Singular Perturbation Theory and Geophysics."

The Pi Mu Epsilon Fraternity held sessions for contributed papers on Tuesday at 3:15 p.m. and on Wednesday at 10:40 a.m. in Room 150 of the Science Building. A banquet was held Tuesday at 6:30 p.m. in Rooms 213 and 214 of the Erb Memorial Union. At this banquet, Professor C. O. Oakley of Haverford College spoke on "Mathematics for Those Who Won't Use It." A Dutch-treat breakfast meeting for Pi Mu Epsilon members was held on Wednesday at 8:00 a.m. in Rooms 115–116 of the Erb Memorial Union.

The Governing Council of Mu Alpha Theta, the national high school and junior college mathematics club, met on Wednesday at 9:00 a.m. in Room 213 of the Erb Memorial Union.

The National Academy of Sciences-National Academy of Engineering presented a panel discussion on Thursday at 8:00 P.M. in the ballroom of the Erb Memorial Union on "The Use of Hard-won Information and Insight," led by Acting President F. J. Weyl. This was a discussion of the recently published report of the Academies' Joint Committee on Scientific and Technical Communication (SATCOM). Panelists were Professor W. J. LeVeque, Chairman of the Commission on a National Information System in Mathematics, Dr. D. L. Thomsen, Jr., a member of this Commission, and Professor J. L. Hodges, Jr., of the Department of Statistics, University of California, Berkeley.

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ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting at the University of Oregon, Eugene, Oregon, on August 24, 1969, elected to membership the sixteenth set of applicants for academic membership (for election of the other fifteen sets, see the April and November issues for 1962–69). Approval for election was given to the following two applicants for academic membership:

Maine Maritime Academy, Castine, Maine
University of Santa Clara, Santa Clara, California

HENRY L. ALDER, *Secretary*

MARCH MEETING OF THE FLORIDA SECTION

The annual Spring meeting of the Florida Section of the MAA was held on March 21–22, 1969, at Florida Atlantic University in Boca Raton. There were 184 persons registered of whom 151 were members of the Association. In conjunction with the Section meeting, a Mathematics Articulation Conference was held devoted to a discussion of ways of improving communication and cooperation between the junior colleges and universities of Florida.

The following papers were presented:

1. *On Banach spaces whose quotients in their biduals are separable*, by R. D. McWilliams, Florida State University.
2. *Coercive inequalities for sectors in the plane*, by J. M. Newman, Florida Atlantic Uuiversity.
3. *The basic inclusion-exclusion principle*, by J. M. Freeman, Florida Atlantic University.
4. *A linearization of Lipschitz homeomorphisms*, by L. Janos, University of Florida.
5. *Order characterizations of unconditional bases*, by C. W. McArthur, Florida State University.
6. *A remark on subdomains in a subfield*, by E. Magarian*, Stetson University and J. Mott, Florida State University.
7. *Extensions of torsion free groups by torsion groups*, by H. Simon, University of Miami.
8. *On Prufer subrings of an integral domain*, by J. Mott, Florida State University (introduced by H. E. Taylor).
9. *An optimization approach to cluster analysis*, by R. L. Patterson and M. Padron*, University of Florida.
10. *A construction for room squares*, by R. C. Mullin, Florida Atlantic University.
11. *On the existence of room squares*, by E. Nemuth, Florida Atlantic University (introduced by R. C. Mullin).
12. *Computer related instruction in mathematics departments of U. S. Colleges and Universities*, by E. P. Miles, Florida State University.
13. *The derivative of the Sin x and Cos x, a possible pedagogical approach*, by S. G. Sadler, University of Florida.
14. *Some approximate solutions of the heat equation based on the method of lines*, by A. Zafarrulah, Florida State University (introduced by D. B. Goodner).
15. *Impure algorithms*, by A. F. Hunter, Florida Presbyterian College (introduced by R. C. Meacham).
16. *The Klein four group as a Z-lie algebra L and as an L-module*, by P. C. Morris, Florida State University.
17. *A new definition of a reduced form*, by E. H. Hadlock* and T. O. Moore, University of Florida.
18. *Linear programming for General Education? Sure!* by Otis Ulm, St. Petersburg Junior College.
19. *Pre-college mathematics in the junior college*, by B. Alwin, B. Hackworth, M. Nott, St. Petersburg Junior College (introduced by Bill Rice).

20. *Mathematics laboratories*, by K. Thompson, Florida Junior College at Jacksonville, and T. Watts, Miami-Dade Junior College, South Campus.

Three invited addresses were presented as follows: Infinity in Mathematics, by J. DeGroot, University of Amsterdam and University of Florida; Inequalities for the Derivatives of Polynomials, by R. Boas, Northwestern University; Qualifications for Mathematics Teachers of Undergraduate Mathematics, by John Jewett, Oklahoma State University.

Professor A. R. Bednarek presided at the business meeting held following a luncheon at noon Saturday. The following officers were elected for 1969–70: Chairman, Professor Herman Meyer, University of Miami; Vice-chairman, Professor William Rice, St. Petersburg Junior College; Secretary-Treasurer, Professor Howard E. Taylor, Florida State University.

H. E. TAYLOR, *Secretary*

MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the MAA was held at The University of Michigan, Ann Arbor on March 29, 1969. One hundred twenty-three persons were in attendance.

At the business meeting the following officers were elected: Chairman, Professor A. B. Clarke, Western Michigan University; Vice-Chairman, Professor J. E. Folkert, Hope College; Secretary-Treasurer, Professor H. T. Slaby, Wayne State University. The section voted to honor the five students from institutions in this section who rank highest in the Putnam Competition by awarding them memberships in the MAA. The section also voted to increase the honorarium for the director of the Michigan Mathematics Prize Competition and his assistants to $1,500.

The following papers were presented:

1. *Argand and analogy*, by P. S. Jones, The University of Michigan.
2. *Applications of a principle of duality in asymptotes*, by G. van Zwalenberg, Calvin College.
3. *Topological equivalents of Ulam non-measurability*, by S. Mrowka, Western Michigan University.
4. *Related partial differential equations*, by J. W. Dettman, Oakland University.
5. *An experimental approach to the central limit theorem*, by E. Tanis and Miss Deanna Gross, Hope College.
6. *Looking at special cases—the beginning of mathematical discovery*, by H. S. Shapiro, The University of Michigan.
7. Panel discussion: *Bridging the gap between elementary calculus and concentration mathematics—part II*; moderator, A. B. Clarke, Western Michigan University; panelists, W. V. Caldwell, The University of Michigan (Flint), R. G. Douglas, The University of Michigan, D. W. Hall, Michigan State University, P. C. Shields, Wayne State University, and D. G. Vander Jagt, Grand Valley State College.
8. *Number theory revisited*, by Irving Kaplansky, University of Chicago (by invitation).
9. *Mathematical abstraction*, by C. F. Brumfiel, University of Michigan.
10. *Optimal distance configuration*, by L. M. Kelly, Michigan State University.
11. *Misuses of the 2-Sample t-distribution*, by M. Stoline, Western Michigan University.
12. *Coloring a dodecahedron with four colors*, by D. Ballard, Albion College.

H. T. SLABY, *Secretary-Treasurer*

JUNE MEETING OF THE NORTHEASTERN SECTION

The biennial spring meeting of the Northeastern Section of the MAA was held at Williams College, Williamstown, Massachusetts on June 21, 1969. Forty-five people

attended the meeting, including thirty-eight members of the Association. Professor William Crawford, Chairman of the Section, presided at both the morning and afternoon sessions.

The following program was presented:

1. *Some historical comments on Diophantine problems mod P*, by K. F. Ireland, Brown University.
2. *Some geometric problems in complex analysis*, by T. H. MacGregor, SUNY at Albany.
3. *A conjecture of Artin*, by L. J. Goldstein, Yale University.
4. *Gauss*, by K. O. May, University of Toronto.

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CALENDAR OF FUTURE MEETINGS

Fifty-third Annual Meeting, San Antonio, Texas, January 24–26, 1970.

Fifty-first Summer Meeting, University of Wyoming, Laramie, August 24–26, 1970.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, May 2, 1970.

FLORIDA, Rollins College, Winter Park, March 20–21, 1970.

ILLINOIS, Loyola University, Chicago, May 8–9, 1970.

INDIANA

IOWA, Grinnell College, Grinnell, April 17, 1970.

KANSAS, Kansas State Teachers College, Emporia, March 1970.

KENTUCKY, University of Kentucky, Lexington, Spring 1970.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 20–21, 1970.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Wagner College, Staten Island, Spring 1970.

MICHIGAN, Wayne State University, Detroit, April 4, 1970.

MISSOURI, Central Missouri State College, Warrensburg, May 2, 1970.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 24–25, 1970.

NEW JERSEY

NORTH CENTRAL

NORTHEASTERN

NORTHERN CALIFORNIA, Diablo Valley College, Concord, February 7, 1970.

OHIO, Bowling Green State University, Bowling Green, Spring 1970.

OKLAHOMA-ARKANSAS, Southwestern State College, Weatherford, Oklahoma, March 1970.

PACIFIC NORTHWEST, Pacific Lutheran University, Tacoma, Washington, June 19–22, 1970.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, May 8–9, 1970.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, March 20–21, 1970.

SOUTHERN CALIFORNIA, University of California, Irvine, March 21, 1970.

SOUTHWESTERN, University of Texas at El Paso, March 27–28, 1970.

TEXAS, Sam Houston State College, Huntsville, April 10–11, 1970.

UPPER NEW YORK STATE

WISCONSIN, University of Wisconsin, Waukesha, May 1970.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Chicago, December 26–31, 1970.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1970.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Ohio State University, June 22–25, 1970.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC

CENTRAL ASSOCIATION OF SCIENCE AND MATHEMATICS TEACHERS, Chicago, November 26–28, 1970.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., April 1–4, 1970.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hilton Hotel, Washington, D. C., April 20–22, 1970.

PI MU EPSILON

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of Denver, Colorado, June 1970.

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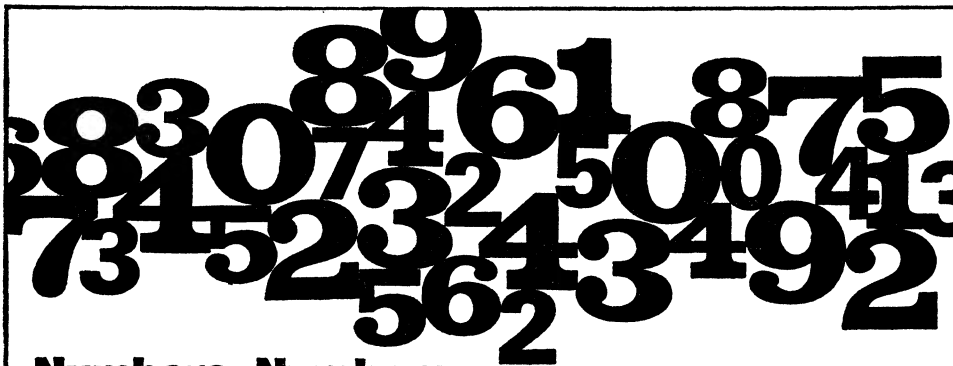

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